Inner products

Def An inner product on vector space V is a mapping $\vec{V} \times \vec{V} \to \mathbb{R}$ (or $\vec{V} \times \vec{V} \to \mathbb{C}$) such that

bilinear (i) $\vec{v} \mapsto \langle \vec{v}, \vec{w} \rangle$ is linear for every $\vec{w}$

symmetric (ii) $\langle \vec{v}, \vec{w} \rangle = \langle \vec{w}, \vec{v} \rangle$ (or $\overline{\langle \vec{w}, \vec{v} \rangle}$ when $\mathbb{C}$)

positive (iii) $\langle \vec{v}, \vec{v} \rangle \geq 0$ and $\langle \vec{v}, \vec{v} \rangle = 0$ only when $\vec{v} = 0$

Examples of mappings: (1) $V = \mathbb{R}^n$ $\quad \langle \vec{v}, \vec{w} \rangle = v^T w$ (dot product)

(2) $V = \mathbb{R}^2$ $\quad \langle \vec{v}, \vec{w} \rangle = \det [\vec{v}, \vec{w}]$ (not inner product)

(3) $V = \mathbb{R}^n$ with basis $\mathcal{B} = \{\vec{b}_1, \vec{b}_2 \dots \vec{b}_n\}$

$$\langle \vec{v}, \vec{w} \rangle := \mathrm{Rep}_{\mathcal{B}}(\vec{v}) \cdot \mathrm{Rep}_{\mathcal{B}}(\vec{w})$$

$$= \mathrm{Rep}_{\mathcal{B}}(\vec{v})^T \, \mathrm{Rep}_{\mathcal{B}}(\vec{w})$$

$$= (P_{\mathcal{E}\mathcal{B}} \vec{v})^T \cdot P_{\mathcal{E}\mathcal{B}} \vec{w}$$

$$= \vec{v}^T P^T P \vec{w}$$

Turns out that this is the most general form of the inner product on $\mathbb{R}^n$

(4) $V = \mathcal{P}_n$ $\quad \langle p, q \rangle = [p(0), p(1), p(2)] \begin{bmatrix} q(0) \\ q(1) \\ q(2) \end{bmatrix}$

This is ~~not~~ inner product if $n = 1, 2, 3$ but not if $n \geq 4$

(use this space for extra work if needed)

5 $V = \mathcal{P}_n$ $\quad \langle p, q \rangle = \int_{-1}^{1} p(x) q(x)\, dx$

6 $V = C[0,1]$ $\quad \langle f, g \rangle = \int_{0}^{1} f(x) g(x)\, dx$

7 $V = \mathcal{M}_{m \times n}$ $\quad \langle A, B \rangle = tr(AB^T)$

Norm of a vector: $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$ depends on inner product

Theorem [Cauchy-Schwarz-Buniakovsky]

$$|\langle \vec{v}, \vec{w} \rangle| \leq \|\vec{v}\| \, \|\vec{w}\|$$

Proof: (At least one of the previous proofs works!) $\vec{w} = 0 \Rightarrow EZ$

Suppose $\vec{w} \neq 0$

$$\|\vec{v} + t\vec{w}\|^2 \geq 0$$

Use bi-linearity:

$$\langle \vec{v} + t\vec{w}, \vec{v} + t\vec{w} \rangle \geq 0$$

and symmetry

$$\|\vec{v}\|^2 + 2t\langle \vec{v}, \vec{w} \rangle + t^2 \|w\|^2 \geq 0$$

Non-zero!

$$at^2 + bt + c \geq 0 \text{ iff } b^2 - 4ac < 0$$

Def "angle between vectors"

$\boxed{\text{Def } \vec{v}, \vec{w} \text{ are orthogonal if } \langle \vec{v}, \vec{w} \rangle = 0}$

Orthogonal and orthonormal bases

Def $\{\vec{v}_1, \dots, \vec{v}_n\}$ are orthogonal if $\langle \vec{v}_i ; \vec{v}_j \rangle = 0$ for $i \neq j$

orthonormal if $\langle v_i ; v_j \rangle = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

Theorem Orthogonal non-zero vectors are linearly independent

Proof. If $c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_n \vec{v}_n = 0$ then multiply by $\vec{v}_k$

$$\langle \vec{v}_k, c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_n \vec{v}_n \rangle = 0$$

$$c_k \|\vec{v}_k\|^2 = 0$$

So $c_k = 0$ or $\vec{v}_k = 0$

Def Orthogonal basis of V: $\{\vec{v}_1, \dots, \vec{v}_n\}$ such that
(a) non-zero
(b) orthogonal
(c) $V = \text{span}\{\vec{v}_1 \dots \vec{v}_n\}$

Why orthogonal bases are convenient?

Example: Find coordinates of $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ in basis $\vec{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $\vec{b}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ $\vec{b}_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$

Solution

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

We do not have to solve the system of equations!

In orthonormal basis $\{\vec{v}_1, \dots, \vec{v}_k\}$ coordinates are $\begin{bmatrix} \langle \vec{v}, \vec{w}_1 \rangle \\ \vdots \\ \langle v, \vec{w}_k \rangle \end{bmatrix}$ just like in $\mathbb{R}^n$

(use this space for extra work if needed)

Problem of "best approximation": If V is vector space with inner product $\langle\, ,\, \rangle$ and $W \subset V$ is a subspace, find a point $w_0 \in W$ which is closest to a given $\vec{v} \in V$

Examples: (a) Find a distance of $\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ to the line span $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

(b) Find a distance of $\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ to the plane $\begin{cases} x+y+z+w=0 \\ x-y+z-w=0 \end{cases}$ in $\mathbb{R}^4$

(c) Find a polynomial $p \in \mathcal{P}_3$ which approximates $f(x) = \sin x$ by minimizing distance $\int_{-\pi}^{\pi} (p(x) - f(x))^2 dx$

Solution: Let $\vec{w}_0$ be such that $\vec{v} - \vec{w}_0$ is orthogonal to every $\vec{w} \in W$

Then
$$\|\vec{v} - \vec{w}\|^2 = \|\vec{v} - \vec{w}_0 + \vec{w}_0 - \vec{w}\|^2 =$$
$$= \|\vec{v} - \vec{w}_0\|^2 + 2\langle \vec{v} - \vec{w}_0, \vec{w}_0 - \vec{w} \rangle + \|\vec{w}_0 - \vec{w}\|^2$$
$$= \|\vec{v} - \vec{w}_0\|^2 + \|\vec{w}_0 - \vec{w}\|^2 \geq \|\vec{v} - \vec{w}_0\|^2$$

So $\vec{w}_0$ is the solution of the problem of best approx.

$\boxed{\vec{w}_0 = P_W(\vec{v}) \text{ is called the orthogonal projection}}$

(use this space for extra work if needed)

How to find orthogonal projections?

This is much easier if we have orthogonal basis for W!

Example: $W = \text{span}\{\vec{w}_1\}$ - one dimensional subspace of V

Given $\vec{v} \in V$

Find $\vec{w}_0 \in W$ such that $\vec{v} - \vec{w}_0 \perp W$

Solution:

Find $\vec{w}_0 = t \cdot \vec{w}_1$ such that $\vec{v} - t\vec{w}_1 \perp \vec{w}_1$

$$\langle \vec{v} - t\vec{w}_1, w_1 \rangle = 0$$

$$\langle \vec{v}, \vec{w}_1 \rangle = t \langle w_1, w_1 \rangle \qquad t = \frac{\langle \vec{v}_1, \vec{w}_1 \rangle}{\langle w_1, w_1 \rangle}$$

$$P_W(\vec{v}) = \frac{\langle \vec{v}, \vec{w}_1 \rangle}{\langle w_1, w_1 \rangle} \cdot \vec{w}_1$$

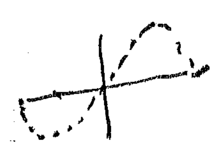
Sub-**Example** Orthogonal projection onto the span $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\} \subset \mathbb{R}^4$

$$P_W(\vec{v}) = \frac{\left\langle \vec{v}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\rangle}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{v_1 + v_2 + v_3 + v_4}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$$

Sub-**Example**: Orthogonal projection of $L_2(-1,1)$ onto span$\{x\}$

$$P_W(f) = \frac{\int_{-1}^{1} x f(x)\,dx}{\int_{-1}^{1} x^2\,dx} \cdot x \qquad P_W(\sin \pi x) = \frac{3}{\pi} \cdot x$$



$\frac{2}{3} \rightarrow \int_{-1}^{1} x^2\,dx$

$$P_W(\cos \pi x) = 0$$

(use this space for extra work if needed)

$$\boxed{P_W(x^3) = \frac{3}{5} x} \qquad P_W(x^2) = 0$$

Exercises ① Find the orthogonal projection onto a line

$$\left.\begin{aligned} x+y+z+w&=0\\ x-y+z-w&=0\\ x+2y+3z+4w&=0 \end{aligned}\right\} \text{ in } \mathbb{R}^4$$

② Find an orthogonal projection of $f(x)=x$ onto $W=\operatorname{span}\{\sin \pi x\}$ in $V=L_2(-1,1)$ with inner product $\langle f,g\rangle = \int_{-1}^{1} f(x)\,g(x)\,dx$

Example $W=\operatorname{span}\{\vec{w}_1,\vec{w}_2\}$ – two dimensional subspace of $\vec{V}$

Given $\vec{v}\in V$, find $\vec{w}_0\in W$ such that $\vec{v}-\vec{w}_0 \perp W$

Solution: $\vec{w}_0 = s\cdot\vec{w}_1 + t\,\vec{w}_2$

$$\vec{v}-\vec{w}_0 \perp \vec{w}_1$$
$$\vec{v}-\vec{w}_0 \perp \vec{w}_2$$

So
$$\left.\begin{aligned} \langle \vec{v}-s\vec{w}_1-t\vec{w}_2, \vec{w}_1\rangle &= 0\\ \langle \vec{v}-s\vec{w}_1-t\vec{w}_2, \vec{w}_2\rangle &= 0 \end{aligned}\right\} \text{System of equations for unknown } s,t$$

IF $\{\vec{w}_1,\vec{w}_2\}$ are orthogonal, this becomes much simpler!

$$s=\frac{\langle\vec{v},\vec{w}_1\rangle}{\langle w_1,w_1\rangle} \qquad t=\frac{\langle\vec{v},\vec{w}_2\rangle}{\langle\vec{w}_2,\vec{w}_2\rangle}$$

Answer for orthogonal $\vec{w}_1,\vec{w}_2$ ONLY

$$P_W(\vec{v}) = \frac{\langle\vec{v},\vec{w}_1\rangle}{\langle\vec{w}_1,\vec{w}_1\rangle}\cdot\vec{w}_1 + \frac{\langle\vec{v},\vec{w}_2\rangle}{\langle\vec{w}_2,w_2\rangle}\cdot\vec{w}_2$$

(use this space for extra work if needed)

Sub-Example: Find orthogonal projection onto a plane $x+y+z=0$ in $\mathbb{R}^3$

Solution: Choose nice basis $\vec{b}_1 = \begin{bmatrix} -2 \\ +1 \\ 1 \end{bmatrix}$ $\vec{b}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ $\vec{b}_1 \cdot \vec{b}_2 = 0$

$$P_W(\vec{v}) = \frac{\vec{v}\cdot\vec{b}_1}{b_1 \cdot b_1}\vec{b}_1 + \frac{v \cdot b_2}{b_2 \cdot b_2}\bar{b}_2$$

$$= \frac{-2v_1 + v_2 + v_3}{6} \cdot \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} + \frac{v_2 - v_3}{2}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-2v_1 + v_2 + v_3}{6} \\ \frac{-2v_1 + 4v_2 - 2v_3}{6} \\ \frac{-2v_1 - 2v_2 + 4v_3}{6} \end{bmatrix} = \frac{1}{6}\begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

Sub-Example: Find orthogonal projection in $L_2[-1,1]$ onto span$\{1, x\}$ (inner product $\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\,dx$)

Solution:

$$P_W(f) = \frac{1}{2}\int_{-1}^{1} f(x)\,dx + \left(\frac{3}{2}\int_{-1}^{1} x f(x)\,dx\right)\cdot x$$

For example

$P_W(\sin \pi x) = \frac{2}{\pi}x$ $\qquad$ $P_W(x^2) = \frac{1}{3}$

$P_W(\cos \pi x) = 0$ $\qquad$ $P_W(x^3) = \frac{3}{5}x$

$P_W(1+x+x^2+x^3) = ?$

(use this space for extra work if needed)

Exercises

① Find the orthogonal projection onto a plane

$$\left.\begin{array}{l} x+y+z+w=0 \\ x-y+z-w=0 \end{array}\right\} \text{ in } \mathbb{R}^4$$

② Find an orthogonal projection of $f(x)=x$ onto $W = \text{span}\{\sin(\pi x), \cos(\pi x)\}$

More generally: If W has <u>orthogonal</u> basis $\{\vec{w}_1, \vec{w}_2, \ldots \vec{w}_k\}$ then orthogonal projection onto W is

$$P_W(\vec{v}) = \sum_{j=1}^{k} \frac{\langle \vec{v}, \vec{w}_j \rangle}{\langle \vec{w}_j, \vec{w}_j \rangle} \cdot \vec{w}_j$$

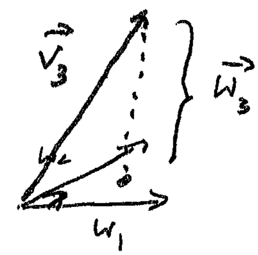
<u>Proof</u>: Same as before!

(use this space for extra work if needed)

How to find an orthogonal basis

Gram - Schmidt orthogonalization

Suppose $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ is a basis of $\bar{W}$. Then the following $\{\bar{w}_1 \dots \bar{w}_k\}$ is an orthogonal basis:



$$\vec{w}_1 = \vec{v}_1$$

$$\bar{w}_2 = \vec{v}_2 - \frac{\langle \bar{v}_2, w_1 \rangle}{\langle \bar{w}_1, \bar{w}_1 \rangle} \cdot \vec{w}_1$$

$$\vec{w}_3 = \vec{v}_3 - \frac{\langle \vec{v}_3, \bar{w}_1 \rangle}{\langle w_1, w_1 \rangle} \cdot \vec{w}_1 - \frac{\langle \vec{v}_3, \vec{w}_2 \rangle}{\langle w_2, w_2 \rangle} \vec{w}_2$$

$$\vec{w}_4 = \vec{v}_4 - \frac{\langle \bar{v}_4, w_1 \rangle}{\langle w_1, w_1 \rangle} \bar{w}_1 - \frac{\langle \bar{v}_4, \bar{w}_2 \rangle}{\langle \bar{w}_2, \vec{w}_2 \rangle} \vec{w}_2 - \frac{\langle \vec{v}_4, \vec{w}_3 \rangle}{\langle w_3, w_3 \rangle} \cdot \vec{w}_3$$

$$\vdots$$

$$\vec{w}_k = \vec{v}_k - \underbrace{\sum_{j=1}^{k-1} \frac{\langle \vec{v}_k, \vec{w}_j \rangle}{\langle \vec{w}_j, \vec{w}_j \rangle} \vec{w}_j}$$

Notice that we use $\vec{w}_1, \vec{w}_2, \dots$ here, not $\vec{v}_1, \vec{v}_2 \dots$

Example $W = \{ x + y + z + w = 0 \}$ has basis $\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$

To find orthogonal projection onto W, we wish to have orthogonal basis

$$w_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \qquad \bar{w}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ -1 \\ \frac{1}{2} \end{bmatrix}$$

lets change it to $\bar{w}_2 = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$

$$\vec{w}_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} - \frac{1}{6} \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 \\ -1 \\ 1/3 \\ 1/3 \end{bmatrix}$$

lets take $\vec{w}_3 = \begin{bmatrix} 1 \\ -3 \\ 1 \\ 1 \end{bmatrix}$

(use this space for extra work if needed)

Example $W = \mathcal{P}_2$ (quadratic polynomials) in $L_2(-1,1)$ with $\langle f,g \rangle = \int_{-1}^{1} f(x)g(x)dx$

Find an orthogonal basis

Solution Lets start with standard basic $e_0 = 1$ $e_1 = x$ $e_2 = x^2$

$$f_0(x) = e_0(x) = 1$$

$$f_1(x) = x - \frac{0}{2} f_0(x) = x$$

$$f_2(x) = x^2 - \frac{\quad}{2} f_0(x) - \frac{\quad}{2/3} f_1(x)$$

$$= x^2 - \frac{2}{3}$$

In $\mathcal{P}_3$:

$$f_3(x) = x^3 - \frac{\quad}{2} f_0(x) - \frac{\quad}{2/3} f_1(x) - \frac{\quad}{\quad} f_2(x)$$

$$= x^3 - \frac{0}{2} 1 - \frac{2/5}{2/3} x - \frac{0}{2/5} \left(x^2 - 2/3\right)$$

$$f_3(x) = x^3 - \frac{3}{5} x$$

$$f_4(x) = ?$$

These are called orthogonal polynomials (with weight function $w(x) = \begin{cases} 1 & |x| \le 1 \\ 0 & \end{cases}$)

(use this space for extra work if needed)

Exercises on Gram-Schmidt

1. Find orthogonal basis of $W = \{x+y+z+w=0\} \subset \mathbb{R}^4$ starting with vectors (basis)

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \right\}.$$

Why your answer differs from Example on pg 9?

2. Find orthogonal basis in $\mathcal{P}_2$ with respect to inner product $\langle f, g \rangle = \int_0^1 f(x) g(x) \, dx$

3. Find orthogonal basis in $\mathcal{P}_2$ with respect to inner product

$$\langle f, g \rangle := f(0)g(0) + f(1)g(1) + f(-1)g(-1)$$

(use this space for extra work if needed)