Three.I Isomorphisms

Linear Algebra

Jim Hefferon

http://joshua.smcvt.edu/linearalgebra

Definition

Example We have the intuition that the vector spaces $\mathbb{R}^2$ and $\mathcal{P}_1$ are "the same," in that they are two-component spaces. For instance

$$\begin{pmatrix}1\\2\end{pmatrix} \quad \text{is just like} \quad 1+2x,$$

$$\text{and} \quad \begin{pmatrix}-3\\1/2\end{pmatrix} \quad \text{is just like} \quad -3+(1/2)x,$$

etc. What makes the spaces alike, not just the sets, is that the association persists through the operations: this illustrates addition

$$\begin{pmatrix}1\\2\end{pmatrix}+\begin{pmatrix}-3\\1/2\end{pmatrix}=\begin{pmatrix}-2\\5/2\end{pmatrix}$$

$$\text{is just like} \quad (1+2x)+(-3+(1/2)x)=-2+(5/2)x$$

and this illustrates scalar multiplication.

$$3\begin{pmatrix}1\\2\end{pmatrix}=\begin{pmatrix}3\\6\end{pmatrix} \quad \text{is just like} \quad 3(1+2x)=3+6x$$

Example Similarly, we can link each two-tall vector with a linear polynomial.

$$\begin{pmatrix} a \\ b \end{pmatrix} \quad \longleftrightarrow \quad a + bx$$

Example Similarly, we can link each two-tall vector with a linear polynomial.

$$\begin{pmatrix} a \\ b \end{pmatrix} \quad \longleftrightarrow \quad a + bx$$

This association holds through the vector space operations of addition

$$\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 \\ b_1 + b_2 \end{pmatrix}$$

$$\longleftrightarrow \quad (a_1 + b_1x) + (a_2 + b_2x) = (a_1 + a_2) + (b_1 + b_2)x$$

and scalar multiplication.

$$r\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ra \\ rb \end{pmatrix} \quad \longleftrightarrow \quad r(a + bx) = (ra) + (rb)x$$

We say that the association *preserves the structure* of the spaces.

Example We can think of $\mathcal{M}_{2\times 2}$ as "the same" as $\mathbb{R}^4$ if we associate in this way.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \longleftrightarrow \quad \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

For instance, these are corresponding elements.

$$\begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} \quad \longleftrightarrow \quad \begin{pmatrix} 1 \\ -1 \\ 2 \\ -2 \end{pmatrix}$$

Example We can think of $\mathcal{M}_{2\times 2}$ as "the same" as $\mathbb{R}^4$ if we associate in this way.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \longleftrightarrow \quad \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

For instance, these are corresponding elements.

$$\begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} \quad \longleftrightarrow \quad \begin{pmatrix} 1 \\ -1 \\ 2 \\ -2 \end{pmatrix}$$

This association persists under addition.

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{pmatrix}$$

$$\longleftrightarrow \quad \begin{pmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{pmatrix} + \begin{pmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 \\ b_1 + b_2 \\ c_1 + c_2 \\ d_1 + d_2 \end{pmatrix}$$

Here is an example of addition being preserved under this association.

$$\begin{pmatrix}1 & -1\\ 2 & -2\end{pmatrix} + \begin{pmatrix}0 & 4\\ 3 & -3\end{pmatrix} = \begin{pmatrix}1 & 3\\ 5 & -5\end{pmatrix}$$

$$\longleftrightarrow \quad \begin{pmatrix}1\\ -1\\ 2\\ -2\end{pmatrix} + \begin{pmatrix}0\\ 4\\ 3\\ -3\end{pmatrix} = \begin{pmatrix}1\\ 3\\ 5\\ -5\end{pmatrix}$$

Here is an example of addition being preserved under this association.

$$\begin{pmatrix}1 & -1\\ 2 & -2\end{pmatrix} + \begin{pmatrix}0 & 4\\ 3 & -3\end{pmatrix} = \begin{pmatrix}1 & 3\\ 5 & -5\end{pmatrix}$$

$$\longleftrightarrow \quad \begin{pmatrix}1\\ -1\\ 2\\ -2\end{pmatrix} + \begin{pmatrix}0\\ 4\\ 3\\ -3\end{pmatrix} = \begin{pmatrix}1\\ 3\\ 5\\ -5\end{pmatrix}$$

The association also persists through scalar multiplication.

$$r \cdot \begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}ra & rb\\ rc & rd\end{pmatrix} \quad \longleftrightarrow \quad r \cdot \begin{pmatrix}a\\ b\\ c\\ d\end{pmatrix} = \begin{pmatrix}ra\\ rb\\ rc\\ rd\end{pmatrix}$$

This illustrates.

$$2 \cdot \begin{pmatrix}1 & -1\\ 2 & -2\end{pmatrix} = \begin{pmatrix}2 & -2\\ 4 & -4\end{pmatrix} \quad \longleftrightarrow \quad 2 \cdot \begin{pmatrix}1\\ -1\\ 2\\ -1\end{pmatrix} = \begin{pmatrix}2\\ -2\\ 4\\ -4\end{pmatrix}$$

Isomorphism

1.3 *Definition* An *isomorphism* between two vector spaces V and W is a map $f\colon V \to W$ that

1) is a correspondence: f is one-to-one and onto;
2) *preserves structure:* if $\vec{v}_1, \vec{v}_2 \in V$ then

$$f(\vec{v}_1 + \vec{v}_2) = f(\vec{v}_1) + f(\vec{v}_2)$$

and if $\vec{v} \in V$ and $r \in \mathbb{R}$ then

$$f(r\vec{v}) = rf(\vec{v})$$

(we write $V \cong W$, read "V is isomorphic to W", when such a map exists).

How-to

To verify that $f\colon V\to W$ is an isomorphism, do these four.

- To show that f is one-to-one, assume that $\vec{v}_1,\vec{v}_2\in V$ are such that $f(\vec{v}_1)=f(\vec{v}_2)$ and derive that $\vec{v}_1=\vec{v}_2$.
- To show that f is onto, let $\vec{w}$ be an element of W and find a $\vec{v}\in V$ such that $f(\vec{v})=\vec{w}$.
- To show that f preserves addition, check that for all $\vec{v}_1,\vec{v}_2\in V$ we have $f(\vec{v}_1+\vec{v}_2)=f(\vec{v}_1)+f(\vec{v}_2)$.
- To show that f preserves scalar multiplication, check that for all $\vec{v}\in V$ and $r\in\mathbb{R}$ we have $f(r\cdot\vec{v})=r\cdot f(\vec{v})$.

The first two cover condition (1), that the spaces correspond, that for each member of W there exactly one assocated member of V. The latter two cover (2), that the map preserves structure. For these two, the intuition is in the discussion above. (Later section cover these two at length.)

Example The space of quadratic polynomials $\mathcal{P}_2$ is isomorphic to $\mathbb{R}^3$ under this map.

$$f(a_0 + a_1 x + a_2 x^2) = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$$

Here are two examples of the action of f.

$$f(1 + 2x + 3x^2) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad f(3 + 4x^2) = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$$

To verify that f is an isomorphism we must check condition (1), that f is a correspondence, and condition (2), that f preserves structure.

The first part of the definition's clause (1) is that f is one-to-one. We suppose $f(\vec{v}_1) = f(\vec{v}_2)$, that the function yields the same output on two inputs $f(a_0 + a_1x + a_2x^2) = f(b_0 + b_1x + b_2x^2)$. From that, we must derive that the two inputs are equal. The definition of f gives

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix}$$

and two column vectors are equal only if their entries are equal $a_0 = b_0$, $a_1 = b_1$, and $a_2 = b_2$. Thus the original inputs are equal $\vec{v}_1 = a_0 + a_1x + a_2x^2 = b_0 + b_1x + b_2x^2 = \vec{v}_2$. So f is one-to-one.

The first part of the definition's clause (1) is that f is one-to-one. We suppose $f(\vec{v}_1) = f(\vec{v}_2)$, that the function yields the same output on two inputs $f(a_0 + a_1x + a_2x^2) = f(b_0 + b_1x + b_2x^2)$. From that, we must derive that the two inputs are equal. The definition of f gives

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix}$$

and two column vectors are equal only if their entries are equal $a_0 = b_0$, $a_1 = b_1$, and $a_2 = b_2$. Thus the original inputs are equal $\vec{v}_1 = a_0 + a_1x + a_2x^2 = b_0 + b_1x + b_2x^2 = \vec{v}_2$. So f is one-to-one.

The second part of (1) is that f is onto. We consider an element of the codomain

$$\vec{w} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} \in \mathbb{R}^3$$

and produce an element of the domain that maps to it. Observe that $\vec{w}$ is the image under f of the member $\vec{v} = a_0 + a_1x + a_2x^2$ of the domain. Thus f is onto.

The definition's clause (2) also has two halves. First we show that f preserves addition. Consider f acting on the sum of two elements of the domain.

$$f(\,(a_0 + a_1x + a_2x^2) + (b_0 + b_1x + b_2x^2)\,)$$
$$= f(\,(a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2\,)$$

The definition of f gives

$$= \begin{pmatrix} a_0 + b_0 \\ a_1 + b_1 \\ a_2 + b_2 \end{pmatrix}$$

and that equals

$$= \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix}$$

which gives

$$= f(\,a_0 + a_1x + a_2x^2) + f(b_0 + b_1x + b_2x^2\,)$$

as required.

We finish by checking that f preserves scalar multiplication. This is similar to the check for addition.

$$\begin{aligned}
f(r\cdot(a_0+a_1x+a_2x^2)) &= f(\,(ra_0)+(ra_1)x+(ra_2)x^2\,) \\
&= \begin{pmatrix} ra_0 \\ ra_1 \\ ra_2 \end{pmatrix} \\
&= r\cdot\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} \\
&= r\cdot f(\,a_0+a_1x+a_2x^2\,)
\end{aligned}$$

We finish by checking that f preserves scalar multiplication. This is similar to the check for addition.

$$\begin{aligned}
f(r\cdot(a_0+a_1x+a_2x^2)) &= f(\,(ra_0)+(ra_1)x+(ra_2)x^2\,) \\
&= \begin{pmatrix} ra_0 \\ ra_1 \\ ra_2 \end{pmatrix} \\
&= r\cdot\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} \\
&= r\cdot f(\,a_0+a_1x+a_2x^2\,)
\end{aligned}$$

So the function f is an isomorphism. Because there is an isomorphism, the two spaces are isomorphic $\mathcal{P}_2 \cong \mathbb{R}^3$.

Example Consider these two vector spaces (under the natural operations)

$$V=\{\begin{pmatrix}a&b\\c&0\end{pmatrix}\mid a,b,c\in\mathbb{R}\}\qquad W=\{\begin{pmatrix}x&y&z\end{pmatrix}\mid x,y,z\in\mathbb{R}\}$$

and consider this function.

$$\begin{pmatrix}a&b\\c&0\end{pmatrix}\stackrel{f}{\longmapsto}\begin{pmatrix}b&2a&a+c\end{pmatrix}$$

Here is an example of the map's action.

$$\begin{pmatrix}3&2\\1&0\end{pmatrix}\stackrel{f}{\longmapsto}\begin{pmatrix}2&6&4\end{pmatrix}$$

We will verify that f is an isomorphism.

To show that f is one-to-one, suppose that

$$f(\begin{pmatrix} a_1 & b_1 \\ c_1 & 0 \end{pmatrix}) = f(\begin{pmatrix} a_2 & b_2 \\ c_2 & 0 \end{pmatrix})$$

Then $(b_1 \;\; 2a_1 \;\; a_1 + c_1) = (b_2 \;\; 2a_2 \;\; a_2 + c_2)$. The first entries give that $b_1 = b_2$, the second entries that $a_1 = a_2$, and with that the third entries give that $c_1 = c_2$.

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & 0 \end{pmatrix} = \begin{pmatrix} a_2 & b_2 \\ c_2 & 0 \end{pmatrix}$$

To show that f is one-to-one, suppose that

$$f(\begin{pmatrix} a_1 & b_1 \\ c_1 & 0 \end{pmatrix}) = f(\begin{pmatrix} a_2 & b_2 \\ c_2 & 0 \end{pmatrix})$$

Then $(b_1 \;\; 2a_1 \;\; a_1 + c_1) = (b_2 \;\; 2a_2 \;\; a_2 + c_2)$. The first entries give that $b_1 = b_2$, the second entries that $a_1 = a_2$, and with that the third entries give that $c_1 = c_2$.

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & 0 \end{pmatrix} = \begin{pmatrix} a_2 & b_2 \\ c_2 & 0 \end{pmatrix}$$

To show that f is onto, consider a member of W.

$$\vec{w} = (x \;\; y \;\; z)$$

We must find a $\vec{v}$ so that $f(\vec{v}) = \vec{w}$. The map sends the upper right entry of the input to the first entry of the output, so the upper right of $\vec{v}$ is x. Similarly, the upper left of $\vec{v}$ is $(1/2)y$. With that, the lower left is $z - (1/2)y$.

$$(x \;\; y \;\; z) = f(\begin{pmatrix} y/2 & x \\ z - y/2 & 0 \end{pmatrix})$$

To show that f preserves addition, assume

$$f(\begin{pmatrix} a_1 & b_1 \\ c_1 & 0 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 \\ c_2 & 0 \end{pmatrix}) = f(\begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & 0 \end{pmatrix})$$

which equals $(b_1 + b_2 \;\; 2(a_1 + a_2) \;\; (a_1 + a_2) + (c_1 + c_2))$. In turn, that equals this.

$$(b_1 \;\; 2a_1 \;\; a_1 + c_1) + (b_2 \;\; 2a_2 \;\; a_2 + c_2) = f(\begin{pmatrix} a_1 & b_1 \\ c_1 & 0 \end{pmatrix}) + f(\begin{pmatrix} a_2 & b_2 \\ c_2 & 0 \end{pmatrix})$$

To show that f preserves addition, assume

$$f(\begin{pmatrix} a_1 & b_1 \\ c_1 & 0 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 \\ c_2 & 0 \end{pmatrix}) = f(\begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & 0 \end{pmatrix})$$

which equals $(b_1 + b_2 \;\; 2(a_1 + a_2) \;\; (a_1 + a_2) + (c_1 + c_2))$. In turn, that equals this.

$$(b_1 \;\; 2a_1 \;\; a_1 + c_1) + (b_2 \;\; 2a_2 \;\; a_2 + c_2) = f(\begin{pmatrix} a_1 & b_1 \\ c_1 & 0 \end{pmatrix}) + f(\begin{pmatrix} a_2 & b_2 \\ c_2 & 0 \end{pmatrix})$$

Preservation of scalar multiplication is similar.

$$\begin{aligned} f(r \cdot \begin{pmatrix} a & b \\ c & 0 \end{pmatrix}) &= f(\begin{pmatrix} ra & rb \\ rc & 0 \end{pmatrix}) \\ &= (rb \;\; 2ra \;\; ra + rc) \\ &= r \cdot (b \;\; 2a \;\; a + c) \\ &= r \cdot f(\begin{pmatrix} a & b \\ c & 0 \end{pmatrix}) \end{aligned}$$

Preservation is special

Many functions do not preserve addition and scalar multiplication. For instance, $f\colon \mathbb{R}^2 \to \mathbb{R}^2$

$$f(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} x^2 \\ y^2 \end{pmatrix}$$

does not preserve addition since the sum done one way

$$f(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix}) = f(\begin{pmatrix} 3 \\ 0 \end{pmatrix}) = \begin{pmatrix} 9 \\ 0 \end{pmatrix}$$

gives a different result than the sum done the other way.

$$f(\begin{pmatrix} 1 \\ 0 \end{pmatrix}) + f(\begin{pmatrix} 2 \\ 0 \end{pmatrix}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$$

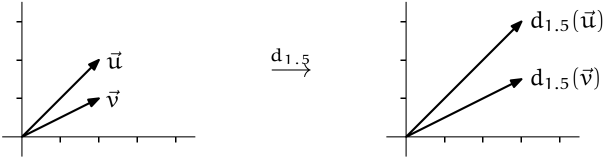
Special case: Automorphisms

1.7 *Definition* An *automorphism* is an isomorphism of a space with itself.

Special case: Automorphisms

1.7 *Definition* An *automorphism* is an isomorphism of a space with itself.

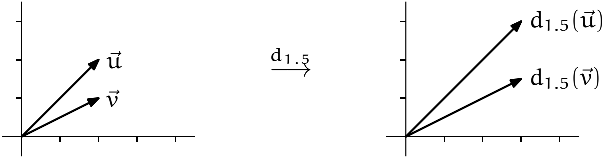
1.8 *Example* A *dilation* map $d_s\colon \mathbb{R}^2 \to \mathbb{R}^2$ that multiplies all vectors by a nonzero scalar s is an automorphism of $\mathbb{R}^2$.



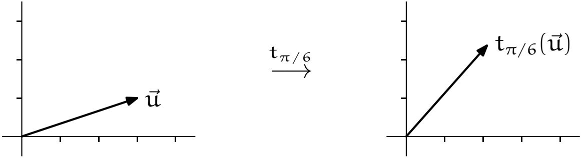
Special case: Automorphisms

1.7 *Definition* An *automorphism* is an isomorphism of a space with itself.

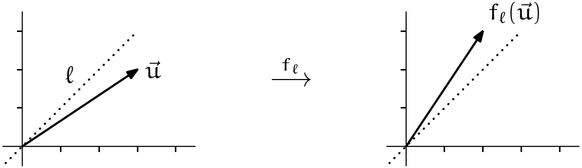
1.8 *Example* A *dilation* map $d_s \colon \mathbb{R}^2 \to \mathbb{R}^2$ that multiplies all vectors by a nonzero scalar s is an automorphism of $\mathbb{R}^2$.



Another automorphism is a *rotation* or *turning map*, $t_\theta \colon \mathbb{R}^2 \to \mathbb{R}^2$ that rotates all vectors through an angle θ.

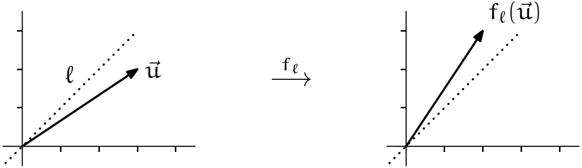


A third type of automorphism of $\mathbb{R}^2$ is a map $f_\ell\colon \mathbb{R}^2 \to \mathbb{R}^2$ that *flips* or *reflects* all vectors over a line ℓ through the origin.



Checking that each is an isomorphism is an exercise.

A third type of automorphism of $\mathbb{R}^2$ is a map $f_\ell\colon \mathbb{R}^2 \to \mathbb{R}^2$ that *flips* or *reflects* all vectors over a line ℓ through the origin.



Checking that each is an isomorphism is an exercise.

Why study automorphisms? Isn't it trivial that the plane is just like itself?

Consider the family of automorphisms t_Θ rotating all vectors counterclockwise. They makes precise the intuition that the plane is uniform — that space near the x-axis is just like space near the y-axis.

So one lesson is that we can use maps to describe relationships between spaces. If the maps are isomorphisms then this relation makes precise the intuition “just like”.

A second lesson is that while there is an obvious automorphism of $\mathbb{R}^2$

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \end{pmatrix}$$

there are reasons to consider maps other than the obvious one.

1.10 *Lemma* An isomorphism maps a zero vector to a zero vector.

1.10 *Lemma* An isomorphism maps a zero vector to a zero vector.

Proof Where $f\colon V \to W$ is an isomorphism, fix some $\vec{v} \in V$. Then $f(\vec{0}_V) = f(0 \cdot \vec{v}) = 0 \cdot f(\vec{v}) = \vec{0}_W$. QED

1.11 *Lemma* For any map $f\colon V \to W$ between vector spaces these statements are equivalent.

(1) f preserves structure

$$f(\vec{v}_1 + \vec{v}_2) = f(\vec{v}_1) + f(\vec{v}_2) \quad \text{and} \quad f(c\vec{v}) = c\, f(\vec{v})$$

(2) f preserves linear combinations of two vectors

$$f(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1 f(\vec{v}_1) + c_2 f(\vec{v}_2)$$

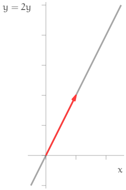
(3) f preserves linear combinations of any finite number of vectors

$$f(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) = c_1 f(\vec{v}_1) + \cdots + c_n f(\vec{v}_n)$$

The book contains the details of both proofs.

Remark Examination of the proofs show that they depend only on clause (2) of the definition, not on that the map is a correspondence. We will say more in the next section.

Example The line through the origin with slope two



$$L = \{ \begin{pmatrix} x \\ y \end{pmatrix} \mid y = 2x \} = \{ \begin{pmatrix} t \\ 2t \end{pmatrix} \mid t \in \mathbb{R} \}$$

is a vector space.

The parametrization

$$L = \{ \begin{pmatrix} t \\ 2t \end{pmatrix} \mid t \in \mathbb{R} \} = \{ t \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} \mid t \in \mathbb{R} \}$$

suggests that L is just like the real line $\mathbb{R}$: there is a point on L

$$1 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

associated with $1 \in \mathbb{R}$, a point associated with $2 \in \mathbb{R}$, etc. We will show that this function is an isomorphism between L and $\mathbb{R}^1$.

$$f(\begin{pmatrix} t \\ 2t \end{pmatrix}) = t$$

To verify that f is one-to-one suppose that f maps two members of L to the same output.

$$f(\, t_1 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} \,) = f(\, t_2 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} \,)$$

By the definition of f we have $t_1 = t_2$ and so the two input members of L are equal.

To check that f is onto consider a member of the codomain, $r \in \mathbb{R}$. There is a member of the domain L that maps to it, namely this one.

$$f(\, r \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} \,) = r$$

To finish, we combine the two structure checks, using the lemma's (2).

$$\begin{aligned} f(\, c_1 \cdot \begin{pmatrix} t_1 \\ 2t_1 \end{pmatrix} + c_2 \cdot \begin{pmatrix} t_2 \\ 2t_2 \end{pmatrix} \,) &= f(\, (c_1 t_1 + c_2 t_2) \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} \,) \\ &= c_1 t_1 + c_2 t_2 = c_1 \cdot f(\, \begin{pmatrix} t_1 \\ 2t_1 \end{pmatrix} \,) + c_2 \cdot f(\, \begin{pmatrix} t_2 \\ 2t_2 \end{pmatrix} \,) \end{aligned}$$

Dimension characterizes isomorphism

2.1 *Lemma* The inverse of an isomorphism is also an isomorphism.

2.1 *Lemma* The inverse of an isomorphism is also an isomorphism.

Proof Suppose that V is isomorphic to W via $f\colon V \to W$. An isomorphism is a correspondence between the sets so f has an inverse function $f^{-1}\colon W \to V$ that is also a correspondence.

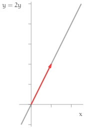
2.1 *Lemma* The inverse of an isomorphism is also an isomorphism.

Proof Suppose that V is isomorphic to W via $f\colon V \to W$. An isomorphism is a correspondence between the sets so f has an inverse function $f^{-1}\colon W \to V$ that is also a correspondence.

We will show that because f preserves linear combinations, so also does f^{-1}. Suppose that $\vec{w}_1, \vec{w}_2 \in W$. Because it is an isomorphism, f is onto and there are $\vec{v}_1, \vec{v}_2 \in V$ such that $\vec{w}_1 = f(\vec{v}_1)$ and $\vec{w}_2 = f(\vec{v}_2)$. Then

$$\begin{aligned} f^{-1}(c_1 \cdot \vec{w}_1 + c_2 \cdot \vec{w}_2) &= f^{-1}\big(\,c_1 \cdot f(\vec{v}_1) + c_2 \cdot f(\vec{v}_2)\,\big) \\ &= f^{-1}(\,f\big(c_1\vec{v}_1 + c_2\vec{v}_2)\,\big) = c_1\vec{v}_1 + c_2\vec{v}_2 = c_1 \cdot f^{-1}(\vec{w}_1) + c_2 \cdot f^{-1}(\vec{w}_2) \end{aligned}$$

since $f^{-1}(\vec{w}_1) = \vec{v}_1$ and $f^{-1}(\vec{w}_2) = \vec{v}_2$. With that, by Lemma 1.11 's second statement, this map preserves structure. QED



Example We saw earlier that this line through the origin subspace of $\mathbb{R}^2$

$$L = \{ \begin{pmatrix} t \\ 2t \end{pmatrix} \mid t \in \mathbb{R} \}$$

is isomorphic to $\mathbb{R}^1$ via this function.

$$f(t \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}) = t$$

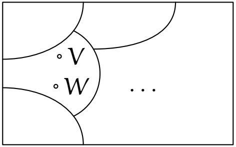
The inverse $f^{-1} \colon \mathbb{R} \to L$ given by

$$f^{-1}(r) = r \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} r \\ 2r \end{pmatrix}$$

is also an isomorphism.

2.2 *Theorem* Isomorphism is an equivalence relation between vector spaces.

The book contains the proof; here is a diagram of what it tells us: the collection of all finite-dimensional vector spaces is partitioned into classes. Two spaces are in the same class if they are isomorphic.



The next result characterizes these classes.

2.3 *Theorem* Vector spaces are isomorphic if and only if they have the same dimension.

The proof is these two lemmas.

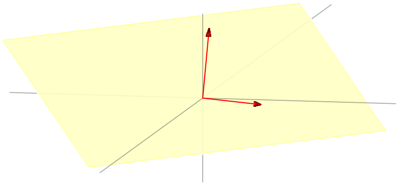
2.3 *Theorem* Vector spaces are isomorphic if and only if they have the same dimension.

The proof is these two lemmas.

2.4 *Lemma* If spaces are isomorphic then they have the same dimension.

2.5 *Lemma* If spaces have the same dimension then they are isomorphic.

Example The plane $2x - y + z = 0$ through the origin in $\mathbb{R}^3$ is a vector space (under the natural operations).



Describe the space as a span by taking that to be a one-equation linear system and parametrizing $x = (1/2)y - (1/2)z$.

$$P = \{ \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix} y + \begin{pmatrix} -1/2 \\ 0 \\ 1 \end{pmatrix} z \mid y, z \in \mathbb{R} \}$$

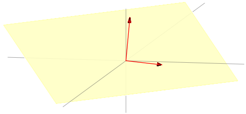
Clearly that two-vector set is linearly independent, so it is a basis.

$$B = \langle \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1/2 \\ 0 \\ 1 \end{pmatrix} \rangle$$

The basis has two vectors so this is a dimension 2 space.

The book contains the full proofs of the two. The proof of the first lemma depends on showing that under isomorphic the image of a domain basis is a range basis. The proof of the second lemma depends on giving an explicit isomorphism from an n-dimensional domain to $\mathbb{R}^n$. We will illustrate the two, using the prior example.

Example Consider again the plane

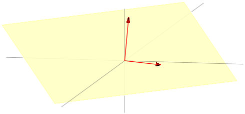


$$P = \{ \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix} \cdot y + \begin{pmatrix} -1/2 \\ 0 \\ 1 \end{pmatrix} \cdot z \mid y, z \in \mathbb{R} \}$$

Second lemma first. It's proof shows that this is an isomorphism: the map $f\colon P \to \mathbb{R}^2$ that associates each element $\vec{v} \in P$ with its representation $\mathrm{Rep}_B(\vec{v}) \in \mathbb{R}^2$. Here is an example of its action on a basis vector picked at random.

$$\vec{v}_1 = \begin{pmatrix} 7/2 \\ 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix} \cdot 3 + \begin{pmatrix} -1/2 \\ 0 \\ 1 \end{pmatrix} \cdot (-4) \quad \stackrel{f}{\longmapsto} \quad \mathrm{Rep}_B(\vec{v}_1) = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$$

Example Consider again the plane



$$P = \{ \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix} \cdot y + \begin{pmatrix} -1/2 \\ 0 \\ 1 \end{pmatrix} \cdot z \mid y, z \in \mathbb{R} \}$$

Second lemma first. It's proof shows that this is an isomorphism: the map $f\colon P \to \mathbb{R}^2$ that associates each element $\vec{v} \in P$ with its representation $\mathrm{Rep}_B(\vec{v}) \in \mathbb{R}^2$. Here is an example of its action on a basis vector picked at random.

$$\vec{v}_1 = \begin{pmatrix} 7/2 \\ 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix} \cdot 3 + \begin{pmatrix} -1/2 \\ 0 \\ 1 \end{pmatrix} \cdot (-4) \quad \stackrel{f}{\longmapsto} \quad \mathrm{Rep}_B(\vec{v}_1) = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$$

Another.

$$\vec{v}_2 = \begin{pmatrix} -17/4 \\ 1/2 \\ 9 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix} \cdot (1/2) + \begin{pmatrix} -1/2 \\ 0 \\ 1 \end{pmatrix} \cdot 9 \quad \stackrel{f}{\longmapsto} \quad \mathrm{Rep}_B(\vec{v}_2) = \begin{pmatrix} 1/2 \\ 9 \end{pmatrix}$$

The first lemma's proof shows that any isomorphism takes bases to bases: starting with basis vectors $\vec{\beta}_i$ for the domain and applying an isomorphism f gives basis vectors $f(\vec{\beta}_i)$ for the range.

For the isomorphism from the prior slide we have

$$\vec{\beta}_1 = \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix} \cdot 1 + \begin{pmatrix} -1/2 \\ 0 \\ 1 \end{pmatrix} \cdot 0 \quad \stackrel{f}{\longmapsto} \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

$$\vec{\beta}_2 = \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix} \cdot 0 + \begin{pmatrix} -1/2 \\ 0 \\ 1 \end{pmatrix} \cdot 1 \quad \stackrel{f}{\longmapsto} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

which together make the basis $\mathcal{E}_2$ for $\mathbb{R}^2$.

2.7 *Corollary* Each finite-dimensional vector space is isomorphic to one and only one of the $\mathbb{R}^n$.

2.7 *Corollary* Each finite-dimensional vector space is isomorphic to one and only one of the $\mathbb{R}^n$.

Thus the real spaces $\mathbb{R}^n$ form a set of canonical representatives of the isomorphism classes — every isomorphism class contains one and only one $\mathbb{R}^n$.