Printed: April 21, 2022 Name ______________________

Differential Equations MATH 2073 Final-2018. Key

Instructions. Simplify your answers when appropriate. Be sure to show your work so that it is clear how you got your answers.

1. Find the solution of the initial value problem $y' + 6t^2y^2 = 0$, $y(0) = 1$.

   This is separable equation. $-\int \frac{dy}{y^2} = \int 6t^2 dt$ so $1/y = 2t^3 + C$ and $y = \frac{1}{2t^3+C}$. Initial condition gives $C = 1$ so the answer is $\boxed{y(t) = \frac{1}{1+2t^3}}$

2. Solve the initial value problem $y' - 2y = 4t,\ y(0) = 0$.

   This is linear equation with the integrating factor $\mu(t) = e^{-2t}$, so the general solution is $y(t) = \frac{1}{\mu(t)} \int 4t\mu(t)dt = e^{2t} \int 4te^{-2t} = e^{2t}(-2te^{-2t} - 2e^{-2t} + C) = Ce^{2t} - 1 - 2t$

   $\boxed{\text{Answer: } y = e^{2t} - 1 - 2t}$

3. Find the general solution of the equation $y^{(4)} - y = 10e^t$.

   The characteristic equation $r^4 - 1 = 0$ factors as $r^4 - 1 = (r^2 - 1)(r^2 + 1) = (r - 1)(r + 1)(r - i)(r + i) = 0$ with roots $\pm 1, \pm i$. The general solution is $y = C_1e^t + C_2e^{-t} + C_3 \sin t + C_4 \cos t$.

   Seek particular solution $y_p = Ate^t$. This gives $y_p = \frac{5}{2}te^t$ Answer: $\boxed{y = C_1e^t + C_2e^{-t} + C_3 \sin t + C_4 \cos t + \frac{5}{2}te^t}$

4. Find the general solution of the equation $y''' + 4y' = 10e^t$ We first find the general solution of the homogeneous equation $y''' + 4y' = 0$. Characteristic equation $r^3 + 4r = 0$ factors as $r(r - 2i)(r + 2i) = 0$. The roots are $0, -2i, 2i$. So the general solution is $y = C_1 + C_2 \cos 2t + C_3 \sin 2t$

   Seek $y_p = Ae^t$. This gives $y_p = 2e^t$ Answer: $\boxed{y = C_1 + C_2 \cos 2t + C_3 \sin 2t + 2e^t}$

5. Find the general solution of the nonhomogeneous equation $y'' - 4y' + 3y = 10 \cos t$.

   The homogeneous equation has solution $y(t) = C_1e^t + C_2e^{2t}$. Using method of undetermined parameters, we seek the particular solution $y_* = A \cos t + B \sin t$. Calculation gives $A = 1, B = -2$.

   Answer $\boxed{y(t) = C_1e^t + C_2e^{2t} + \cos t - 2 \sin t}$.

6. Find the general solution of the homogeneous Euler equation $x^2y'' + xy' - y = 0$ for $x > 0$. *Hint. Recall that we search for the solutions of the form $y = x^r$.* The characteristic equation is $r(r - 1) + r - 1 = 0$. This has roots $r = \pm 1$ The general solution of the homogeneous equation is $\boxed{y = C_1\frac{1}{x} + C_2x}$

7. Find the power series solution $y = \sum_{n=0}^{\infty} a_nx^n$ of equation $y'' - xy' - y = 0$ with the initial value $y(0) = 1, y'(0) = 0$. Steps in the solution:

   (a) Express xy' and y'' as the power series in powers of x^n.
   $xy' = \sum_{n=1}^{\infty} na_nx^n$
   $y'' = \sum_{n=0}^{\infty}(n + 2)(n + 1)a_{n+2}x^n$

   (b) Determine the recurrence relation for coefficients a_n. Simplify your answer!

   $$(n + 2)(n + 1)a_{n+2} - na_n - a_n = 0$$

   This simplifies to

   $$a_{n+2} = \frac{a_n}{n + 2}$$

(c) Compute the first six terms of the power series $y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \dots$ and notice the pattern.

Since $a_0 = 1, a_1 = 0$, the recursion gives

$$a_2 = \frac{a_o}{2} = \frac{1}{2},\ a_3 = \frac{0}{3},\ a_4 = \frac{a_2}{4} = \frac{1}{2\cdot 4},\ a_5 = \frac{a_3}{5} = \frac{0}{3\cdot 5},\ a_6 = \frac{a_4}{6} = \frac{1}{2\cdot 4\cdot 6}$$

The first six terms are $\boxed{y = 1 + \frac{x^2}{2} + \frac{x^4}{2\cdot 4} + \frac{x^6}{2\cdot 4\cdot 6} + \dots} = 1 + \frac{x^2}{2} + \frac{x^4}{8} + \frac{x^6}{48} + \dots$.

The pattern is

$$a_{2k} = \frac{1}{2\times 4\times\cdots\times(2k)} = \frac{1}{2^k k!}$$

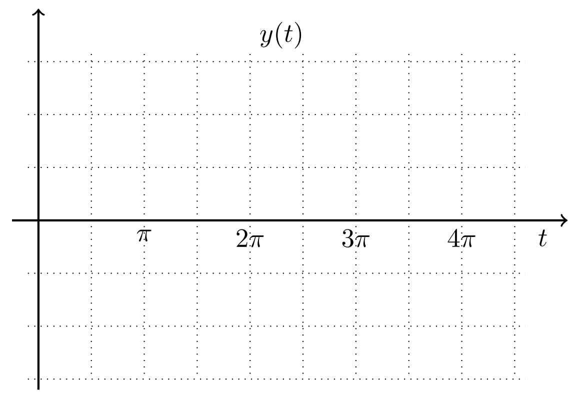
$$a_{2k+1} = 0$$

The answer is

$$y = \sum_{k=0}^{\infty} \frac{x^{2k}}{2^k k!} = \sum_{k=0}^{\infty} \frac{\left(\frac{x^2}{2}\right)^k}{k!} = e^{x^2/2}$$

8. Solve the differential equation $y'' + y = -u_\pi(t),\ y(0) = 1, y'(0) = 0$. (Here $u_\pi(t)$ is the unit step function.)

Sketch the graph of the solution.

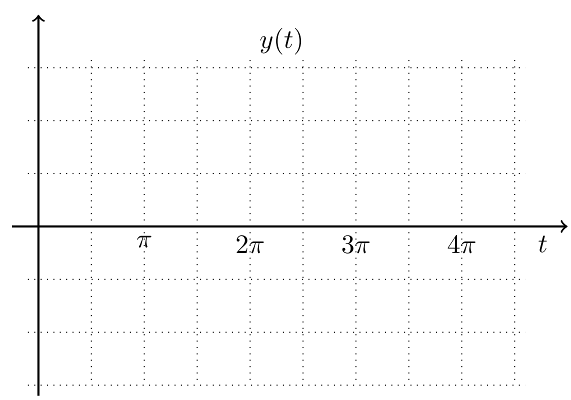


Solution: The Laplace transform is $Y(s) = \frac{s}{s^2+1} - \frac{e^{-\pi s}}{s(s^2+1)} = \frac{s}{s^2+1} - e^{-\pi s}\left(\frac{1}{s} - \frac{s}{s^2+1}\right)$. Therefore the solution is $y(t) = \cos t - u_\pi(t)(1 - \cos(t-\pi)) = \cos t - u_\pi(t)(1 + \cos t)$. Thus

$$y(t) = \begin{cases} \cos t & t < \pi \\ -1 & t > \pi \end{cases}.$$

9. Solve the differential equation $y'' + y = \delta(t - \pi), y(0) = 0, y'(0) = 1$. (Here $\delta(t)$ is the unit impulse function.)

Sketch the graph of the solution.



Solution: The Laplace transform is $Y(s) = \frac{1}{s^2+1} + \frac{e^{-\pi s}}{s^2+1}$. Therefore the solution is $y(t) = \sin t + u_\pi(t)\sin(t-\pi) = (1 - u_\pi(t))\sin t$. Thus

$$y(t) = \left\{ \begin{array}{cc} \sin t & \text{if } t \le \pi \\ 0 & \text{if } t > \pi \end{array} \right.$$

10. Consider the system of differential equations

$$\begin{aligned} u_1' &= 2u_1 + u_2, \\ u_2' &= u_1 + 2u_2 \end{aligned}$$

with initial condition $u_1(0) = 2, u_2(0) = 4$.

Determine solution $u_1(t)$ of this system. *Hint. Transform the system into a single equation of order 2. (Alternatively, you can use the Laplace transform.)*

Standard solution from the book: From first equation, $u_2 = u_1' - 2u_1$, so the second equation becomes $u_1'' - 2u_1' = u_1 + 2(u_1' - 2u_1)$. This simplifies to

$$u_1'' - 4u_1' + 3u_1 = 0$$

The initial conditions are $u_1(0) = 2$ and $u_1'(0) = 2u_1(0) + u_2(0) = 8$.

The characteristic equation is $r^2 - 4r + 3 = (r-1)(r-3) = 0$ so the general solution is $u_1 = C_1 e^t + C_2 e^{3t}$. Using the initial condition

$$C_1 + C_2 = 2, \ C_1 + 3C_2 = 8$$

We determine that $C_2 = 3$ and $C_1 = -1$.

Answer: $\boxed{u_1(t) = 3e^{3t} - e^t}$ and $u_2(t) = 3e^{3t} + e^t$

Non-standard solutio:

Adding the equations we get $u_1' + u_2' = 3u_1 + 3u_2$. So we have an equation $y' = 3y$ with initial condition $y(0) = 6$ for $y = u_1 + u_2$. The solution of this equation is $y = Ce^{3t}$ with $C = 6$.

Subtracting the equations we get $u_1' - u_2' = u_1 - u_2$. So we have an equation $y' = y$ with $y(0) = -2$ for $y = u_1 - u_2$. The solution of this equation is $y = Ce^t$ with $C = -2$.

So $u_1 + u_2 = 6e^{3t}$ and $u_1 - u_2 = -2e^t$. Adding the equations we get

Answer: $\boxed{u_1(t) = 3e^{3t} - e^t}$

Laplace transform solution: Taking the Laplace transforms of both sides of the equations we get

$$sU_1(s) - 2 = 2U_1 + U_2 \text{ and } sU_2 - 4 = U_1 + 2U_2$$

where $U_1 = U_1(s) = \mathcal{L}(u_1)$.

Solving the system of equations

$$\begin{aligned} (s-2)U_1 - U_2 &= 2 \\ -U_1 + (s-2)U_2 &= 4 \end{aligned}$$

we get

$$U_1 = \frac{\begin{vmatrix} 2 & -1 \\ 4 & s-2 \end{vmatrix}}{\begin{vmatrix} s-2 & -1 \\ -1 & s-2 \end{vmatrix}} = \frac{2s}{(s-2)^2 - 1^2} = \frac{2s}{((s-2)-1)((s-2)+1)} = \frac{2s}{(s-3)(s-1)} = \frac{3}{s-3} - \frac{1}{s-1}$$

From the table of Laplace transforms we read out the answer $\boxed{u_1 = 3e^{3t} - e^t}$

Table of Laplace transforms $\mathcal{L}(f) = \int_0^\infty e^{-ts} f(t)dt$

$f(t)$	$F(s) = \mathcal{L}(f)$	(domain of $F(s)$)
1	$\frac{1}{s}$	$(s > 0)$
t^n	$\frac{n!}{s^{n+1}}$	$(s > 0)$
e^{at}	$\frac{1}{s-a}$	$(s > a)$
$\cos at$	$\frac{s}{s^2+a^2}$	$(s > 0)$
$\sin at$	$\frac{a}{s^2+a^2}$	$(s > 0)$
$u_c(t)$	$\frac{e^{-cs}}{s}$	$(s > 0)$
$\delta(t-a)$	e^{-as}	$(s > 0)$

Laplace transform formulas

$$\mathcal{L}(f_1(t) + cf_2(t)) = \mathcal{L}(f_1(t)) + c\mathcal{L}(f_2(t))$$

$$\mathcal{L}\left(f^{(n)}(t)\right) = s^n F(s) - s^{n-1} f(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0)$$

$$\mathcal{L}\left(e^{at} f(t)\right) = F(s-a)$$

$$\mathcal{L}(t^k f(t)) = (-1)^k \frac{d^k}{ds^k} F(s)$$

$$\mathcal{L}\left(u_c(t) f(t-c)\right) = e^{-cs} F(s)$$

$$\mathcal{L}\left(\int_o^t f(\tau) g(t-\tau)\, d\tau\right) = F(s) \cdot G(s)$$

$$\mathcal{L}\left(f(at)\right) = \frac{1}{a} F\left(\frac{s}{a}\right)$$

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