Three.II Homomorphisms

Linear Algebra

Jim Hefferon

http://joshua.smcvt.edu/linearalgebra

Definition

Homomorphism

1.1 *Definition* A function between vector spaces $h\colon V \to W$ that preserves addition

$$\text{if } \vec{v}_1, \vec{v}_2 \in V \text{ then } h(\vec{v}_1 + \vec{v}_2) = h(\vec{v}_1) + h(\vec{v}_2)$$

and scalar multiplication

$$\text{if } \vec{v} \in V \text{ and } r \in \mathbb{R} \text{ then } h(r \cdot \vec{v}) = r \cdot h(\vec{v})$$

is a *homomorphism* or *linear map*.

Example Of these two maps $h, g\colon \mathbb{R}^2 \to \mathbb{R}$, the first is a homomorphism while the second is not.

$$\begin{pmatrix} x \\ y \end{pmatrix} \stackrel{h}{\longmapsto} 2x - 3y \qquad \begin{pmatrix} x \\ y \end{pmatrix} \stackrel{g}{\longmapsto} 2x - 3y + 1$$

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The map h respects addition

$$\begin{aligned} h(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}) = h(\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}) &= 2(x_1 + x_2) - 3(y_1 + y_2) \\ &= (2x_1 - 3y_1) + (2x_2 - 3y_2) = h(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}) + h(\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}) \end{aligned}$$

and scalar multiplication.

$$r \cdot h(\begin{pmatrix} x \\ y \end{pmatrix}) = r \cdot (2x - 3y) = 2rx - 3ry = (2r)x - (3r)y = h(r \cdot \begin{pmatrix} x \\ y \end{pmatrix})$$

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In contrast, g does not respect addition.

$$g(\begin{pmatrix} 1 \\ 4 \end{pmatrix} + \begin{pmatrix} 5 \\ 6 \end{pmatrix}) = -17 \qquad g(\begin{pmatrix} 1 \\ 4 \end{pmatrix}) + g(\begin{pmatrix} 5 \\ 6 \end{pmatrix}) = -16$$

We proved these two while studying isomorphisms.

1.6 *Lemma* A homomorphism sends the zero vector to the zero vector.

1.7 *Lemma* The following are equivalent for any map $f\colon V \to W$ between vector spaces.

(1) f is a homomorphism

(2) $f(c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2) = c_1 \cdot f(\vec{v}_1) + c_2 \cdot f(\vec{v}_2)$ for any $c_1, c_2 \in \mathbb{R}$ and $\vec{v}_1, \vec{v}_2 \in V$

(3) $f(c_1 \cdot \vec{v}_1 + \cdots + c_n \cdot \vec{v}_n) = c_1 \cdot f(\vec{v}_1) + \cdots + c_n \cdot f(\vec{v}_n)$ for any $c_1, \ldots, c_n \in \mathbb{R}$ and $\vec{v}_1, \ldots, \vec{v}_n \in V$

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Example Between any two vector spaces the zero map $Z\colon V \to W$ given by $Z(\vec{v}) = \vec{0}_W$ is a linear map. Using (2): $Z(c_1\vec{v}_1 + c_2\vec{v}_2) = \vec{0}_W = \vec{0}_W + \vec{0}_W = c_1 Z(\vec{v}_1) + c_2 Z(\vec{v}_2)$.

Example The *inclusion map* $\iota\colon \mathbb{R}^2 \to \mathbb{R}^3$

$$\iota(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

is a homomorphism.

$$\begin{aligned}
\iota(c_1 \cdot \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + c_2 \cdot \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}) &= \iota(\begin{pmatrix} c_1x_1 + c_2x_2 \\ c_1y_1 + c_2y_2 \end{pmatrix}) \\
&= \begin{pmatrix} c_1x_1 + c_2x_2 \\ c_1y_1 + c_2y_2 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} c_1x_1 \\ c_1y_1 \\ 0 \end{pmatrix} + \begin{pmatrix} c_2x_2 \\ c_2y_2 \\ 0 \end{pmatrix} \\
&= c_1 \cdot \iota(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}) + c_2 \cdot \iota(\begin{pmatrix} x_2 \\ y_2 \end{pmatrix})
\end{aligned}$$

Example The derivative is a transformation on polynomial spaces. For instance, consider $d/dx\colon \mathcal{P}_2 \to \mathcal{P}_1$ given by

$$d/dx\,(ax^2 + bx + c) = 2ax + b$$

(examples are $d/dx\,(3x^2 - 2x + 4) = 6x - 2$ and $d/dx\,(x^2 + 1) = 2x$).

It is a homomorphism.

$$\begin{aligned}
d/dx\,\bigl(r_1(a_1x^2 + b_1x + c_1) + r_2(a_2x^2 + b_2x + c_2)\bigr) \\
&= d/dx\,\bigl((r_1a_1 + r_2a_2)x^2 + (r_1b_1 + r_2b_2)x + (r_1c_1 + r_2c_2)\bigr) \\
&= 2(r_1a_1 + r_2a_2)x + (r_1b_1 + r_2b_2) \\
&= (2r_1a_1x + r_1b_1) + (2r_2a_2x + r_2b_2) \\
&= r_1 \cdot d/dx\,(a_1x^2 + b_1x + c_1) + r_2 \cdot d/dx\,(a_2x^2 + b_2x + c_2)
\end{aligned}$$

Example The *trace* of a square matrix is the sum down the upper-left to lower-right diagonal. Thus $\mathrm{Tr}\colon \mathcal{M}_{2\times 2} \to \mathbb{R}$ is this.

$$\mathrm{Tr}(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = a + d$$

It is linear.

$$\begin{aligned}
\mathrm{Tr}(\, r_1 \cdot \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + r_2 \cdot \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\,) \\
&= \mathrm{Tr}(\begin{pmatrix} r_1 a_1 + r_2 a_2 & r_1 b_1 + r_2 b_2 \\ r_1 c_1 + r_2 c_2 & r_1 d_1 + r_2 d_2 \end{pmatrix}) \\
&= (r_1 a_1 + r_2 a_2) + (r_1 d_1 + r_2 d_2) \\
&= r_1 (a_1 + d_1) + r_2 (a_2 + d_2) \\
&= r_1 \cdot \mathrm{Tr}(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}) + r_2 \cdot \mathrm{Tr}(\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix})
\end{aligned}$$

1.9 *Theorem* A homomorphism is determined by its action on a basis: if V is a vector space with basis $\langle\vec{\beta}_1,\ldots,\vec{\beta}_n\rangle$, if W is a vector space, and if $\vec{w}_1,\ldots,\vec{w}_n\in W$ (these codomain elements need not be distinct) then there exists a homomorphism from V to W sending each $\vec{\beta}_i$ to $\vec{w}_i$, and that homomorphism is unique.

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Example The book has the proof. Here is an illustration. Consider a map $h\colon\mathbb{R}^2\to\mathbb{R}^2$ with this action on a basis.

$$\begin{pmatrix}1\\1\end{pmatrix}\overset{h}{\longmapsto}\begin{pmatrix}1\\3\end{pmatrix}\qquad\begin{pmatrix}1\\0\end{pmatrix}\overset{h}{\longmapsto}\begin{pmatrix}2\\0\end{pmatrix}$$

The effect of the map on any vector $\vec{v}$ at all is determined by those two facts. Represent that vector $\vec{v}$ with respect to the basis.

$$\begin{pmatrix}-1\\5\end{pmatrix}=5\cdot\begin{pmatrix}1\\1\end{pmatrix}-6\cdot\begin{pmatrix}1\\0\end{pmatrix}$$

Compute $h(\vec{v})$ using the definition of homomorphism.

$$h(\vec{v})=h(\,5\cdot\begin{pmatrix}1\\1\end{pmatrix}-6\cdot\begin{pmatrix}1\\0\end{pmatrix}\,)=5\cdot\begin{pmatrix}1\\3\end{pmatrix}-6\cdot\begin{pmatrix}2\\0\end{pmatrix}=\begin{pmatrix}-7\\15\end{pmatrix}$$

Example Consider $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ with this effect on the standard basis.

$$f(\vec{e}_1) = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \quad f(\vec{e}_2) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad f(\vec{e}_3) = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$$

Because this is the standard basis, the effect of the map on any vector $\vec{v} \in \mathbb{R}^3$ is especially easy to compute. For instance,

$$\text{Rep}_{\mathcal{E}_3,\mathcal{E}_3}(\begin{pmatrix} -5 \\ 0 \\ 10 \end{pmatrix}) = \begin{pmatrix} -5 \\ 0 \\ 10 \end{pmatrix}$$

and so we have this.

$$f(\begin{pmatrix} -5 \\ 0 \\ 10 \end{pmatrix}) = -5 \cdot \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 10 \cdot \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 25 \\ -20 \\ 5 \end{pmatrix}$$

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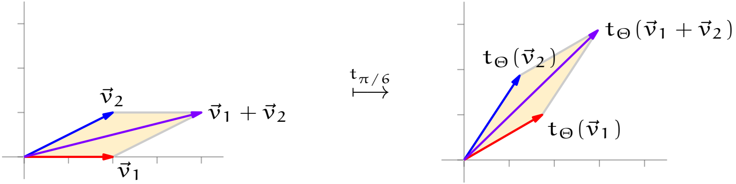
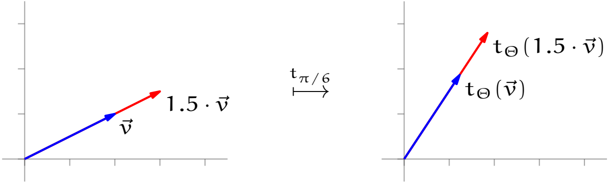
1.10 *Definition* Let V and W be vector spaces and let $B=\langle\vec{\beta}_1,\ldots,\vec{\beta}_n\rangle$ be a basis for V. A function defined on that basis $f\colon B\to W$ is *extended linearly* to a function $\hat{f}\colon V\to W$ if for all $\vec{v}\in V$ such that $\vec{v}=c_1\vec{\beta}_1+\cdots+c_n\vec{\beta}_n$, the action of the map is $\hat{f}(\vec{v})=c_1\cdot f(\vec{\beta}_1)+\cdots+c_n\cdot f(\vec{\beta}_n)$.

Example Consider the action $t_\Theta\colon \mathbb{R}^2 \to \mathbb{R}^2$ of rotating all vectors in the plane through an angle Θ. These drawings show that this map satisfies the addition

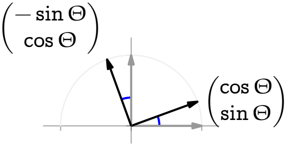


and scalar multiplication conditions.



We will develop the formula for t_Θ.

Fix a basis for the domain $\mathbb{R}^2$; the standard basis $\mathcal{E}_2$ is convenient. We want the basis vectors mapped as here.



$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

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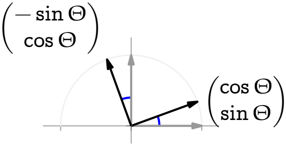

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Extend linearly.

$$\begin{aligned} t_\theta(\begin{pmatrix} x \\ y \end{pmatrix}) &= t_\theta(x \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}) \\ &= x \cdot t_\theta(\begin{pmatrix} 1 \\ 0 \end{pmatrix}) + y \cdot t_\theta(\begin{pmatrix} 0 \\ 1 \end{pmatrix}) \\ &= x \cdot \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} + y \cdot \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \\ &= \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix} \end{aligned}$$

Example One basis of the space of quadratic polynomials $\mathcal{P}_2$ is $B = \langle x^2, x, 1 \rangle$. Define the *evaluation map* $\text{eval}_3 \colon \mathcal{P}_2 \to \mathbb{R}$ by specifying its action on that basis

$$x^2 \overset{\text{eval}_3}{\longmapsto} 9 \quad x \overset{\text{eval}_3}{\longmapsto} 3 \quad 1 \overset{\text{eval}_3}{\longmapsto} 1$$

and then extending linearly.

$$\begin{aligned} \text{eval}_3(ax^2 + bx + c) &= a \cdot \text{eval}_3(x^2) + b \cdot \text{eval}_3(x) + c \cdot \text{eval}_3(1) \\ &= 9a + 3b + c \end{aligned}$$

For instance, $\text{eval}_3(x^2 + 2x + 3) = 9 + 6 + 3 = 18$.

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For instance, $\text{eval}_3(x^2 + 2x + 3) = 9 + 6 + 3 = 18$.

On the basis elements, we can describe the action of this map as: plugging the value 3 in for x. That remains true when we extend linearly, so $\text{eval}_3(\,p(x)\,) = p(3)$.

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Example In $\mathbb{R}^3$ the function f_{yz} that reflects vectors over the yz-plane

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \stackrel{f_{yz}}{\longmapsto} \begin{pmatrix} -x \\ y \\ z \end{pmatrix}$$

is a linear transformation.

$$f_{yz}(r_1 \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + r_2 \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}) = f_{yz}(\begin{pmatrix} r_1x_1 + r_2x_2 \\ r_1y_1 + r_2y_2 \\ r_1z_1 + r_2z_2 \end{pmatrix}) = \begin{pmatrix} -(r_1x_1 + r_2x_2) \\ r_1y_1 + r_2y_2 \\ r_1z_1 + r_2z_2 \end{pmatrix}$$

$$= r_1 \begin{pmatrix} -x_1 \\ y_1 \\ z_1 \end{pmatrix} + r_2 \begin{pmatrix} -x_2 \\ y_2 \\ z_2 \end{pmatrix} = r_1 f_{yz}(\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}) + r_2 f_{yz}(\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix})$$

1.17 *Lemma* For vector spaces V and W, the set of linear functions from V to W is itself a vector space, a subspace of the space of all functions from V to W.

We denote the space of linear maps from V to W by $\mathcal{L}(V, W)$.

The book contains the proof.

Example We can combine the two homomorphisms $f, g\colon \mathcal{P}_1 \to \mathbb{R}^2$

$$f(a_0 + a_1x) = \begin{pmatrix} a_0 + a_1 \\ 0 \end{pmatrix} \qquad g(a_0 + a_1x) = \begin{pmatrix} 4a_1 \\ a_1 \end{pmatrix}$$

into a function $2f + 3g\colon \mathcal{P}_1 \to \mathbb{R}^2$ whose action is this.

$$(2f + 3g)\,(a_0 + a_1x) = \begin{pmatrix} 2a_0 + 14a_1 \\ 3a_1 \end{pmatrix}$$

The point of the lemma is that $2f + 3g$ is also a homomorphism; the check is routine. The collection of homomorphisms from $\mathcal{P}_1$ to $\mathbb{R}^2$ is closed under linear combinations of those homomorphisms — it is a vector space.

Example Consider $\mathcal{L}(\mathbb{R},\mathbb{R}^2)$. A member of $\mathcal{L}(\mathbb{R},\mathbb{R}^2)$ is a linear map. A linear map is determined by its action on a basis of the domain space. Fix these bases.

$$B_{\mathbb{R}} = \mathcal{E}_1 = \langle 1 \rangle \qquad B_{\mathbb{R}^2} = \mathcal{E}_2 = \langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rangle$$

Thus the functions that are elements of $\mathcal{L}(\mathbb{R},\mathbb{R}^2)$ are determined by c_1 and c_2 here.

$$1 \overset{t}{\longmapsto} c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

We could write each such map as $h = h_{c_1,c_2}$. There are two parameters and thus $\mathcal{L}(\mathbb{R},\mathbb{R}^2)$ is a dimension 2 space.

Range space and null space

2.1 *Lemma* Under a homomorphism, the image of any subspace of the domain is a subspace of the codomain. In particular, the image of the entire space, the range of the homomorphism, is a subspace of the codomain.

The book has the proof; we instead consider an example.

Example Let $f\colon \mathbb{R}^2 \to \mathcal{M}_{2\times 2}$ be

$$\begin{pmatrix} a \\ b \end{pmatrix} \stackrel{f}{\longmapsto} \begin{pmatrix} a & a+b \\ 2b & b \end{pmatrix}$$

(the check that it is a homomorphism is routine). One subspace of the domain is the x axis.

$$S = \{\begin{pmatrix} a \\ 0 \end{pmatrix} \mid a \in \mathbb{R}\}$$

The image under f of the x axis is a subspace of of the codomain $\mathcal{M}_{2\times 2}$.

$$f(S) = \{\begin{pmatrix} a & a \\ 0 & 0 \end{pmatrix} \mid a \in \mathbb{R}\}$$

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Another subspace of $\mathbb{R}^2$ is $\mathbb{R}^2$ itself. The image of $\mathbb{R}^2$ under f is this subspace of $\mathcal{M}_{2\times 2}$.

$$f(\mathbb{R}^2) = \{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \cdot c_1 + \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} \cdot c_2 \mid c_1, c_2 \in \mathbb{R} \}$$

Example For any angle θ, the function $t_\theta\colon \mathbb{R}^2 \to \mathbb{R}^2$ that rotates vectors counterclockwise through an angle θ is a homomorphism.

In the domain $\mathbb{R}^2$ each line through the origin is a subspace. The image of that line under this map is another line through the origin, a subspace of the codomain $\mathbb{R}^2$.

Range space

2.2 *Definition* The *range space* of a homomorphism $h\colon V \to W$ is

$$\mathscr{R}(h) = \{h(\vec{v}) \mid \vec{v} \in V\}$$

sometimes denoted $h(V)$. The dimension of the range space is the map's *rank*.

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Example This map from $\mathcal{M}_{2\times 2}$ to $\mathbb{R}^2$ is linear.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \stackrel{h}{\longmapsto} \begin{pmatrix} a+b \\ 2a+2b \end{pmatrix}$$

The range space is a line through the origin.

$$\{\begin{pmatrix} t \\ 2t \end{pmatrix} \mid t \in \mathbb{R}\}$$

Every member of that set is the image of a 2×2 matrix.

$$\begin{pmatrix} t \\ 2t \end{pmatrix} = h(\begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix})$$

The map's rank is 1.

Example The derivative map $d/dx\colon \mathcal{P}_4 \to \mathcal{P}_4$ is linear. Its range is $\mathscr{R}(d/dx) = \mathcal{P}_3$. (Verifying that every member of $\mathcal{P}_3$ is the derivative of some member of $\mathcal{P}_4$ is easy.) The rank of this derivative function is the dimension of $\mathcal{P}_3$, namely 4.

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Example Projection $\pi\colon \mathbb{R}^3 \to \mathbb{R}^2$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \end{pmatrix}$$

is a linear map; the check is routine. The range space is $\mathscr{R}(\pi) = \mathbb{R}^2$ because given a vector $\vec{w} \in \mathbb{R}^2$

$$\vec{w} = \begin{pmatrix} a \\ b \end{pmatrix}$$

we can find a $\vec{v} \in \mathbb{R}^3$ that maps to it, specifically any $\vec{v}$ with a first component a and second component b. Thus the rank of π is 2.

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Example Projection $\pi\colon \mathbb{R}^3 \to \mathbb{R}^2$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \end{pmatrix}$$

is a linear map; the check is routine. The range space is $\mathscr{R}(\pi) = \mathbb{R}^2$ because given a vector $\vec{w} \in \mathbb{R}^2$

$$\vec{w} = \begin{pmatrix} a \\ b \end{pmatrix}$$

we can find a $\vec{v} \in \mathbb{R}^3$ that maps to it, specifically any $\vec{v}$ with a first component a and second component b. Thus the rank of π is 2.

In the book's next section, on computing linear maps, we will do more examples of determining the range space.

Many-to-one

In moving from isomorphisms to homomorphisms we dropped the requirement that the maps be onto and one-to-one. But any homomorphism $h\colon V \to W$ is onto its range space $\mathscr{R}(h)$, so dropping the onto condition has, in a way, no effect on the range. It doesn't allow any essentially new maps.

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In contrast, consider the effect of dropping the one-to-one condition. With that, an output vector $\vec{w} \in W$ may have many associated inputs, many $\vec{v} \in V$ such that $h(\vec{v}) = \vec{w}$.

Recall that for any function $h\colon V \to W$, the set of elements of V that map to $\vec{w} \in W$ is the *inverse image* $h^{-1}(\vec{w}) = \{\vec{v} \in V \mid h(\vec{v}) = \vec{w}\}$.

The structure of the inverse image sets will give us insight into the definition of homomorphism.

Example Projection $\pi\colon \mathbb{R}^2 \to \mathbb{R}$ onto the x axis is linear.

$$\pi(\begin{pmatrix} x \\ y \end{pmatrix}) = x$$

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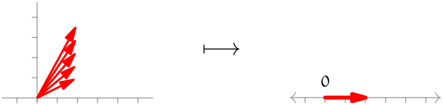
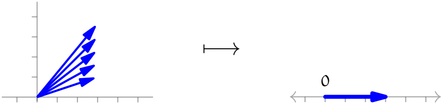
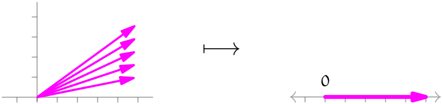
Here are some elements of $\pi^{-1}(2)$. Think of these as "2 vectors."



Think of elements of $\pi^{-1}(3)$ as "3 vectors."



These elements of $\pi^{-1}(5)$ are "5 vectors."



These drawings give us a way to make the definition of homomorphism more concrete. Consider preservation of addition.

$$\pi(\vec{u}) + \pi(\vec{v}) = \pi(\vec{u} + \vec{v})$$

If $\vec{u}$ is such that $\pi(\vec{u}) = 2$, and $\vec{v}$ is such that $\pi(\vec{v}) = 3$, then $\vec{u} + \vec{v}$ will be such that the sum $\pi(\vec{u} + \vec{v}) = 5$.

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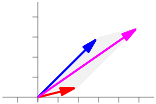
If $\vec{u}$ is such that $\pi(\vec{u}) = 2$, and $\vec{v}$ is such that $\pi(\vec{v}) = 3$, then $\vec{u} + \vec{v}$ will be such that the sum $\pi(\vec{u} + \vec{v}) = 5$. That is, a "2 vector" plus a "3 vector" is a "5 vector." Red plus blue makes magenta.

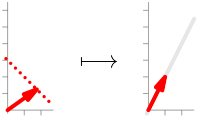


A similar interpretation holds for preservation of scalar multiplication: the image of an "$r \cdot 2$ vector" is r times 2.

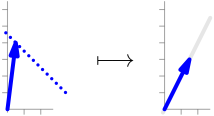
Example This function $h\colon \mathbb{R}^2 \to \mathbb{R}^2$ is linear.

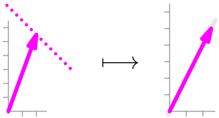
$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x+y \\ 2x+2y \end{pmatrix}$$

Here are elements of $h^{-1}(\begin{pmatrix} 1 \\ 2 \end{pmatrix})$. (Only one inverse image element is shown as a vector, most are indicated with dots.)



Here are some elements of $h^{-1}(\begin{pmatrix} 1.5 \\ 3 \end{pmatrix})$ and $h^{-1}(\begin{pmatrix} 2.5 \\ 5 \end{pmatrix})$.

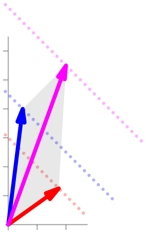




The way that the range space vectors add

$$\begin{pmatrix}1\\2\end{pmatrix}+\begin{pmatrix}1.5\\3\end{pmatrix}=\begin{pmatrix}2.5\\5\end{pmatrix}$$

is reflected in the domain: red plus blue makes magenta.



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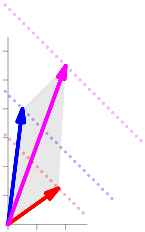
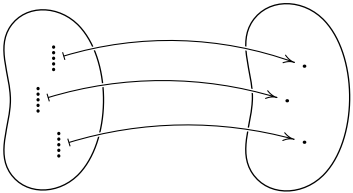


That is, preservation of addition is: $h(\vec{v}_1)+h(\vec{v}_2)=h(\vec{v}_1+\vec{v}_2)$.

Homomorphisms organize the domain

So the intuition is that a linear map organizes its domain into inverse images,

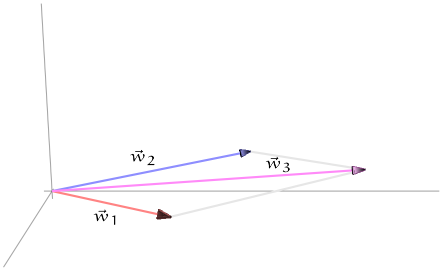


such that those sets reflect the structure of the range.

Example Projection $\pi\colon \mathbb{R}^3 \to \mathbb{R}^2$ is a homomorphism.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \end{pmatrix}$$

Here we draw the range $\mathbb{R}^2$ as the xy-plane inside of $\mathbb{R}^3$.



In the range the parallelogram shows a vector addition $\vec{w}_1 + \vec{w}_2 = \vec{w}_3$.

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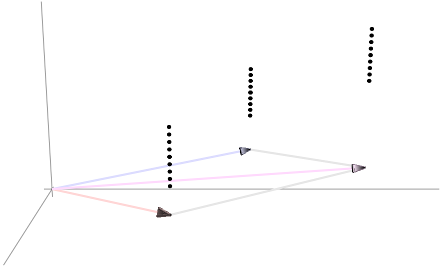

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The diagram shows some of the points in each inverse image $\pi^{-1}(\vec{w}_1)$, $\pi^{-1}(\vec{w}_2)$, and $\pi^{-1}(\vec{w}_3)$.

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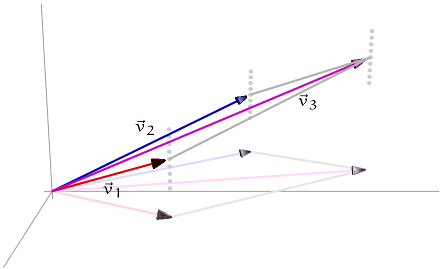

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The diagram shows some of the points in each inverse image $\pi^{-1}(\vec{w}_1)$, $\pi^{-1}(\vec{w}_2)$, and $\pi^{-1}(\vec{w}_3)$. The sum of a vector $\vec{v}_1 \in \pi^{-1}(\vec{w}_1)$ and a vector $\vec{v}_2 \in \pi^{-1}(\vec{w}_2)$ equals a vector $\vec{v}_3 \in \pi^{-1}(\vec{w}_3)$. A $\vec{w}_1$ vector plus a $\vec{w}_2$ vector equals a $\vec{w}_3$ vector.

This interpretation of the definition of homomorphism also holds when the spaces are not ones that we can sketch.

Example Let $h\colon \mathcal{P}_2 \to \mathbb{R}^2$ be

$$ax^2 + bx + c \mapsto \begin{pmatrix} b \\ b \end{pmatrix}$$

and consider these three members of the range such that $\vec{w}_1 + \vec{w}_2 = \vec{w}_3$

$$\vec{w}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \vec{w}_2 = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \quad \vec{w}_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

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The inverse image of $\vec{w}_1$ is $h^{-1}(\vec{w}_1) = \{a_1x^2 + 1x + c_1 \mid a_1, c_1 \in \mathbb{R}^2\}$. Members of this set are "$\vec{w}_1$ vectors."

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Any $\vec{v}_1 \in h^{-1}(\vec{w}_1)$ plus any $\vec{v}_2 \in h^{-1}(\vec{w}_2)$ equals a $\vec{v}_3 \in h^{-1}(\vec{w}_3)$: a quadratic with an x coefficient of 1 plus a quadratic with an x coefficient of -1 equals a quadratic with an x coefficient of 0.

Null space

In each of those examples, the homomorphism $h\colon V \to W$ shows how to view the domain V as organized into the inverse images $h^{-1}(\vec{w})$.

In the examples these inverse images are all the same, but shifted. So if we describe one of them then we understand how the domain is divided. Vector spaces have a distinguished element, $\vec{0}$. So we next consider the inverse image $h^{-1}(\vec{0})$.

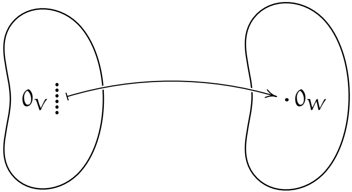
2.10 *Lemma* For any homomorphism the inverse image of a subspace of the range is a subspace of the domain. In particular, the inverse image of the trivial subspace of the range is a subspace of the domain.

The book has the verification.

2.11 *Definition* The *null space* or *kernel* of a linear map $h\colon V \to W$ is the inverse image of $\vec{0}_W$.

$$\mathscr{N}(h) = h^{-1}(\vec{0}_W) = \{\vec{v} \in V \mid h(\vec{v}) = \vec{0}_W\}$$

The dimension of the null space is the map's *nullity*.



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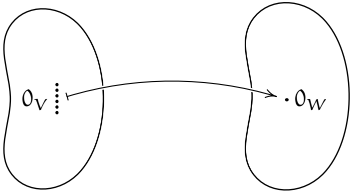


Note Strictly, the trivial subspace of the codomain is not $\vec{0}_W$, it is $\{\vec{0}_W\}$, and so we may think to write the nullspace as $h^{-1}(\{\vec{0}_W\})$. But we have defined the two sets $h^{-1}(\vec{w})$ and $h^{-1}(\{\vec{w}\})$ to be equal and the first is easier to write.

Example Consider the derivative $d/dx\colon \mathcal{P}_2 \to \mathcal{P}_1$. This is the nullspace; note that it is a subset of the domain

$$\mathscr{N}(d/dx) = \{ax^2 + bx + c \mid 2ax + b = 0\}$$

(the '0' there is the zero polynomial $0x + 0$). Now, $2ax + b = 0$ if and only if they have the same constant coefficient $b = 0$, the same x coefficient of $a = 0$, and the same coefficient of x^2 (which gives no restriction). So this is the nullspace, and the nullity is 1.

$$\mathscr{N}(d/dx) = \{ax^2 + bx + c \mid a = 0,\ b = 0,\ c \in \mathbb{R}\} = \{c \mid c \in \mathbb{R}\}$$

Example The function $h\colon \mathbb{R}^2 \to \mathbb{R}^1$ given by

$$\begin{pmatrix} a \\ b \end{pmatrix} \mapsto 2a + b$$

has this null space and so its nullity is 1.

$$\mathscr{N}(h) = \{\begin{pmatrix} a \\ b \end{pmatrix} \mid 2a + b = 0\} = \{\begin{pmatrix} -1/2 \\ 1 \end{pmatrix} b \mid b \in \mathbb{R}\}$$

Example The homomorphism $f\colon \mathcal{M}_{2\times 2} \to \mathbb{R}^2$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \stackrel{f}{\longmapsto} \begin{pmatrix} a+b \\ c+d \end{pmatrix}$$

has this null space

$$\begin{aligned} \mathscr{N}(f) &= \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a+b=0 \text{ and } c+d=0 \} \\ &= \{ \begin{pmatrix} -b & b \\ -d & d \end{pmatrix} \mid b, d \in \mathbb{R} \} \end{aligned}$$

and a nullity of 2.

Example The dilation function $d_3\colon \mathbb{R}^2 \to \mathbb{R}^2$

$$\begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} 3a \\ 3b \end{pmatrix}$$

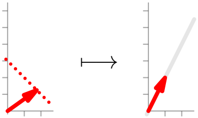
has $\mathscr{N}(d_3) = \{\vec{0}\}$. A trivial space has an empty basis so d_3's nullity is 0.

Rank plus nullity

Recall the example map $h\colon \mathbb{R}^2 \to \mathbb{R}^2$

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + y \\ 2x + 2y \end{pmatrix}$$

whose range space $\mathscr{R}(h)$ is the line $y = 2x$ and whose domain is organized into lines, $\mathscr{N}(h)$ is the line $y = -x$. There, an entire line's worth of domain vectors collapses to the single range point.



In moving from domain to range, this maps drops a dimension. We can account for it by thinking that each output point absorbs a one-dimensional set.

2.14 *Theorem* A linear map's rank plus its nullity equals the dimension of its domain.

The book contains the proof.

Example Consider this map $h\colon \mathbb{R}^3 \to \mathbb{R}$.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \stackrel{h}{\longmapsto} x/2 + y/5 + z$$

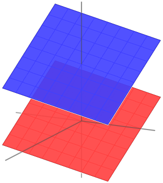
The null space is this plane.

$$\mathscr{N}(h) = h^{-1}(0) = \{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x/2 + y/5 + z = 0 \}$$

Other inverse image sets are also planes.

$$h^{-1}(1) = \{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x/2 + y/5 + z = 1 \} = \{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid z = 1 - x/2 - y/5 \}$$

This shows the inverse images $h^{-1}(0)$ and $h^{-1}(1)$ lined up on the z axis.



So h breaks the domain into stacked planes — any two inverse images $h^{-1}(r_1)$ and $h^{-1}(r_2)$ are collections of domain vectors whose endpoints form a plane. The only difference between these 2-dimensional subsets is where they sit in the stack, shown here as where they intersect z axis.

That is, h partitions the 3-dimensional domain into 2-dimensional sets, leaving 1 dimension of freedom, which matches the dimension of the map's range.

Example Projection $\pi\colon \mathbb{R}^3 \to \mathbb{R}^2$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \mapsto \begin{pmatrix} a \\ b \end{pmatrix}$$

takes a 3-dimensional domain to a 2-dimensional range. Its null space is the z-axis, so its nullity is 1.

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This example shows the idea of the proof particularly clearly. Take the basis $B_N = \langle \vec{e}_3 \rangle$ for the null space.

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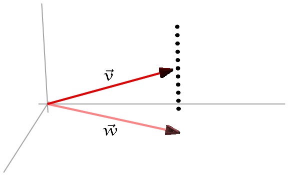
This example shows the idea of the proof particularly clearly. Take the basis $B_N = \langle \vec{e}_3 \rangle$ for the null space. Expand that to the basis $\mathcal{E}_3$ for the entire domain. On an input vector the action of π is

$$c_1\vec{e}_1 + c_2\vec{e}_2 + c_3\vec{e}_3 \quad \mapsto \quad c_1\vec{e}_1 + c_2\vec{e}_2 + \vec{0}$$

and so the domain is organized by π into inverse images that are vertical lines, one-dimensional sets like the null space.



Example The derivative function $d/dx\colon \mathcal{P}_2 \to \mathcal{P}_1$

$$ax^2 + bx + c \mapsto 2a \cdot x + b$$

has this range space

$$\mathscr{R}(d/dx) = \{d \cdot x + e \mid d, e \in \mathbb{R}\} = \mathcal{P}_1$$

(the linear polynomial $dx + e \in \mathcal{P}_1$ is the image of any antiderivative $(d/2)x^2 + ex + C$, where $C \in \mathbb{R}$). This is its null space.

$$\mathscr{N}(d/dx) = \{0x^2 + 0x + c \mid c \in \mathbb{R}\} = \{c \mid c \in \mathbb{R}\}$$

The rank is 2 while the nullity is 1, and they add to the domain's dimension 3.

Example The dilation function $d_3 : \mathbb{R}^2 \to \mathbb{R}^2$

$$\begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} 3a \\ 3b \end{pmatrix}$$

has range space $\mathbb{R}^2$ and a trivial nullspace $\mathscr{N}(d_3) = \{\vec{0}\}$. So its rank is 2 and its nullity is 0.

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The book's next section is on computing linear maps, and we will compute more null spaces there.

2.18 *Lemma* Under a linear map, the image of a linearly dependent set is linearly dependent.

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Proof Suppose that $c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{0}_V$ with some c_i nonzero. Apply h to both sides: $h(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) = c_1 h(\vec{v}_1) + \cdots + c_n h(\vec{v}_n)$ and $h(\vec{0}_V) = \vec{0}_W$. Thus we have $c_1 h(\vec{v}_1) + \cdots + c_n h(\vec{v}_n) = \vec{0}_W$ with some c_i nonzero. QED

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Example The trace function $\text{Tr}\colon \mathcal{M}_{2\times 2} \to \mathbb{R}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto a + d$$

is linear. This set of matrices is dependent.

$$S = \{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \}$$

The three matrices map to 1, 0, and 2 respectively. The set $\{1, 0, 2\} \subseteq \mathbb{R}$ is linearly dependent.

A one-to-one homomorphism is an isomorphism

2.20 *Theorem* Where V is an n-dimensional vector space, these are equivalent statements about a linear map $h\colon V \to W$.

(1) h is one-to-one

(2) h has an inverse from its range to its domain that is a linear map

(3) $\mathscr{N}(h) = \{\vec{0}\,\}$, that is, $\text{nullity}(h) = 0$

(4) $\text{rank}(h) = n$

(5) if $\langle\vec{\beta}_1, \dots, \vec{\beta}_n\rangle$ is a basis for V then $\langle h(\vec{\beta}_1), \dots, h(\vec{\beta}_n)\rangle$ is a basis for $\mathscr{R}(h)$

The book has the proof.