One.II Linear Geometry

Linear Algebra

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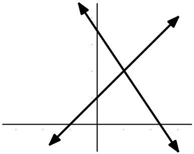
Geometry

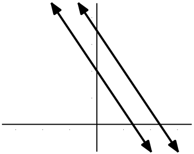
We can draw two-unknown equations as lines. Then the three possibilities for solution sets become clear.

Unique solution



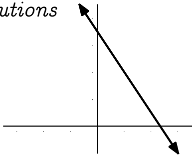
$$\begin{aligned} 3x + 2y &= 7 \\ x - y &= -1 \end{aligned}$$

No solutions



$$\begin{aligned} 3x + 2y &= 7 \\ 3x + 2y &= 4 \end{aligned}$$

Infinitely many solutions



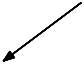
$$\begin{aligned} 3x + 2y &= 7 \\ 6x + 4y &= 14 \end{aligned}$$

This is a nice restatement of the possibilities; the geometry gives us insight into what can happen with linear systems.

Vectors in space

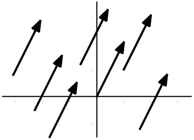
Vectors

A *vector* is an object consisting of a magnitude and a direction.



For instance, a vector can model a displacement.

Two vectors with the same magnitude and same direction, such as all of these, are equal.



For instance, each of the above could model a displacement of one over and two up.

Denote the vector that extends from (a_1, a_2) to (b_1, b_2) by

$$\begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix}$$

so the "one over, two up" vector would be written in this way.

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

We often picture a vector

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

as starting at the origin. From there $\vec{v}$ extends to (v_1, v_2) and we may refer to it as "the point $\vec{v}$" so that we may call each of these $\mathbb{R}^2$.

$$\{(x_1, x_2) \mid x_1, x_2 \in \mathbb{R}\} \qquad \{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mid x_1, x_2 \in \mathbb{R}\}$$

These definitions extend to higher dimensions. The vector that starts at $(a_1, \ldots, a_n)$ and ends at $(b_1, \ldots, b_n)$ is represented by this column

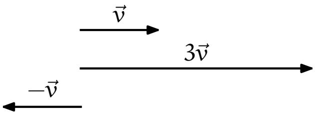
$$\begin{pmatrix} b_1 - a_1 \\ \vdots \\ b_n - a_n \end{pmatrix}$$

and two vectors are equal if they have the same representation. Also, we aren't too careful about distinguishing between a point and the vector which, when it starts at the origin, ends at that point.

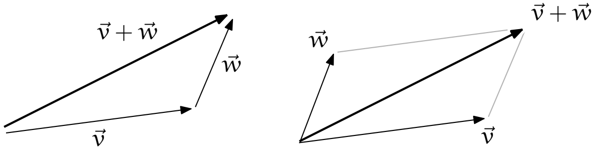
$$\mathbb{R}^n = \{ \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \mid v_1, \ldots, v_n \in \mathbb{R} \}$$

Vector operations

Scalar multiplication makes a vector longer or shorter, including possibly flipping it around.



Where $\vec{v}$ and $\vec{w}$ represent displacements, the vector sum $\vec{v}+\vec{w}$ represents those displacements combined.

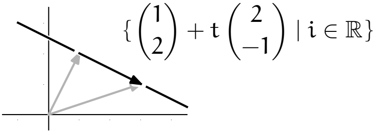


The second drawing shows the *parallelogram rule* for vector addition.

Lines

The line in $\mathbb{R}^2$ through $(1, 2)$ and $(3, 1)$ is comprised of the vectors in this set

$$\{\begin{pmatrix}1\\2\end{pmatrix} + t\begin{pmatrix}2\\-1\end{pmatrix} \mid t \in \mathbb{R}\}$$



(that is, it is comprised of the endpoints of those vectors). The vector associated with the parameter t

$$\begin{pmatrix}2\\-1\end{pmatrix}$$

is a *direction vector* for the line. Lines in higher dimensions work the same way.

Planes

The plane in $\mathbb{R}^3$ through the points $(1,0,5)$, $(2,1,-3)$, and $(-2,4,0.5)$ consists of (endpoints of) the vectors in this set.

$$\{\begin{pmatrix}1\\0\\5\end{pmatrix}+t\begin{pmatrix}1\\1\\-8\end{pmatrix}+s\begin{pmatrix}-3\\4\\-4.5\end{pmatrix}\mid t,s\in\mathbb{R}\}$$

For each column vector associated with a parameter

$$\begin{pmatrix}1\\1\\-8\end{pmatrix}=\begin{pmatrix}2\\1\\-3\end{pmatrix}-\begin{pmatrix}1\\0\\5\end{pmatrix}\qquad\begin{pmatrix}-3\\4\\-4.5\end{pmatrix}=\begin{pmatrix}-2\\4\\0.5\end{pmatrix}-\begin{pmatrix}1\\0\\5\end{pmatrix}$$

its whole body, not just its endpoint, lies in the plane.

A set of the form $\{\vec{p}+t_1\vec{v}_1+t_2\vec{v}_2+\cdots+t_k\vec{v}_k\mid t_1,\ldots,t_k\in\mathbb{R}\}$ where $\vec{v}_1,\ldots,\vec{v}_k\in\mathbb{R}^n$ and $k\leqslant n$ is a k-*dimensional linear surface* (or k-*flat*).

Length and angle measures

Length

2.1 *Definition* The *length* of a vector $\vec{v} \in \mathbb{R}^n$ is the square root of the sum of the squares of its components.

$$|\vec{v}\,| = \sqrt{v_1^2 + \cdots + v_n^2}$$

Example The length of

$$\begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix}$$

is $\sqrt{1+4+9} = \sqrt{14}$.

For any nonzero vector $\vec{v}$, the length one vector with the same direction is $\vec{v}/|\vec{v}|$. We say that this *normalizes* $\vec{v}$ to unit length.

Dot product

2.3 *Definition* The *dot product* (or *inner product* or *scalar product*) of two n-component real vectors is the linear combination of their components.

$$\vec{u} \bullet \vec{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

Example The dot product of two vectors

$$\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ -3 \\ 4 \end{pmatrix} = 3 - 3 - 4 = -4$$

is a scalar, not a vector.

The dot product of a vector with itself $\vec{v} \bullet \vec{v} = v_1^2 + \cdots + v_n^2$ is the square of the vector's length.

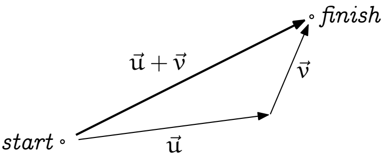
Triangle Inequality

2.5 *Theorem* For any $\vec{u}, \vec{v} \in \mathbb{R}^n$,

$$|\vec{u} + \vec{v}| \leqslant |\vec{u}| + |\vec{v}|$$

with equality if and only if one of the vectors is a nonnegative scalar multiple of the other one.

This is the source of the familiar saying, "The shortest distance between two points is in a straight line."



2.5 *Proof* Since all the numbers are positive, the inequality holds if and only if its square holds.

$$\begin{aligned} |\vec{u}+\vec{v}|^2 &\leqslant (\,|\vec{u}\,|+|\vec{v}\,|\,)^2 \\ (\,\vec{u}+\vec{v}\,)\cdot(\,\vec{u}+\vec{v}\,) &\leqslant |\vec{u}\,|^2+2\,|\vec{u}\,|\,|\vec{v}\,|+|\vec{v}\,|^2 \\ \vec{u}\cdot\vec{u}+\vec{u}\cdot\vec{v}+\vec{v}\cdot\vec{u}+\vec{v}\cdot\vec{v} &\leqslant \vec{u}\cdot\vec{u}+2\,|\vec{u}\,|\,|\vec{v}\,|+\vec{v}\cdot\vec{v} \\ 2\,\vec{u}\cdot\vec{v} &\leqslant 2\,|\vec{u}\,|\,|\vec{v}\,| \end{aligned}$$

That, in turn, holds if and only if the relationship obtained by multiplying both sides by the nonnegative numbers $|\vec{u}\,|$ and $|\vec{v}\,|$

$$2\,(\,|\vec{v}\,|\,\vec{u}\,)\cdot(\,|\vec{u}\,|\,\vec{v}\,) \leqslant 2\,|\vec{u}\,|^2\,|\vec{v}\,|^2$$

and rewriting

$$0 \leqslant |\vec{u}\,|^2\,|\vec{v}\,|^2 - 2\,(\,|\vec{v}\,|\,\vec{u}\,)\cdot(\,|\vec{u}\,|\,\vec{v}\,) + |\vec{u}\,|^2\,|\vec{v}\,|^2$$

is true. But factoring shows that it is true

$$0 \leqslant (\,|\vec{u}\,|\,\vec{v}-|\vec{v}\,|\,\vec{u}\,)\cdot(\,|\vec{u}\,|\,\vec{v}-|\vec{v}\,|\,\vec{u}\,)$$

since it only says that the square of the length of the vector $|\vec{u}\,|\,\vec{v}-|\vec{v}\,|\,\vec{u}$ is not negative.

As for equality, it holds when, and only when, $|\vec{u}\,|\,\vec{v}-|\vec{v}\,|\,\vec{u}$ is $\vec{0}$. The check that $|\vec{u}\,|\,\vec{v}=|\vec{v}\,|\,\vec{u}$ if and only if one vector is a nonnegative real scalar multiple of the other is easy. QED

Cauchy-Schwarz Inequality

2.6 *Corollary* For any $\vec{u}, \vec{v} \in \mathbb{R}^n$,

$$|\vec{u} \cdot \vec{v}| \leqslant |\vec{u}|\,|\vec{v}|$$

with equality if and only if one vector is a scalar multiple of the other.

2.6 *Proof* The Triangle Inequality's proof shows that $\vec{u} \cdot \vec{v} \leqslant |\vec{u}|\,|\vec{v}|$ so if $\vec{u} \cdot \vec{v}$ is positive or zero then we are done. If $\vec{u} \cdot \vec{v}$ is negative then this holds.

$$|\vec{u} \cdot \vec{v}| = -(\vec{u} \cdot \vec{v}) = (-\vec{u}) \cdot \vec{v} \leqslant |-\vec{u}|\,|\vec{v}| = |\vec{u}|\,|\vec{v}|$$

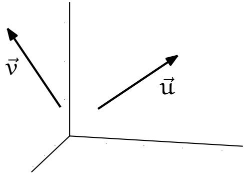
The equality condition is Exercise 19 .

Angle measure

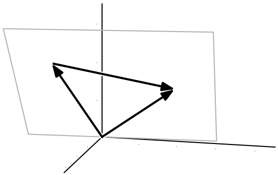
Definition The *angle* between two vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$ is this.

$$\theta = \arccos\left(\frac{\vec{u} \cdot \vec{v}}{|\vec{u}\,|\,|\vec{v}\,|} \right)$$

We motivate that definition with two vectors in $\mathbb{R}^3$.



If neither is a multiple of the other then they determine a plane, because if we put them in canonical position then the origin and the endpoints make three noncolinear points. Consider the triangle formed by $\vec{u}$, $\vec{v}$, and $\vec{u} - \vec{v}$.



Apply the Law of Cosines: $|\vec{u}-\vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2\,|\vec{u}|\,|\vec{v}|\cos\theta$ where θ is the angle that we want to find. The left side gives

$$\begin{aligned}(u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2 \\ = (u_1^2 - 2u_1v_1 + v_1^2) + (u_2^2 - 2u_2v_2 + v_2^2) + (u_3^2 - 2u_3v_3 + v_3^2)\end{aligned}$$

while the right side gives this.

$$(u_1^2 + u_2^2 + u_3^2) + (v_1^2 + v_2^2 + v_3^2) - 2\,|\vec{u}|\,|\vec{v}|\cos\theta$$

Canceling squares u_1^2 $\ldots$, v_3^2 and dividing by 2 gives the formula.

2.8 *Corollary* Vectors from $\mathbb{R}^n$ are orthogonal, that is, perpendicular, if and only if their dot product is zero. They are parallel if and only if their dot product equals the product of their lengths.