Printed: November 27, 2019 Name ______________________________

MATH 6012 Quiz8-2019 Answer: Key

Show your work.

1. Write the recursion $y_{n+1} = 3y_n + 2y_{n-1}$ in matrix form. (Do not solve!)

   Answer: Put $\vec{x}_n = \begin{bmatrix} y_{n+1} \\ y_n \end{bmatrix}$. Then $\vec{x}_{n+1} = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} \vec{x}_n$

2. Use matrix diagonalization to solve the recursion

Useful formula: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

$$x_{n+1} = 2x_n + y_n$$
$$y_{n+1} = x_n + 2y_n$$

   with initial values $x_0 = 4, y_0 = 2$.

   Answer: In vector form, $\vec{x}_{n+1} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \vec{x}_n$ with $\vec{x}_0 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$.

   Eigenvalues: $\det(A - \lambda I) = (\lambda - 2)^2 - 1 = (\lambda - 3)(\lambda - 1)$. So $\lambda_1 = 3$ with eigenvector $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\lambda_2 = 1$ with $v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

   We can now write the solution: $\vec{x}_n = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^n \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3^n & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 3^n & 1 \\ 3^n & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3^{n+1} + 1 \\ 3^{n+1} - 1 \end{bmatrix}$

   Answer: $x_n = 3^{n+1} + 1$, $y_n = 3^{n+1} - 1$

3. Use matrix exponentials (diagonalization) to solve the system of differential equations

$$u_1' = 2u_1 + u_2,$$
$$u_2' = u_1 + 2u_2$$

   with initial condition $u_1(0) = 4, u_2(0) = 2$. *Hint: use your diagonalization from the previous problem*

   Answer: Using diagonalization from previous problem, $\begin{bmatrix} u_1(t) \\ u_t(t) \end{bmatrix} = \exp\left(t \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right) \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{3t} & 0 \\ 0 & e^t \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} e^{3t} & e^t \\ e^{3t} & -e^t \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3e^{3t} + e^t \\ 3e^{3t} - e^t \end{bmatrix}$

   Answer: $u_1(t) = 3e^{3t} + e^t$ and $u_2(t) = 3e^{3t} - e^t$

   Answer: Standard solution from the DE book: From first equation, $u_2 = u_1' - 2u_1$, so the second equation becomes $u_1'' - 2u_1' = u_1 + 2(u_1' - 2u_1)$. This simplifies to

$$u_1'' - 4u_1' + 3u_1 = 0$$

   The initial conditions are $u_1(0) = 4$ and $u_1'(0) = 2u_1(0) + u_2(0) = 10$.

   The characteristic equation is $r^2 - 4r + 3 = (r - 1)(r - 3) = 0$ so the general solution is $u_1 = C_1e^t + C_2e^{3t}$. Using the initial condition

$$C_1 + C_2 = 4, \; C_1 + 3C_2 = 10$$

We determine that $C_1 = 3$ and $C_1 = 1$.

Answer: $\boxed{u_1(t) = 3e^{3t} + e^t}$ and $u_2(t) = 3e^{3t} - e^t$

Answer: Ad-hoc method:

Adding the equations we get $u_1' + u_2' = 3u_1 + 3u_2$. So we have an equation $y' = 3y$ with initial condition $y(0) = 6$ for $y = u_1 + u_2$. The solution of this equation is $y = Ce^{3t}$ with $C = 6$.

Subtracting the equations we get $u_1' - u_2' = u_1 - u_2$. So we have an equation $y' = y$ with $y(0) = 2$ for $y = u_1 - u_2$. The solution of this equation is $y = Ce^t$ with $C = 2$.

So $u_1 + u_2 = 6e^{3t}$ and $u_1 - u_2 = 2e^t$.

Adding the equations we get $2u_1 = 6e^{3t} + 2e^t$

Answer: $\boxed{u_1(t) = 3e^{3t} + e^t}$

Answer: Laplace transform solution: Taking the Laplace transforms of both sides of the equations we get

$$sU_1(s) - 4 = 2U_1 + U_2 \text{ and } sU_2 - 2 = U_1 + 2U_2$$

where $U_1(s) = \mathcal{L}(u_1)$. Solving the system of equations

$$\begin{aligned} (s-2)U_1 - U_2 &= 4 \\ -U_1 + (s-2)U_2 &= 2 \end{aligned}$$

we get

$$U_1 = \frac{\begin{vmatrix} 4 & -1 \\ 2 & s-2 \end{vmatrix}}{\begin{vmatrix} s-2 & -1 \\ -1 & s-2 \end{vmatrix}} = \frac{4s-6}{(s-2)^2 - 1^2} = \frac{4s-6}{((s-2)-1)((s-2)+1)} = \frac{4s-6}{(s-3)(s-1)} = \frac{3}{s-3} + \frac{1}{s-1}$$

From the table of Laplace transforms we read out the answer $\boxed{u_1 = 3e^{3t} + e^t}$