Printed: February 11, 2023 Name ______________________________

Dynamical Systems Exam-1

A Key

Instructions. In the differential equations below, $x = x(t)$ is a function of independent variable t. Simplify your answers when appropriate. Be sure to show your work so that it is clear how you got your answers.

1. Determine explicit solution of the initial value problem $x' = tx$, $x(0) = -3$.

   Solution This equation can be solved as linear, or as separable.

$$\frac{dx}{x} = tdt$$

$$\ln|x| = t^2/2 + c$$

   The general solution is $x = Ce^{t^2/2}$

   Answer: $\boxed{x = -3e^{t^2/2}}$

2. Find the general solution of the equation $x' = e^t + x$

   Solution This is linear equation $x' - x = e^t$ with the integrating factor e^{-t}.

$$e^{-t}x' - ye^{-t} = 1$$

$$(e^{-t}x)' = 1$$

$$e^{-t}x = t + C$$

   So the answer is $x = te^t + Ce^t$ where C is an arbitrary constant.

3. Find the general solution of the equation $x'' - 2x' - 3x = 10\cos t$. This is a linear equation with general solution $\boxed{x(t) = C_1e^{-t} + C_2e^{3t} - 2\cos t - \sin t}$.

4. Solve the (non-homogeneous) equation $x'' - 4x' + 4x = 4$ with the initial values $x(0) = 0$, $x'(0) = 0$.

   Answer: $\boxed{x = 1 - e^{2t} + 2te^{2t}}$

5. Solve the equation $x'' - 4x' + 8x = 4 + 8t$.

   Answer: $\boxed{x = C_1e^{2t}\cos(2t) + C_2e^{2t}\sin(2t) + t + 1}$

Dynamical Systems Exam-1

B Key

Instructions. In the differential equations below, $x = x(t)$ is a function of independent variable t. Simplify your answers when appropriate. Be sure to show your work so that it is clear how you got your answers.

1. Determine explicit solution of the initial value problem $x' = 4tx$, $x(0) = -2$. **Solution** This equation can be solved as linear, or as separable.

$$\frac{dx}{x} = 4tdt$$

$$\ln|x| = 2t^2 + c$$

The general solution is $x = Ce^{2t^2}$ Answer: $\boxed{x = -2e^{2t^2}}$

2. Find the general solution of the equation $x' = 2e^t + x$

Solution This is linear equation $x' - x = 2e^t$ with the integrating factor e^{-t}.

$$e^{-t}x' - ye^{-t} = 2$$

$$(e^{-t}x)' = 2$$

$$e^{-t}x = 2t + C$$

So the answer is $x = 2te^t + Ce^t$ where C is an arbitrary constant.

3. Find the general solution of the equation $x'' - 3x' + 2x = 10\cos t$. This is a linear equation with general solution $\boxed{x(t) = C_1e^t + C_2e^{2t} + \cos t - 3\sin t}$.

4. Solve the equation $x'' + 2x' + 5x = 5t - 3$.

Answer: $\boxed{x = C_1e^{-t}\cos(2t) + C_2e^{-t}\sin(2t) + t - 1}$

5. Solve the (non-homogeneous) equation $x'' - 2x' + x = 1$ with the initial values $x(0) = 0$, $x'(0) = 0$.

Answer: $\boxed{x = 1 - e^t + te^t}$

Dynamical Systems Exam-1

C Key

Instructions. In the differential equations below, $x = x(t)$ is a function of independent variable t. Simplify your answers when appropriate. Be sure to show your work so that it is clear how you got your answers.

1. Determine explicit solution of the initial value problem $x' = 6tx$, $x(0) = -5$.

   Solution This equation can be solved as linear, or as separable.

$$\frac{dx}{x} = 6tdt$$

$$\ln|x| = 3t^2 + c$$

   Exponentiating, the general solution is $x = Ce^{3t^2}$, where $C = \pm e^c$. The last step is to compute C from the initial condition $x(0) = -5$. Answer: $\boxed{x = -5e^{3t^2}}$

2. Find the general solution of the equation $x' = 3e^t + x$

   Solution This is linear equation $x' - x = 3e^t$ with the integrating factor $\mu = e^{-t}$. Multiplying the equation by μ, get

$$e^{-t}x' - ye^{-t} = 3$$

$$(e^{-t}x)' = 3$$

$$e^{-t}x = 3t + C$$

   So the answer is $x = 3te^t + Ce^t$ where C is an arbitrary constant.

3. Find the general solution of the equation $x'' + 2x' - 3x = 10\cos t$.

   This is linear equation with constant coefficients. Characteristic equation $r^2 + 2r - 3$ has two roots: $r_1 = -3, r_2 = 1$. So the general solution is $x = C_1e^{-3t} + C_2e^t + x_p$. We seek a particular solution of the form $x_p = A\cos t + B\sin t$. Computing the derivatives, we have

$$x_p' = -A\sin t + B\cos t, \quad x_p'' = -A\cos t - B\sin t$$

   Putting those into the equation,

$$\begin{aligned} x_p'' + 2x_p' - 3x_p &= (-A\cos t - B\sin t) + 2(-A\sin t + B\cos t) - 3(A\cos t + B\sin t) \\ &= -A\cos t - B\sin t - 2A\sin t + 2B\cos t - 3A\cos t - 3B\sin t \\ &= (-4A + 2B)\cos t + (-2A - 4B)\sin t = 10\cos t \end{aligned}$$

   It remains to solve the system of equations

$$\begin{aligned} -4A + 2B &= 10 \\ -2A - 4B &= 0 \end{aligned}$$

   Since from the second equation $A = -2B$ from the first equation we get $8B + 2B = 10$ or $B = 1$.

   $\boxed{\text{Answer: } x = C_1e^{-3t} + C_2e^t + \sin t - 2\cos t}$

4. Solve the equation $x'' - 2x' + 5x = 8 + 5t$.

   Characteristic equation $r^2 - 2r + 5 = 0$ has roots $1 \pm 2i$. We seek particular solution $x_p = A + Bt$ with $x_p' = B$ and $x_p'' = 0$.

   Answer: $\boxed{x = C_1e^t\cos(2t) + C_2e^t\sin(2t) + 2 + t}$

5. Solve the equation $x'' + 2x' + x = 1$ with the initial values $x(0) = 0$, $x'(0) = 0$.

   Characteristic equation $r^2 + 2r + 1 = 0$ has double root $r = -1$. We seek particular solution $x_p = A$ and compute $A = 1$.

   The general solution is $x = C_1e^{-t} + C_2te^{-t} + 1$. From $x(0) = 0$ we compute $C_1 = -1$. From $x'(0) = 0$ we get $C_2 = C_1$.

   Answer: $\boxed{x = 1 - e^{-t} - te^{-t}}$

Dynamical Systems Exam-1

x Key

Instructions. In the differential equations below, $x = x(t)$ is a function of independent variable t. Simplify your answers when appropriate. Be sure to show your work so that it is clear how you got your answers.

1. Determine explicit solution of the initial value problem $x' = tx$, $x(0) = -2$.

   Solution This equation can be solved as linear, or as separable.

$$\frac{dx}{x} = tdt$$

$$\ln|x| = t^2/2 + c$$

   The general solution is $x = Ce^{t^2/2}$

   Answer: $\boxed{x = -2e^{t^2/2}}$

2. Find the general solution of the equation $x' = e^t + x$

   Solution This is linear equation $x' - x = e^t$ with the integrating factor e^{-t}.

$$e^{-t}x' - ye^{-t} = 1$$

$$(e^{-t}x)' = 1$$

$$e^{-t}x = t + C$$

   So the answer is $x = te^t + Ce^t$ where C is an arbitrary constant.

3. Find the general solution of the equation $x'' + 4x' + 3x = 10\cos t$. This is a linear equation with general solution $\boxed{x(t) = C_1e^{-t} + C_2e^{-3t} + \cos t + 2\sin t}$.

4. Solve the (non-homogeneous) equation $x'' + 4x' + 4x = -4$ with the initial values $x(0) = 0$, $x'(0) = 0$.

   Answer: $\boxed{x = 2e^{-2t}t + e^{-2t} - 1}$

5. Solve the equation $x'' - 2x' + 10x = 10t - 12$.

   Answer: $\boxed{x = C_1e^t\cos(3t) + C_2e^t\sin(3t) + t - 1}$