Printed: November 8, 2019   Name ________________________________

MATH 6012 Exam-2-2019 Answer: Key

1. Find the characteristic polynomial of the matrix $A = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 5 & 0 \\ -2 & 0 & 7 \end{pmatrix}$, and determine the eigenvalues of A. *(You should not need it, but for 2 pts you can buy one eigenvalue from me!)*

   Answer:

$$\det(A-\lambda I) = \det \begin{pmatrix} 3-\lambda & 1 & 1 \\ 0 & 5-\lambda & 0 \\ -2 & 0 & 7-\lambda \end{pmatrix} = (1-\lambda) \det\det \begin{pmatrix} 3-\lambda & 1 \\ -2 & 7-\lambda \end{pmatrix}$$
$$= (1-\lambda)((3-\lambda)(7-\lambda)+2) = (5-\lambda)(\lambda^2 - 10\lambda + 23)$$

   The characteristic polynomial can be expanded into $-\lambda^3 + 15\lambda^2 - 73\lambda + 115$, but this is a less useful form.

   The roots of the quadratic factor are $\frac{10 \pm \sqrt{100-92}}{2} = 5 \pm \sqrt{2}$

   The eigenvalues, in increasing order, are $\boxed{\lambda_1 = 5-\sqrt{2},\ \lambda_2 = 5,\ \lambda_3 = 5+\sqrt{2}}$.

2. Find an orthogonal basis for the vector space $\mathcal{P}_2$ of quadratic polynomials with the inner product $\langle p, q \rangle = \int_0^2 p(t)q(t)\,dt$. (Simplify your answer.)

   Answer: Applying the Gram-Schmidt orthogonalization to the standard basis $\{1, t, t^2\}$ we get $p_0(t) = 1$, $p_1(t) = t - \frac{\langle t, p_0 \rangle}{\langle p_0, p_0 \rangle} p_0(t) = t - 1$

   $p_2(t) = t^2 - \frac{\langle t^2, p_0 \rangle}{\langle p_0, p_0 \rangle} p_0(t) - \frac{\langle t^2, p_1 \rangle}{\langle p_1, p_1 \rangle} p_1(t) = t^2 - \frac{4/3}{2/3}(t-1) - \frac{8/3}{2} = \boxed{t^2 - 2t + \frac{2}{3}}$

3. Consider vector space $V = L_2[0,1]$ consisting of piece-wise continuous functions on the interval $[0,1]$, with the inner product $\langle f, g \rangle = \int_0^1 f(x)g(x)dx$. Find the "best approximation" of the step function

$$h(x) = \begin{cases} 1 & 0 \le x \le 1/2 \\ 0 & 1/2 < x \le 1 \end{cases}$$

   by an affine function $a + b\cos(\pi x)$ of $\cos(\pi x)$. That is, determine the orthogonal projection of h onto the two-dimensional subspace $W = span\{1, \cos(\pi x)\} \subset V$.

   Trig: $\boxed{\cos^2(\pi x)dx = \frac{1+\cos(2\pi x)}{2}}$
   Derivatives: $\boxed{\sin'(\pi x) = \pi\cos(\pi x)}$ $\boxed{\cos'(\pi x) = -\pi\sin(\pi x)}$.

   Answer: First we check that $\int_0^1 1 \times \cos(\pi x)dx = \frac{1}{\pi}\sin(\pi x)\big|_{x=0}^{x=1} = 0$ so the two functions are orthogonal.

   Next, compute the orthogonal projection: We need

$$\langle 1, h \rangle = \int_0^{1/2} 1dx = 1/2, \quad \langle 1, 1 \rangle = \int_0^1 1dx = 1,$$

$$\langle \cos(\pi x), h \rangle = \int_0^{1/2} \cos(\pi x)dx = \frac{1}{\pi}\sin(\pi/2) = \frac{1}{\pi},$$

$$\langle \cos(\pi x), \cos(\pi x) \rangle = \int_0^1 \cos^2(\pi x)dx = \int_0^1 \frac{1+\cos(2\pi x)}{2}dx = \frac{1}{2}$$

   So $g(x) = \frac{\langle 1, h \rangle}{\langle 1, 1 \rangle} + \frac{\langle \cos(\pi x), h \rangle}{\langle \cos(\pi x), \cos(\pi x) \rangle}\cos(\pi x) = \boxed{\frac{1}{2} + \frac{2\cos(\pi x)}{\pi}}$.

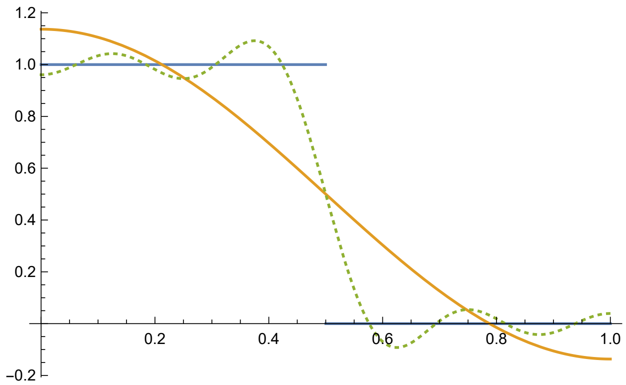

Figure 1: Graphs of h, its projection g. For comparison I included the graph of the projection onto the 5-dimensional subspace spanned by orthogonal functions $\{1, \cos(\pi x), \cos(3\pi x), \cos(5\pi x), \cos(7\pi x)\}$ (dotted line)

Useful formula: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$

4. If $A = \begin{bmatrix} -4 & -5 \\ 10 & 11 \end{bmatrix}$, use the diagonalization technique to determine the formulas for the four entries of the matrix $A^n = \begin{bmatrix} a_{11}(n) & a_{12}(n) \\ a_{21}(n) & a_{22}(n) \end{bmatrix}$ as functions of n.

Answer: Characteristic polynomial is $\det(A - \lambda I) = \lambda^2 - 7\lambda + 6 = (\lambda - 6)(\lambda - 1)$. The eigenvalues are $\lambda_1 = 1, \lambda_2 = 6$ with the eigenvectors $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

So $A^n = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & 6^n \end{bmatrix} \times \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 - 6^n & 1 - 6^n \\ -2 + 2 \times 6^n & 2 \times 6^n - 1 \end{bmatrix}$