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Exercises for applications of matrix diagonalization

1. Use matrix diagonalization to solve the recursion

$$y_{n+1} = 5y_n - 6y_{n-1}, \quad y_0 = 2, y_1 = 5$$

Answer: With $\vec{x}_n = \begin{bmatrix} y_{n+1} \\ y_n \end{bmatrix}$ we have $y_n = \begin{bmatrix} 0 & 1 \end{bmatrix} \vec{x}_n$ and $\vec{x}_{n+1} = A\vec{x}_n$ with $A = \begin{bmatrix} 5 & -6 \\ 1 & 0 \end{bmatrix}$. So $\vec{x}_n = A^n \begin{bmatrix} 5 \\ 2 \end{bmatrix}$. It remains to determine A^n.

Eigenvalues: $\det(A - \lambda I) = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3)$

Eigenvectors: $\lambda_1 = 2$ has $\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\lambda_2 = 3$ has eigenvector $\vec{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

So $A^n = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2^n & 0 \\ 0 & 3^n \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2^n & 0 \\ 0 & 3^n \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}$

We now plug this into the formula for y_n:

$y_n = [01] \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2^n & 0 \\ 0 & 3^n \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 2^n & 0 \\ 0 & 3^n \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 2^n \\ 3^n \end{bmatrix} = 2^n + 3^n$

Answer: $y_n = 2^n + 3^n$

2. Use matrix diagonalization to solve the differential equation

$$y'' = 5y' - 6y, \quad y(0) = 2, y'(0) = 5$$

Answer: With $\vec{x}(t) = \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}$ we have $y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \vec{x}(t)$ and $x'(t) = Ax(t)$ with $A = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix}$ and $x(0) = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$

So $y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} e^{At} \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ and it remains to compute matrix exponential $e^{At} = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n$.

To do so, we determine A^n.

Eigenvalues: $\det(A - \lambda I) = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3)$

Eigenvectors: $\lambda_1 = 2$ has $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\lambda_2 = 3$ has eigenvector $\vec{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

So $A^n = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2^n & 0 \\ 0 & 3^n \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2^n & 0 \\ 0 & 3^n \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$

Matrix exponential: $e^{At} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \sum_{n=0}^{\infty} \frac{t^n}{n!} \begin{bmatrix} 2^n & 0 \\ 0 & 3^n \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$

We now plug this into the formula for $y(t)$:

$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} e^{At} \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = e^{2t} + e^{3t}$

Answer: $y(t) = e^{2t} + e^{3t}$

3. Use matrix diagonalization to solve the vector recursion

$$\begin{aligned} x_{n+1} &= -x_n + y_n \\ y_{n+1} &= x_n - y_n \\ x_0 &= 0 \\ y_0 &= 2 \end{aligned}$$

Answer: In vector form $\vec{x}_n = \begin{bmatrix} x_n \\ y_n \end{bmatrix}$, this recursion is $\vec{x}_{n+1} = A\vec{x}_n$ with $A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ and $\vec{x}_0 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

The eigenvalues and eigenvectors of A are $\lambda_1 = -2$ $\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\lambda_2 = 0$ with $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

So for $n > 0$ we have $A^n = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} (-2)^n & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ and

$\vec{x}_n = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} (-2)^n & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -(-2)^n & 0 \\ (-2)^n & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Answer: $x_n = -(-2)^n = (-1)^{n+1}2^n$, $y_n = (-1)^n 2^n$ for $n \geq 1$ but not for $n = 0$!

4. Use matrix diagonalization to solve the system of differential equations

$$\begin{aligned} u' &= -u + v \\ v' &= u - v \\ u(0) &= 0 \\ v(0) &= 2 \end{aligned}$$

Answer: In vector form, $\vec{x}(t) = \begin{bmatrix} u(t) \\ v(t) \end{bmatrix}$ satisfies $\vec{x}' = A\vec{x}$ with $A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ and $\vec{x}(0) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$.

Previous calculations give

$e^{At} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{-2t} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ so

$\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{-2t} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -e^{-2t} & 1 \\ e^{-2t} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Answer: $u(t) = 1 - e^{-2t}$, $v(t) = 1 + e^{-2t}$