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Dynamical Systems Exam-2

A Key

Instructions. Simplify your answers when appropriate. Be sure to show your work so that it is clear how you got your answers.

A-1. Use the Laplace transform to solve the initial value problem $y'' - 2y' + 5y = 0$, $y(0) = 2, y'(0) = 4$.

$s^2Y(s) - 2s - 4 - 2(sY(s) - 2) + 5Y(s) = 0$ so

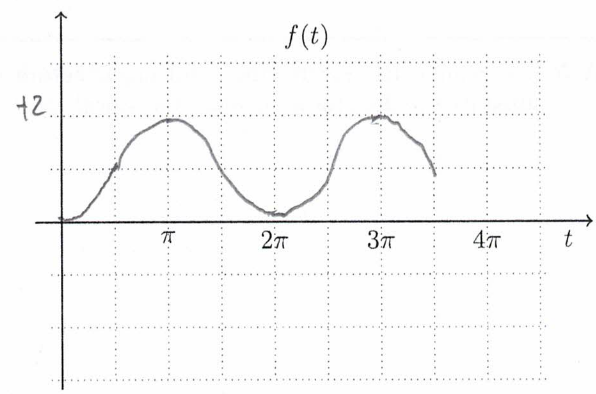
$$Y(s) = \frac{2s}{s^2 - 2s + 5} = \frac{2s}{(s-1)^2 + 4} = 2\frac{s-1}{(s-1)^2+4} + \frac{2}{(s-1)^2+4} = 2\mathcal{L}\left[e^t\cos(2t)\right] + \mathcal{L}\left[e^t\sin(2t)\right]$$

From the tables/formulas we read out the answer:

$\boxed{y(t) = e^t \sin(2t) + 2e^t\cos(2t)}$

A-2. Determine the function with given Laplace transform, and sketch its graph.

(a) $F(s) = \frac{1}{s(s^2+1)}$



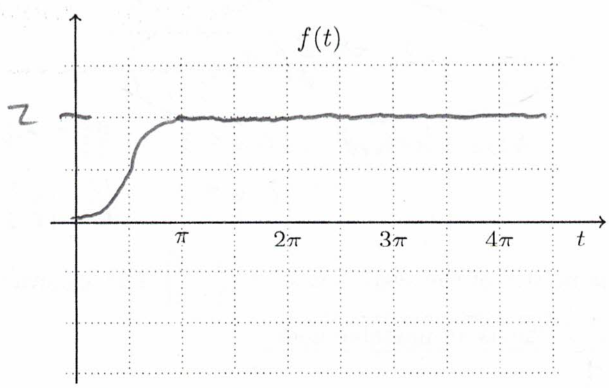
$F(s) = \frac{1}{s} - \frac{s}{s^2+1}$ so $\boxed{f(t) = 1 - \cos t}$.

(b) $F(s) = \frac{1+e^{-\pi s}}{s(s^2+1)}$ From previous problem, $F(s) = \frac{1}{s} - \frac{s}{s^2+1} + e^{-\pi s}\left(\frac{1}{s} - \frac{s}{s^2+1}\right)$.

We get $f(t) = 1 - \cos t + u_\pi(t)(1 - \cos(t - \pi)) = 1 - \cos t + u_\pi(t)(1 + \cos t)$.

To graph the function, write

$$f(t) = \begin{cases} 1 - \cos t & t < \pi \\ 2 & t > \pi \end{cases}$$



A-3. Solve the fourth-order differential equation $y^{(4)} - y = 0$ with the initial conditions $y(0) = 1$ $y'(0) = 0$, $y''(0) = -1$, $y'''(0) = 0$ Laplace transform with careful algebra should be convenient.

$s^4Y(s) - s^3 + s + Y(s) = 0$, so $Y(s) = \frac{s^3 - s}{(s^2-1)(s^2+1)} = \frac{s(s^2-1)}{(s^2-1)(s^2+1)} = \frac{s}{s^2+1}$

Answer: $\boxed{y(t) = \cos t}$

A-4. Compute the **Laplace transform** $Y(s)$ of the solution $y(t)$ to the initial value problem $y'' + y = g(t)$, $y(0) = 2$, $y'(0) = 3$, with discontinuous right hand side function

$$g(t) = \begin{cases} 0 & t < \pi \\ 3 & \text{for } \pi \le t \le 2\pi \\ 2 & \text{for } t > 2\pi \end{cases}$$

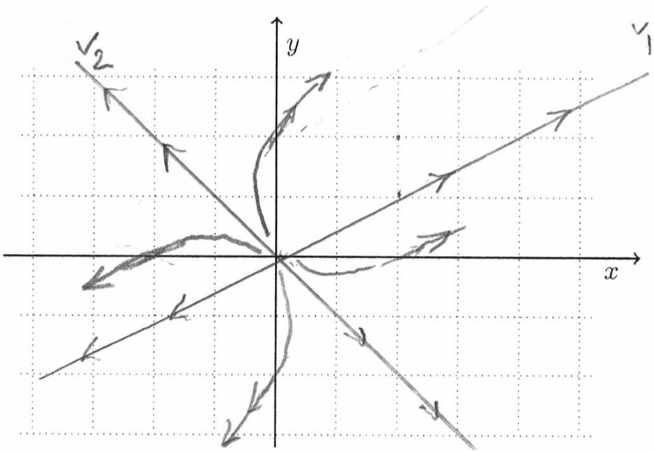
(Do not simplify your answer. Do not solve for $y(t)$.) $g(t) = 3H(t-\pi) - H(t-2\pi)$ has the Laplace transform $G(s) = 3e^{-\pi s}/s - e^{-2\pi s}/s$. So $\boxed{Y(s) = \frac{2s+3}{s^2+1} + 3\frac{e^{-\pi s}}{s(s^2+1)} - \frac{e^{-2\pi s}}{s(s^2+1)}}$

A-5. Determine the eigenvalues and eigenvectors if the eigenvalues are real, classify the system (state whether stable or unstable node, stable or unstable spiral, center, saddle point) and sketch the phase plane of the linear system.

$$\frac{dx}{dt} = 3x + 2y$$
$$\frac{dy}{dt} = x + 2y$$

$\lambda^2 - 5\lambda + 4 = 0$

$\lambda_1 = 4 \quad \lambda_2 = 1$



$x=0 \;\; y=2 \Rightarrow \frac{d\vec{x}}{dt} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$

Also checked $x=0 \;\; y=-2$ $\quad \frac{d\vec{x}}{dt} = -\begin{bmatrix} 4 \\ 4 \end{bmatrix}$

$x=2 \;\; y=0$ $\quad \frac{d\vec{x}}{dt} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$

The matrix of the system is $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$ The eigenvalues are $\lambda_1 = 4, \lambda_2 = 1$. Eigenvectors are $\vec{v}_1 = \cancel{[2, 1]}$, and $\vec{v}_2 = \cancel{[-1, 1]}$. $\boxed{\text{This is an unstable node.}}$

$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\quad \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

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Dynamical Systems Exam-2

B Key

Instructions. Simplify your answers when appropriate. Be sure to show your work so that it is clear how you got your answers.

B-1. Use the Laplace transform to solve the initial value problem $y'' + 2y' + 5y = 0$, $y(0) = 4, y'(0) = 2$.

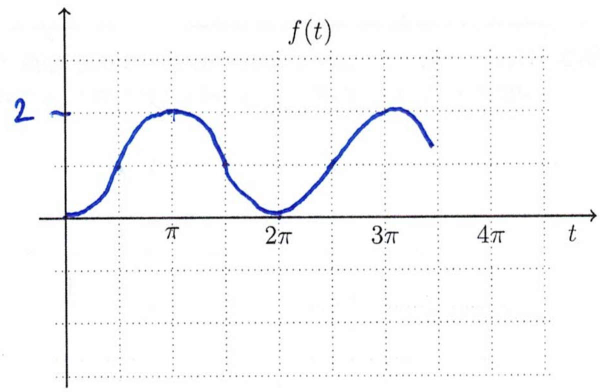
$s^2Y(s) - 4s - 2 + 2(sY(s) - 4) + 5Y(s) = 0$ so

$$Y(s) = \frac{4s+10}{s^2+2s+5} = \frac{4s+10}{(s+1)^2+4} = 4\frac{s+1}{(s+1)^2+4} + 3\frac{2}{(s+1)^2+4}$$

$\boxed{y(t) = 3e^{-t}\sin(2t) + 4e^{-t}\cos(2t)}$

B-2. Determine the function with given Laplace transform, and sketch its graph.

(a) $F(s) = \frac{1}{s(s^2+1)}$



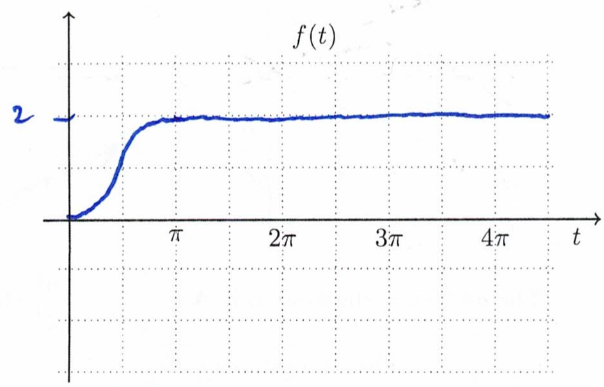
$F(s) = \frac{1}{s} - \frac{s}{s^2+1}$ so $\boxed{f(t) = 1 - \cos t}$.

(b) $F(s) = \frac{1+e^{-\pi s}}{s(s^2+1)}$ From previous problem, $F(s) = \frac{1}{s} - \frac{s}{s^2+1} + e^{-\pi s}\left(\frac{1}{s} - \frac{s}{s^2+1}\right)$.

We get $f(t) = 1 - \cos t + u_\pi(t)(1 - \cos(t-\pi)) = 1 - \cos t + u_\pi(t)(1 + \cos t)$.

To graph the function, write

$$f(t) = \begin{cases} 1 - \cos t & t < \pi \\ 2 & t > \pi \end{cases}$$



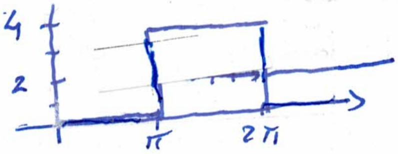
B-3. Solve the fourth-order differential equation $y^{(4)} - y = 0$ with the initial conditions $y(0) = 0$ $y'(0) = 1$, $y''(0) = 0$, $y'''(0) = -1$ Laplace transform with careful algebra should be convenient.

$s^4Y(s) - s^2 + 1 + Y(s) = 0$, so $Y(s) = \frac{s^2-1}{(s^2-1)(s^2+1)} = \frac{1}{s^2+1}$

Answer: $\boxed{y(t) = \sin t}$

B-4. Compute the **Laplace transform** $Y(s)$ of the solution $y(t)$ to the initial value problem $y'' + y = g(t)$, $y(0) = 5$, $y'(0) = 2$, with discontinuous right hand side function

$$g(t) = \begin{cases} 0 & t < \pi \\ 4 & \text{for } \pi \le t \le 2\pi \\ 2 & \text{for } t > 2\pi \end{cases}$$



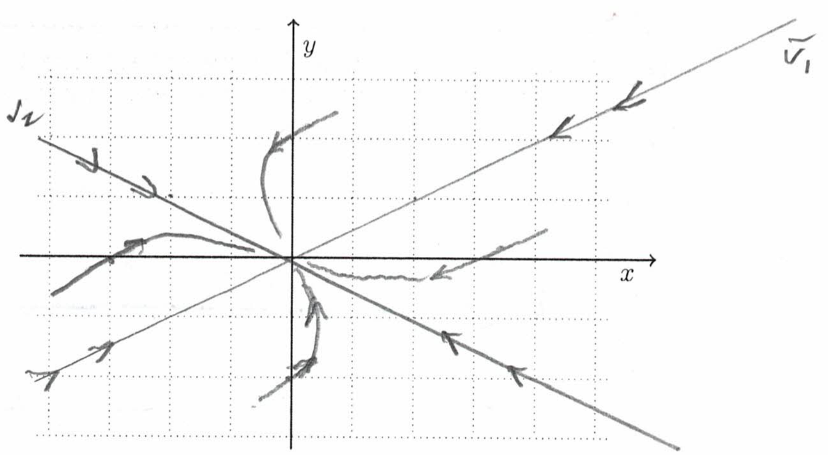
(Do not simplify your answer. Do not solve for $y(t)$.) $g(t) = 4H(t-\pi) - 2H(t-2\pi)$ has the Laplace transform $G(s) = 4e^{-\pi s}/s - 2e^{-2\pi s}/s$. So $\boxed{Y(s) = \frac{5s+2}{s^2+1} + 4\frac{e^{-\pi s}}{s(s^2+1)} - 2\frac{e^{-2\pi s}}{s(s^2+1)}}$

B-5. Determine the eigenvalues and eigenvectors if the eigenvalues are real, classify the system (state whether stable or unstable node, stable or unstable spiral, center, saddle point) and sketch the phase plane of the linear system.

$$\frac{dx}{dt} = -3x - 4y$$
$$\frac{dy}{dt} = -x - 3y$$

char. eqtn: $\lambda^2 + 6\lambda + 5 = 0$

$(\lambda+1)(\lambda+5) = 0$



The matrix of the system is $\mathbf{A} = \begin{bmatrix} -3 & -4 \\ -1 & -3 \end{bmatrix}$ the eigenvalues are $\lambda_1 = -5$, $\lambda_2 = -1$. The eigenvectors are $\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. This is a $\boxed{\text{stable node}}$

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Dynamical Systems Exam-2

C Key

Instructions. Simplify your answers when appropriate. Be sure to show your work so that it is clear how you got your answers.

C-1. Use the Laplace transform to solve the initial value problem $y'' - 4y' + 5y = 0$, $y(0) = 2, y'(0) = 1$.

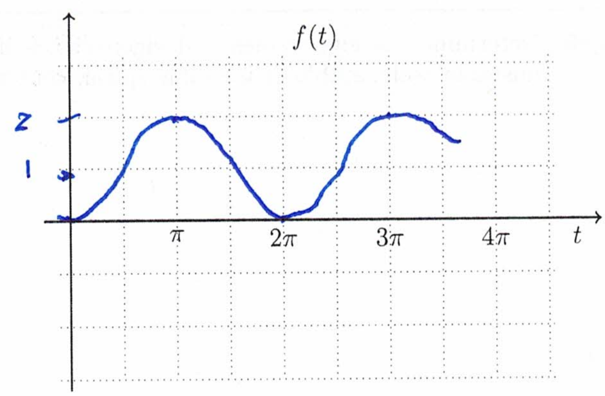
$s^2Y(s) - 2s - 1 - 4(sY(s) - 2) + 5Y(s) = 0$ so

$$Y(s) = \frac{2s-7}{s^2-4s+5} = \frac{2s-7}{(s-2)^2+1} = \frac{2(s-2)}{(s-2)^2+1} - \frac{3}{(s-2)^2+1}$$

$\boxed{y(t) = 2e^{2t}\cos(t) - 3e^{2t}\sin(t)}$

C-2. Determine the function with given Laplace transform, and sketch its graph.

(a) $F(s) = \frac{1}{s(s^2+1)}$



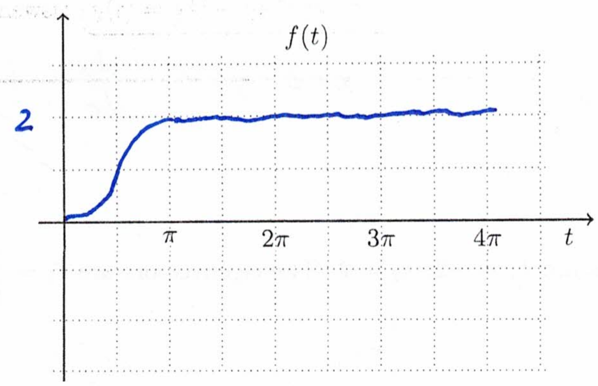
$F(s) = \frac{1}{s} - \frac{s}{s^2+1}$ so $\boxed{f(t) = 1 - \cos t}$.

(b) $F(s) = \frac{1+e^{-\pi s}}{s(s^2+1)}$ From previous problem, $F(s) = \frac{1}{s} - \frac{s}{s^2+1} + e^{-\pi s}\left(\frac{1}{s} - \frac{s}{s^2+1}\right)$.

We get $f(t) = 1 - \cos t + u_\pi(t)(1 - \cos(t-\pi)) = 1 - \cos t + u_\pi(t)(1 + \cos t)$.

To graph the function, write

$$f(t) = \begin{cases} 1 - \cos t & t < \pi \\ 2 & t > \pi \end{cases}$$



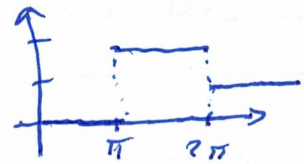
C-3. Solve the fourth-order differential equation $y^{(4)} - y = 0$ with the initial conditions $y(0) = 0$ $y'(0) = 1$, $y''(0) = 0$, $y'''(0) = 1$ Laplace transform with careful algebra should be convenient.

$s^4Y(s) - s^2 - 1 + Y(s) = 0$, so $Y(s) = \frac{s^2+1}{(s^2-1)(s^2+1)} = \frac{1}{s^2-1} = \frac{1}{(s-1)(s+1)} = \frac{1/2}{s-1} - \frac{1/2}{s+1}$

Answer: $\boxed{y(t) = \frac{1}{2}e^t - \frac{1}{2}e^{-t} = \sinh t}$

C-4. Compute the **Laplace transform** $Y(s)$ of the solution $y(t)$ to the initial value problem $y'' + y = g(t)$, $y(0) = 2$, $y'(0) = 3$, with discontinuous right hand side function

$$g(t) = \begin{cases} 0 & t < \pi \\ 2 & \text{for } \pi \le t \le 2\pi \\ 1 & \text{for } t > 2\pi \end{cases}$$

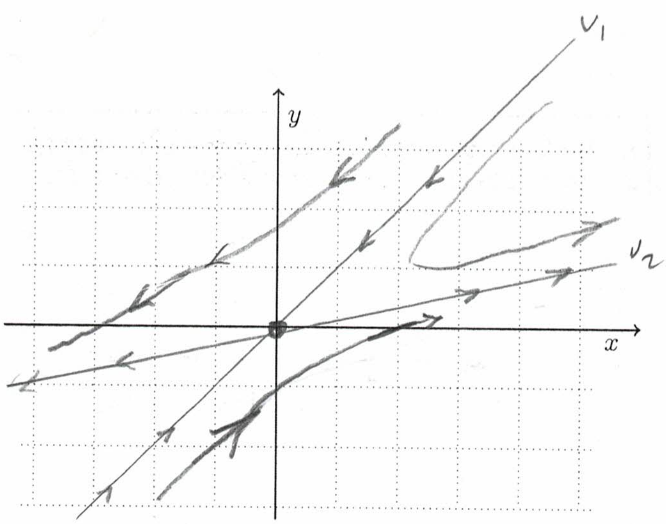


(Do not simplify your answer. Do not solve for $y(t)$.) $g(t) = 2H(t-\pi) - H(t-2\pi)$ has the Laplace transform $G(s) = 2e^{-\pi s}/s - e^{-2\pi s}/s$. So $\boxed{Y(s) = \frac{2s+3}{s^2+1} + \frac{2e^{-\pi s}}{s(s^2+1)} - \frac{e^{-2\pi s}}{s(s^2+1)}}$

C-5. Determine the eigenvalues and eigenvectors if the eigenvalues are real, classify the system (state whether stable or unstable node, stable or unstable spiral, center, saddle point) and sketch the phase plane of the linear system.

$$\frac{dx}{dt} = 3x - 5y$$
$$\frac{dy}{dt} = x - 3y$$

Char. eqtn: $\lambda^2 - 4 = 0$



$x = 2$ $y = 0$ $\frac{d\bar{x}}{dt} = \frac{6}{2}$

The matrix of this system is $\mathbf{A} = \begin{bmatrix} 3 & -5 \\ 1 & -3 \end{bmatrix}$ The eigenvalues are $\lambda_1 = -2$, $\lambda_2 = 2$. The eigenvectors are $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

This is a saddle point.

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D Key

Dynamical Systems Exam-2

Instructions. Simplify your answers when appropriate. Be sure to show your work so that it is clear how you got your answers.

D-1. Use the Laplace transform to solve the initial value problem $y'' + 4y' + 5y = 0$, $y(0) = 2, y'(0) = 1$.

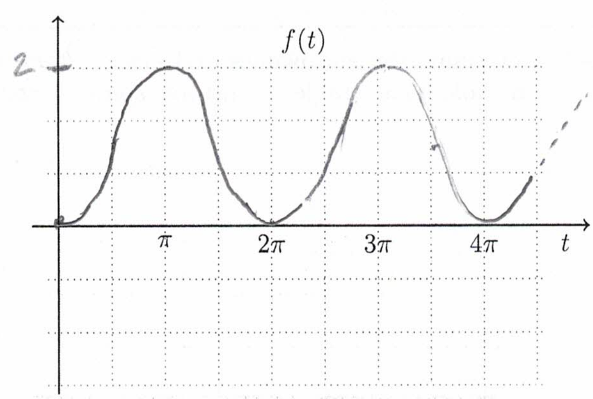
$s^2Y(s) - 2s - 1 + 4(sY(s) - 2) + 5Y(s) = 0$ so

$$Y(s) = \frac{2s+9}{s^2+4s+5} = \frac{2s+9}{(s+2)^2+1} = \frac{2(s+2)}{(s+2)^2+1} + \frac{5}{(s+2)^2+1}$$

Combining formula $F(s+2) = \mathcal{L}[e^{-2t}f(t)]$ with the Laplace transforms of the trig functions we see that $\mathcal{L}\left[e^{-2t}\cos t\right](s) = \frac{s+2}{(s+2)^2+1}$ and $\mathcal{L}\left[e^{-2t}\sin t\right](s) = \frac{1}{(s+2)^2+1}$. We read out the answer: $\boxed{y(t) = 5e^{-2t}\sin(t) + 2e^{-2t}\cos(t)}$

D-2. Determine the function with given Laplace transform, and sketch its graph.

(a) $F(s) = \frac{1}{s(s^2+1)}$



$F(s) = \frac{1}{s} - \frac{s}{s^2+1}$ so $\boxed{f(t) = 1 - \cos t}$.

$f(0)=0$

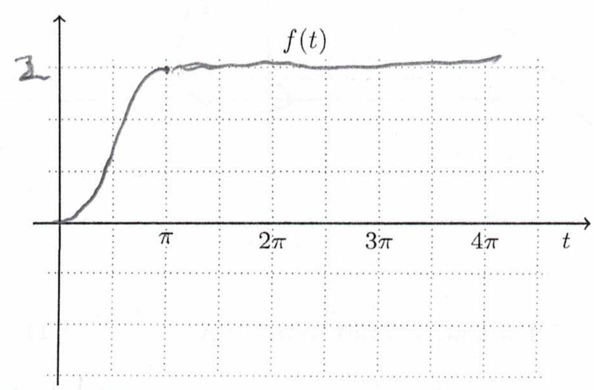
$f'(0)=0$

(b) $F(s) = \frac{1+e^{-\pi s}}{s(s^2+1)}$ From previous problem, $F(s) = \frac{1}{s} - \frac{s}{s^2+1} + e^{-\pi s}\left(\frac{1}{s} - \frac{s}{s^2+1}\right)$.

We get $f(t) = 1 - \cos t + u_\pi(t)(1 - \cos(t-\pi)) = 1 - \cos t + u_\pi(t)(1 + \cos t)$.

To graph the function, write

$$f(t) = \begin{cases} 1 - \cos t & t < \pi \\ 2 & t > \pi \end{cases}$$

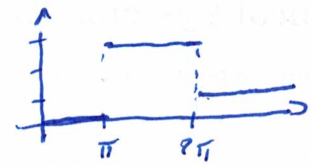


D-3. Solve the fourth-order differential equation $y^{(4)} - y = 0$ with the initial conditions $y(0) = 1$ $y'(0) = 0$, $y''(0) = 1$, $y'''(0) = 0$ Laplace transform with careful algebra should be convenient.

$s^4Y(s) - s^3 - s + Y(s) = 0$, so $Y(s) = \frac{s^3+s}{(s^2-1)(s^2+1)} = \frac{s(s^2+1)}{(s^2-1)(s^2+1)} = \frac{s}{s^2-1} = \frac{1/2}{s-1} + \frac{1/2}{s+1}$

Answer: $\boxed{y(t) = \frac{1}{2}e^t + \frac{1}{2}e^{-t} = \cosh t}$

D-4. Compute the **Laplace transform** $Y(s)$ of the solution $y(t)$ to the initial value problem $y'' + y = g(t)$, $y(0) = 5$, $y'(0) = 3$, with discontinuous right hand side function

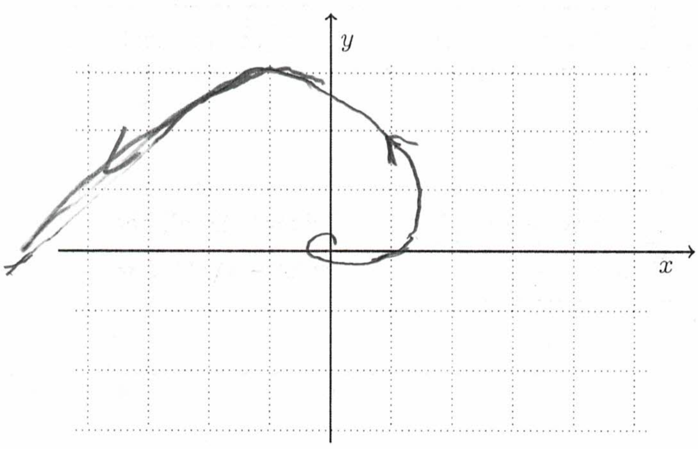


$$g(t) = \begin{cases} 0 & t < \pi \\ 3 & \text{for } \pi \le t \le 2\pi \\ 1 & \text{for } t > 2\pi \end{cases}$$

(Do not simplify your answer. Do not solve for $y(t)$.) $g(t) = 3H(t-\pi) - 2H(t-2\pi)$ has the Laplace transform $G(s) = 3e^{-\pi s}/s - 2e^{-2\pi s}/s$. So $\boxed{Y(s) = \frac{5s+3}{s^2+1} + 3\frac{2e^{-\pi s}}{s(s^2+1)} - 2\frac{e^{-2\pi s}}{s(s^2+1)}}$

D-5. Determine the eigenvalues and eigenvectors if the eigenvalues are real, classify the system (state whether stable or unstable node, stable or unstable spiral, center, saddle point) and sketch the phase plane of the linear system.

$$\frac{dx}{dt} = 4x - 2y$$
$$\frac{dy}{dt} = x + 2y$$



The matrix of this system is $\mathbf{A} = \begin{bmatrix} 4 & -2 \\ 1 & 2 \end{bmatrix}$ The eigenvalues are $\lambda = 3 \pm i$. This is an $\boxed{\text{unstable spiral, moving counter-clockwise}}$

char. polynomial $\det \begin{bmatrix} 4-\lambda & -2 \\ 1 & 2-\lambda \end{bmatrix} = (4-\lambda)(2-\lambda) + 2 = \lambda^2 - 6\lambda + 10$