Printed: September 20, 2019 Name ______________________________

Use extra paper but staple this page on top of your answers.

Take-home MATH 6012 Quiz2-2019$_A$ due Mo, Sept 16 at 2:31PM
Answer: Key

Select the total of **three** *questions to turn in: one on linear independence, one on bases, one on coordinates. You may work with others to figure out how to do questions, and you are welcome to look for answers in the book, online, by talking to someone who had the course before, etc. However, you must write the answers on your own. You must also show your work.*

I **Linear independence:** Turn in solution of **one** of the following

Q 1.1: Check whether polynomials $p_1(t) = t^2, p_2(t) = (1-t)^2, p_3(t) = (1+t)^2$ are linearly independent.
Answer: There are many ways of solving this question: one can compute the derivatives, one can express the problem in the standard coordinates, or one can choose enough values of t.
Here is one of the solutions by the latter method for a slightly different question: show that $q_1(t) = t^3, q_2(t) = (1-t)^3, q_3(t) = (1+t)^3$ are linearly independent.
Consider

$$f(t) = C_1t^3 + C_2(1-t)^3 + C_3(1+t)^3$$

and suppose that $f(t) = 0$ for all t. Then $f(0) = 0$ and $f(1) = 0$ and $f(-1) = 0$ so we get the following system of equations:

$$\begin{aligned} C_2 + C_3 &= 0 && (1)\\ C_1 + 8C_3 &= 0 && (2)\\ -C_1 - 8C_2 &= 0 && (3) \end{aligned}$$

The matrix of this system is $A = \begin{bmatrix} 0 & 1 & 1\\ 1 & 0 & 8\\ -1 & -8 & 0 \end{bmatrix}$. Now

$$\det A = \det \begin{bmatrix} 0 & 1 & 1\\ 1 & 0 & 8\\ 0 & -8 & 8 \end{bmatrix} = -\det \begin{bmatrix} 1 & 1\\ -8 & 8 \end{bmatrix} = -16 \neq 0$$

So A is invertible. We now use the invertibility as follows:
Write $\vec{C} = \begin{bmatrix} C_1\\ C_2\\ C_3 \end{bmatrix}$. The system of equations (1-3) in vector notation is $A\vec{C} = \vec{0}$. Since A is invertible, the equation $A\vec{C} = \vec{0}$ has the unique solution $\vec{C} = A^{-1}\vec{0} = \vec{0}$. So $\vec{C} = \begin{bmatrix} C_1\\ C_2\\ C_3 \end{bmatrix} = \begin{bmatrix} 0\\ 0\\ 0 \end{bmatrix}$.
This shows that $C_1 = C_2 = C_3 = 0$ is the only choice of the coefficients for the linear combination to make $f(t) = 0$ for all t.

Q 1.2: Check whether functions $f_1(x) = \frac{1}{x+1}, f_2(x) = \frac{1}{x-1}, f_3(x) = \frac{x}{x^2-1}$ are linearly independent **Answer:** They are not linearly independent as $f_3 = (f_1 + f_2)/2$.

Q 1.3: Check whether functions $g_0(x) = 1, g_1(x) = x, g_2(x) = xe^x, g_3(x) = x^2e^x$ are linearly independent.
Answer: Suppose

$$c_0g_0 + c_1g_1 + c_2g_2 + c_3g_3 = 0 \text{ for all real } x. \qquad (*)$$

Evaluating this expression at $x = 0$ we get $c_0 = 0$.
So we are left with

$$c_1 g_1 + c_2 g_2 + c_3 g_3 = 0$$

which is the same as

$$c_1 x + c_2 x e^x + c_3 x^2 e^x = 0$$

Since this holds for all x, we can divide by $x \neq 0$ and get

$$c_1 + c_2 e^x + c_3 x e^x = 0$$

Taking the limit as $x \to -\infty$ we get $c_1 = 0$.
So we are left with $c_2 e^x + c_3 x e^x = 0$. Dividing by e^x we get $c_2 + c_3 x = 0$, and by continuity this must hold also at $x = 0$. Evaluating at $x = 0$, we get $c_2 = 0$.
So we are left with $x c_3 = 0$. Putting $x = 1$ we get $c_3 = 0$.
In summary, we showed that if (**??**) holds then $c_0 = c_1 = c_2 = c_3 = 0$ i.e. the functions are linearly independent.

II **Bases:** Turn in solution of **one** of the following

Q 2.1: Give two different bases for $\Re^3$. Verify that each is a basis.
Answer: There is an infinite number of answers. Perhaps the easiest for proofs are choices easy to convert into in echelon form like $\mathcal{B}_1 = \langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \rangle$, with the matrix is already in echelon form and $\mathcal{B}_2 = \langle \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \rangle$ where the rows can be permuted into echelon form.

Q 2.2: Find a basis for the subspace $M = \{a + bx + cx^2 + dx^3 :$ such that $a + b + c - d = 0\}$ of $\mathcal{P}_3$. Verify that it is a basis.
Answer: There are many answers. Since the condition for $p \in M$ is $p(x) = a + bx + cx^2 + (a + b + c)x^3 = a(1 + x^2) + b(x + x^3) + c(x^2 + x^3)$, it is clear that

$$M = span\{1 + x^3, x + x^3, x^2 + x^3\}$$

This is a basis, after we check that these polynomials are linearly independent: Suppose

$$c_1(1 + x^3) + c_2(x + x^3) + c_3(x^2 + x^3) = 0$$

Then $c_1 + c_2 x + c_3 x^2 + (c_1 + c_2 + c_3)x^3 = 0 + 0x + 0x^2 + 0x^3$ so comparing the coefficients at $1, x, x^2$ we get $c_1 = c_2 = c_3 = 0$, proving linear independence. (We can also invoke a theorem from the book that every $p \in M$ is a unique linear combination of $\{1 + x^3, x + x^3, x^2 + x^3\}$, so this is a basis.)

Q 2.3: Find a basis for the subspace

$$W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ such that } a - d = 0 \right\}$$

of $\mathcal{M}_{2\times 2}$. Verify that it is a basis.
Answer: Lets write W as a span. If $A \in W$ then

$$A = \begin{bmatrix} a & b \\ c & a \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

So

$$W = span \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$$

It is not hard to verify that these three matrices are linearly independent so W is a 3-dimensional subspace of $\mathcal{M}_{2\times 2}$ with the above basis.

Q 2.4: Find a basis for the subspace

$$P = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ such that } x + 2y = 0 \right\}$$

of $\mathbb{R}^3$.

Answer: Lets write P as the span: if $\vec{v} \in P$ then $\vec{v} = \begin{bmatrix} -2y \\ y \\ z \end{bmatrix} = y \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. So

$$P = span \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

i.e. as expected, P is a plane through origin in $\mathbb{R}^3$.

III **Coordinates:** Turn in solution of **one** of the following

Q 3.1: Represent vector $\vec{v} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ with respect to each of the two bases.

$$\mathrm{B}_1 = \left\langle \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\rangle, \ \mathcal{B}_2 = \left\langle \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\rangle$$

Answer: This problem asks for solutions of two systems of equations: $Rep_{\mathcal{B}_1}(\vec{v}) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

So we get

$$Rep_{\mathcal{B}_1}(\vec{v}) = \frac{1}{2} \begin{bmatrix} -3 \\ 5 \end{bmatrix}$$

For the second basis, $Rep_{\mathcal{B}_2}(\vec{v}) = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ where

$$\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

which gives

$$Rep_{\mathcal{B}_2}(\vec{v}) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Q 3.2: Consider the vector space V of all symmetric 2 by 2 matrices with the basis $\mathcal{B} = \langle A_1, A_2, A_3 \rangle$ where $A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, $A_3 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

i. Find the coordinates of the matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ in this basis.

Answer: The routine way to answer this question is to solve the system of equations for the new coordinates. Here is a "slick way" to get the asnwer: $A_2 - A_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $A_3 - A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $A_3 - A_1 = I$ so $A = 2A_1 - A_3 + (A_2 - A_1) = A_1 + A_2 - A_3$. $[A]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

ii. Which matrix C has coordinates $Rep_{\mathcal{B}}(C) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$?

Answer: This is much easier than the previous question:
$B = A_1 + 2A_2 + 3A_3 = \begin{bmatrix} 9 & 6 \\ 6 & 11 \end{bmatrix}$

Q 3.3: Consider the following basis $\mathcal{B} = \langle 1, 1+t, (1+t)^2 \rangle$ of the vector space $\mathcal{P}_2$ of quadratic polynomials.

i. Which polynomial p has coordinates $Rep_{\mathcal{B}}(p) = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$? Simplify your answer.

Answer: $p(t) = 3 + 2(1+t) + (1+t)^2 = 6 + 4t + t^2$

ii. What are the coordinates of the monomial t^2 in basis $\mathcal{B}$?

Answer: $t^2 = (t+1-1)^2 = (t+1)^2 - 2(t+1) + 1$ so the coordinates are $[t^2]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

Q 3.4: Consider the subspace $\mathbf{H}$ of $\mathbb{R}^4$ spanned by the vectors $\vec{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{b}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \end{bmatrix}$, $\vec{b}_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix}$. Assume (without checking) that $\mathcal{B} = \langle \vec{b}_1, \vec{b}_2, \vec{b}_3 \rangle$ is linearly independent so that $\mathcal{B}$ is a basis of $\mathbf{H}$. Find the coordinates of vector $\vec{v} = \begin{bmatrix} 5 \\ 7 \\ 11 \\ 11 \end{bmatrix}$ in basis $\mathcal{B}$.

Answer: We need to solve the system of equations $c_1\vec{b}_1 + c_2\vec{b}_2 + c_3\vec{b}_3 = \vec{v}$ for the unknown coordinates c_1, c_2, c_3 of vector $\vec{v}$ in basis $\mathcal{B}$. That is

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 11 \\ 11 \end{bmatrix}$$

The augmented matrix of this system is

$$\begin{bmatrix} 1 & 1 & 1 & 5 \\ 1 & 2 & 2 & 7 \\ 1 & 2 & 3 & 11 \\ 1 & 2 & 3 & 11 \end{bmatrix}$$

After row reduction we get

$$\begin{bmatrix} 1 & 1 & 1 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solving the resulting system $c_1 + c_2 + c_3 = 5$, $c_2 + c_3 = 2$, $c_3 = 4$, we get $c_1 = 3, c_2 = -2, c_3 = 4$. so the coordinates of $\vec{v}$ in basis $\mathcal{B}$ are given by $Rep_{\mathcal{B}}(\vec{v}) = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$