Printed: February 25, 2023 Name ______________________________

Dynamical Systems Quiz-6 Key

Instructions. Simplify your answers when appropriate. Be sure to indicate how you got your answers.

1. Determine the Laplace transform $Y(s)$ of the solution of the following initial value problem

   $y'' - y' - 6y = 0, \quad y(0) = 2, \quad y'(0) = -1.$ Answer $\boxed{Y(s) = \frac{2s-3}{s^2-s-6}}$.

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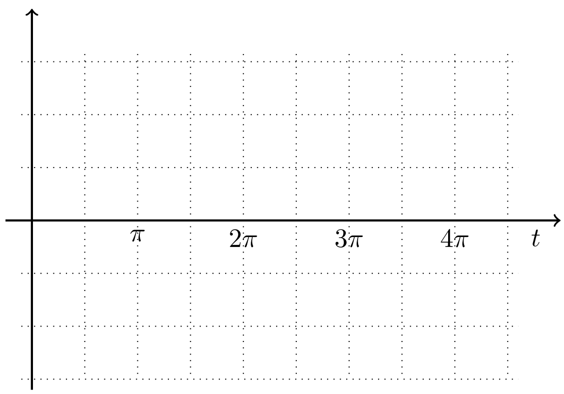
   Let's also determine the solution. $Y(s) = \frac{7}{5(s+2)} + \frac{3}{5(s-3)}$ so $\boxed{y(t) = \frac{3}{5}e^{3t} + \frac{7}{5}e^{-2t}}$

2. Sketch the graph of the function $f(t) = \sin t + H(t-\pi)\sin(t-\pi) = \sin t - H(t-\pi)\sin t = \begin{cases} \sin t & t < \pi \\ \sin t - \sin t = 0 & t > \pi \end{cases}$



3. Compute the **Laplace transform** $Y(s)$ of the solution $y(t)$ to the initial value problem $y'' + y = g(t)$, $y(0) = 2$, $y'(0) = 3$, with discontinuous right hand side function

$$g(t) = \begin{cases} 0 & \text{for } t < \pi \\ 1 & \text{for } \pi \le t \le 2\pi \\ 2 & \text{for } t > 2\pi \end{cases}$$

   (Do not simplify your answer. Do not solve for $y(t)$.) *Hint: You can express $g(t)$ as a linear combination of Heaviside functions* $g(t) = H(t-\pi) + H(t-2\pi)$ has the Laplace transform $G(s) = e^{-\pi s}/s + e^{-2\pi s}/s$. So $\boxed{Y(s) = \frac{2s+3}{s^2+1} + \frac{e^{-\pi s}}{s(s^2+1)} + \frac{e^{-2\pi s}}{s(s^2+1)}}$

   The solution, which you were not asked to find, is $y(t) = 2\cos t + 3\sin t + H(t-\pi)(1-\cos(t-\pi)) + H(t-2\pi)(1-\cos(t-2\pi)) = 2\cos t + 3\sin t + H(t-\pi)(1+\cos t) + H(t-2\pi)(1-\cos t) = \begin{cases} 3\sin t + 2\cos t & t < \pi \\ 1 + 3\sin t + 3\cos t & \pi < t < 2\pi \\ 2 + 3\sin t + 2\cos t & t > 2\pi \end{cases}$

Table of Laplace transforms $\mathcal{L}(f) = \int_0^\infty e^{-ts} f(t)dt$

$f(t)$	$F(s) = \mathcal{L}(f)$	(domain of $F(s)$)
1	$\frac{1}{s}$	$(s > 0)$
t^n	$\frac{n!}{s^{n+1}}$	$(s > 0)$
e^{at}	$\frac{1}{s-a}$	$(s > a)$
$\cos at$	$\frac{s}{s^2+a^2}$	$(s > 0)$
$\sin at$	$\frac{a}{s^2+a^2}$	$(s > 0)$
$H(t-c)$	$\frac{e^{-cs}}{s}$	$(s > 0)$
$\delta(t-a)$	e^{-as}	$(s > 0)$

Laplace transform formulas

$$\mathcal{L}(f_1(t) + cf_2(t)) = \mathcal{L}(f_1(t)) + c\mathcal{L}(f_2(t))$$

$$\mathcal{L}\left(f^{(n)}(t)\right) = s^n F(s) - s^{n-1} f(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0)$$

$$\mathcal{L}\left(e^{at} f(t)\right) = F(s-a)$$

$$\mathcal{L}(t^k f(t)) = (-1)^k \frac{d^k}{ds^k} F(s)$$

$$\mathcal{L}\left(H(t-c)f(t-c)\right) = e^{-cs} F(s)$$

$$\mathcal{L}\left(\int_o^t f(\tau)g(t-\tau)\, d\tau\right) = F(s) \cdot G(s)$$

$$\mathcal{L}\left(f(at)\right) = \frac{1}{a} F\left(\frac{s}{a}\right)$$

Trig identities

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \sin\beta\cos\alpha$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\cos^2\alpha + \sin^2\alpha = 1$$