One more version of Quiz 3

1. Find a basis and the dimension for the following subspace V of $\mathcal{M}_{2\times 2}$:

$$V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \quad a+b+c+d=0 \right\}$$

Answer: Since $a = -b-c-d$ we see that if $A \in V$ then

$$A = \begin{bmatrix} -b-c-d & b \\ c & d \end{bmatrix} = b \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

and this representation is unique. This gives a basis

$$\mathcal{B} = \left\langle \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \right\rangle$$

Other solutions are possible, but they alsways have 3 vectors so the dimension is 3.

Is identity matrix I_2 in V? Is vector $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ in V? Is polynomial x^2 in V? Is function $\sin \pi x$ in V?

Answer: None are in V. Only the first question "makes sense"!

2. Suppose that $T : \mathcal{M}_{2\times 2} \to \mathcal{P}_2$ is a linear map which to a matrix A assigns quadrtic polynomial

$$p(x) = \begin{bmatrix} 1 & x \end{bmatrix} \times A \times \begin{bmatrix} 1 \\ x \end{bmatrix}$$

   (a) Confirm (i.e., convince yourself!) that this is linear map. **Answer:** This is for your own satisfaction!

   However, I will write an explicit formula which can be used to verify linearity and which is used several times below.

   $\boxed{\text{If } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ and } p = T(A), \text{ then } p(x) = a + (b+c)x + dx^2.}$

(b) Find the matrix representation of T, using the standard basis of $\mathcal{M}_{2\times 2}$ and the standard basis of $\mathcal{P}_2$. **Answer:** The standard basis of $\mathcal{M}_{2\times 2}$ is

$$\mathcal{B} = \left\langle \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\rangle$$

The order here is arbitrary but needs to be fixed for the solution. The actual answer depends on this order!

The standard basis of $\mathcal{P}_2$ is $\mathcal{C} = \langle 1, x, x^2 \rangle$. Again, the answer depend on the order in which we write these monomials, and it might be different if we take $\langle x^2, x, 1 \rangle$ as a basis. Because of that you need to specify the order before you write down your solution. My answer is with respect to bases $\mathcal{B}$ and $\mathcal{C}$ written above.

There are two steps to find the matrix: compute polynomials that correspond to matrices in $\mathcal{B}$, and then expand the answers in basis $\mathcal{C}$. Both steps are straightforward: First, the calculation,

$$T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = 1, \quad T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = x, \quad T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = x, \quad T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = x^2$$

The second step is to expand these polynomials in basis $\mathcal{C}$.

$$Rep_{\mathcal{C}}(1) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, Rep_{\mathcal{C}}(x) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, Rep_{\mathcal{C}}(x^2) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

This gives the columns of the matrix we seek:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(Can you explain why we have four columns, not three?)

There are other more ad-hoc ways of getting this answer. The solution I wrote up is the "routine" solution that works for all vector spaces.

(c) Find a basis for $Range(T)$, and determine the $rank(T)$. Is polynomial x^2 in $Range(T)$?

Answer: #1: Lets use matrix representation: Since A has pivot in every row, T is onto. So the range of T is the entire codoman $\mathcal{P}_2$, with basis $\mathcal{C}$. The rank of T is 3.

Answer: #2: Lets guess that T is onto and then prove it from the definition. Given a polynomial $p = a + bx + cx^2$, we can find many matrices A such that $T(A) = p$. For example, $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ or $A = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix}$ works. So T is onto, and we also see that T is not one-to-one, because I've shown two different matrices A that are mapped to the same polynomial p).

(d) Find a basis for the null space $Null(T)$ and its dimension.

Answer: #1: Lets work this out using coordinates: The null space of T consists of 2×3 matrices that are mapped to 0. So their coordinates are mapped to 0 by matrix A. So we solve the equation $A\vec{x} = 0$. Matrix A is in echelon form with free variable x_3 and $x_1 = 0, x_2 = -x_3, x_4 = 0$

So the solution is $\vec{x} = Span \left\{ \begin{bmatrix} 0 \\ t \\ -t \\ 0 \end{bmatrix} \right\}$ with arbitrary t.

These are the coordinates in basis $\mathcal{B}$ of all matrices of the form $\begin{bmatrix} 0 & t \\ -t & 0 \end{bmatrix}$.

So the Null space of T is one-dimensional, and is spanned by matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. (This is of course one of many possible bases for the null space.)

Answer: #2: Lets use the definition: For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we have $T(A) = a + (b + c)x + dx^2$. So $T(A) = 0$ iff $a = 0$, $b + c = 0$, and $d = 0$. So $A = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} = a \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ i.e. the null space of T is is spanned by matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. (This is of course one of many possible bases for the null space.)

(e) State whether the map T is onto and whether it is one-to-one. (Justify your answers!) **Answer:** T is onto but not one-to-one. You need to hunt for justification in the blue text above or below.

Another version of Quiz 3

1. Find a basis and the dimension for the following subspace V of $\mathcal{P}_3$:

$$V = \left\{ p \in \mathcal{P}_3 : \quad \int_0^1 p(x)dx = 0 \right\}$$

   - Is identity matrix I_2 in V? **Answer:** No this is not a polynomials
   - Is vector $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ in V? **Answer:** No this is not a polynomials
   - Is polynomial x^2 in V? **Answer:** No, $\int_0^1 x^2 dx = 1/3 \neq 0$
   - Is function $\sin \pi x$ in V? **Answer:** No this is not a polynomials

   Answer: Lets look at p in the standard coordinates. Write $p(x) = a + bx + cx^2 + dx^3$. Then the condition for $p \in V$ says that $\int_0^1 (a + bx + cx^2 + dx^3)dx = 0$. So $a + b/2 + c/3 + d/4 = 0$ so $a = -b/2 - c/3 - d/4$ and $p(x) = b(x - 1/2) + c(x^2 - 1/3) + d(x^3 - 1/4)$

   This representation is unique, so a basis for V is $\mathcal{B} = \left\{x - 1/2, x^2 - 1/3, x^3 - 1/4\right\}$ and the dimension $\dim V = 3$

   There are other bases, like, $\mathcal{B}_1 = \left\{2x - 1, (x - 1)(3x - 1), (2x - 1)^3\right\}$

2. Suppose that $T : \mathcal{P}_3 \to \mathcal{P}_3$ is the following linear transformation which to a polynomial p assigns polynomial $q = T(p)$ given by $q(x) = xp''(x) - 2p'(x)$.

   (a) Find the matrix representation of T in the standard basis of $\mathcal{P}_3$.

   Answer: Corrected Wednesday, October 2, 2019 16:48

   Compute $T(1) = 0$, $T(x) = -2$, $T(x^2) = -2x$, $T(x^3) = 0$. So the matrix representation of T is $A = \begin{bmatrix} 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

   (b) Find a basis for $Range(T)$, and determine $rank(T)$. Is polynomial $(x - 2)^2$ in $Range(T)$?[1]

   Answer: The column space of A is the span of the middle two columns. So the range of T is the span of $1, x$, i.e. all linear polynomials.

   In particular, $(x - 2)^2$ is not such a linear combination. So $(x - 2)^2$ is not in the range of T.

   (c) Find a basis for the null space $Null(T)$ and its dimension.

   Answer: If $p = a + bx + cx^2 + dx^3$ then $T(p) = 0$ iff $\begin{bmatrix} 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0$. The first and last variables are free, and $b = c = 0$. So $Null(T)$ consists of polynomials $a + dx^3$. A basis of $Null(T)$ is $\langle 1, x^3 \rangle$ and the dimension is 2.

   (d) State whether the map T is onto and whether it is one-to-one. (Justify your answers!)

   Answer: The map is not onto because the range space is not all of the codomain. The map is not one-to-one because the null space is not trivial.

[1] That is, can we seek a quadratic polynomial as a particular solution of the differential equation $xy'' - 2y' = (x - 2)^2$?