Last week ~~[illegible]~~ of classes Appl. Lin. Alg

Recall the spectral theorem: every symmetric (real, $n \times n$) matrix has orthogonal diagonalization

If $A = A^T$ then there exists an orthogonal matrix U and a diagonal matrix Λ such that

$$A = U \Lambda U^T$$

Procedure:

1) Find eigenvalues of A
2) Find basis for each eigenspace
3) Apply Gram-Schmidt to each basis
4) Normalize to get orthonormal basis of $\mathbb{R}^n$

U

Quadratic forms: $Q(x_1 \dots x_n)$ is a linear combination of $x_i x_j$

Matrix representation: $Q(\vec{x}) = \vec{x}^T A \vec{x}$ with symmetric A

Examples: $x_1 x_2 \quad = \vec{x}^T \begin{bmatrix} & \\ & \end{bmatrix} \vec{x}$

$4x_1^2 + 9x_2^2$

$2x_1x_2 + 4x_2x_3 + 13x_3^2$

$$\vec{x}^T \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \vec{x} = \vec{x}^T \begin{bmatrix} 1 & 3 \\ 3 & 3 \end{bmatrix} \vec{x}$$

Relation to inner products: $\langle \vec{x}, \vec{y} \rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i y_j a_{ij}$ with $a_{ij} = a_{ji}$

$\langle \vec{x}, \vec{x} \rangle \geq 0 \qquad = \vec{x}^T A \vec{y}$

Positive definite quadratic forms: $Q(\vec{x}) \geq 0$ ~~(and 0~~

Multivariate calculus

$f(x,y) \approx f(x_0,y_0) + (x-x_0)\frac{\partial f}{\partial x}(x_0,y_0) + (y-y_0)\frac{\partial f}{\partial y}(x_0,y_0) + \frac{1}{2}(x-x_0)^2 \frac{\partial^2 f}{\partial x^2} + (x-x_0)(y-y_0)\frac{\partial^2 f}{\partial x \partial y} + \dots$

Principal axes theorem There is an orthogonal change of variable $\vec{x} = \mathcal{U}\vec{y}$, which transform $\vec{x}^T A \vec{x}$ into a quadratic form $\vec{y}^T \Lambda \vec{y}$ with no cross-product terms

$\vec{y} = [x]_{\mathcal{U}}$

Proof $x^T A \vec{x} = \vec{y}^T P^T P \Lambda P^T P \vec{y}$

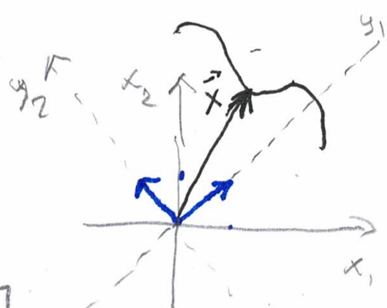
Corollary Positive quadratic forms have positive eigenvalues for A

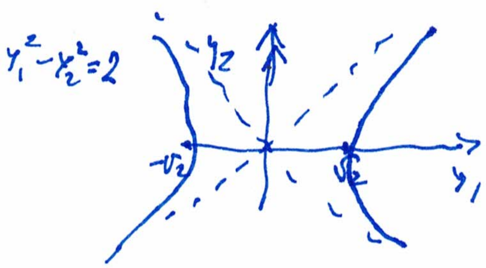
Using principal axes to draw conic curves : change of basis

Easy graphs $4x_1^2 + 9x_2^2 = 36$

$4x_1^2 - 9x_2^2 = 36$

Example $x_1 x_2 = 1$ $\vec{x} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \vec{y}$



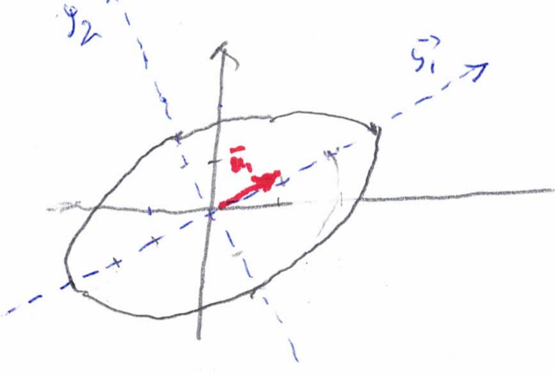


$\boxed{\vec{y} = Rep_{\mathcal{U}}(\vec{x})}$

Example Sketch the curve $8x_1^2 - 4x_1x_2 + 5x_2^2 = 36$

$A = \begin{bmatrix} 8 & -2 \\ -2 & 5 \end{bmatrix}$ Eigenvalues $\lambda_1 = 4$ $\lambda_2 = 9$ So the graph is ellipse

New coords $\vec{u}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} -1 \\ +2 \end{bmatrix}$ $\vec{u}_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ New eqtn $4y_1^2 + 9y_2^2 = 36$



Find all local minima / maxima of a function

Note Min: matrix of partial derivatives has <u>positive</u> eigenvalues

Max: <u>negative</u> eigenvalues

(zero $\Rightarrow$ unresolved case

neg <u>and</u> positive $\Rightarrow$ not a max not a min

<u>Example</u>

$$f(x,y) = \exp\left(-\tfrac{1}{3}x^3 + x - y^2\right)$$

<u>Critical points</u>: $\dfrac{\partial f}{\partial x} = 0 \qquad \dfrac{\partial f}{\partial y} = 0$

<u>Exercises from internet</u>: Classify critical points of

$$4 + x^3 + y^3 - 3xy$$

$$3x^2y + y^3 - 3x^2 - 3y^2 + 2$$

$$ye^x - 3x - y + 2$$

<u>Note</u> There are shortcuts for functions of 2 variables

$$\det(A - \lambda I) = \lambda^2 - \underbrace{\operatorname{tr} A}_{\lambda_1+\lambda_2} \cdot \lambda + \underbrace{\det A}_{\lambda_1 \cdot \lambda_2}$$

Must be positive if min or max

<u>Note</u> Three variables are less obvious:

$$\det(A - \lambda I) = -\lambda^3 + \lambda^2 \underbrace{\operatorname{tr} A}_{(\lambda_1+\lambda_2+\lambda_3)} - \underbrace{\dots}_{(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3)} \lambda + \underbrace{\det A}_{\lambda_1\lambda_2\lambda_3}$$

$$(\lambda_1-\lambda)(\lambda_2-\lambda)(\lambda_3-\lambda)$$

How to see if $\lambda_1 > 0$ $\lambda_2 > 0$ $\lambda_3 > 0$

Example: $x^2 + y^2 - 2yz + z^2$

$\frac{\partial f}{\partial x} = 2x$

$x = 0 \quad y = z$ critical line

$\frac{\partial f}{\partial y} = 2y - 2z$

$\frac{\partial f}{\partial z} = 2z - 2y$

Hessian

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & -2 \\ 0 & -2 & 2 \end{bmatrix}$$

$\lambda_1 = 2 \quad \lambda_2 = 0 \quad \lambda_3 = 4$ unresolved

In fact, min as $f(x,y,z) = x^2 + (y-z)^2$