Printed: September 20, 2019 Name ____________________

MATH 6012 Quiz3-2019$_{\text{A}}$ Answer: Key

Show your work.

1. Find a basis and the dimension for the following subspace of $\mathbb{R}^4$:

$$R = \left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} : \quad x + y + z + w = 0 \right\}$$

Answer: With y, z, w free, $\vec{v} \in$R iff

$$\vec{v} = \begin{bmatrix} -y - z - w \\ y \\ z \\ w \end{bmatrix} = y \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + w \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

This representation of R as a span of three vectors is unique, so a basis for R is

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

and $\dim R = 3$

2. Suppose that matrix $A = \begin{bmatrix} 0 & 1 & 3 \\ 2 & 3 & 4 \\ -2 & -1 & 2 \end{bmatrix}$ represents a linear map $T : \mathbb{R}^m \to \mathbb{R}^n$ with $T(\vec{x}) = A\vec{x}$.

(i) Find a basis for $Range(T)$, and $rank(T)$

Answer: The answers are based on calculation of $REF(A)$, which is here augmented with arbitrary right hand side vector $\vec{b} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

$$\begin{bmatrix} 0 & 1 & 3 & a \\ 2 & 3 & 4 & b \\ -2 & -1 & 2 & c \end{bmatrix} \xrightarrow{\rho_1 \leftrightarrow \rho_2} \xrightarrow{\rho_1 + \rho_3} \xrightarrow{-2\rho_2 + \rho_3} \xrightarrow{(1/2)\rho_1} \xrightarrow{-(3/2)\rho_2 + \rho_1} \begin{bmatrix} 1 & 0 & -5/2 & -(3/2)a + (1/2)b \\ 0 & 1 & 3 & a \\ 0 & 0 & 0 & -2a + b + c \end{bmatrix}$$

This shows that there are codomain triples

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3$$

for which the system $A\vec{x} = \vec{b}$ does not have a solution. Specifically the system only has a solution if $-2a + b + c = 0$. This gives the following description of the range

$$Range(T) = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a = (b + c)/2 \right\} = \left\{ \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} b + \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix} c : \quad b, c \in \mathbb{R} \right\}$$

Note there are many other answers. For example, from $REF(A)$ we see that the columns of A are linearly dependent but the first two columns of A are linearly independent. So the Range of T is the span of the first two columns on A, i.e. $Range(T) = Span \left\{ \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} \right\}$,

The map's rank is the range's dimension, 2

(ii) Find a basis for the null space $Null(T)$ and its dimension,

Answer: Setting $a = b = c = 0$ in the calculation gives infinitely many solutions. Paramatrizing using the free variable z leads to this description of the nullspace.

$$Nullspace(T) = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : \quad y = -3z \text{ and } x = (5/2)z \right\} = \left\{ \begin{bmatrix} 5/2 \\ -3 \\ 1 \end{bmatrix} z : \quad z \in \mathbb{R} \right\}$$

The nullity is the dimension of that null space, 1.

(iii) State whether the map T is onto and whether it is one-to-one. (Justify your answers!)

Answer: (iv) The map is not onto because the range space is not all of the codomain. The map is not one-to-one because the null space is not trivial.