# MATH 6012 Exam-1-2019 **Answer: Key**

1. Find the cosine of the angle between vectors $\vec{u}+\vec{v}$ and $\vec{u}-\vec{v}$ if $\vec{u} = \begin{bmatrix}1\\0\\1\\0\end{bmatrix}$ and $\vec{v} = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$.

   **Answer:** $\vec{s} = \vec{u}+\vec{v} = \begin{bmatrix}2\\1\\2\\1\end{bmatrix}$ and $\vec{d} = \vec{u}-\vec{v} = \begin{bmatrix}0\\-1\\0\\-1\end{bmatrix}$ so

$$\cos\theta = \frac{\vec{s}\cdot\vec{d}}{\|\vec{s}\| \times \|\vec{d}\|} = \frac{-2}{\sqrt{10}\sqrt{2}} = -\sqrt{5}/5$$

   The angle is obtuse, with $\theta \approx 2.03444$ radians, i.e., about $116.565°$.

2. Use the definition to show that functions $g_1(x) = 1, g_2(x) = x, g_3(x) = x(e^x + e^{-x})$ are linearly independent.

   **Answer:** Suppose

$$c_1g_1 + c_2g_2 + c_3g_3 = 0 \text{ for all real } x. \tag{*}$$

   Our goal is to show that this implies $c_1 = c_2 = c_3 = 0$.

   **Routine solution:** Denote by $f(x)$ the left hand side of (*). Then $f(0) = c_1 = 0$, $f'(0) = c_2+2c_3 = 0$, $f''(0) = 0$, $f'''(0) = 6c_3 = 0$. This gives a system of 4 equations for 3 unknown coefficients $c_1, c_2, c_3$:

$$\begin{aligned} c_1 &= 0 \\ c_2 + 2c_3 &= 0 \\ 0 &= 0 \\ 6c_3 &= 0 \end{aligned}$$

   Clearly, all $c_j = 0$.

   In summary, we showed that if (*) holds then we must have $c_0 = c_1 = c_2 = c_3 = 0$, i.e. ,the functions are linearly independent.

   There are numerous other solutions. **An ad-hoc method:** Evaluating (*) expression at $x = 0$ we get $c_1 = 0$, as $g_2(0) = g_3(0) = 0$.
   So (*) becomes $c_2g_2 + c_3g_3 = 0$, i.e.

$$c_2x + c_3x(e^x + e^{-x}) = 0$$

   Dividing by $x$, we get

$$c_2 + c_3(e^x + e^{-x}) = 0 \tag{**}$$

   Differentiating (**) at $x = 1$ we get $c_3(e - \frac{1}{e}) = 0$, so $c_3 = 0$. Inserting this back into (**) we see that $c_2 = 0$, too.

3. Matrix $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \end{bmatrix}$ is row equivalent to matrix $B = \begin{bmatrix} 1 & 0 & -1 & -2 & -3 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Use this information to find a basis (and the dimension) for the null space $Null(A) = \left\{ \vec{x} : A\vec{x} = \vec{0} \right\}$.

   **Answer:** Equation $A\vec{x} = \vec{0}$ is equivalent to $B\vec{x} = \vec{0}$, and $B$ is in echelon form, so we can read out the solution. Basic variables are $x_1, x_2$. Free variables are $x_3 = u, x_4 = s, x_5 = t$. We get

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} u + 2s + 3t \\ -2u - 3s - 4t \\ u \\ s \\ t \end{bmatrix} = u \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

   So the solution set is the span of 3 linearly independent vectors in $\mathbb{R}^5$:

$$\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

   The dimension of the $Null(A)$ is 3.

4. Consider the following basis $\mathcal{B} = \langle 1, 1 - t, (1 - t)^2 \rangle$ of the vector space $\mathcal{P}_2$ of quadratic polynomials. (You do not need to check that this is a basis of $\mathcal{P}_2$)

   (a) Which polynomial $p$ has coordinates $Rep_{\mathcal{B}}(p) = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$? Simplify your answer.

   **Answer:** From the definition of coordinates, we have $p(t) = 3 + 2(1 - t) + (1 - t)^2 = 6 - 4t + t^2$

   (b) What are the coordinates of the monomial $t^2$ in basis $\mathcal{B}$?

   **Answer:** $t^2 = (t - 1 + 1)^2 = (t - 1)^2 + 2(t - 1) + 1 = (t - 1)^2 - 2(1 - t) + 1$ so the coordinates are $[t^2]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

5. Suppose $V = span\{1, \cos x, \sin x\}$ and $W = span\{1, x, \cos x, \sin x\}$. Let $S : V \to W$ be a mapping which to a function $f(x)$ assigns its definite integral, the function $g(x) = \int_0^x f(t)dt$. Without checking you can assume that $S$ is a linear mapping and that the above sets of functions are linearly independent, so they form respective bases of the spaces $V$ and $W$.

   (a) Find the matrix representation of $S$ with respect to the above bases. **Answer:**

$$\int_0^x 1dt = x, \int_0^x \cos t dt = \sin x, \int_0^x \sin t dt = 1 - \cos x$$

   So the columns of the matrix representation are the expansions of these functions in the second basis, i.e.

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

   (b) Is $S$ one-to-one? Justify your answer. **Answer:** #1: Yes, the columns of $A$ are linearly independent - this is easier seen after swapping the last two! **Answer:** #2: If $\int_0^x f(t)dt = \int_0^x g(t)dt$ and $f, g$ are continuous then by differentiation we get $f = g$. So yes, it is one-to-one

   (c) Is $S$ onto? Justify your answer. **Answer:** #1: No, the domensions do not match. The dimension of range of $S$ can be at most 3. **Answer:** #2: If $f(x) = a + b\cos x + c\sin x$ then $S(f)(x) = ax + b\sin x + c - c\cos x$ so the range of range of $S$ is span of functions $x, \sin x, \cos x - 1$ and is three dimensional, not the four dimensional space $W$.

6. Find the inverse of $A = \begin{bmatrix} 1 & b & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ for arbitrary $b \in \mathbb{R}$.

**Answer:** $A^{-1} = \begin{pmatrix} 1 & -b & 2b-3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$