Differential Equations MATH 2073 Final-2019A Key

Instructions. Simplify your answers when appropriate. Be sure to show your work so that it is clear how you got your answers.

1. Solve the initial value problem $y' = 2x + y$; $y(0) = 2$.

   Solution

   $\frac{dy}{dx} - y = 2x$, so integrating factor is $\mu(x) = e^{-x}$. The general solution is $y = Ce^x + 2e^x \int_0^x ue^{-u}\,du = Ce^x - 2x - 2$ and the initial value is satisfied by $y = 4e^x - 2x - 2$.

2. Solve the initial value problem $y' = 6yx$; $y(0) = 6$

   Solution $\frac{dy}{y} = 6x\,dx$, so $\ln|y| = 3x^2 + C$. The general solution is $y = Ce^{3x^2}$ and the initial value is satisfied by $y = 6e^{3x^2}$.

3. Find the solution of the equation $y'' - 2y' = 4$ with the initial value $y(0) = 1, y'(0) = 1$.

   Solution I The Laplace transform of y satisfies the equation $s^2Y - s - 1 - 2sY + 2 = 4/s$ Therefore

   $$Y(s) = \frac{s^2 - s + 4}{s^2(s-2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-2}$$

   This gives $As^2 - 2As + sB - 2B + Cs^2 = s^2 - s + 4$. Therefore $A + C = 1, -2B = 4, B - 2A = -1$. So

   $$Y(s) = -\frac{1}{2}\frac{1}{s} - 2\frac{1}{s^2} + \frac{3}{2}\frac{1}{s-2}$$

   The answer is $\boxed{y(t) = -\frac{1}{2} - 2t + \frac{3}{2}e^{2t}}$.

   Solution II The homogeneous equation has the characteristic equation $r^2 - 2r = 0$ so the solution is $y(t) = C_1e^{2t} + C_2 + y_*(t)$. By undetermined coefficients method, the particular solution takes form $y_* = At$. Putting this into the equation, $0 - 2A = 4$, so $A = -2$. Thus the general solution is $y(t) = C_1e^{2t} + C_2 - 2t$.

   Now we use the initial values: $y(0) = C_1 + C_2 = 1, y'(0) = 2C_1 - 2 = 1$. So $C_1 = 3/2, C_2 = -1/2$, and the solution is $\boxed{y(t) = -\frac{1}{2} - 2t + \frac{3}{2}e^{2t}}$.

   Solution III Substitute $u = y'$. This reduces the order of the equation to $u' - 2u = 4$. This equation of order 1 can be solved either by integrating factor method, or by general theory of n-th order linear equations, or by separation of variables. Choosing the latter method, $u' = 2(u+2)$, so $\int \frac{du}{u+2} = \int 2dt$, and $\ln(u+2) = c + 2t$. This gives $u = C_1e^{2t} - 2$. Since $u = y'$, this means that $y' = C_1e^{2t} - 2$. Using the initial condition $y'(0) = 1$ we get $C_1 = 3$ and $y' = 3e^{2t} - 2$. Integrating this, we get $y(t) = \frac{3}{2}e^{2t} - 2t + C_2$. From $y(0) = 1$ we determine $C_2 = 1 - 3/2 = -1/2$. So the answer is $\boxed{y = \frac{3}{2}e^{2t} - 2t - \frac{1}{2}}$.

4. Find the solution of the equation

   $$y''' + 4y' = 0 \text{ with initial conditions } y(0) = 1,\ y'(0) = 2,\ y''(0) = 4$$

   Solution: Characteristic equation $r^3 + 4r = 0$ factors as $r(r - 2i)(r + 2i) = 0$. The roots are $0, -2i, 2i$. So the general solution is $y = C_1 + C_2\cos 2x + C_3\sin 2x$. The initial conditions determine the constants. Answer: $\boxed{y = 2 - \cos(2x) + \sin(2x)}$

5. Find the general solution of the equation $y^{(4)} - y = 4e^t$.

   The characteristic equation $r^4 - 1 = 0$ factors as $r^4 - 1 = (r^2 - 1)(r^2 + 1) = (r-1)(r+1)(r-i)(r+i) = 0$ with roots $\pm 1, \pm i$. The general solution is $y = C_1e^t + C_2e^{-t} + C_3\sin t + C_4\cos t$.

   Seek particular solution $y_p = Ate^t$. This gives $y_p = te^t$ Answer: $\boxed{y = C_1e^t + C_2e^{-t} + C_3\sin t + C_4\cos t + te^t}$

6. Find the general solution of the homogeneous Euler equation $x^2y'' - 6y = 0$ for $x > 0$. *Hint. Recall that we search for the solutions of the form $y = x^r$.*

   The characteristic equation is $r(r-1) - 6 = 0$. This factors as $(r-3)(r+2) = 0$ The general solution of the homogeneous equation is $\boxed{y = C_1\frac{1}{x^2} + C_2x^3}$

7. Find the power series solution $y = \sum_{n=0}^{\infty} a_n x^n$ of equation $y'' - xy' - y = 0$ with the initial value $y(0) = 1, y'(0) = 0$. Steps in the solution:

(a) Express xy' and y'' as the power series in powers of x^n.
$xy' = \sum_{n=1}^{\infty} n a_n x^n$
$y'' = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$

(b) Determine the recurrence relation for coefficients a_n. Simplify your answer!

$$(n+2)(n+1)a_{n+2} - na_n - a_n = 0$$

This simplifies to

$$a_{n+2} = \frac{a_n}{n+2}$$

(c) Compute the first six terms of the power series $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \dots$ and notice the pattern.
Since $a_0 = 1, a_1 = 0$, the recursion gives

$$a_2 = \frac{a_o}{2} = \frac{1}{2},\ a_3 = \frac{0}{3},\ a_4 = \frac{a_2}{4} = \frac{1}{2 \cdot 4},\ a_5 = \frac{a_3}{5} = \frac{0}{3 \cdot 5},\ a_6 = \frac{a_4}{6} = \frac{1}{2 \cdot 4 \cdot 6}$$

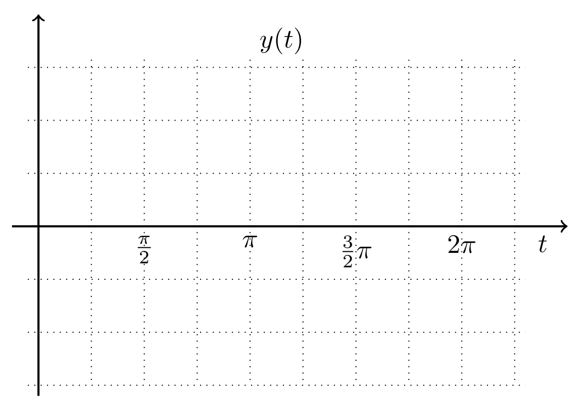
The first six terms are $\boxed{y = 1 + \frac{x^2}{2} + \frac{x^4}{2 \cdot 4} + \frac{x^6}{2 \cdot 4 \cdot 6} + \dots} = 1 + \frac{x^2}{2} + \frac{x^4}{8} + \frac{x^6}{48} + \dots$.
The pattern is

$$a_{2k} = \frac{1}{2 \times 4 \times \dots \times (2k)} = \frac{1}{2^k k!}$$
$$a_{2k+1} = 0$$

The answer is

$$y = \sum_{k=0}^{\infty} \frac{x^{2k}}{2^k k!} = \sum_{k=0}^{\infty} \frac{\left(\frac{x^2}{2}\right)^k}{k!} = e^{x^2/2}$$

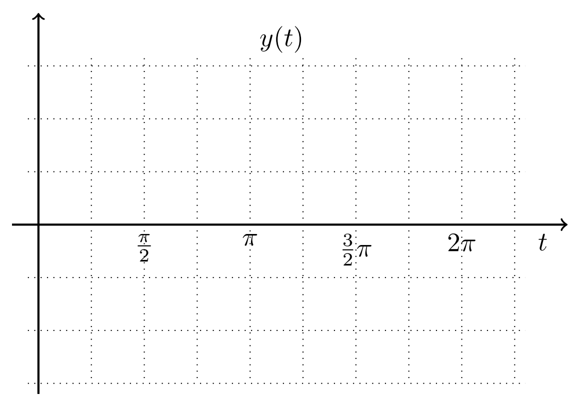
8. Solve the differential equation $y'' + 4y = 4u_\pi(t)$, $y(0) = 1, y'(0) = 0$. (Here $u_\pi(t)$ is the unit step function.)
Sketch the graph of the solution.



Solution: The Laplace transform is $Y(s) = \frac{s}{s^2+4} + \frac{4e^{-\pi s}}{s(s^2+4)} = \frac{s}{s^2+4} + e^{-\pi s}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)$. Therefore the solution is $y(t) = \cos 2t + u_\pi(t)(1 - \cos 2(t - \pi)) = \cos t + u_\pi(t)(1 - \cos t)$. Thus

$$y(t) = \begin{cases} \cos 2t & t < \pi \\ 1 & t > \pi \end{cases}.$$

9. Solve the differential equation $y'' + y = \delta(t-\pi), y(0) = 0, y'(0) = 1$. (Here $\delta(t)$ is the unit impulse function.) Sketch the graph of the solution.



Solution: The Laplace transform is $Y(s) = \frac{1}{s^2+1} + \frac{e^{-\pi s}}{s^2+1}$. Therefore the solution is $y(t) = \sin t + u_\pi(t)\sin(t-\pi) = (1 - u_\pi(t))\sin t$. Thus

$$y(t) = \begin{cases} \sin t & \text{if } t \le \pi \\ 0 & \text{if } t > \pi \end{cases}$$

10. Consider the system of differential equations

$$\begin{aligned} u_1' &= 2u_1 + u_2, \\ u_2' &= u_1 + 2u_2 \end{aligned}$$

with initial condition $u_1(0) = 4, u_2(0) = 2$.

Determine solution $u_1(t)$ of this system. *Hint. Transform the system into a single equation of order 2. (Alternatively, you can use the Laplace transform.)*

Standard solution from the book: From first equation, $u_2 = u_1' - 2u_1$, so the second equation becomes $u_1'' - 2u_1' = u_1 + 2(u_1' - 2u_1)$. This simplifies to

$$u_1'' - 4u_1' + 3u_1 = 0$$

The initial conditions are $u_1(0) = 4$ and $u_1'(0) = 2u_1(0) + u_2(0) = 10$.

The characteristic equation is $r^2 - 4r + 3 = (r-1)(r-3) = 0$ so the general solution is $u_1 = C_1e^t + C_2e^{3t}$. Using the initial condition

$$C_1 + C_2 = 4, \ C_1 + 3C_2 = 10$$

We determine that $C_1 = 3$ and $C_1 = 1$.

Answer: $\boxed{u_1(t) = 3e^{3t} + e^t}$ and $u_2(t) = 3e^{3t} - e^t$

Non-standard solution:

Adding the equations we get $u_1' + u_2' = 3u_1 + 3u_2$. So we have an equation $y' = 3y$ with initial condition $y(0) = 6$ for $y = u_1 + u_2$. The solution of this equation is $y = Ce^{3t}$ with $C = 6$.

Subtracting the equations we get $u_1' - u_2' = u_1 - u_2$. So we have an equation $y' = y$ with $y(0) = 2$ for $y = u_1 - u_2$. The solution of this equation is $y = Ce^t$ with $C = 2$.

So $u_1 + u_2 = 6e^{3t}$ and $u_1 - u_2 = 2e^t$.

Adding the equations we get $2u_1 = 6e^{3t} + 2e^t$

Answer: $\boxed{u_1(t) = 3e^{3t} + e^t}$

Laplace transform solution: Taking the Laplace transforms of both sides of the equations we get

$$sU_1(s) - 4 = 2U_1 + U_2 \text{ and } sU_2 - 2 = U_1 + 2U_2$$

where $U_1(s) = \mathcal{L}(u_1)$. Solving the system of equations

$$\begin{aligned} (s-2)U_1 - U_2 &= 4 \\ -U_1 + (s-2)U_2 &= 2 \end{aligned}$$

we get

$$U_1 = \frac{\begin{vmatrix} 4 & -1 \\ 2 & s-2 \end{vmatrix}}{\begin{vmatrix} s-2 & -1 \\ -1 & s-2 \end{vmatrix}} = \frac{4s-6}{(s-2)^2 - 1^2} = \frac{4s-6}{((s-2)-1)((s-2)+1)} = \frac{4s-6}{(s-3)(s-1)} = \frac{3}{s-3} + \frac{1}{s-1}$$

From the table of Laplace transforms we read out the answer $\boxed{u_1 = 3e^{3t} + e^t}$