One.III Reduced Echelon Form

Linear Algebra

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Gauss-Jordan reduction

Here is an extension of Gauss's method with some advantages.

Example Start as usual by getting echelon form.

$$\begin{array}{rcrcrcr} x & + & y & - & z & = & 2 \\ 2x & - & y & & & = & -1 \\ x & - & 2y & + & 2z & = & -1 \end{array} \xrightarrow[-1\rho_1+\rho_3]{-2\rho_1+\rho_2} \begin{array}{rcrcrcr} x & + & y & - & z & = & 2 \\ & - & 3y & + & 2z & = & -5 \\ & - & 3y & + & 3z & = & -3 \end{array}$$

$$\xrightarrow{-1\rho_2+\rho_3} \begin{array}{rcrcrcr} x & + & y & - & z & = & 2 \\ & - & 3y & + & 2z & = & -5 \\ & & & & z & = & 2 \end{array}$$

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Now, instead of doing back substitution, continue using row operations. First make all the leading entries one.

$$\xrightarrow{(-1/3)\rho_2} \begin{array}{rcrcrcr} x & + & y & - & z & = & 2 \\ & & y & - & (2/3)z & = & 5/3 \\ & & & & z & = & 2 \end{array}$$

Finish by using the leading entries to eliminate upwards, until we can read off the solution.

$$\begin{array}{rcl} x+y-z &=& 2 \\ y-(2/3)z &=& 5/3 \\ z &=& 2 \end{array} \quad \xrightarrow[(2/3)\rho_3+\rho_2]{\rho_3+\rho_1} \quad \begin{array}{rcl} x+y &=& 4 \\ y &=& 3 \\ z &=& 2 \end{array}$$

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$$\xrightarrow{-\rho_2+\rho_1} \quad \begin{array}{rcl} x &=& 1 \\ y &=& 3 \\ z &=& 2 \end{array}$$

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$$
\begin{array}{rcl}
x + y - z &=& 2 \\
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\end{array}
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\xrightarrow{-\rho_2+\rho_1}
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$$

Using one entry to clear out the rest of a column is *pivoting* on that entry.

Example With this system

$$\begin{aligned} x - y \qquad\quad - 2w &= 2 \\ x + y + 3z + \ \ w &= 1 \\ -y + \ \ z - \ \ w &= 0 \end{aligned}$$

we start by getting echelon form.

$$\xrightarrow{-1\rho_1+\rho_2} \left(\begin{array}{cccc|c} 1 & -1 & 0 & -2 & 2 \\ 0 & 2 & 3 & 3 & -1 \\ 0 & -1 & 1 & -1 & 0 \end{array}\right) \xrightarrow{(1/2)\rho_2+\rho_3} \left(\begin{array}{cccc|c} 1 & -1 & 0 & -2 & 2 \\ 0 & 2 & 3 & 3 & -1 \\ 0 & 0 & 5/2 & 1/2 & -1/2 \end{array}\right)$$

We turn the leading entries to 1's.

$$\xrightarrow[(2/5)\rho_3]{(1/2)\rho_2} \left(\begin{array}{cccc|c} 1 & -1 & 0 & -2 & 2 \\ 0 & 1 & 3/2 & 3/2 & -1/2 \\ 0 & 0 & 1 & 1/5 & -1/5 \end{array}\right)$$

Now eliminate upwards.

$$\xrightarrow{-(3/2)\rho_3+\rho_2} \left(\begin{array}{cccc|c} 1 & -1 & 0 & -2 & 2 \\ 0 & 1 & 0 & 6/5 & -1/5 \\ 0 & 0 & 1 & 1/5 & -1/5 \end{array}\right) \xrightarrow{\rho_2+\rho_1} \left(\begin{array}{cccc|c} 1 & 0 & 0 & -4/5 & 9/5 \\ 0 & 1 & 0 & 6/5 & -1/5 \\ 0 & 0 & 1 & 1/5 & -1/5 \end{array}\right)$$

The final augmented matrix

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -4/5 & 9/5 \\ 0 & 1 & 0 & 6/5 & -1/5 \\ 0 & 0 & 1 & 1/5 & -1/5 \end{array}\right)$$

gives the parametrized description of the solution set.

$$\{\begin{pmatrix} 9/5 \\ -1/5 \\ -1/5 \\ 0 \end{pmatrix} + \begin{pmatrix} 4/5 \\ -6/5 \\ -1/5 \\ 1 \end{pmatrix} w \mid w \in \mathbb{R}\}$$

Gauss-Jordan reduction

This extension of Gauss's Method is the *Gauss-Jordan Method* or *Gauss-Jordan reduction*.

1.3 *Definition* A matrix or linear system is in *reduced echelon form* if, in addition to being in echelon form, each leading entry is a 1 and is the only nonzero entry in its column.

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The cost of using Gauss-Jordan reduction to solve a system is the additional arithmetic. The benefit is that we can just read off the solution set description.

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Proof For any matrix A, the effect of swapping rows is reversed by swapping them back, multiplying a row by a nonzero k is undone by multiplying by $1/k$, and adding a multiple of row i to row j (with $i \neq j$) is undone by subtracting the same multiple of row i from row j.

$$A \xrightarrow{\rho_i \leftrightarrow \rho_j} \xrightarrow{\rho_j \leftrightarrow \rho_i} A \qquad A \xrightarrow{k\rho_i} \xrightarrow{(1/k)\rho_i} A \qquad A \xrightarrow{k\rho_i + \rho_j} \xrightarrow{-k\rho_i + \rho_j} A$$

(The third case requires that $i \neq j$.) QED

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We say that matrices that reduce to each other are equivalent with respect to the relationship of row reducibility. The next result justifies this, using the definition of an equivalence.

1.6 *Lemma* Between matrices, 'reduces to' is an equivalence relation.

The book has the full proof. For the intuition, consider this Gauss's method application.

$$\begin{pmatrix}1&2&-1\\2&4&-5\end{pmatrix}\xrightarrow{-2\rho_1+\rho_2}\begin{pmatrix}1&2&-1\\0&0&-3\end{pmatrix}$$

While our experience with Gauss's method leads us to feel that the second matrix in some way "comes after" the first, in fact the two are interreducible. Here are some other 2×3 matrices that are interreducible with those two.

$$\begin{pmatrix}1&2&-1\\0&0&1\end{pmatrix}\quad\begin{pmatrix}2&4&-2\\2&4&-1\end{pmatrix}\quad\begin{pmatrix}1&2&-1\\3&6&-6\end{pmatrix}$$

1.7 *Definition* Two matrices that are interreducible by elementary row operations are *row equivalent*.

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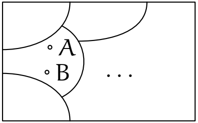
The diagram below shows the collection of all matrices as a box. Inside that box each matrix lies in a class. Matrices are in the same class if and only if they are interreducible. The classes are disjoint — no matrix is in two distinct classes. We have partitioned the collection of matrices into *row equivalence classes*.



Linear Combination Lemma

How Gauss's method acts

Example Consider this reduction.

$$\left(\begin{array}{cc|c} 1 & 3 & 5 \\ 2 & 4 & 8 \end{array}\right) \xrightarrow{-2\rho_1+\rho_2} \left(\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & -2 & -2 \end{array}\right) \xrightarrow{-(1/2)\rho_2} \left(\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & 1 & 1 \end{array}\right)$$

$$\xrightarrow{-3\rho_2+\rho_1} \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right)$$

Denote the matrices as $A \to D \to G \to B$. The steps take us through these row combinations.

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \xrightarrow{-2\rho_1+\rho_2} \begin{pmatrix} \delta_1 = \alpha_1 \\ \delta_2 = -2\alpha_1 + \alpha_2 \end{pmatrix}$$

$$\xrightarrow{-(1/2)\rho_2} \begin{pmatrix} \gamma_1 = \alpha_1 \\ \gamma_2 = \alpha_1 - (1/2)\alpha_2 \end{pmatrix}$$

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So: Gauss's method acts by taking linear combinations of rows.

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Proof Given the set $c_{1,1}x_1 + \cdots + c_{1,n}x_n$ through $c_{m,1}x_1 + \cdots + c_{m,n}x_n$ of linear combinations of the x's, consider a combination of those

$$d_1(c_{1,1}x_1 + \cdots + c_{1,n}x_n) + \cdots + d_m(c_{m,1}x_1 + \cdots + c_{m,n}x_n)$$

where the d's are scalars along with the c's. Distributing those d's and regrouping gives

$$= (d_1c_{1,1} + \cdots + d_mc_{m,1})x_1 + \cdots + (d_1c_{1,n} + \cdots + d_mc_{m,n})x_n$$

which is also a linear combination of the x's. QED

2.4 *Corollary* Where one matrix reduces to another, each row of the second is a linear combination of the rows of the first.

This states formally what is illustrated in the example showing how Gauss's method acts. The book contains the proof.

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Example The book contains the full proof. This matrix illustrates.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 0 & 8 \end{pmatrix}$$

Consider writing the second row as a combination of the others.

$$\begin{pmatrix} 0 & 5 & 6 & 7 \end{pmatrix} = c_1 \cdot (1 \;\; 2 \;\; 3 \;\; 4) + c_3 \cdot (0 \;\; 0 \;\; 0 \;\; 8)$$

We start by looking at the first row. With the matrix in echelon form, the second row's leading entry 5 lies to the right of the first row's 1. The equation of entries in 1's column, the first column, gives that $c_1 = 0$.

$$0 = c_1 \cdot 1 + c_3 \cdot 0$$

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Now for the third row. Again because of echelon from, the third row's leading entry lies to the right of the 5. The equation of entries in the 5's column gives an impossibility.

$$5 = c_3 \cdot 0$$

Summarizing the prior two lemmas: Gauss's method takes linear combinations of the rows, systematically eliminating any linear relationship among those rows.

Example In this non-echelon form matrix the third row is the sum of the first and second.

$$\begin{pmatrix} 1 & -1 & 3 \\ 2 & 0 & 4 \\ 3 & -1 & 7 \end{pmatrix}$$

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But after Gauss's method

$$\xrightarrow[-3\rho_1+\rho_3]{-2\rho_1+\rho_3} \begin{pmatrix} 1 & -1 & 3 \\ 0 & 2 & -2 \\ 0 & 2 & -2 \end{pmatrix} \xrightarrow{-\rho_2+\rho_3} \begin{pmatrix} 1 & -1 & 3 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

the only linear relationship among the nonzero rows

$$c_1 \cdot (1 \;\; -1 \;\; 3) = c_2 \cdot (0 \;\; 2 \;\; -2)$$

is the trivial relationship $c_1 = c_2 = 0$, since the equation of first entries gives that $c_1 = 0$ and then the equation of second entries gives $c_2 = 0$.

2.6 *Theorem* Each matrix is row equivalent to a unique reduced echelon form matrix.

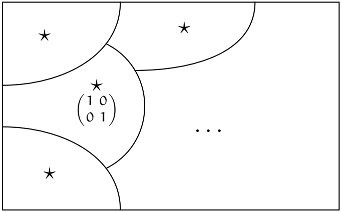
Example The book contains the full proof. This Gauss-Jordan reduction shows that the matrix on the left is row equivalent to the reduced echelon form matrix on the right.

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 0 \\ 3 & 6 & 1 \end{pmatrix} \xrightarrow[-3\rho_1+\rho_3]{-2\rho_1+\rho_2} \xrightarrow{-\rho_2+\rho_3} \xrightarrow{(1/2)\rho_2} \xrightarrow{-\rho_2+\rho_1} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

By the theorem the matrix on the left is not row equivalent to any of these.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

So the reduced echelon form is a canonical form for row equivalence: the reduced echelon form matrices are representatives of the classes.



Example To decide if these two are row equivalent

$$\begin{pmatrix} 3 & 2 & 0 \\ 1 & -1 & 2 \\ 4 & 1 & 2 \end{pmatrix} \qquad \begin{pmatrix} 3 & 1 & -2 \\ 6 & 2 & -4 \\ 1 & 0 & 2 \end{pmatrix}$$

use Gauss-Jordan elimination to bring each to reduced echelon form and see if they are equal.

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use Gauss-Jordan elimination to bring each to reduced echelon form and see if they are equal. The results are

$$\xrightarrow[-(4/3)\rho_1+\rho_3]{-(1/3)\rho_1+\rho_2} \xrightarrow{-1\rho_2+\rho_3} \xrightarrow[-(3/5)\rho_2]{(1/3)\rho_1} \xrightarrow{-(2/3)\rho_2+\rho_1} \begin{pmatrix} 1 & 0 & 4/5 \\ 0 & 1 & -6/5 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\xrightarrow[-(1/3)\rho_1+\rho_3]{-2\rho_1+\rho_2} \xrightarrow{\rho_2\leftrightarrow\rho_3} \xrightarrow[-3\rho_2]{(1/3)\rho_1} \xrightarrow{-(1/3)\rho_2+\rho_1} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -8 \\ 0 & 0 & 0 \end{pmatrix}$$

and therefore the original matrices are not row equivalent.