Quantum Field Theory 1

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Abstract

These notes are based on the courses on Quantum Field Theory given at TU Dortmund and U Cincinnati. The main goal of these lectures is to calculate a relativistic QED cross section from first principles, and to gain a basic understanding of radiative corrections.

Contents

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0 Introductory Remarks

Large parts of these lecture notes rely heavily on the textbooks Refs. [1–4]. The "mostly minus" metric, $\eta = \text{diag}(1, -1, -1, -1)$ is used throughout.

Text set in smaller type face was not part of the actual lecture and should be regarded as supplementary.

- Historically:
  - Problems with one-particle solutions
  - Interpret solutions as fields, "second quantization"
  - Starting point: "classical Lagrangian"
- Problems:
  - Why canonical quantization?
  - Why fields?
- Answer: QFT is natural synthesis of QM and STR
- Remarks / Plan:
  - Logical structure of QFT
  - Understand modern approaches
  - Higher order / renormalization
  - Exercises
  - Read as much as possible on your own

1 Relativistic Quantum Mechanics

Goals of this section: Understand the concept of a particle as a state in Hilbert space. Understand Lorentz transformation properties of particle states, as foundation to build Lorentz-invariant interations. Physics application: CP transformation and electric dipole moments.

Reminder: Quantum Mechanics

i) *Physical states* are representated by normalized vectors in Hilbert space $\mathcal{H}$:
$\Phi, \Psi \in \mathcal{H}, \xi, \eta \in \mathbb{C} \Rightarrow \xi\Phi + \eta\Psi \in \mathcal{H}$.
Scalar product:

$$(\Phi, \Psi) = (\Psi, \Phi)^* = \langle \Phi | \Psi \rangle \,, \tag{1.1}$$

$$(\Phi, \xi_1 \Psi_1 + \xi_2 \Psi_2) = \xi_1 (\Phi, \Psi_1) + \xi_2 (\Phi, \Psi_2) \,, \tag{1.2}$$

$$(\eta_1 \Phi_1 + \eta_2 \Phi_2, \Psi) = \eta_1^* (\Phi_1, \Psi) + \eta_2^* (\Phi_2, \Psi) \,, \tag{1.3}$$

$$||\Psi|| \equiv (\Psi, \Psi) \geq 0 \,. \tag{1.4}$$

NB: Actually, states correspond to rays in $\mathcal{H}$: Ψ and $\Psi' = \xi\Psi$ are equivalent for $\xi \in \mathbb{C}$ with $|\xi| = 1$.

ii) *Observables* are represented by Hermitian linear operators A, $\Psi \to A\Psi$.

$$A(\eta\Phi + \xi\Psi) = \eta A\Phi + \xi A\Psi \,, \tag{1.5}$$

$$A^\dagger = A, \text{where}(\Phi, A^\dagger\Psi) \equiv (A\Phi, \Psi) = (\Psi, A\Phi)^* \,. \tag{1.6}$$

Eigenvectors and eigenvalues:
$A\Psi = \alpha\Psi$. If A is Hermitian, then $\alpha \in \mathbb{R}$, and eigenvectors with different eigenvalues are orthogonal.

iii) *Probability*:

$$P(\Psi \to \Psi_n) = |(\Psi, \Psi_n)|^2 \,, \tag{1.7}$$

$$\sum_n P(\Psi \to \Psi_n) = 1 \,. \tag{1.8}$$

1.1 Symmetries

Change of point of view, without changing results of measurements. Observer $\mathcal{O}$ sees system in states $\Psi, \Psi_1, \Psi_2, \ldots$. Observer $\mathcal{O}$ sees same system in states $\Psi', \Psi'_1, \Psi'_2, \ldots$. They must find same probabilities:

$$P(\Psi \to \Psi_n) = P(\Psi' \to \Psi'_n) \,. \tag{1.9}$$

Theorem (Wigner 1930): $\Psi \to \Psi'$ described by operator U, where U is either unitary and linear,

$$(U\Phi, U\Psi) = (\Phi, \Psi) \,, \tag{1.10}$$

$$U(\xi\Phi + \eta\Psi) = \xi U\Phi + \eta U\Psi \,, \tag{1.11}$$

or antiunitary and antilinear,

$$(U\Phi, U\Psi) = (\Phi, \Psi)^* \,, \tag{1.12}$$

$$U(\xi\Phi + \eta\Psi) = \xi^* U\Phi + \eta^* U\Psi \,. \tag{1.13}$$

NB: The adjoint of an antilinear operator is defined as

$$(\Phi, A^\dagger\Psi) \equiv (A\Phi, \Psi)^* = (\Psi, A\Phi) \,. \tag{1.14}$$

Trivial symmetry transformation $\Psi \to \Psi$ corresponds to $U = \mathbb{1}$ (linear!). Infinitesimal transformation: $U = \mathbb{1} + i\epsilon t$, $\epsilon > 0$. t must be Hermitian and linear, thus potentially is an observable. (All known observables arise in this way.)

Symmetries have a natural group structure:

$$T_1, T_2 \in S \Rightarrow T_2 T_1 \in S \,; \quad T \in S \Rightarrow T^{-1} \in S \,; \quad \mathbb{1} \in S \,. \tag{1.15}$$

This structure is mirrored by the corresponding unitary operators:

$$U(T_2)U(T_1)\Psi = U(T_2T_1)\Psi\,. \tag{1.16}$$

The $U(T)$ *furnish a representation* of S.

In fact, since states are represented by rays rather than vectors in Hilbert space, the $U(T)$ furnish *projective representations*. See Ref. [1] for more details.

Lie groups

$$T(\bar\theta)T(\theta) = T(f(\bar\theta,\theta))\,, \tag{1.17}$$
$$T(0) = \mathbb{1} \Rightarrow f^a(\theta,0) = f^a(0,\theta) = \theta^a\,. \tag{1.18}$$

Represent as a power series in finite neighborhood of identity:

$$U(T(\theta)) = \mathbb{1} + i\theta^a t_a + \tfrac{1}{2}\theta^b\theta^c t_{bc} + \dots\,, \tag{1.19}$$
$$U(T(\theta))U(T(\bar\theta)) = U(T(f(\bar\theta,\theta)))\,. \tag{1.20}$$

Expand Eq. (1.20) about $\theta^a, \bar\theta^a$. Eq. (1.18) implies

$$f^a(\bar\theta,\theta) = \theta^a + \bar\theta^a + f^a_{bc}\bar\theta^b\theta^c + \dots \tag{1.21}$$

(no terms quadratic in θ or $\bar\theta$ can appear.) Thus, Eq. (1.20) becomes

$$\begin{aligned}
&\left[1 + i\bar\theta^a t_a + \tfrac{1}{2}\bar\theta^b\bar\theta^c t_{bc} + \dots\right] \times \left[1 + i\theta^a t_a + \tfrac{1}{2}\theta^b\theta^c t_{bc} + \dots\right]\\
&\quad = 1 + i\big(\theta^a + \bar\theta^a f^a_{bc}\bar\theta^b\theta^c + \dots\big)t_a + \tfrac{1}{2}\big(\theta^b + \bar\theta^b\big)\big(\theta^c + \bar\theta^c\big)t_{bc}\\
\Leftrightarrow &1 + i\bar\theta^a t_a + i\theta^a t_a - \bar\theta^b\theta^c t_b t_c + \tfrac{1}{2}\big(\bar\theta^b\bar\theta^c + \theta^b\theta^c\big)t_{bc} + \dots\\
&\quad = 1 + i\bar\theta^a t_a + i\theta^a t_a + i f^a_{bc}\bar\theta^b\theta^c t_a + \tfrac{1}{2}\big(\bar\theta^b\bar\theta^c + \theta^b\theta^c\big)t_{bc} + \tfrac{1}{2}\big(\underbrace{\theta^b\bar\theta^c}_{=\theta^c\bar\theta^b} + \bar\theta^b\theta^c\big)t_{bc} + \dots\\
\Leftrightarrow &t_{bc} = -t_b t_c - i f^a_{bc} t_a\\
\Rightarrow &0 = t_{cb} - t_{bc} = t_b t_c - t_c t_b + i f^a_{bc} t_a - i f^a_{cb} t_a\\
\Leftrightarrow &[t_b, t_c] = i C^a_{bc} t_a\,,
\end{aligned} \tag{1.22}$$

where $C^a_{bc} = f^a_{cb}t_a - f^a_{bc}$ are the *structure constants*. (In the derivation above we repeatedly used the fact that the t_{bc} are symmetric.) Important special case:

$$f^a(\theta,\bar\theta) = \theta^a + \bar\theta^a\,. \tag{1.23}$$

Then $f^a_{bc} = 0$, and

$$[t_b, t_c] = 0 \tag{1.24}$$

(*Abelian group*). In this case it is easy to show (use (1.20), (1.23)) that

$$U(T(\theta)) = \big[U(T(\theta/N))\big]^N = \lim_{N\to\infty}\left[1 + \frac{i}{N}\theta^a t_a\right]^N = \exp\big(i\theta^a t_a\big)\,. \tag{1.25}$$

1.2 Quantum Lorentz transformations

Reminder: Poincaré group $x^\mu = (t, x^1, x^2, x^3)$ with $c = 1$ are coordinates in inertial system. Einstein's principle of relativity:

$$\eta_{\mu\nu} dx'^\mu dx'^\nu = \eta_{\mu\nu} dx^\mu dx^\nu \tag{1.26}$$

or

$$\eta_{\mu\nu} = \frac{\partial x'^\mu}{\partial x^\rho} \frac{\partial x'^\nu}{\partial x^\sigma} \eta_{\rho\sigma} \,, \tag{1.27}$$

where $\eta_{11} = \eta_{22} = \eta_{33} = -1$, $\eta_{00} = +1$ are non-zero.

If $|d\boldsymbol{x}/dt| = c = 1$, we have $\eta_{\mu\nu} dx^\mu dx^\nu = d\boldsymbol{x}^2 - dt^2 = 0$ and hence also $\eta_{\mu\nu} dx'^\mu dx'^\nu = 0$, so $|d\boldsymbol{x}'/dt'| = 1$, which means the speed of light is the same in all inertial frames. This property is also true for the somewhat more general *conformal transformations*, where instead of Eq. (1.26), we have $\eta_{\mu\nu} dx'^\mu dx'^\nu \propto \eta_{\mu\nu} dx^\mu dx^\nu$.

Lorentz transformations:

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu \tag{1.28}$$

where a^μ is a constant and

$$\eta_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = \eta_{\rho\sigma} \,. \tag{1.29}$$

It follows

$$\left(\Lambda^{-1}\right)^\rho{}_\nu \equiv \eta_{\nu\mu} \eta^{\rho\sigma} \Lambda^\mu{}_\sigma = \Lambda_\nu{}^\rho \tag{1.30}$$

with $\eta^{\mu\nu} = \eta_{\mu\nu}$ the inverse of $\eta_{\mu\nu}$. Transformations with $a^\mu = 0$ are called *homogeneous L.T.*. One can show that $\det \Lambda = \pm 1$, and $\Lambda^0{}_0 \geq +1$ or $\Lambda^0{}_0 \leq -1$.
$\det \Lambda = 1$, $\Lambda^0{}_0 \geq +1$ …*proper orthochronous Lorentz group*.

Perform two L.T.:

$$x''^\mu = \bar{\Lambda}^\mu{}_\rho x'^\rho + \bar{a}^\mu = \bar{\Lambda}^\mu{}_\rho \left(\Lambda^\rho{}_\nu x^\nu + a^\rho\right) + \bar{a}^\mu = \left(\bar{\Lambda}^\mu{}_\rho \Lambda^\rho{}_\nu\right) x^\nu + \left(\bar{\Lambda}^\mu{}_\rho a^\rho + \bar{a}^\mu\right)$$

and hence

$$\Rightarrow T\left(\bar{\Lambda}, \bar{a}\right) T\left(\Lambda, a\right) = T\left(\bar{\Lambda}\Lambda, \bar{\Lambda} a + \bar{a}\right) \,, \tag{1.31}$$

$$\Rightarrow T^{-1}\left(\Lambda, a\right) = T\left(\Lambda^{-1}, -\Lambda^{-1} a\right) \,. \tag{1.32}$$

The (proper, orthochronous) L.T. induce unitary transformations in Hilbert space:

$$\Psi \to U(\Lambda, a) \Psi \,, \tag{1.33}$$

with

$$U\left(\bar{\Lambda}, \bar{a}\right) U\left(\Lambda, a\right) = U\left(\bar{\Lambda}\Lambda, \bar{\Lambda} a + \bar{a}\right) \,. \tag{1.34}$$

The Poincaré algebra Near the identity ($\Lambda^\mu{}_\nu = \delta^\mu{}_\nu$, $a^\mu = 0$) we have

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu\,, \qquad a^\mu = \epsilon^\mu\,. \tag{1.35}$$

The Lorentz condition (1.29) implies

$$\eta_{\rho\sigma} = \eta_{\mu\nu}\big(\delta^\mu{}_\rho + \omega^\mu{}_\rho\big)\big(\delta^\nu{}_\sigma + \omega^\nu{}_\sigma\big) = \eta_{\rho\sigma} + \omega_{\rho\sigma} + \omega_{\sigma\rho} + \mathcal{O}(\omega^2)\,, \tag{1.36}$$

and hence

$$\omega_{\rho\sigma} = -\omega_{\sigma\rho}\,. \tag{1.37}$$

L.T. contain $6+4$ independent parameters. We choose $U(1,0) = \mathbb{1}$ and write

$$U(1+\omega,\epsilon) = \mathbb{1} - \tfrac{i}{2}\omega_{\rho\sigma}J^{\rho\sigma} + i\epsilon_\rho P^\rho + \dots \tag{1.38}$$

with Hermitian generators

$$J^{\rho\sigma\dagger} = J^{\rho\sigma} = -J^{\sigma\rho}\,; \qquad P^{\rho\dagger} = P^\rho\,. \tag{1.39}$$

$\boldsymbol{P} \equiv \big(P^1, P^2, P^3\big)$ is the momentum; $\boldsymbol{J} \equiv \big(J^{23}, J^{31}, J^{12}\big)$ is the angular momentum; $H \equiv P^0$ is the energy (Hamiltonian). How do $J^{\mu\nu}$, P^μ transform? Consider

$$\begin{aligned}
&U(\Lambda,a)U(1+\omega,\epsilon)U^{-1}(\Lambda,a)\\
&\overset{(1.31),(1.32)}{=} U(\Lambda(1+\omega),\Lambda\epsilon + a)U(\Lambda^{-1},-\Lambda^{-1}a)\\
&= U(\Lambda(1+\omega)\Lambda^{-1}, -\Lambda(1+\omega)\Lambda^{-1}a + \Lambda\epsilon + a)\\
&= U(1+\Lambda\omega\Lambda^{-1}, \Lambda\epsilon - \Lambda\omega\Lambda^{-1}a)\,.
\end{aligned} \tag{1.40}$$

To first order in ω, ϵ

$$\begin{aligned}
&U(\Lambda,a)\big[\mathbb{1} - \tfrac{i}{2}\omega_{\rho\sigma}J^{\rho\sigma} + i\epsilon_\rho P^\rho\big]U^{-1}(\Lambda,a)\\
&= \mathbb{1} - \tfrac{i}{2}(\Lambda\omega\Lambda^{-1})_{\mu\nu}J^{\mu\nu} + i(\Lambda\epsilon - \Lambda\omega\Lambda^{-1}a)_\mu P^\mu\\
&= \mathbb{1} - \tfrac{i}{2}\Lambda_{\mu\rho}\omega^\rho{}_\sigma(\Lambda^{-1})^\sigma{}_\nu J^{\mu\nu} + i\Lambda_{\mu\rho}\epsilon^\rho P^\mu - i\Lambda_{\mu\rho}\omega^\rho{}_\sigma(\Lambda^{-1})^\sigma{}_\lambda a^\lambda P^\mu\\
&\overset{(1.30)}{=} \mathbb{1} - \tfrac{i}{2}\Lambda_\mu{}^\rho\Lambda_\nu{}^\sigma\omega_{\rho\sigma}J^{\mu\nu} + i\Lambda_\mu{}^\rho\epsilon_\rho P^\mu - i\underbrace{\Lambda_\mu{}^\rho\Lambda_\lambda{}^\sigma\omega_{\rho\sigma}}_{\frac{1}{2}(\Lambda_\mu{}^\rho\Lambda_\lambda{}^\sigma - \Lambda_\lambda{}^\rho\Lambda_\mu{}^\sigma)\omega_{\rho\sigma}} a^\lambda P^\mu\\
&= \mathbb{1} - \tfrac{i}{2}\omega_{\rho\sigma}\Lambda_\mu{}^\rho\Lambda_\nu{}^\sigma\big(J^{\mu\nu} + a^\nu P^\mu - a^\mu P^\nu\big) + i\epsilon_\rho\Lambda_\mu{}^\rho P^\mu\,,
\end{aligned}$$

and hence

$$U(\Lambda,a)J^{\rho\sigma}U^{-1}(\Lambda,a) = \Lambda_\mu{}^\rho\Lambda_\nu{}^\sigma\big(J^{\mu\nu} - a^\mu P^\nu + a^\nu P^\mu\big)\,, \tag{1.41}$$

$$U(\Lambda,a)P^\rho U^{-1}(\Lambda,a) = \Lambda_\mu{}^\rho P^\mu\,. \tag{1.42}$$

Special case $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$, $a^\mu = \epsilon^\mu$ yields the *Lie algebra of the Poincaré group*:

$$\big[J^{\mu\nu}, J^{\rho\sigma}\big] = i\big(\eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\sigma\mu}J^{\rho\nu} + \eta^{\sigma\nu}J^{\rho\mu}\big)\,, \tag{1.43}$$

$$\big[P^\mu, J^{\rho\sigma}\big] = i\big(\eta^{\mu\rho}P^\sigma - \eta^{\mu\sigma}P^\rho\big)\,, \tag{1.44}$$

$$\left[P^\mu, P^\rho\right] = 0\,. \tag{1.45}$$

$\boldsymbol{P}$, $\boldsymbol{J}$, H are conserved (i.e. commute with H). $\boldsymbol{K} = (J^{01}, J^{02}, J^{03})$ is not conserved.

It is easy to see that the commutation relations in three-vector notation are

$$[J_i, J_j] = i\epsilon_{ijk}J_k\,, \tag{1.46}$$
$$[J_i, K_j] = i\epsilon_{ijk}K_k\,, \tag{1.47}$$
$$[K_i, K_j] = -i\epsilon_{ijk}J_k\,, \tag{1.48}$$
$$[J_i, P_j] = i\epsilon_{ijk}P_k\,, \tag{1.49}$$
$$[K_i, P_j] = -iH\delta_{ij}\,, \tag{1.50}$$
$$[P_i, P_j] = [J_i, H] = [P_i, H] = [H, H] = 0\,, \tag{1.51}$$
$$[K_i, H] = -iP_i\,. \tag{1.52}$$

1.3 One-particle states

All components of P^μ commute, so choose eigenstates of P^μ to classify one-particle states:

$$P^\mu|p,\sigma\rangle = p^\mu|p,\sigma\rangle\,. \tag{1.53}$$

The index σ comprises further d.o.f., discrete for one-particle states. Eq. (1.25) implies

$$U(1,a)|p,\sigma\rangle = e^{ip\cdot a}|p,\sigma\rangle\,. \tag{1.54}$$

Consider now $U(\Lambda, 0) \equiv U(\Lambda)$:

$$\begin{aligned} P^\mu U(\Lambda)|p,\sigma\rangle &= U(\Lambda)\left[U^{-1}(\Lambda)P^\mu U(\Lambda)\right]|p,\sigma\rangle \\ &\overset{(1.42)}{=} U(\Lambda)\left[(\Lambda^{-1})_\rho{}^\mu P^\rho\right]|p,\sigma\rangle = \Lambda^\mu{}_\rho p^\rho U(\Lambda)|p,\sigma\rangle\,, \end{aligned} \tag{1.55}$$

and hence

$$U(\Lambda)|p,\sigma\rangle = \sum_{\sigma'} C_{\sigma'\sigma}(\Lambda, p)|\Lambda p, \sigma'\rangle\,. \tag{1.56}$$

Need to find the structure of the $C_{\sigma'\sigma}$ in irreducible (block-diagonal) representations of the Lorentz group.

Proper orthochronous L.T. conserve only $p^2 = \eta_{\mu\nu}p^\mu p^\nu$, and $\text{sign}(p^0)$ if $p^2 \geq 0$. For each p, and (for $p^2 \geq 0$) for each sign of p^0, choose a *"standard" four-momentum* k^μ and write

$$p^\mu = L^\mu{}_\nu(p)k^\nu\,, \tag{1.57}$$

where $L^\mu{}_\nu(p)$ is a p-dependent "standard" L.T.. Define the states with momentum p as

$$|p,\sigma\rangle \equiv N(p)U(L(p))|k,\sigma\rangle \tag{1.58}$$

($N(p)$ is a normalization factor, to be determined later). Apply $U(\Lambda)$:

$$U(\Lambda)|p,\sigma\rangle = N(p)U(\Lambda L(p))|k,\sigma\rangle = N(p)U(L(\Lambda p))\big[U(\underbrace{L^{-1}(\Lambda p)\Lambda L(p)}_{k\to p\to\Lambda p\to k})\big]|k,\sigma\rangle \,. \tag{1.59}$$

$L^{-1}(\Lambda p)\Lambda L(p)$ is an element of the subgroup of L.T. that leaves k^μ invariant; $W^\mu{}_\nu k^\nu = k^\mu$; *little group*. Eq. (1.56) implies

$$U(W)|k,\sigma\rangle = \sum_{\sigma'} D_{\sigma'\sigma}(W)|k,\sigma'\rangle \,. \tag{1.60}$$

The $D(W)$ furnish a representation of the little group:

$$\begin{aligned}\sum_{\sigma'} D_{\sigma'\sigma}(\bar{W}W)|k,\sigma'\rangle &= U(\bar{W}W)|k,\sigma\rangle = U(\bar{W})U(W)|k,\sigma\rangle \\ = U(\bar{W})\sum_{\sigma''} D_{\sigma''\sigma}(W)|k,\sigma''\rangle &= \sum_{\sigma'\sigma''} D_{\sigma''\sigma}(W)D_{\sigma'\sigma''}(\bar{W})|k,\sigma'\rangle \,,\end{aligned} \tag{1.61}$$

and hence

$$D_{\sigma'\sigma}(\bar{W}W) = \sum_{\sigma''} D_{\sigma'\sigma''}(\bar{W})D_{\sigma''\sigma}(W) \,. \tag{1.62}$$

For $W(\Lambda,p) \equiv L^{-1}(\Lambda p)\Lambda L(p)$, Eq. (1.59) implies

$$U(\Lambda)|p,\sigma\rangle = N(p)\sum_{\sigma'} D_{\sigma'\sigma}(W(\Lambda,p))U(L(\Lambda p))|k,\sigma'\rangle \,, \tag{1.63}$$

and with Eq. (1.58)

$$\boxed{U(\Lambda)|p,\sigma\rangle = \frac{N(p)}{N(\Lambda p)}\sum_{\sigma'} D_{\sigma'\sigma}(W(\Lambda,p))|\Lambda p,\sigma'\rangle \,.} \tag{1.64}$$

"Method of induced representations"

What about $N(p)$? We can choose, as usual,

$$\langle k',\sigma'|k,\sigma\rangle = \delta^3(\boldsymbol{k'}-\boldsymbol{k})\delta_{\sigma'\sigma} \,. \tag{1.65}$$

(From Eq. (1.60) it then follows that the representation $D(W)$ is unitary, $D^\dagger(W) = D^{-1}(W)$.) For arbitrary momenta we find

$$\begin{aligned}(\Psi_{p',\sigma'},\Psi_{p,\sigma}) &\overset{(1.58)}{=} N(p)(U^{-1}(L(p))\Psi_{p',\sigma'},\Psi_{k,\sigma}) \\ &\overset{(1.63)}{=} N(p)\big(N(p')\sum_{\sigma''} D(W(L^{-1}(p),p'))_{\sigma''\sigma}U(L(L^{-1}(p),p')))\underbrace{\Psi_{k',\sigma''},\Psi_{k,\sigma}}_{\delta_{\sigma''\sigma}\delta^3(\boldsymbol{k'}-\boldsymbol{k})}\big) \\ &= N(p)N^*(p')D(W(L^{-1}(p),p'))^*_{\sigma\sigma'}\delta^3(\boldsymbol{k'}-\boldsymbol{k}) \,,\end{aligned} \tag{1.66}$$

where $k' = L^{-1}(p')p'$. Moreover, we have $k = L^{-1}(p)p$; hence, $\delta^3(\boldsymbol{k}' - \boldsymbol{k}) \propto \delta^3(\boldsymbol{p}' - \boldsymbol{p})$. For $p' = p$, we have $W(L^{-1}(p), p) = L^{-1}(L^{-1}(p)p)L^{-1}(p)L(p) = L^{-1}(k) = 1$, and so

$$(\Psi_{p',\sigma'}, \Psi_{p,\sigma}) = |N(p)|^2 \delta_{\sigma'\sigma}\delta^3\big(\boldsymbol{k}' - \boldsymbol{k}\big)\,. \tag{1.67}$$

The Lorentz-invariant integral of a scalar function $f(p)$ over p with $p^2 = M^2 \geq 0$ and $p^0 > 0$ is

$$\begin{aligned}\int d^4p\,\delta(p^2 - M^2)\theta(p^0)f(p) &= \int d^3\boldsymbol{p}\,dp^0\,\delta\big((p^0)^2 - \boldsymbol{p}^2 - M^2\big)\theta(p^0)f(p^0, \boldsymbol{p})\\ &= \int d^3\boldsymbol{p}\frac{f(\sqrt{\boldsymbol{p}^2 + M^2}, \boldsymbol{p})}{2\sqrt{\boldsymbol{p}^2 + M^2}}\,.\end{aligned} \tag{1.68}$$

The invariant volume element "on the mass shell" is

$$d\Omega_{\text{L.I.}} = \frac{d^3\boldsymbol{p}}{\sqrt{\boldsymbol{p}^2 + M^2}}\,. \tag{1.69}$$

By definition,

$$F(\boldsymbol{p}) = \int d^3\boldsymbol{p}'\delta^3(\boldsymbol{p}' - \boldsymbol{p})F(\boldsymbol{p}') = \int \frac{d^3\boldsymbol{p}'}{\sqrt{\boldsymbol{p}'^2 + M^2}}\sqrt{\boldsymbol{p}'^2 + M^2}\,\delta^3(\boldsymbol{p}' - \boldsymbol{p})F(\boldsymbol{p}')\,, \tag{1.70}$$

so the invariant delta function is

$$\sqrt{\boldsymbol{p}'^2 + M^2}\,\delta^3(\boldsymbol{p}' - \boldsymbol{p}) = p^0\delta^3(\boldsymbol{p}' - \boldsymbol{p})\,. \tag{1.71}$$

Since p, p' and k, k' are related by $L(p)$, we have

$$p^0\delta^3(\boldsymbol{p}' - \boldsymbol{p}) = k^0\delta^3(\boldsymbol{k}' - \boldsymbol{k})\,, \tag{1.72}$$

and Eq. (1.67) becomes

$$(\Psi_{p',\sigma'}, \Psi_{p,\sigma}) = |N(p)|^2\delta_{\sigma'\sigma}\frac{p^0}{k^0}\delta^3(\boldsymbol{p}' - \boldsymbol{p})\,. \tag{1.73}$$

We will choose $N(p) = \sqrt{k^0/p^0}$, such that

$$(\Psi_{p',\sigma'}, \Psi_{p,\sigma}) = \delta_{\sigma'\sigma}\delta^3(\boldsymbol{p}' - \boldsymbol{p})\,. \tag{1.74}$$

Careful – e.g. Peskin chooses different convention!

Massive particles Standard momentum $k^\mu = (M, 0, 0, 0)$. Rotation group $SO(3)$ leaves k^μ invariant. The representations of spin j are

$$\begin{aligned} D^{(j)}_{\sigma'\sigma}(1 + \theta) &= \delta_{\sigma'\sigma} + \tfrac{i}{2}\theta_k\big(J^{(j)}_k\big)_{\sigma'\sigma}\,,\\ \big(J^{(j)}_1 + J^{(j)}_2\big)_{\sigma'\sigma} &= \delta_{\sigma'\sigma\pm1}\sqrt{(j \mp \sigma)(j \pm \sigma + 1)}\,,\\ \big(J^{(j)}_3\big)_{\sigma'\sigma} &= \sigma\delta_{\sigma'\sigma}\,,\end{aligned} \tag{1.75}$$

where $\sigma = -j, \dots, j-1, j$.

NB: For rotations $\mathcal{R}$ we have $W(\mathcal{R}, p) = \mathcal{R}$, so relativistic states transform under rotations like in non-relativistic quantum mechanics! Examples:

spin 0:	Higgs, pions, B mesons
spin 1/2:	quarks, leptons
spin 1:	$W^\pm$, Z^0, J/Ψ

Massless particles Standard momentum $k^\mu = (\kappa, 0, 0, \kappa)$. Invariant under rotations about z axis, and combined rotations and boosts: J_3, $A = J_2 + K_1$, $B = -J_1 + K_2$ ("$ISO(2)$ group of Euclidean translations"). J_3: helicity (integer or half integer, for topological reasons – read the very interesting chapter 2.7 in [5] on this topic).

$$U(\Lambda)\Psi_{p,\sigma} = \sqrt{\frac{(\Lambda p)^0}{p^0}}\exp(i\sigma\theta(\Lambda, p))\Psi_{\Lambda p,\sigma}\,. \tag{1.76}$$

Examples:

spin 1/2:	(massless) neutrinos
spin 1:	photon, gluons
spin 2:	graviton

1.4 Space inversion and time reversal

$$\mathcal{P}^\mu{}_\nu = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \mathcal{T}^\mu{}_\nu = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}. \tag{1.77}$$

Question: Are there $\mathrm{P} \equiv U(\mathcal{P}, 0)$, $\mathrm{T} \equiv U(\mathcal{T}, 0)$? Eq. (1.34) implies

$$\mathrm{P}U(\Lambda, a)\mathrm{P}^{-1} = U(\mathcal{P}\Lambda\mathcal{P}^{-1}, \mathcal{P}a)\,, \tag{1.78}$$

$$\mathrm{T}U(\Lambda, a)\mathrm{T}^{-1} = U(\mathcal{T}\Lambda\mathcal{T}^{-1}, \mathcal{T}a)\,. \tag{1.79}$$

Wu et al. 1957 [6]: parity violation in nuclear β decay; Christenson et al. 1964 [7]: time reversal ($= CP$) violation in K decays. P and T are symmetries only if one neglects the weak interaction.

Consider (1.78), (1.79) for infinitesimal Λ, a. In analogy to above we find

$$\mathrm{P}iJ^{\rho\sigma}\mathrm{P}^{-1} = i\mathcal{P}_\mu{}^\rho\mathcal{P}_\nu{}^\sigma J^{\mu\nu}\,, \tag{1.80}$$

$$\mathrm{P}iP^\rho\mathrm{P}^{-1} = i\mathcal{P}_\mu{}^\rho P^\mu\,, \tag{1.81}$$

$$\mathrm{T}iJ^{\rho\sigma}\mathrm{T}^{-1} = i\mathcal{T}_\mu{}^\rho\mathcal{T}_\nu{}^\sigma J^{\mu\nu}\,, \tag{1.82}$$

$$\mathrm{T}iP^\rho\mathrm{T}^{-1} = i\mathcal{P}_\mu{}^\rho P^\mu\,. \tag{1.83}$$

Are P,T unitary/linear or antiunitary/antilinear? Set $\rho = 0$ in (1.81): $\mathrm{P}iH\mathrm{P}^{-1} = iH$. Assume P is antiunitary, then $\mathrm{P}H\mathrm{P}^{-1} = -H \Leftrightarrow H\mathrm{P}^{-1} = -\mathrm{P}^{-1}H$. Let $H|\Psi\rangle = E|\Psi\rangle$, then $H\mathrm{P}^{-1}|\Psi\rangle = -\mathrm{P}^{-1}H|\Psi\rangle = -E\mathrm{P}^{-1}|\Psi\rangle$, hence stability of matter requires that P be unitary and linear.

Now set $\rho = 0$ in (1.83): $\mathrm{T}iH\mathrm{T}^{-1} = -iH$. Assume T is unitary, then $\mathrm{T}H\mathrm{T}^{-1} = -H\ldots$ etc. as before; hence stability of matter requires that T be antiunitary and antilinear. In summary, we have

$$\mathrm{P}H\mathrm{P}^{-1} = H\,, \qquad \mathrm{T}H\mathrm{T}^{-1} = H\,. \tag{1.84}$$

It is straightforward to verify the following transformation properties in three-vector notation:

$$\mathrm{P}\boldsymbol{J}\mathrm{P}^{-1} = +\boldsymbol{J}\,, \tag{1.85}$$

$$\mathrm{P}\boldsymbol{K}\mathrm{P}^{-1} = -\boldsymbol{K}\,, \tag{1.86}$$

$$\mathrm{P}\boldsymbol{P}\mathrm{P}^{-1} = -\boldsymbol{P}\,, \tag{1.87}$$

$$\mathrm{T}\boldsymbol{J}\mathrm{T}^{-1} = -\boldsymbol{J}\,, \tag{1.88}$$

$$\mathrm{T}\boldsymbol{K}\mathrm{T}^{-1} = +\boldsymbol{K}\,, \tag{1.89}$$

$$\mathrm{T}\boldsymbol{P}\mathrm{T}^{-1} = -\boldsymbol{P}\,. \tag{1.90}$$

Action on (massive) one-particle states Parity: $|k,\sigma\rangle$ is eigenstate of $\boldsymbol{P}$, H, J^3: $\boldsymbol{P}|k,\sigma\rangle = 0$; $H|k,\sigma\rangle = M|k,\sigma\rangle$, $J^3|k,\sigma\rangle = \sigma|k,\sigma\rangle$. Eq. (1.80), (1.81) imply that $\mathrm{P}|k,\sigma\rangle$ is eigenstate with the same eigenvalues, hence $\mathrm{P}|k,\sigma\rangle = \eta_\sigma|k,\sigma\rangle$, with $|\eta_\sigma| = 1$. Eq.s (1.60) and (1.75) imply

$$\begin{aligned}
&(J_1 \pm iJ_2)|k,\sigma\rangle = \sqrt{(j\mp\sigma)(j\pm\sigma+1)}|k,\sigma\pm 1\rangle \\
&\Rightarrow \mathrm{P}(J_1 \pm iJ_2)\mathrm{P}^{-1}\mathrm{P}|k,\sigma\rangle = \sqrt{(j\mp\sigma)(j\pm\sigma+1)}\mathrm{P}|k,\sigma\pm 1\rangle \\
&\Rightarrow \eta_\sigma(J_1 \pm iJ_2)k,\sigma\rangle = \eta_{\sigma\pm1}\sqrt{(j\mp\sigma)(j\pm\sigma+1)}|k,\sigma\pm 1\rangle \\
&\Rightarrow \eta_\sigma = \eta_{\sigma\pm1}\,.
\end{aligned}$$

The σ-independent phase η is called *intrinsic parity*. One can show [1] that

$$\mathrm{P}|p,\sigma\rangle = \eta|\mathcal{P}p,\sigma\rangle\,. \tag{1.91}$$

Time reversal: $\boldsymbol{P}\mathrm{T}|k,\sigma\rangle = 0$; $H\mathrm{T}|k,\sigma\rangle = M\mathrm{T}|k,\sigma\rangle$; $J_3\mathrm{T}|k,\sigma\rangle = -\sigma\mathrm{T}|k,\sigma\rangle$. Therefore, $\mathrm{T}|k,\sigma\rangle = \zeta_\sigma\mathrm{T}|k,-\sigma\rangle$. Similar to above one can show that $-\zeta_\sigma = \zeta_{\sigma\pm1}$; we write $\zeta_\sigma = (-1)^{j-\sigma}\zeta$, with $|\zeta| = 1$. Then we have $\mathrm{T}|k,\sigma\rangle = (-1)^{j-\sigma}\zeta\mathrm{T}|k,-\sigma\rangle$.

The phase ζ has no physical meaning, as we can always get rid of it by a phase tranformation: Let $|k,\sigma\rangle \to |k,\sigma\rangle' = \zeta^{1/2}|k,\sigma\rangle$. Then $\mathrm{T}|k,\sigma\rangle' = \zeta^{*1/2}\mathrm{T}|k,\sigma\rangle = \zeta^{*1/2}\zeta(-1)^{j-\sigma}|k,-\sigma\rangle = (-1)^{j-\sigma}|k,-\sigma\rangle'$. One can show [1] that

$$\mathrm{T}|p,\sigma\rangle = \zeta(-1)^{j-\sigma}|\mathcal{P}p,-\sigma\rangle\,. \tag{1.92}$$

1.5 T invariance and electric dipole moments

Note that

$$\mathrm{T}^2|p,\sigma\rangle = \mathrm{T}\zeta(-1)^{j-\sigma}|\mathcal{P}p,-\sigma\rangle = \zeta^*(-1)^{j-\sigma}\zeta(-1)^{j+\sigma}|p,\sigma\rangle = (-1)^{2j}|p,\sigma\rangle\,. \tag{1.93}$$

Let $|\Psi\rangle$ be the state vector of a system of non-interacting particles with an odd number of particles with half-integer spin. Then

$$\mathrm{T}^2|\Psi\rangle = -|\Psi\rangle\,. \tag{1.94}$$

This holds true for interacting systems as long as the interaction is T-invariant (e.g. static electric fields). Now assume that $|\Psi\rangle$ is eigenstate of H; it follows that $\mathrm{T}|\Psi\rangle$ is an eigenstate of H. Is it the same state? If yes: $\mathrm{T}|\Psi\rangle = \alpha|\Psi\rangle$, with $|\alpha| = 1$, but then

$$\mathrm{T}^2|\Psi\rangle = \mathrm{T}\alpha|\Psi\rangle = \alpha^*\mathrm{T}|\Psi\rangle = |\alpha|^2|\Psi\rangle = |\Psi\rangle\,, \tag{1.95}$$

a contradiction to (1.94). Hence $|\Psi\rangle$ must be at least two-fold degenerate, even in static electric fields. Any electric dipole moment (EDM) would entirely remove that degeneracy, hence EDMs are forbidden by T invariance.

E.g. neutron: $|d_n/e| < 2.9 \times 10^{-26}$ cm
electron: $|d_e/e| < 1.1 \times 10^{-29}$ cm

2 Scattering theory

2.1 In and out states

A state of free particles transforms as a product of free one-particle states $|p, \sigma, n\rangle$ (here, n is a particle index):

$$\begin{aligned} U(\Lambda, a)|p_1\sigma_1 n_1; p_2\sigma_2 n_2; \ldots\rangle &= \exp[ia\cdot(\Lambda p_1 + \Lambda p_2 + \ldots)] \\ &\times \sqrt{\frac{(\Lambda p_1)^0(\Lambda p_2)^0\ldots}{p_1^0 p_2^0\ldots}} \sum_{\sigma_1'\sigma_2'\cdots} D^{(j_1)}_{\sigma_1'\sigma_1}(W(\Lambda, p_1))D^{(j_2)}_{\sigma_2'\sigma_2}(W(\Lambda, p_2))\cdots \\ &\times |\Lambda p_1\sigma_1' n_1; \Lambda p_2\sigma_2' n_2; \ldots\rangle\,. \end{aligned} \tag{2.1}$$

The normalization of the states is

$$\begin{aligned} &\langle p_1'\sigma_1'n_1'; p_2'\sigma_2'n_2'; \ldots|p_1\sigma_1 n_1; p_2\sigma_2 n_2; \ldots\rangle \\ &= \delta^3(\boldsymbol{p}_1' - \boldsymbol{p}_1)\delta_{\sigma_1'\sigma_1}\delta_{n_1'n_1}\delta^3(\boldsymbol{p}_2' - \boldsymbol{p}_2)\delta_{\sigma_2'\sigma_2}\delta_{n_2'n_2}\cdots \pm \text{permutations}\,. \end{aligned} \tag{2.2}$$

For obvious reasons, we introduce the short-hand notation $\langle\alpha'|\alpha\rangle = (\Psi_{\alpha'}, \Psi_\alpha) = \delta(\alpha' - \alpha)$, as well as

$$\int d\alpha\ldots \equiv \sum_{n_1\sigma_1, n_2\sigma_2,\ldots} \int d^3\boldsymbol{p}_1 d^3\boldsymbol{p}_2 \ldots\,. \tag{2.3}$$

The completeness relation then reads

$$\Psi = \int d\alpha \Psi_\alpha(\Psi_\alpha, \Psi)\,. \tag{2.4}$$

From Eq. (2.1) with $\Lambda = 1$, $a^\mu = (\tau, 0, 0, 0)$ follows

$$H\Psi_\alpha = E_\alpha\Psi_\alpha\,, \tag{2.5}$$

where $E_\alpha = p_1^0 + p_2^0 + \ldots$ (no interaction terms).

We are interested in scattering experiments. For $t \to -\infty$: particles far apart, not interacting; for finite t: interaction; for $t \to \infty$: particles far apart, not interacting.

We have two sets of particles whose states transform as in Eq. (2.1): "in" and "out" states Ψ_α^+ and Ψ_α^- contain particles α if observations are made at time $t \to -\infty$ and $t \to +\infty$, respectively. We use the *Heisenberg picture*: states are time independent; operators are time dependent.

Assume observer $\mathcal{O}$ sees collision at time $t = 0$, while observer $\mathcal{O}'$ uses clock with $t' = 0$ for $t = \tau$, hence $t' = t - \tau$. $\mathcal{O}$ sees state Ψ; $\mathcal{O}'$ sees state $U(1, -\tau)\Psi = \underbrace{\exp(-iH\tau)}_{\text{time translation operator}} \Psi$.

In and out states appear like free states with particle content α if measurements are made at $t \to \mp\infty$:

$$\lim_{t\to\mp\infty} \exp(-iHt)\Psi_\alpha^\pm = \lim_{t\to\mp\infty} \exp(-iE_\alpha t)\Psi_\alpha^\pm \,. \tag{2.6}$$

(In reality need wave packets.) Assume we can split H into a free part H_0 and an interaction term V,

$$H = H_0 + V \,, \tag{2.7}$$

with eigenstates of H_0

$$H_0\Phi_\alpha = E_\alpha\Phi_\alpha \,, \qquad (\Phi_\alpha, \Phi_{\alpha'}) = \delta(\alpha' - \alpha) \,. \tag{2.8}$$

Define Ψ_α as eigenstates of the full Hamiltonian H,

$$H\Psi_\alpha = E_\alpha\Psi_\alpha \,, \tag{2.9}$$

where

$$\lim_{t\to\mp\infty} \exp(-iHt)\Psi_\alpha^\pm = \lim_{t\to\mp\infty} \exp(-iH_0 t)\Phi_\alpha \,. \tag{2.10}$$

Then we have

$$\Psi_\alpha^\pm = \Omega(\mp)\Phi_\alpha \,, \tag{2.11}$$

with

$$\Omega(\tau) = \exp(+iH\tau)\exp(-iH_0\tau) \,. \tag{2.12}$$

It follows that

$$(\Psi_\alpha^\pm, \Psi_{\alpha'}^\pm) = (\Phi_\alpha, \Phi_{\alpha'}) = \delta(\alpha' - \alpha) \,. \tag{2.13}$$

2.2 The S-matrix

Experimentalists prepare state Ψ_α^+ with defined particle content α for $t \to -\infty$; after collision measure state Ψ_β^- with particle content β, for $t \to +\infty$. The transition amplitude is

$$S_{\beta\alpha} = (\Psi_\beta^-, \Psi_\alpha^+) \,. \tag{2.14}$$

$S_{\beta\alpha}$ is the *S-matrix*. Without interaction, $S_{\beta\alpha} = \delta(\beta - \alpha)$; hence the reaction rate is

$$R(\alpha \to \beta) = |S_{\beta\alpha} - \delta(\beta - \alpha)|^2 \,. \tag{2.15}$$

The S-matrix is unitary:

$$\int d\beta S_{\beta\gamma}^* S_{\beta\alpha} \stackrel{(2.15)}{=} \int d\beta (\Psi_\gamma^+, \Psi_\beta^-)(\Psi_\beta^-, \Psi_\alpha^+) \stackrel{(2.4)}{=} (\Psi_\gamma^+, \Psi_\alpha^+) = \delta(\gamma - \alpha) \,, \tag{2.16}$$

and thus $S^\dagger S = \mathbb{1}$. Similarly, $SS^\dagger = \mathbb{1}$.

Define the S-operator by

$$(\Phi_\beta, S\Phi_\alpha) \equiv S_{\beta\alpha} = (\Psi^-_\beta, \Psi^+_\alpha) \tag{2.17}$$

with free states Φ_α. Eq. (2.11) then gives

$$S = \Omega^\dagger(+\infty)\Omega(-\infty) = U(+\infty, -\infty)\,, \tag{2.18}$$

where

$$U(\tau, \tau_0) \equiv \Omega^\dagger(\tau)\Omega(\tau_0) = \exp(iH_0\tau)\exp(-iH(\tau-\tau_0))\exp(-iH_0\tau_0)\,. \tag{2.19}$$

2.3 Decay rates and cross sections

Time and translation invariance imply conservation of total energy and momentum, respectively; hence, we can write

$$S_{\beta\alpha} = \delta(\beta-\alpha) - 2\pi i\delta(E_\beta - E_\alpha)\delta^3(\boldsymbol{p}_\beta - \boldsymbol{p}_\alpha)M_{\beta\alpha}\,. \tag{2.20}$$

What is the transition probability $|S_{\beta\alpha}|^2$? Imagine the system in a finite box (Volume V) for a finite time T (later $V, T \to \infty$). Then

$$\boldsymbol{p} = \frac{2\pi}{L}(n_1, n_2, n_3)\,, \tag{2.21}$$

with $n_i \in \mathbb{N}$ and $L^3 = V$, and the delta functions become

$$\delta^3_V(\boldsymbol{p}' - \boldsymbol{p}) \equiv \frac{1}{(2\pi)^3}\int\limits_V d^3x e^{i(\boldsymbol{p}'-\boldsymbol{p})\cdot\boldsymbol{x}} = \frac{V}{(2\pi)^3}\delta_{\boldsymbol{p}',\boldsymbol{p}}\,. \tag{2.22}$$

Hence, the "box states" have factor $(V/(2\pi)^3)^N$ in scalar product (N is the number of particles in the box state). Therefore, define normalized states

$$\Psi^{\text{BOX}}_\alpha \equiv \left(\frac{(2\pi)^3}{V}\right)^{N_\alpha/2}\Psi_\alpha\,, \tag{2.23}$$

with

$$(\Psi^{\text{BOX}}_\alpha, \Psi^{\text{BOX}}_\beta) = \delta_{\alpha\beta}\,. \tag{2.24}$$

The time delta function becomes

$$\delta_T(E_\alpha - E_\beta) \equiv \frac{1}{2\pi}\int\limits_{-T/2}^{T/2} dt\, e^{i(E_\alpha - E_\beta)t}\,. \tag{2.25}$$

The transition probability into a specific final state is

$$P(\alpha\to\beta) = \left|S^{\text{BOX}}_{\beta\alpha}\right|^2 = \left(\frac{(2\pi)^3}{V}\right)^{N_\alpha+N_\beta}\left|S_{\beta\alpha}\right|^2\,. \tag{2.26}$$

The number of one-particle box states in momentum volume element d^3p is $V d^3p/(2\pi)^3$ (see Eq. (2.22)). We define $d\beta = d^3p'_1 \cdots d^3p'_{N_\beta}$, such that the number of states in $d\beta$ is

$$d\mathcal{N}_\beta = \left(\frac{V}{(2\pi)^3}\right)^{N_\beta} d\beta\,. \tag{2.27}$$

Hence the total probability for the transition into the range $d\beta$ is

$$dP(\alpha \to \beta) = P(\alpha \to \beta) d\mathcal{N}_\beta = \left(\frac{(2\pi)^3}{V}\right)^{N_\alpha} \left|S_{\beta\alpha}\right|^2 d\beta\,. \tag{2.28}$$

Interpretation of the squares of delta functions:

$$\left[\delta^3_V(\boldsymbol{p}_\beta - \boldsymbol{p}_\alpha)\right]^2 = \delta^3_V(\boldsymbol{p}_\beta - \boldsymbol{p}_\alpha)\delta^3_V(0) = \delta^3_V(\boldsymbol{p}_\beta - \boldsymbol{p}_\alpha)\frac{V}{(2\pi)^3}\,, \tag{2.29}$$

$$\left[\delta_T(E_\beta - E_\alpha)\right]^2 = \delta_T(E_\beta - E_\alpha)\delta_T(0) = \delta_T(E_\beta - E_\alpha)\frac{T}{2\pi}\,, \tag{2.30}$$

$$\tag{2.31}$$

and Eq. (2.28) becomes

$$dP(\alpha \to \beta) \overset{(2.20)}{=} (2\pi)^2 \left(\frac{(2\pi)^3}{V}\right)^{N_\alpha - 1} \frac{T}{2\pi} \left|M_{\beta\alpha}\right|^2 \delta^3_V(\boldsymbol{p}_\beta - \boldsymbol{p}_\alpha)\delta_T(E_\beta - E_\alpha) d\beta\,. \tag{2.32}$$

The transition probability is proportional to T; the coefficient is the *differential transition rate* $d\Gamma$. For $V, T \to \infty$

$$d\Gamma(\alpha \to \beta) \equiv dP(\alpha \to \beta)/T = (2\pi)^{3N_\alpha - 2} V^{1 - N_\alpha} \left|M_{\beta\alpha}\right|^2 \delta^4(p_\beta - p_\alpha) d\beta\,, \tag{2.33}$$

where (for $\alpha \neq \beta$)

$$S_{\beta\alpha} \equiv -2\pi i \delta^4(p_\beta - p_\alpha) M_{\beta\alpha}\,. \tag{2.34}$$

$N_\alpha = 1$: Decay rate

Here, the volume cancels and

$$d\Gamma(\alpha \to \beta) = 2\pi \left|M_{\beta\alpha}\right|^2 \delta^4(p_\beta - p_\alpha) d\beta\,. \tag{2.35}$$

$N_\alpha = 2$: Collision of two particles

Rate is proportional to $1/V$, density of one particle at the position of the other particle. Usually, one measures the rate per *flux* Φ_α of incoming particles,

$$\Phi_\alpha = \frac{u_\alpha}{V}\,, \tag{2.36}$$

where u_α is the relative velocity between the two particles. This is called the *differential cross section*

$$d\sigma(\alpha \to \beta) \equiv \frac{d\Gamma(\alpha \to \beta)}{\Phi_\alpha} = (2\pi)^4 u_\alpha^{-1} \left|M_{\beta\alpha}\right|^2 \delta^4(p_\beta - p_\alpha) d\beta \,. \tag{2.37}$$

The cases $N_\alpha \geq 3$ play a role in astrophysics, cosmology, and chemistry, but rarely in particle physics.

What is u_α? From Eq. (2.1) and the unitarity of the matrices $D(W)$ one can show [1] that

$$\sum_{\text{spins}} M_{\beta\alpha}\big|^2 \prod_\beta E \prod_\alpha E \equiv R_{\beta\alpha} \tag{2.38}$$

is a Lorentz scalar. Hence the decay rate (2.35), summed over particle spins, can be written as

$$\sum_{\text{spins}} d\Gamma(\alpha \to \beta) = 2\pi E_\alpha^{-1} R_{\beta\alpha} \delta^4(p_\beta - p_\alpha) \frac{d\beta}{\prod_\beta E} \,. \tag{2.39}$$

This is Lorentz invariant apart from factor E_α^{-1} – the faster the particle moves, the slower it decays (time dilation).

Similarly, we write the spin-summed differential cross section as

$$\sum_{\text{spins}} d\sigma(\alpha \to \beta) = (2\pi)^4 u_\alpha^{-1} E_1^{-1} E_2^{-1} R_{\beta\alpha} \delta^4(p_\beta - p_\alpha) \frac{d\beta}{\prod_\beta E} \,. \tag{2.40}$$

To make this a Lorentz-invariant function, one usually defines u_α such that $u_\alpha E_1 E_2$ is a Lorentz scalar. Moreover, in rest frame of one particle, u_α is the velocity of the other particle. Hence

$$u_\alpha = \frac{\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}}{E_1 E_2} \,. \tag{2.41}$$

(If particle 1 is at rest, $\boldsymbol{p}_1 = 0$, $E_1 = m_1$, and $p_1 \cdot p_2 = m_1 E_2$. Therefore,

$$u_\alpha = \frac{\sqrt{E_2^2 - m_2^2}}{E_2} = \frac{|\boldsymbol{p}_2|}{E_2} \,, \tag{2.42}$$

the velocity of particle 2.)

Phase space To calculate the *phase space factor* $\delta^4(p_\beta - p_\alpha) d\beta$, we work in the center-of-mass frame, such that

$$\boldsymbol{p}_\alpha = 0 \,. \tag{2.43}$$

For a final state with momenta $p_1', p_2', \ldots$ we have

$$\delta^4(p_\beta - p_\alpha) d\beta = \delta^3(\boldsymbol{p}_1' + \boldsymbol{p}_2' + \ldots) \delta(E_1' + E_2' + \ldots - E) d^3\boldsymbol{p}_1' d^3\boldsymbol{p}_2' \ldots \,, \tag{2.44}$$

where E is the total energy of the initial state. The integral over, e.g., $\boldsymbol{p}_1'$ can be easily solved using the momentum delta function:

$$\delta^4(p_\beta - p_\alpha) d\beta \to \delta(E_1' + E_2' + \ldots - E) d^3\boldsymbol{p}_2' \ldots \,, \tag{2.45}$$

where now everywhere

$$\boldsymbol{p}_1' = -\boldsymbol{p}_2' - \boldsymbol{p}_3' - \dots . \tag{2.46}$$

What about $\delta(E)$? The easiest case is $N_\beta = 2$:

$$\begin{aligned}\delta^4(p_\beta - p_\alpha)d\beta &\to \delta(E_1' + E_2' - E\dots)d^3\boldsymbol{p}_2' \\ &= \delta(\sqrt{|\boldsymbol{p}_1'|^2 + m_1'^2} + \sqrt{|\boldsymbol{p}_1'|^2 + m_2'^2} - E)|\boldsymbol{p}_1'|^2 d|\boldsymbol{p}_1'|d\Omega\,,\end{aligned} \tag{2.47}$$

where $\boldsymbol{p}_2' = -\boldsymbol{p}_1'$ and the solid angle is $d\Omega = \sin\theta d\theta d\phi$. Now we use

$$\delta(f(x)) = \frac{\delta(x - x_0)}{|f'(x_0)|}\,, \tag{2.48}$$

where x_0 is a simple zero of f. In our case, the zero of the argument of the energy delta function is $k' \equiv |\boldsymbol{p}_1'|$, where

$$k' = \frac{\sqrt{(E^2 - m_1'^2 - m_2'^2)^2 - 4m_1'^2 m_2'^2}}{2E} \equiv \frac{\lambda(E^2, m_1'^2, m_2'^2)}{2E}\,, \tag{2.49}$$

and, therefore,

$$E_1' = \sqrt{k'^2 + m_1'^2} = \frac{E^2 - m_2'^2 + m_1'^2}{2E}\,, \tag{2.50}$$

$$E_2' = \sqrt{k'^2 + m_2'^2} = \frac{E^2 - m_1'^2 + m_2'^2}{2E}\,, \tag{2.51}$$

and the derivative

$$\frac{d}{d|\boldsymbol{p}_1'|}\left(\sqrt{|\boldsymbol{p}_1'|^2 + m_1'^2} + \sqrt{|\boldsymbol{p}_1'|^2 + m_2'^2} - E\right)\Big|_{|\boldsymbol{p}_1'|=k'} = \frac{k'}{E_1'} + \frac{k'}{E_2'} = \frac{k'E}{E_1'E_2'}\,. \tag{2.52}$$

Finally, we have

$$\delta^4(p_\beta - p_\alpha)d\beta \to \frac{k'E_1'E_2'}{E}d\Omega\,. \tag{2.53}$$

In particular, the differential decay rate of a particle at rest with energy E into two particles is

$$\boxed{\frac{d\Gamma(\alpha\to\beta)}{d\Omega} = \frac{2\pi k' E_1' E_2'}{E}|M_{\beta\alpha}|^2\,,} \tag{2.54}$$

and the differential cross section for $2 \to 2$ scattering (in the c.m.s.) is

$$\boxed{\frac{d\sigma(\alpha\to\beta)}{d\Omega} = \frac{(2\pi)^4 k' E_1' E_2'}{E u_\alpha}|M_{\beta\alpha}|^2 = \frac{(2\pi)^4 k' E_1' E_2' E_1 E_2}{E^2 k}|M_{\beta\alpha}|^2\,,} \tag{2.55}$$

where $k \equiv |\boldsymbol{p}_1| = |\boldsymbol{p}_2|$.

Here, we have used

$$u_\alpha = \frac{\sqrt{(E_1E_2+k^2)^2-(E_1^2-k^2)(E_2-k^2)}}{E_1E_2} \tag{2.56}$$
$$= \frac{\sqrt{E_1^2E_2^2+2kE_1E_2+k^4-E_1^2E_2^2-k^4+k^2(E_1^2+k^2)}}{E_1E_2} \tag{2.57}$$
$$= \frac{kE}{E_1E_2}\,. \tag{2.58}$$

2.4 Perturbation Theory

Calculate the S-matrix (in the form (2.18)) as a power series in the interaction term V (see Eq. (2.7)). Differentiate Eq. (2.19) w.r.t. t:

$$\begin{aligned} i\frac{d}{dt}U(t,t') &= i\frac{d}{dt}\Big[\exp(iH_0t)\exp(-iH(t-t'))\exp(-iH_0t')\Big] \\ &= i\Big[\exp(iH_0t)(iH_0-iH)\exp(-iH(t-t'))\exp(-iH_0t')\Big] \\ &= \Big[\exp(iH_0t)V\exp(-iH_0t)\Big]\Big[\exp(iH_0t)\exp(-iH(t-t'))\exp(-iH_0t')\Big] \\ &\equiv V_I(t)U(t,t')\,. \end{aligned} \tag{2.59}$$

Here, the index "I" denotes the *interaction picture* (time dependence of operators given by free Hamiltonian H_0). The initial condition for U is

$$U(t,t) = \mathbb{1}\,. \tag{2.60}$$

Eq.s (2.59) and (2.60) are equivalent to the integral equation

$$U(t,t') = \mathbb{1} - i\int_{t'}^{t} d\tau\, V_I(\tau)U(\tau,t')\,. \tag{2.61}$$

Solve through iteration:

$$\begin{aligned} U(t,t') = \mathbb{1} &- i\int_{t'}^{t} d\tau_1\, V_I(\tau_1) + (-i)^2\int_{t'}^{t} d\tau_1\int_{t'}^{\tau_1} d\tau_2\, V_I(\tau_1)V_I(\tau_2) \\ &+ (-i)^3\int_{t'}^{t} d\tau_1\int_{t'}^{\tau_1} d\tau_2\int_{t'}^{\tau_2} d\tau_3\, V_I(\tau_1)V_I(\tau_2)V_I(\tau_2) + \dots\,. \end{aligned} \tag{2.62}$$

This may be rewritten using the *time-ordered product*, s.th. operators with larger (later) time argument appear to the left of the earlier ones:

$$T\{V(\tau)\} = V(\tau)\,, \tag{2.63}$$

$$T\{V(\tau_1), V(\tau_2)\} = \theta(\tau_1 - \tau_2)V(\tau_1)V(\tau_2) + \theta(\tau_2 - \tau_1)V(\tau_2)V(\tau_1)\,, \tag{2.64}$$

etc., with $n!$ terms in the time-ordered product of n operators V. Each term, integrated between t' and t, gives the same integral as the n-th term in Eq. (2.62). Hence we have

$$U(t, t') = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{t'}^{t} d\tau_1 \ldots d\tau_n T\{V(\tau_1)\cdots V(\tau_n)\}\,. \tag{2.65}$$

From Eq. (2.65) we obtain the *Dyson series* for the S-matrix (F. Dyson 1949 [8]):

$$S = \mathbb{1} + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} d\tau_1 \ldots d\tau_n T\{V(\tau_1)\cdots V(\tau_n)\}\,. \tag{2.66}$$

Short-hand notation:

$$S = T\exp\left(-i\int_{-\infty}^{\infty} dt\, V(t)\right)\,. \tag{2.67}$$

However, in general the series does not converge; it may be regarded as an asymptotic series in some coupling constants.

What about Lorentz invariance? One can show [1] that S commutes with the generators of the Lorentz group if

$$V(t) = \int d^3x \mathcal{H}(\boldsymbol{x}, t)\,, \tag{2.68}$$

where $\mathcal{H}(x)$ is a scalar "Hamilton density", in the sense that

$$U(\Lambda, a)\mathcal{H}(x)U^{-1}(\Lambda, a) = \mathcal{H}(\Lambda x + a)\,. \tag{2.69}$$

We can then write

$$\boxed{S = \mathbb{1} + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int d^4x_1 \ldots d^4x_n T\{\mathcal{H}(x_1)\cdots\mathcal{H}(x_n)\}\,.} \tag{2.70}$$

The time ordering of two events x_1, x_2 is Lorentz invariant unless theor difference is space-like, $(x_1 - x_2)^2 < 0$. If all $\mathcal{H}(x)$ commute at space-like distances, no special inertial system is introduced:

$$\boxed{[\mathcal{H}(x), \mathcal{H}(x')] =)\text{for}(x - x') \leq 0\,.} \tag{2.71}$$

3 Creation and Annihilation operators

Goal: construction of a scalar interaction density

3.1 Fock space of many-particle states

Denote the state containing N particles with momenta $\boldsymbol{p}_i$, spins σ_i, of type n_i, by $|\boldsymbol{p}_1, \sigma_1, n_1; \ldots; \boldsymbol{p}_N, \sigma_N, n_N\rangle$. All known particles are either *bosons* or *fermions*; i.e., for identical particles we have

$$|\ldots \boldsymbol{p}\sigma n \ldots \boldsymbol{p}'\sigma' n \ldots\rangle = \pm|\ldots \boldsymbol{p}'\sigma' n \ldots \boldsymbol{p}\sigma n \ldots\rangle \,. \tag{3.1}$$

We choose the normalization accordingly:

$$N = 0: \qquad \langle 0|0\rangle = 1 \,, \tag{3.2}$$
$$N = 1: \qquad \langle q'|q\rangle \equiv \langle \boldsymbol{p}'\sigma' n'|\boldsymbol{p}\sigma n\rangle = \delta^3(\boldsymbol{p}' - \boldsymbol{p})\delta_{\sigma'\sigma}\delta_{n'n} \equiv \delta(q' - q) \,, \tag{3.3}$$
$$N = 2: \qquad \langle q_1' q_2'|q_1 q_2\rangle = \delta(q_1' - q_1)\delta(q_2' - q_2) \pm \delta(q_2' - q_1)\delta(q_1' - q_2) \,, \tag{3.4}$$

and in general

$$\langle q_1' q_2' \ldots q_M'|q_1 q_2 \ldots q_N\rangle = \delta_{NM} \sum_{\mathcal{P}} \delta_{\mathcal{P}} \prod_i \delta(q_i' - q_{\mathcal{P}i}) \,, \tag{3.5}$$

where $\mathcal{P}$ is a permutation operator. We define the *creation operator* $a^\dagger(\boldsymbol{p}\sigma n)$ by

$$a^\dagger(q)|q_1 q_2 \ldots q_N\rangle = |q q_1 q_2 \ldots q_N\rangle \,. \tag{3.6}$$

Thus, we can obtain the N-particle state from the vacuum state $|0\rangle$ as

$$|q_1 q_2 \ldots q_N\rangle = a^\dagger(q_1)a^\dagger(q_2)\cdots a^\dagger(q_N)|0\rangle \,. \tag{3.7}$$

The adjoint *annihilation operator* $a(\boldsymbol{p}\sigma n)$ removes a particle from the state. In particular,

$$a(q)|q\rangle = |0\rangle \,, \tag{3.8}$$
$$a(q)|0\rangle = 0 \,. \tag{3.9}$$

One can show [1] that these operators satisfy the following *(anti-)commutation relations*

$$[a(q), a^\dagger(q')]_\mp \equiv a(q)a^\dagger(q') \mp a^\dagger(q')a(q) = \delta(q' - q) \,, \tag{3.10}$$
$$[a(q), a(q')]_\mp = [a^\dagger(q), a^\dagger(q')]_\mp = 0 \,. \tag{3.11}$$

Theorem: Every operator $\mathcal{O}$ can be represented as a sum of products of creation and annihilation operators:

$$\begin{aligned} \mathcal{O} = \sum_{N=0}^{\infty} \sum_{M=0}^{\infty} \int & dq_1' \cdots dq_N' dq_1 \cdots dq_M \\ & \times a^\dagger(q_1') \cdots a^\dagger(q_N') a(q_M) \cdots a(q_1) \\ & \times C_{NM}(q_1', \ldots, q_N', q_1, \ldots, q_M) \,. \end{aligned} \tag{3.12}$$

Example: Any additive operator F (momentum, charge, …)

$$F|q_1,\ldots,q_N\rangle = (f(q_1)+\ldots+f(q_N))|q_1,\ldots,q_N\rangle \tag{3.13}$$

can be written as in Eq. (3.12) with $N=M=1$:

$$F = \int dq\, a^\dagger(q)a(q)f(q)\,. \tag{3.14}$$

E.g. for $f(q)\equiv 1$ this is the *number operator* $N=\int dq\, a^\dagger(q)a(q)$. In particular, the free Hamiltonian always has the form

$$H_0 = \int dq\, a^\dagger(q)a(q)E(q)\,, \tag{3.15}$$

with $E(q)=E(\boldsymbol{p},\sigma,n)=\sqrt{\boldsymbol{p}^2+m_n^2}$.

E.g. for identical bosons:

$$\begin{aligned}
&H_0|p_1p_2\rangle = H_0 a^\dagger(\boldsymbol{p}_1)a^\dagger(\boldsymbol{p}_2)|0\rangle\\
&\quad= \int d^3\boldsymbol{q}\,E(\boldsymbol{q})a^\dagger(\boldsymbol{q})a(\boldsymbol{q})a^\dagger(\boldsymbol{p}_1)a^\dagger(\boldsymbol{p}_2)|0\rangle\\
&\quad\overset{(3.10)}{=} \int d^3\boldsymbol{q}\,E(\boldsymbol{q})\big[a^\dagger(\boldsymbol{q})a^\dagger(\boldsymbol{p}_1)a(\boldsymbol{q})a^\dagger(\boldsymbol{p}_2) + a^\dagger(\boldsymbol{q})\delta^3(\boldsymbol{q}-\boldsymbol{p}_1)a^\dagger(\boldsymbol{p}_2)\big]|0\rangle\\
&\quad\overset{(3.10),(3.9)}{=} \int d^3\boldsymbol{q}\,E(\boldsymbol{q})\big[0 + a^\dagger(\boldsymbol{q})a^\dagger(\boldsymbol{p}_1)\delta^3(\boldsymbol{q}-\boldsymbol{p}_2) + a^\dagger(\boldsymbol{q})\delta^3(\boldsymbol{q}-\boldsymbol{p}_1)a^\dagger(\boldsymbol{p}_2)\big]|0\rangle\\
&\quad= E(\boldsymbol{p}_2)a^\dagger(\boldsymbol{p}_2)a^\dagger(\boldsymbol{p}_1)|0\rangle + E(\boldsymbol{p}_1)a^\dagger(\boldsymbol{p}_1)a^\dagger(\boldsymbol{p}_2)|0\rangle\\
&\quad\overset{(3.11)}{=} \big(E(\boldsymbol{p}_1)+E(\boldsymbol{p}_2)\big)|p_1p_2\rangle\,.
\end{aligned}$$

Transformation properties Write the state in Eq. (2.1) as $a^\dagger(\boldsymbol{p}_1\sigma_1n_1)a^\dagger(\boldsymbol{p}_2\sigma_2n_2)\cdots|0\rangle$. The vacuum state is Lorentz invariant, $U(\Lambda,\alpha)|0\rangle=|0\rangle$. It follows that

$$U(\Lambda,\alpha)a^\dagger(\boldsymbol{p}\sigma n)U^{-1}(\Lambda,\alpha) = e^{i\alpha\cdot(\Lambda p)}\sqrt{\frac{(\Lambda p)^0}{p^0}}\sum_{\sigma'}D^{(j)}_{\sigma'\sigma}(W(\Lambda,p))a^\dagger(\boldsymbol{p}_\Lambda\sigma' n)\,. \tag{3.16}$$

(Here, $\boldsymbol{p}_\Lambda$ is the spatial part of Λp.) Similarly,

$$\mathrm{C}a^\dagger(\boldsymbol{p},\sigma,n)\mathrm{C}^{-1} = \xi_n a^\dagger(\boldsymbol{p},\sigma,n^c)\,, \tag{3.17}$$

$$\mathrm{P}a^\dagger(\boldsymbol{p},\sigma,n)\mathrm{P}^{-1} = \eta_n a^\dagger(-\boldsymbol{p},\sigma,n)\,, \tag{3.18}$$

$$\mathrm{T}a^\dagger(\boldsymbol{p},\sigma,n)\mathrm{T}^{-1} = \zeta_n(-1)^{j-\sigma}a^\dagger(-\boldsymbol{p},-\sigma,n)\,. \tag{3.19}$$

3.2 Cluster decomposition and structure of interaction

Consider processes $\alpha_1 \to \beta_1, \alpha_1 \to \beta_1, \ldots, \alpha_N \to \beta_N$ happening far apart from each other. We expect

$$S_{\beta_1+\ldots+\beta_N,\alpha_1+\ldots+\alpha_N} \to S_{\beta_1,\alpha_1} \cdots S_{\beta_N,\alpha_N}\,, \tag{3.20}$$

implying uncorrelated experimental results. One can show [1] that the S-matrix satisfies the *cluster decomposition priniciple* if

$$\begin{aligned} H = \sum_{N,M=0}^{\infty} \int & dq_1' \cdots dq_N' dq_1 \cdots dq_M \\ & \times a^\dagger(q_1') \cdots a^\dagger(q_N') a(q_M) \cdots a(q_1) \\ & \times \delta^3(\boldsymbol{p}_1' + \ldots + \boldsymbol{p}_N' - \boldsymbol{p}_1 - \ldots - \boldsymbol{p}_M) \\ & \times h_{NM}(q_1', \ldots, q_N', q_1, \ldots, q_M)\,. \end{aligned} \tag{3.21}$$

with a single momentum-conservation delta function.

(Counter-)Example:

$$S_{\boldsymbol{p}_1'\boldsymbol{p}_2',\boldsymbol{p}_1\boldsymbol{p}_2} \supset \delta(E_1' + E_2' - E_1 - E_2)\delta^3(\boldsymbol{p}_1' + \boldsymbol{p}_2' - \boldsymbol{p}_1 - \boldsymbol{p}_2)\delta^3(\boldsymbol{p}_2' - \boldsymbol{p}_2)V(\boldsymbol{p}_1', \boldsymbol{p}_2', \boldsymbol{p}_1, \boldsymbol{p}_2) \tag{3.22}$$

Fourier transformation:

$$S_{\boldsymbol{x}_1'\boldsymbol{x}_2',\boldsymbol{x}_1\boldsymbol{x}_2} = \int d\boldsymbol{p}_1' d\boldsymbol{p}_2' d\boldsymbol{p}_1 d\boldsymbol{p}_2 \delta^3(\boldsymbol{p}_2' - \boldsymbol{p}_2) V(\boldsymbol{p}_1', \boldsymbol{p}_2', \boldsymbol{p}_1, \boldsymbol{p}_2) e^{i\boldsymbol{p}_1'\cdot\boldsymbol{x}_1'} e^{i\boldsymbol{p}_2'\cdot\boldsymbol{x}_2'} e^{i\boldsymbol{p}_1\cdot\boldsymbol{x}_1} e^{i\boldsymbol{p}_2\cdot\boldsymbol{x}_2} S_{\boldsymbol{p}_1'\boldsymbol{p}_2',\boldsymbol{p}_1\boldsymbol{p}_2}\,. \tag{3.23}$$

Evaluating the first delta function only gives

$$S_{\boldsymbol{x}_1'\boldsymbol{x}_2',\boldsymbol{x}_1\boldsymbol{x}_2} = \int d\boldsymbol{p}_2' d\boldsymbol{p}_1 d\boldsymbol{p}_2 \delta^3(\boldsymbol{p}_2' - \boldsymbol{p}_2) V(\boldsymbol{p}_1', \boldsymbol{p}_2', \boldsymbol{p}_1, \boldsymbol{p}_2) e^{i\boldsymbol{p}_2'\cdot(\boldsymbol{x}_2'-\boldsymbol{x}_1')} e^{-i\boldsymbol{p}_1\cdot(\boldsymbol{x}_1-\boldsymbol{x}_1')} e^{-i\boldsymbol{p}_2\cdot(\boldsymbol{x}_2-\boldsymbol{x}_1')} S_{\boldsymbol{p}_1'\boldsymbol{p}_2',\boldsymbol{p}_1\boldsymbol{p}_2} \to 0 \tag{3.24}$$

for $x_i^{(\prime)} - x_i^{(\prime)} \to \infty$, employing the Riemann-Lebesgue lemma. Evaluating the second delta function gives

$$S_{\boldsymbol{x}_1'\boldsymbol{x}_2',\boldsymbol{x}_1\boldsymbol{x}_2} = \int d\boldsymbol{p}_1 d\boldsymbol{p}_2 \delta^3(\boldsymbol{p}_2' - \boldsymbol{p}_2) V(\boldsymbol{p}_1', \boldsymbol{p}_2', \boldsymbol{p}_1, \boldsymbol{p}_2) e^{-i\boldsymbol{p}_1\cdot(\boldsymbol{x}_1-\boldsymbol{x}_1')} e^{-i\boldsymbol{p}_2\cdot(\boldsymbol{x}_2-\boldsymbol{x}_2')} S_{\boldsymbol{p}_1'\boldsymbol{p}_2',\boldsymbol{p}_1\boldsymbol{p}_2} \tag{3.25}$$

which remains finite for $x_1 - x_2 \to \infty$.

4 Quantum fields

Goal: Construct a scalar Hamilton density $\mathcal{H}$ out of creation and annihilation operators.
Problem: L.T. (3.20) is momentum dependent.
Solution: Combine $a^\dagger$ and a into fields[1]

$$\Psi_\ell^+(x) = \sum_\sigma \int d^3p\, u_\ell(x; \boldsymbol{p}, \sigma) a(\boldsymbol{p}, \sigma)\,, \tag{4.1}$$

[1] I will suppress the particle type label from now on. It is straightorward to include if needed.

$$\Psi_\ell^-(x) = \sum_\sigma \int d^3p\, v_\ell(x;\boldsymbol{p},\sigma)a^\dagger(\boldsymbol{p},\sigma)\,, \tag{4.2}$$

such that

$$U(\Lambda,a)\Psi_\ell^\pm U^{-1}(\Lambda,a) = \sum_{\bar\ell} D_{\ell\bar\ell}(\Lambda^{-1})\Psi_{\bar\ell}^\pm(\Lambda x + a)\,. \tag{4.3}$$

The matrices $D_{\ell\bar\ell}$ again furnish a representation of the Lorentz group.

E.g. Scalar fields: $D(\Lambda) = 1$, $\quad \Psi_\ell^\pm = \Psi^\pm$;
Vector fields: $D(\Lambda)^\mu{}_\nu = \Lambda^\mu{}_\nu$, $\quad \Psi_\ell^\pm = \Psi_\mu^\pm$.

In this case, we can write the interaction as

$$\mathcal{H}(x) = \sum_{NM}\sum_{\ell_1'\cdots\ell_N'}\sum_{\ell_1\cdots\ell_M} g_{\ell_1'\cdots\ell_N'\ell_1\cdots\ell_M}\Psi_{\ell_1'}^-(x)\cdots\Psi_{\ell_N'}^-(x)\Psi_{\ell_1}^+(x)\cdots\Psi_{\ell_M}^+(x)\,. \tag{4.4}$$

This is Lorentz invariant if the $g_{\ell_1'\cdots}$ are Lorentz covariant. E.g. $g^{\mu\nu}\Psi_\mu^-(x)\Psi_\nu^+(x)$; const. $\times$ $\Psi^-(x)\Psi^+(x)$, …

Now write (3.20), (3.20)† as

$$U(\Lambda,b)a(\boldsymbol{p},\sigma)U^{-1}(\Lambda,b) = \exp(-i(\Lambda p)\cdot b)\sqrt{\frac{(\Lambda p)^0}{p^0}}\sum_{\bar\sigma} D_{\sigma\bar\sigma}^{(j)}(W^{-1}(\Lambda,p))a(\boldsymbol{p}_\Lambda,\bar\sigma)\,, \tag{4.5}$$

$$U(\Lambda,b)a^\dagger(\boldsymbol{p},\sigma)U^{-1}(\Lambda,b) = \exp(+i(\Lambda p)\cdot b)\sqrt{\frac{(\Lambda p)^0}{p^0}}\sum_{\bar\sigma} D_{\sigma\bar\sigma}^{(j)*}(W^{-1}(\Lambda,p))a^\dagger(\boldsymbol{p}_\Lambda,\bar\sigma)\,. \tag{4.6}$$

Then the fields transform as (recall $d^3p/p^0 = d^3(\Lambda p)/(\Lambda p)^0$)

$$\begin{aligned} U(\Lambda,b)\Psi_\ell^+U^{-1}(\Lambda,b) = \sum_{\sigma\bar\sigma}\int d^3(\Lambda p)\, u_\ell(x;\boldsymbol{p},\sigma)\sqrt{\tfrac{p^0}{(\Lambda p)^0}} \\ \times \exp(-i(\Lambda p)\cdot b)D_{\sigma\bar\sigma}^{(j)}(W^{-1}(\Lambda,p))a(\boldsymbol{p}_\Lambda,\bar\sigma)\,, \end{aligned} \tag{4.7}$$

$$\begin{aligned} U(\Lambda,b)\Psi_\ell^-U^{-1}(\Lambda,b) = \sum_{\sigma\bar\sigma}\int d^3(\Lambda p)\, v_\ell(x;\boldsymbol{p},\sigma)\sqrt{\tfrac{p^0}{(\Lambda p)^0}} \\ \times \exp(+i(\Lambda p)\cdot b)D_{\sigma\bar\sigma}^{(j)*}(W^{-1}(\Lambda,p))a^\dagger(\boldsymbol{p}_\Lambda,\bar\sigma)\,, \end{aligned} \tag{4.8}$$

and Eq. (4.3) is satisfied if and only if

$$\begin{aligned} &\sum_{\bar\ell} D_{\ell\bar\ell}(\Lambda^{-1})u_{\bar\ell}(\Lambda x + b;\boldsymbol{p}_\Lambda,\sigma) \\ &\quad = \sqrt{p^0/(\Lambda p)^0}\sum_{\bar\sigma} D_{\sigma\bar\sigma}^{(j)}(W^{-1}(\Lambda,p))\exp(-i(\Lambda p)\cdot b)u_\ell(x;\boldsymbol{p},\sigma)\,, \end{aligned} \tag{4.9}$$

$$\begin{aligned} &\sum_{\bar\ell} D_{\ell\bar\ell}(\Lambda^{-1})v_{\bar\ell}(\Lambda x + b;\boldsymbol{p}_\Lambda,\sigma) \\ &\quad = \sqrt{p^0/(\Lambda p)^0}\sum_{\bar\sigma} D_{\sigma\bar\sigma}^{(j)*}(W^{-1}(\Lambda,p))\exp(+i(\Lambda p)\cdot b)v_\ell(x;\boldsymbol{p},\sigma)\,, \end{aligned} \tag{4.10}$$

or

$$\begin{aligned}\sum_{\bar\sigma} u_{\bar\ell}(\Lambda x + b; \boldsymbol{p}_\Lambda, \bar\sigma) D^{(j)}_{\sigma\bar\sigma}(W(\Lambda, p)) \\ = \sqrt{p^0/(\Lambda p)^0} \sum_\ell D_{\bar\ell\ell}(\Lambda) \exp(-i(\Lambda p)\cdot b) u_\ell(x; \boldsymbol{p}, \sigma)\,,\end{aligned} \tag{4.11}$$

$$\begin{aligned}\sum_{\bar\sigma} v_{\bar\ell}(\Lambda x + b; \boldsymbol{p}_\Lambda, \bar\sigma) D^{(j)*}_{\sigma\bar\sigma}(W(\Lambda, p)) \\ = \sqrt{p^0/(\Lambda p)^0} \sum_\ell D_{\bar\ell\ell}(\Lambda) \exp(+i(\Lambda p)\cdot b) v_\ell(x; \boldsymbol{p}, \sigma)\,.\end{aligned} \tag{4.12}$$

Translations Take $\Lambda = 1$, b arbitrary in Eq.s (4.11), (4.12); it follows

$$u_\ell(x; \boldsymbol{p}, \sigma) = (2\pi)^{-3/2} e^{-ip\cdot x} u_\ell(\boldsymbol{p}, \sigma)\,, \tag{4.13}$$

$$v_\ell(x; \boldsymbol{p}, \sigma) = (2\pi)^{-3/2} e^{+ip\cdot x} v_\ell(\boldsymbol{p}, \sigma)\,. \tag{4.14}$$

Hence the quantum fields are Fourier transforms:

$$\Psi^+_\ell(x) = \sum_\sigma \int d^3p\, u_\ell(\boldsymbol{p}, \sigma) e^{-ip\cdot x} a(\boldsymbol{p}, \sigma)\,, \tag{4.15}$$

$$\Psi^-_\ell(x) = \sum_\sigma \int d^3p\, v_\ell(\boldsymbol{p}, \sigma) e^{ip\cdot x} a^\dagger(\boldsymbol{p}, \sigma)\,, \tag{4.16}$$

and Eq.s (4.11), (4.12) become

$$\sum_{\bar\sigma} u_{\bar\ell}(\boldsymbol{p}_\Lambda, \bar\sigma) D^{(j)}_{\sigma\bar\sigma}(W(\Lambda, p)) = \sqrt{p^0/(\Lambda p)^0} \sum_\ell D_{\bar\ell\ell}(\Lambda) u_\ell(\boldsymbol{p}, \sigma)\,, \tag{4.17}$$

$$\sum_{\bar\sigma} v_{\bar\ell}(\boldsymbol{p}_\Lambda, \bar\sigma) D^{(j)*}_{\sigma\bar\sigma}(W(\Lambda, p)) = \sqrt{p^0/(\Lambda p)^0} \sum_\ell D_{\bar\ell\ell}(\Lambda) v_\ell(\boldsymbol{p}, \sigma)\,. \tag{4.18}$$

Boosts Set $\boldsymbol{p} = 0$, $\Lambda = L(q)$ in Eq.s (4.17), (4.18), where q is some arbitrary four-momentum. We have $L(p) = 1$; $W(\Lambda, p) \equiv L^{-1}(\Lambda p)\Lambda L(p) = L^{-1}(q)L(q) = 1$. Hence

$$u_{\bar\ell}(\boldsymbol{q}, \sigma) = \sqrt{m/q^0} \sum_\ell D_{\bar\ell\ell}(L(q)) u_\ell(0, \sigma)\,, \tag{4.19}$$

$$v_{\bar\ell}(\boldsymbol{q}, \sigma) = \sqrt{m/q^0} \sum_\ell D_{\bar\ell\ell}(L(q)) v_\ell(0, \sigma)\,, \tag{4.20}$$

Therefore, it is sufficient to know the u_ℓ, v_ℓ for $\boldsymbol{p} = 0$.

Rotations Set $\boldsymbol{p} = 0$, $\Lambda = R$ (rotation, $\boldsymbol{p}_\Lambda = 0$) in Eq.s (4.17), (4.18):

$$\sum_{\bar\sigma} u_{\bar\ell}(0, \bar\sigma) D^{(j)}_{\bar\sigma\sigma}(R) = \sum_\ell D_{\bar\ell\ell}(R) u_\ell(0, \sigma)\,, \tag{4.21}$$

$$\sum_{\bar{\sigma}} v_{\bar{\ell}}(0,\bar{\sigma}) D^{(j)*}_{\bar{\sigma}\sigma}(R) = \sum_{\ell} D_{\bar{\ell}\ell}(R) v_\ell(0,\sigma)\,, \tag{4.22}$$

or, for infinitesimal rotations

$$\sum_{\bar{\sigma}} u_{\bar{\ell}}(0,\bar{\sigma}) \boldsymbol{J}^{(j)}_{\bar{\sigma}\sigma} = \sum_{\ell} \mathcal{J}_{\bar{\ell}\ell} u_\ell(0,\sigma)\,, \tag{4.23}$$

$$\sum_{\bar{\sigma}} v_{\bar{\ell}}(0,\bar{\sigma}) \boldsymbol{J}^{(j)*}_{\bar{\sigma}\sigma} = -\sum_{\ell} \mathcal{J}_{\bar{\ell}\ell} v_\ell(0,\sigma)\,. \tag{4.24}$$

Inserting Eq.s (4.15) and (4.16) into Eq. (4.4) and integrating over d^3x yields interaction V as a sum of products of a, $a^\dagger$ with a single delta function $\delta^3(\boldsymbol{p}'_1 + \ldots - \boldsymbol{p}_1 - \ldots)$. Therefore, the cluster decomposition principle is satisfied.

What about the commutation relation (2.71)? They are not in general satisfied:

$$\begin{aligned}
[\Psi^+_\ell(x), \Psi^-_{\bar{\ell}}(y)]_\mp &= \sum_{\sigma\sigma'} \int \frac{d^3p d^3p'}{(2\pi)^3} u_\ell(\boldsymbol{p},\sigma) v_{\bar{\ell}}(\boldsymbol{p}',\sigma') e^{-ip\cdot x} e^{ip'\cdot y} \underbrace{[a(\boldsymbol{p},\sigma), a^\dagger(\boldsymbol{p}',\sigma')]_\mp}_{\delta^3(\boldsymbol{p}-\boldsymbol{p}')\delta_{\sigma\sigma'}} \\
&= \sum_{\sigma} \int \frac{d^3p}{(2\pi)^3} u_\ell(\boldsymbol{p},\sigma) v_{\bar{\ell}}(\boldsymbol{p},\sigma) e^{-ip\cdot(x-y)} \neq 0\,.
\end{aligned}$$

<u>Solution:</u> Write $\Psi_\ell = \kappa_\ell \Psi^+_\ell(x) + \lambda_\ell \Psi^-_\ell(x)$ such that

$$[\Psi_\ell(x), \Psi_{\bar{\ell}}(y)]_\mp = [\Psi_\ell(x), \Psi^\dagger_{\bar{\ell}}(y)]_\mp = 0 \quad \text{for} \quad (x-y)^2 \leq 0\,. \tag{4.25}$$

Existence of antiparticles Assume particle n carries charge Q with value $q(n)$. Then (exercise!)

$$[Q, a(\boldsymbol{p},\sigma,n)] = -q(n) a(\boldsymbol{p},\sigma,n)\,, \tag{4.26}$$

$$[Q, a^\dagger(\boldsymbol{p},\sigma,n)] = q(n) a^\dagger(\boldsymbol{p},\sigma,n)\,. \tag{4.27}$$

We require $[\mathcal{H}(x), Q] = 0$ (charge conservation). This can only be satisfied if

$$[Q, \Psi_\ell(x)] = -q_\ell \Psi_\ell(x)\,, \qquad [Q, \Psi^\dagger_\ell(x)] = q_\ell \Psi_\ell(x)\,, \tag{4.28}$$

because then

$$[Q, \Psi_\ell(x)\Psi^\dagger_\ell(x)] = (q_\ell - q_\ell)\Psi_\ell(x)\Psi^\dagger_\ell(x) = 0\,. \tag{4.29}$$

Eq. (4.28) is satisfied if the particle annihilated by Ψ^+_ℓ carries charge q_ℓ and if the particle annihilated by Ψ^-_ℓ carries charge $-q_\ell$. *For each charged particle there is an antiparticle with the opposite charge.*

Field equations Eq.s (4.17), (4.18) immediately show that Ψ_ℓ satisfies the *Klein-Gordon equation*:

$$(\Box + m^2)\Psi_\ell(x) = 0\,. \tag{4.30}$$

4.1 Causal scalar fields

Simplest case: $D(\Lambda) = 1$. Eq.s (4.23), (4.24) imply that scalar fields describe spin-zero particles. Consider neutral particles first: Choose $u(0) = v(0) = 1/\sqrt{2m}$. Using Eq.s (4.19) and (4.20) we find

$$u(\boldsymbol{p}) = v(\boldsymbol{p}) = \frac{1}{\sqrt{2p^0}}\,, \tag{4.31}$$

so we have

$$\phi^+(x) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2p^0}} e^{-ip\cdot x} a(\boldsymbol{p})\,, \tag{4.32}$$

$$\phi^-(x) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2p^0}} e^{ip\cdot x} a^\dagger(\boldsymbol{p})\,. \tag{4.33}$$

We find

$$[\phi^+(x), \phi^+(y)]_\mp = [\phi^-(x), \phi^-(y)]_\mp = 0\,, \tag{4.34}$$

but

$$\begin{aligned}
[\phi^+(x), \phi^-(y)]_\mp &= \int \frac{d^3p d^3p'}{(2\pi)^3\sqrt{2p^0 2p'^0}}\, e^{-ip\cdot x} e^{ip'\cdot y} \underbrace{[a(\boldsymbol{p}), a^\dagger(\boldsymbol{p}')]_\mp}_{\delta^3(\boldsymbol{p}-\boldsymbol{p}')} \\
&= \frac{1}{(2\pi)^3}\int \frac{d^3p}{2p^0}\, e^{-ip\cdot(x-y)} \equiv \Delta_+(x-y)\,.
\end{aligned} \tag{4.35}$$

$\Delta_+(x)$ is Lorentz invariant; for space-like x it can only depend on $x^2 < 0$. Choose coordinate system such that $x^0 = 0$, $|\boldsymbol{x}| = \sqrt{-x^2}$. Then

$$\begin{aligned}
(2\pi)^3\Delta_+(x) &= \int \frac{d^3p}{2\sqrt{\boldsymbol{p}^2 + m^2}}\, e^{i\boldsymbol{p}\cdot\boldsymbol{x}} \\
&= \int \frac{2\pi|\boldsymbol{p}|^2 d|\boldsymbol{p}| \sin\theta d\theta}{2\sqrt{|\boldsymbol{p}|^2 + m^2}}\, e^{i|\boldsymbol{p}|\cdot\sqrt{-x^2}\cos\theta} \\
&\overset{z=\cos\theta}{=} 2\pi \int_0^\infty d|\boldsymbol{p}| \int_{-1}^1 \frac{|\boldsymbol{p}|^2 dz}{2\sqrt{|\boldsymbol{p}|^2 + m^2}}\, e^{i|\boldsymbol{p}|\cdot\sqrt{-x^2}z} \\
&= 4\pi \int_0^\infty \frac{|\boldsymbol{p}|^2 d|\boldsymbol{p}|}{2\sqrt{|\boldsymbol{p}|^2 + m^2}} \frac{\sin\left(|\boldsymbol{p}|\sqrt{-x^2}\right)}{|\boldsymbol{p}|\sqrt{-x^2}} \\
&\overset{u=|\boldsymbol{p}|/m}{=} \frac{2\pi m}{\sqrt{-x^2}} \int_0^\infty \frac{u du}{\sqrt{u^2+1}} \sin\left(m\sqrt{-x^2}u\right) \neq 0\,.
\end{aligned} \tag{4.36}$$

$\Delta_+(x)$ is even in x^μ for $x^2 < 0$. We write $\phi(x) \equiv \kappa\phi^+(x) + \lambda\phi^-(x)$. For $(x-y)^2 < 0$

$$\begin{aligned}
[\phi(x), \phi^\dagger(y)]_\mp &= [\kappa\phi^+(x) + \lambda\phi^-(x), \kappa^*\phi^-(y) + \lambda^*\phi^+(y)]_\mp \\
&= |\kappa|^2[\phi^+(x), \phi^-(y)]_\mp + |\lambda|^2[\phi^-(x), \phi^+(y)]_\mp \\
&= (|\kappa|^2 \mp |\lambda|^2)\Delta_+(x-y)
\end{aligned} \tag{4.37}$$

and

$$
\begin{aligned}
[\phi(x),\phi(y)]_\mp &= \kappa\lambda\big([\phi^+(x),\phi^-(y)]_\mp + [\phi^-(x),\phi^+(y)]_\mp\big) \\
&= \kappa\lambda(1\mp 1)\Delta_+(x-y)\,.
\end{aligned} \tag{4.38}
$$

We see that the particle must be a *boson*, and $|\kappa| = |\lambda|$. We can choose $\kappa = \lambda = 1$, so

$$
\phi(x) = \phi^+(x) + \phi^-(x) = \phi^\dagger(x)\,. \tag{4.39}
$$

Next, assume that the particle created or annihilated by $\phi(x)$ carries a charge. The charge is only conserved if each term in $\mathcal{H}$ contains same number of a, $a^\dagger$, hence $\mathcal{H}$ cannot be a polynomial in ϕ. We have

$$
[Q,\phi^+(x)]_- = -q\,\phi^+(x)\,, \qquad [Q,\phi^{+\dagger}(x)]_- = +q\,\phi^{+\dagger}(x)\,. \tag{4.40}
$$

We must suppose that there are *two* particles of mass m, and charges $+q$ and $-q$, with corresponding fields $\phi^+(x)$ and $\phi^{+*}(x)$, such that

$$
[Q,\phi^+(x)]_- = -q\,\phi^+(x)\,, \qquad [Q,\phi^{+c}(x)]_- = +q\,\phi^{+c}(x)\,. \tag{4.41}
$$

Then we can write $\Phi(x) \equiv \kappa\phi^+(x) + \lambda\phi^{+c\dagger}(x)$, with

$$
[Q,\Phi(x)]_- = -q\,\Phi(x)\,, \tag{4.42}
$$

and we have

$$
\begin{aligned}
[\Phi(x),\Phi^\dagger(y)]_\mp &= |\kappa|^2[\phi^+(x),\phi^{+\dagger}(y)]_\mp + |\lambda|^2[\phi^{+c\dagger}(x),\phi^{+c}(y)]_\mp \\
&= (|\kappa|^2 \mp |\lambda|^2)\Delta_+(x-y)\,.
\end{aligned} \tag{4.43}
$$

We see that charged scalar particles are also bosons. Absorbing phases, we can write

$$
\boxed{\Phi(x) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2p^0}}\left[a(\boldsymbol{p})e^{-ip\cdot x} + a^{c\dagger}(\boldsymbol{p})e^{ip\cdot x}\right] = \phi^+(x) + \phi^{+c\dagger}(x)\,.} \tag{4.44}
$$

<u>NB:</u>

$$
\begin{aligned}
[\Phi(x),\Phi^\dagger(y)] &= \int \frac{d^3p}{(2\pi)^3 2p^0}\left[e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}\right] \\
&= \Delta_+(x-y) - \Delta_+(y-x) \equiv \Delta(x-y)\,.
\end{aligned} \tag{4.45}
$$

Discrete symmetries

Space inversion Recall Eq. (3.18):

$$
\mathrm{P}a(\boldsymbol{p})\mathrm{P}^{-1} = \eta^* a(-\boldsymbol{p})\,, \tag{4.46}
$$

$$
\mathrm{P}a^{c\dagger}(\boldsymbol{p})\mathrm{P}^{-1} = \eta^c a^{c\dagger}(-\boldsymbol{p})\,. \tag{4.47}
$$

Using this, we find

$$\begin{aligned} \mathrm{P}\phi^+(x)\mathrm{P}^{-1} &= \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2p^0}} \eta^* a(-\boldsymbol{p}) e^{-ip\cdot x} \\ &\overset{\boldsymbol{p}\to-\boldsymbol{p}}{=} \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2p^0}} \eta^* a(\boldsymbol{p}) e^{-ip\cdot \mathcal{P}x} \\ &= \eta^* \phi^+(\mathcal{P}x)\,, \end{aligned} \tag{4.48}$$

and similarly

$$\mathrm{P}\phi^{+c\dagger}(x)\mathrm{P}^{-1} = \eta^c \phi^{+c\dagger}(\mathcal{P}x)\,, \tag{4.49}$$

so we have

$$\mathrm{P}\Phi(x)\mathrm{P}^{-1} = \eta^*\phi^+(\mathcal{P}x) + \eta^c\phi^{+c\dagger}(\mathcal{P}x) \equiv \Phi_{\mathrm{P}}(x)\,. \tag{4.50}$$

The fields $\Phi(x)$ and $\Phi_{\mathrm{P}}(x)$ are both separately causal, but in general

$$[\Phi(x), \Phi_{\mathrm{P}}^\dagger(y)] = (\eta - \eta^{c*})\Delta_+(x-y) \neq 0 \quad \text{for} \quad (x-y)^2 < 0\,. \tag{4.51}$$

We need to require $\eta^c = \eta^*$ for a parity invariant theory. It follows that thie intrinsic parity $\eta\eta^c$ of a state containing a spin-zero particle-antiparticle pair is even, and

$$\mathrm{P}\Phi(x)\mathrm{P}^{-1} = \eta^*\Phi(\mathcal{P}x)\,. \tag{4.52}$$

Similarly, one can show [1] that

$$\mathrm{C}\Phi(x)\mathrm{C}^{-1} = \xi^*\Phi^\dagger(x)\,, \tag{4.53}$$

$$\mathrm{T}\Phi(x)\mathrm{T}^{-1} = \zeta^*\Phi(-\mathcal{P}x)\,. \tag{4.54}$$

4.2 Causal vector fields

Here, $D(\Lambda)^\mu{}_\nu = \Lambda^\mu{}_\nu$. We write

$$\phi^{+\mu}(x) = \sum_\sigma \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2p^0}}\, u^\mu(\boldsymbol{p},\sigma) e^{-ip\cdot x} a(\boldsymbol{p},\sigma)\,, \tag{4.55}$$

$$\phi^{-\mu}(x) = \sum_\sigma \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2p^0}}\, v^\mu(\boldsymbol{p},\sigma) e^{ip\cdot x} a^\dagger(\boldsymbol{p},\sigma)\,, \tag{4.56}$$

where (see (4.19), (4.20))

$$u^\mu(\boldsymbol{p},\sigma) = \sqrt{m/p^0}\, L(p)^\mu{}_\nu u^\nu(0,\sigma)\,, \tag{4.57}$$

$$v^\mu(\boldsymbol{p},\sigma) = \sqrt{m/p^0}\, L(p)^\mu{}_\nu v^\nu(0,\sigma)\,. \tag{4.58}$$

There are essentially two possibilities.

Spin 0:
Only $u^0(0,\sigma) \neq 0$, $v^0(0,\sigma) \neq 0$. This is just a derivative of a scalar field, $\phi^\mu(x) = \partial^\mu\phi(x)$.

Spin 1:
Using Eq.s (4.23), (4.24) and (4.57), (4.58), one can show that we can choose

$$u^\mu(\boldsymbol{p},\sigma) = v^{\mu *}(\boldsymbol{p},\sigma) = \tfrac{1}{\sqrt{2p^0}} e^\mu(\boldsymbol{p},\sigma)\,, \tag{4.59}$$

where

$$e^\mu(\boldsymbol{p},\sigma) \equiv L(p)^\mu{}_\nu e^\nu(0,\sigma)\,, \tag{4.60}$$

and

$$e^\mu(0,0) = \begin{pmatrix} 0\\0\\0\\1 \end{pmatrix}\,, \quad e^\mu(0,+1) = -\frac{1}{\sqrt{2}}\begin{pmatrix} 0\\1\\i\\0 \end{pmatrix}\,, \quad e^\mu(0,0) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0\\1\\-i\\0 \end{pmatrix}\,. \tag{4.61}$$

The fields are then

$$\phi^{+\mu}(x) = \phi^{-\mu\dagger}(x) = \sum_\sigma \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2p^0}}\, e^\mu(\boldsymbol{p},\sigma) e^{-ip\cdot x} a(\boldsymbol{p},\sigma)\,, \tag{4.62}$$

with (anti-)commutator

$$[\phi(x)^{+\mu}, \phi^{-\mu}(y)]_\mp = \int \frac{d^3p}{(2\pi)^3 2p^0} e^{-ip\cdot(x-y)} \Pi^{\mu\nu}(\boldsymbol{p})\,. \tag{4.63}$$

Here,

$$\Pi^{\mu\nu}(\boldsymbol{p}) \equiv \sum_\sigma e^\mu(\boldsymbol{p},\sigma) e^{\nu *}(\boldsymbol{p},\sigma) = -\eta^{\mu\nu} + \frac{p^\mu p^\nu}{m^2}\,. \tag{4.64}$$

(Exercise!) Hence, we can write

$$[\phi^{+\mu}(x), \phi^{-\mu}(y)]_\mp = \Big(-\eta^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{m^2}\Big)\Delta_+(x-y) \neq 0 \quad \text{for} \quad (x-y)^2 < 0\,. \tag{4.65}$$

In analogy to scalar fields, form linear combination …also spin-one particles are bosons. Allowing again for charged particles, we have finally

$$\boxed{v^\mu(x) = \sum_\sigma \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2p^0}} \left[e^\mu(\boldsymbol{p},\sigma) a(\boldsymbol{p},\sigma) e^{-ip\cdot x} + e^{\mu *}(\boldsymbol{p},\sigma) a^{c\dagger}(\boldsymbol{p},\sigma) e^{ip\cdot x}\right].} \tag{4.66}$$

The commutator is

$$[v(x)^\mu, v^{\nu\dagger}(y)] = \Big(-\eta^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{m^2}\Big)\Delta(x-y)\,, \tag{4.67}$$

and

$$\boxed{(\Box + m^2) v^\mu(x) = 0\,.} \tag{4.68}$$

Using Eq. (4.60), we see

$$p_\mu e^\mu(\boldsymbol{p},\sigma) = \eta_{\mu\rho} p^\rho e^\mu(\boldsymbol{p},\sigma) = \eta_{\mu\rho} L^\rho{}_\sigma(p) L^\mu{}_\nu(p) k^\sigma e^\mu(0,\sigma) = \eta_{\sigma\nu} k^\sigma e^\nu(0,\sigma) = 0\,, \qquad (4.69)$$

and therefore

$$\boxed{\partial_\mu v^\mu(x) = 0\,.} \qquad (4.70)$$

Interestingly, these would be the equations of electrodynamics for $m \to 0$ (Lorentz gauge). Is it allowed to take this limit? Assume a Hamilton density of the form $\mathcal{H} = J_\mu v^\mu$, where J_μ is some four-vector current. The squared transition matrix element then has the form

$$\sum_\sigma \left|\langle J_\mu\rangle e^{\mu *}(\boldsymbol{p},\sigma)\right|^2 = \langle J_\mu\rangle\langle J_\nu\rangle^* \Bigg(-\eta^{\mu\nu} + \underbrace{\frac{p^\mu p^\nu}{m^2}}_{\longrightarrow\infty \text{ as } m\to 0} \Bigg)\,. \qquad (4.71)$$

We see that $\langle J_\mu\rangle p^\mu$ has to vanish, or, equivalently, J_μ has to be conserved:

$$\partial_\mu J^\mu = 0\,. \qquad (4.72)$$

Discrete symmetries:

$$\eta^c = \eta^*\,, \quad \xi^c = \xi^*\,, \quad \zeta^c = \zeta^*\,; \qquad (4.73)$$

$$\mathrm{P} v^\mu(x) \mathrm{P}^{-1} = -\eta^* \mathcal{P}^\mu{}_\nu v^\nu(\mathcal{P}x)\,, \qquad (4.74)$$

$$\mathrm{C} v^\mu(x) \mathrm{C}^{-1} = \xi^* v^{\mu\dagger}(x)\,, \qquad (4.75)$$

$$\mathrm{T} v^\mu(x) \mathrm{T}^{-1} = \zeta^* \mathcal{P}^\mu{}_\nu v^\nu(-\mathcal{P}x)\,. \qquad (4.76)$$

Massless vector fields?

Consider the "object"

$$A^\mu(x) = \sum_{\sigma=\pm 1} \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2p^0}} \left[e^\mu(\boldsymbol{p},\sigma) a(\boldsymbol{p},\sigma) e^{-ip\cdot x} + e^{\mu *}(\boldsymbol{p},\sigma) a^\dagger(\boldsymbol{p},\sigma) e^{ip\cdot x} \right]. \qquad (4.77)$$

Here, a, $a^\dagger$ annihilate / create massless spin-one particles. We can choose polarization vectors as follows: recall the standard momentum $k^\mu = (\kappa, 0, 0, \kappa)$, $k^2 = 0$. Define

$$e^\mu(\boldsymbol{k},\pm 1) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ \pm i \\ 0 \end{pmatrix}\,. \qquad (4.78)$$

Denote by $R(\hat{p})$ the rotation that brings the z-axis into the direction of $\boldsymbol{p}$. Then the e^μ for general momenta are (note that a boost in z direction does not affect $e^\mu(\boldsymbol{k},\pm 1)$)

$$e^\mu(\boldsymbol{p},\pm 1) = R(\hat{p})^\mu{}_\nu e^\nu(\boldsymbol{k},\pm 1)\,. \qquad (4.79)$$

In particular, $e^0(\boldsymbol{k}, \pm 1) = 0$ and $\boldsymbol{k} \cdot \boldsymbol{e}(\boldsymbol{k}, \pm 1) = 0$, so

$$e^0(\boldsymbol{p}, \pm 1) = 0 \tag{4.80}$$

and

$$\boldsymbol{p} \cdot \boldsymbol{e}(\boldsymbol{p}, \pm 1) = 0 \,. \tag{4.81}$$

Therefore, $A^\mu(x)$ satiesfies the field equations

$$A^0(x) = 0 \tag{4.82}$$

and

$$\nabla \cdot \boldsymbol{A}(x) = 0 \,. \tag{4.83}$$

This shows immediately that $A^\mu(x)$ cannot be a Lorentz four-vector. In fact, it can be shown [1] that under a L.T.

$$A_\mu(x) \to \Lambda^\nu{}_\mu A_\nu(\Lambda x) + \partial_\nu \Omega(x) \,, \tag{4.84}$$

with a scalar function $\Omega(x)$. One has to construct $\mathcal{H}(x)$ in terms of

$$F_{\mu\nu} = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) \tag{4.85}$$

(this is a Lorentz tensor, due to antisymmetry), or with terms of the form $A_\mu J^\mu$, with $\partial_\mu J^\mu = 0$. *This is the origin of gauge invariance.*

From Eq.s (4.78), (4.79) we obtain the polarization sum

$$\sum_{\sigma=\pm} e^i(\boldsymbol{p}, \sigma) e^{j*}(\boldsymbol{p}, \sigma) = \delta^{ij} - \frac{p^i p^j}{|\boldsymbol{p}|^2} \,. \tag{4.86}$$

4.3 The Dirac algebra

Goal: construction of the spin-1/2 representation of the Lorentz group ("spinor representation").

Start with a set of four 4×4 matrices γ^μ satisfying the *Clifford relation*

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu} \,. \tag{4.87}$$

We then define

$$\mathcal{J}^{\mu\nu} \equiv \tfrac{i}{4}[\gamma^\mu, \gamma^\nu] \,. \tag{4.88}$$

Using Eq. (4.87) and $[[A,B],C] = A\{B,C\} + \{B,C\}A - \{A,C\}B - B\{A,C\}$ it is easy to show that

$$[\mathcal{J}^{\mu\nu}, \gamma^\rho] = \tfrac{i}{4}[[\gamma^\mu, \gamma^\nu], \gamma^\rho] = i\gamma^\mu \eta^{\nu\rho} - i\gamma^\nu \eta^{\mu\rho} \,. \tag{4.89}$$

Using $[A,[B,C]] = [A,B]C - C[A,B] - [A,C]B + B[A,C]$ it is straightforward to verify also Eq. (1.43). It follows

$$D(\Lambda)\gamma^\rho D^{-1}(\Lambda) = \Lambda_\sigma{}^\rho \gamma^\sigma \,, \tag{4.90}$$

so γ^ρ is a Lorentz four-vector. Moreover,

$$D(\Lambda)\mathbb{1}D^{-1}(\Lambda) = \mathbb{1}\,, \tag{4.91}$$

so $\mathbb{1}$ is a Lorentz scalar. Eq. (1.43) shows that

$$D(\Lambda)\mathcal{J}^{\rho\sigma}D^{-1}(\Lambda) = \Lambda_\mu{}^\rho\Lambda_\nu{}^\sigma\mathcal{J}^{\mu\nu}\,, \tag{4.92}$$

so $\mathcal{J}^{\rho\sigma}$ is a Lorentz tensor. We now define

$$\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3\,. \tag{4.93}$$

It is easy to show that

$$\{\gamma_5, \gamma^\mu\} = 0\,, \tag{4.94}$$

and

$$\mathbb{1}\,; \qquad \gamma^\mu\,; \qquad \mathcal{J}^{\mu\nu}\,; \qquad \gamma^\mu\gamma_5\,; \qquad \gamma_5 \tag{4.95}$$

form a basis of the Clifford algebra.

We can introduce a "parity transformation" using

$$\beta \equiv \gamma^0 = \beta^{-1}\,. \tag{4.96}$$

We have

$$\beta\mathbb{1}\beta = \mathbb{1} \qquad \ldots\text{scalar} \tag{4.97}$$

$$\beta\gamma^i\beta = -\gamma^i\,; \quad \beta\gamma^0\beta = \gamma^0 \qquad \ldots\text{vector} \tag{4.98}$$

$$\beta\mathcal{J}^{ij}\beta = \mathcal{J}^{ij}\,; \quad \beta\mathcal{J}^{i0}\beta = \mathcal{J}^{i0} \qquad \ldots\text{tensor} \tag{4.99}$$

$$\beta\gamma^i\gamma_5\beta = \gamma^i\gamma_5\,; \quad \beta\gamma^0\gamma_5\beta = -\gamma^0\gamma_5 \qquad \ldots\text{axial vector} \tag{4.100}$$

$$\beta\gamma_5\beta = -\gamma_5 \qquad \ldots\text{pseudoscalar} \tag{4.101}$$

A useful explicit representation of the Dirac matrices is the so-called *chiral representation*:

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{4.102}$$

with the *Pauli matrices*

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.103}$$

In this representation,

$$\gamma_5 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}. \tag{4.104}$$

The matrices $D(\Lambda)$ in this representation act on four-component *Dirac spinors* (not Lorentz four-vectors!). The generators are (in the chiral representation):

$$\mathcal{J}^{ij} = \frac{1}{2}\epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}, \qquad \mathcal{J}^{j0} = -\frac{i}{2}\begin{pmatrix} \sigma^j & 0 \\ 0 & -\sigma^j \end{pmatrix}. \tag{4.105}$$

Now we choose a basis for the Dirac spinors:

$$
\begin{aligned}
u(0,\tfrac{1}{2}) &= \frac{1}{\sqrt{2}}\begin{pmatrix}\xi\\ \xi\end{pmatrix}\,, \qquad & u(0,-\tfrac{1}{2}) &= \frac{1}{\sqrt{2}}\begin{pmatrix}\eta\\ \eta\end{pmatrix}\,,\\
v(0,\tfrac{1}{2}) &= \frac{1}{\sqrt{2}}\begin{pmatrix}\eta\\ -\eta\end{pmatrix}\,, \qquad & v(0,-\tfrac{1}{2}) &= \frac{1}{\sqrt{2}}\begin{pmatrix}-\xi\\ \xi\end{pmatrix}\,,
\end{aligned}
\tag{4.106}
$$

where

$$
\xi = \begin{pmatrix}1\\ 0\end{pmatrix}\,, \qquad \eta = \begin{pmatrix}0\\ 1\end{pmatrix}\,. \tag{4.107}
$$

These are eigenvectors of the spin-z operator

$$
\mathcal{J}^3 = \mathcal{J}^{12} = \frac{1}{2}\begin{pmatrix}\sigma^3 & 0\\ 0 & \sigma^3\end{pmatrix}\,, \tag{4.108}
$$

with eigenvalues $\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}$. How do the spinors transform under a L.T.?

Reminder: four-momentum

$$
\begin{pmatrix}E\\ 0\\ 0\\ p^3\end{pmatrix} = \exp\big(-i\theta K^3\big)\begin{pmatrix}m\\ 0\\ 0\\ 0\end{pmatrix} = \exp\begin{pmatrix}0&0&0&\theta\\ 0&0&0&0\\ 0&0&0&0\\ \theta&0&0&0\end{pmatrix}\begin{pmatrix}m\\ 0\\ 0\\ 0\end{pmatrix} = \begin{pmatrix}m\cosh\theta\\ 0\\ 0\\ m\sinh\theta\end{pmatrix}\,. \tag{4.109}
$$

We have $E = m\cosh\theta = \frac{m}{2}(e^{\theta}+e^{-\theta})$, $p^3 = m\sinh\theta = \frac{m}{2}(e^{\theta}-e^{-\theta})$, and therefore

$$
e^{\theta} = \frac{E+p^3}{2}\,, \qquad e^{-\theta} = \frac{E-p^3}{2}\,. \tag{4.110}
$$

Now let us perform the same boost for spinors. We write $u(0,\sigma) = (\chi,\chi)^T/\sqrt{2}$, where $\chi = \xi, \eta$; and $\boldsymbol{p} = (0,0,p^3)$:

$$
\begin{aligned}
u(\boldsymbol{p},\sigma) &\overset{(4.19)}{=} \sqrt{\frac{m}{2E}}\exp\left[-\frac{\theta}{2}\begin{pmatrix}\sigma^3 & 0\\ 0 & -\sigma^3\end{pmatrix}\right]\begin{pmatrix}\chi\\ \chi\end{pmatrix}\\
&= \sqrt{\frac{m}{2E}}\begin{pmatrix}e^{-\theta/2} &&&\\ & e^{\theta/2} &&\\ && e^{\theta/2} &\\ &&& e^{-\theta/2}\end{pmatrix}\begin{pmatrix}\chi\\ \chi\end{pmatrix}\\
&= \sqrt{\frac{m}{2E}}\begin{pmatrix}e^{\theta/2}\left(\frac{1-\sigma^3}{2}\right)+e^{-\theta/2}\left(\frac{1+\sigma^3}{2}\right) & 0\\ 0 & e^{\theta/2}\left(\frac{1+\sigma^3}{2}\right)+e^{-\theta/2}\left(\frac{1-\sigma^3}{2}\right)\end{pmatrix}\begin{pmatrix}\chi\\ \chi\end{pmatrix}
\end{aligned}
\tag{4.111}
$$

Now we use $\sqrt{m}e^{\theta/2} = \sqrt{E+p^3}$ and $\sqrt{m}e^{-\theta/2} = \sqrt{E-p^3}$, and find

$$
u(\boldsymbol{p},\sigma) = \frac{1}{\sqrt{2E}}\begin{pmatrix}\left[\sqrt{E+p^3}\left(\frac{1-\sigma^3}{2}\right)+\sqrt{E-p^3}\left(\frac{1+\sigma^3}{2}\right)\right]\chi\\ \left[\sqrt{E+p^3}\left(\frac{1+\sigma^3}{2}\right)+\sqrt{E-p^3}\left(\frac{1-\sigma^3}{2}\right)\right]\chi\end{pmatrix}\,. \tag{4.112}
$$

One can easily show, by an explicit calculation, that

$$u^\dagger(\boldsymbol{p},\sigma)u(\boldsymbol{p},\sigma) = 1\,. \tag{4.113}$$

For $\chi = \xi = (1,0)^T$ and a large boost ($E \approx p^3$), Eq. (4.112) becomes

$$u(\boldsymbol{p},1/2) = \frac{1}{\sqrt{2E}}\begin{pmatrix}\sqrt{E-p^3}\xi\\ \sqrt{E+p^3}\xi\end{pmatrix} \to \begin{pmatrix}0\\ \xi\end{pmatrix}\,. \tag{4.114}$$

For $\chi = \eta = (0,1)^T$

$$u(\boldsymbol{p},-1/2) = \frac{1}{\sqrt{2E}}\begin{pmatrix}\sqrt{E+p^3}\eta\\ \sqrt{E-p^3}\eta\end{pmatrix} \to \begin{pmatrix}\eta\\ 0\end{pmatrix}\,. \tag{4.115}$$

The spinors (4.114), (4.115) are eigenstates of the *helicity operator*

$$h = \frac{\boldsymbol{p}\cdot\boldsymbol{\mathcal{J}}}{|\boldsymbol{p}|} = \frac{1}{2}\hat{p}^k\begin{pmatrix}\sigma^k & 0\\ 0 & \sigma^k\end{pmatrix}\,, \tag{4.116}$$

with eigenvalues $+\frac{1}{2}$ and $-\frac{1}{2}$, respectively. Generally, using

$$P_R \equiv \tfrac{1}{2}\big(1+\gamma_5\big) \overset{(4.93)}{=} \begin{pmatrix}0 & 0\\ 0 & \mathbb{1}\end{pmatrix}\,, \qquad P_L \equiv \tfrac{1}{2}\big(1-\gamma_5\big) \overset{(4.93)}{=} \begin{pmatrix}\mathbb{1} & 0\\ 0 & 0\end{pmatrix}\,, \tag{4.117}$$

we can decompose every spinor Ψ into a "left-handed" and a "right-handed" part:

$$\Psi = (P_R + P_L)\Psi \equiv \Psi_L + \Psi_R\,. \tag{4.118}$$

Lorentz-covariant bilinears

Not all generators in Eq. (4.105) are Hermitian: $(\mathcal{J}^{ij})^\dagger = \mathcal{J}^{ij}$, but $(\mathcal{J}^{0j})^\dagger = -\mathcal{J}^{0j}$. Hence, in general $D(\Lambda)^\dagger \neq D(\Lambda)^{-1}$, and $u^\dagger u$ is not a Lorentz scalar. We can use a "trick" to construct a Lorentz scalar, as follows: Define $\bar{u} \equiv u^\dagger\beta = u^\dagger\gamma^0$. Then

$$\begin{aligned}\bar{u}u = u^\dagger\beta u &\overset{\text{L.T.}}{\to} (D(\Lambda)u)^\dagger\beta D(\Lambda)u\\ &= u^\dagger D(\Lambda)^\dagger\beta D(\Lambda)u\\ &= u^\dagger\exp\big(\tfrac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu\dagger}\big)\beta\exp\big(-\tfrac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}\big)u\\ &\overset{(4.99)}{=} u^\dagger\beta\exp\big(\tfrac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu\dagger}\big)\exp\big(-\tfrac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}\big)u\\ &= \bar{u}u\,.\end{aligned} \tag{4.119}$$

In analogy, we can construct Lorentz four-vectors $\bar{u}\gamma^\mu u$, etc.

The Dirac spinors $u(\boldsymbol{p},\sigma)$, $u(\boldsymbol{p},\sigma)$ satisfy the *Dirac equation* (here, we introduce Feynman's short-hand notation $p_\mu\gamma^\mu = \not{p}$):

$$(\not{p} - m)u(\boldsymbol{p},\sigma) = 0\,, \tag{4.120}$$

$$(\not{p}+m)v(\boldsymbol{p},\sigma)=0\,. \tag{4.121}$$

Consider, e.g., Eq. (4.120) in the rest frame:

$$(\not{p}-m)u(\boldsymbol{p},\sigma)=(m\gamma^0-m)u(0,\sigma)\overset{(4.102),(4.106)}{=}\frac{m}{\sqrt{2}}\begin{pmatrix}-\mathbb{1} & \mathbb{1}\\ \mathbb{1} & -\mathbb{1}\end{pmatrix}\begin{pmatrix}\xi\\ \xi\end{pmatrix}=0\,. \tag{4.122}$$

Perform a Lorentz boost (drop global factor $\sqrt{m/E}$):

$$\begin{aligned}0&=D(\Lambda)(m\gamma^0-m)u(0,\sigma)=[mD(\Lambda)\gamma^0D^{-1}(\Lambda)-m]D(\Lambda)u(0,\sigma)\\ &\overset{(4.90)}{=}[m\Lambda_\mu{}^0\gamma^\mu-m]u(\boldsymbol{p},\sigma)\overset{k=(m,\boldsymbol{0})}{=}[k_\nu\Lambda_\mu{}^\nu\gamma^\mu-m]u(\boldsymbol{p},\sigma)\\ &=(\not{p}-m)u(\boldsymbol{p},\sigma)\,.\end{aligned} \tag{4.123}$$

Similarly for $v(\boldsymbol{p},\sigma)$.

4.4 The Dirac field

Using the Dirac spinors, we can write the *Dirac field* as

$$\boxed{\psi_\ell(x)=\sum_\sigma\int\frac{d^3p}{(2\pi)^{3/2}}\left[u_\ell(\boldsymbol{p},\sigma)a(\boldsymbol{p},\sigma)e^{-ip\cdot x}+v_\ell(\boldsymbol{p},\sigma)a^{c\dagger}(\boldsymbol{p},\sigma)e^{ip\cdot x}\right].} \tag{4.124}$$

Here, the u_ℓ, v_ℓ are given by Eq.s (4.106), (4.19), (4.20). One can show [1] that $a(\boldsymbol{p},\sigma)$ annihilates a spin-1/2 *fermion* with momentum $\boldsymbol{p}$, spin-z component σ, and mass m. $a^{c\dagger}(\boldsymbol{p},\sigma)$ creates the corresponding antiparticle with opposite charge. We also have

$$\boxed{\bar\psi_\ell(x)\equiv\psi_\ell\beta=\sum_\sigma\int\frac{d^3p}{(2\pi)^{3/2}}\left[\bar v_\ell(\boldsymbol{p},\sigma)a^{c}(\boldsymbol{p},\sigma)e^{-ip\cdot x}+\bar u_\ell(\boldsymbol{p},\sigma)a^{\dagger}(\boldsymbol{p},\sigma)e^{ip\cdot x}\right].} \tag{4.125}$$

Properties

Eq.s (4.120), (4.121) immediately imply the *Dirac equation*

$$\boxed{(i\not{\partial}-m)\psi_\ell(x)=0\,.} \tag{4.126}$$

Parity:

$$\begin{aligned}\mathrm{P}\psi^+(x)\mathrm{P}^{-1}&=\mathrm{P}\sum_\sigma\int\frac{d^3p}{(2\pi)^{3/2}}u(\boldsymbol{p},\sigma)a(\boldsymbol{p},\sigma)e^{-ip\cdot x}\mathrm{P}^{-1}\\ &\overset{(3.18)}{=}\sum_\sigma\int\frac{d^3p}{(2\pi)^{3/2}}u(\boldsymbol{p},\sigma)\eta^*a(-\boldsymbol{p},\sigma)e^{-ip\cdot x}\\ &\overset{\boldsymbol{p}\to-\boldsymbol{p}}{=}\eta^*\sum_\sigma\int\frac{d^3p}{(2\pi)^{3/2}}u(-\boldsymbol{p},\sigma)a(\boldsymbol{p},\sigma)e^{-ip\cdot\mathcal{P}x}\,,\end{aligned} \tag{4.127}$$

and similarly

$$\mathrm{P}\psi^{c-}(x)\mathrm{P}^{-1} = \eta^c \sum_\sigma \int \frac{d^3p}{(2\pi)^{3/2}} v(-\boldsymbol{p},\sigma) a^{c\dagger}(\boldsymbol{p},\sigma) e^{ip\cdot\mathcal{P}x} \,. \tag{4.128}$$

Eq. (4.99) implies $\beta\mathcal{J}^{0j}\beta = -\mathcal{J}^{0j}$, therefore

$$u(-\boldsymbol{p},\sigma) = \sqrt{m/p^0}\beta D(L(p))\beta u(0,\sigma)\,, \tag{4.129}$$

$$v(-\boldsymbol{p},\sigma) = \sqrt{m/p^0}\beta D(L(p))\beta v(0,\sigma)\,. \tag{4.130}$$

Using Eq.s (4.102) and (4.103) we see (recall $\beta = \gamma^0$)

$$\beta u(0,\sigma) = u(0,\sigma)\,, \qquad \beta v(0,\sigma) = -v(0,\sigma)\,, \tag{4.131}$$

and so

$$\mathrm{P}\psi^{+}(x)\mathrm{P}^{-1} = \eta^*\beta\psi^+(\mathcal{P}x)\,, \tag{4.132}$$

$$\mathrm{P}\psi^{c-}(x)\mathrm{P}^{-1} = -\eta^c\beta\psi^{c-}(\mathcal{P}x)\,. \tag{4.133}$$

In order that $\mathrm{P}\psi(x)\mathrm{P}^{-1} \propto \psi(\mathcal{P}x)$, we need

$$\eta^c = -\eta^*\,. \tag{4.134}$$

It follows that the intrinsic parity of a fermion-antifermion bound state is odd (e.g. pions). We then have, finally,

$$\mathrm{P}\psi(x)\mathrm{P}^{-1} = \eta^*\beta\psi(\mathcal{P}x)\,. \tag{4.135}$$

Time reversal: A similar calculation shows

$$\mathrm{T}\psi(x)\mathrm{T}^{-1} = \zeta^*\mathcal{C}\psi(-\mathcal{P}x)\,, \tag{4.136}$$

for $\zeta^c = \zeta^*$. Here, $\mathcal{C} \equiv -i\gamma^2\gamma^0$ relates the Dirac matrices to their complex conjugates; we have

$$\Gamma_\mu^* = -\beta\mathcal{C}\Gamma_\mu\mathcal{C}^{-1}\beta \tag{4.137}$$

for $\Gamma_\mu = \gamma_\mu, \mathcal{J}_{\mu\nu}, \gamma_5, \gamma_\mu\gamma_5$.

Charge conjugation: A similar calculation shows

$$\mathrm{C}\psi(x)\mathrm{C}^{-1} = \xi^*\beta\mathcal{C}\psi^*(x)\,, \tag{4.138}$$

for $\xi^c = \xi^*$. If particles and antiparticles are identical, their Dirac field must satisfy the equation

$$\psi(x) = \beta\mathcal{C}\psi^*(x)\,, \tag{4.139}$$

and $\eta = \pm i$, $\xi = \pm 1$ (*Majorana fermions*).

Scalar interaction densities To form Lorentz tensors, we write bilinears (as in Eq. (4.119)). Under space inversion

$$\mathrm{P}[\bar{\psi}(x)M\psi(x)]\mathrm{P}^{-1} = \bar{\psi}(\mathcal{P}x)\beta M\beta\psi(\mathcal{P}x)\,, \tag{4.140}$$

so the bilinears behave like

$$M = \underbrace{\mathbb{1}}_{\text{scalar}}, \underbrace{\gamma^\mu}_{\text{vector}}, \underbrace{\mathcal{J}^{\mu\nu}}_{\text{tensor}}, \underbrace{\gamma^\mu\gamma_5}_{\text{axial vector}}, \underbrace{\gamma_5}_{\text{pseudoscalar}}. \tag{4.141}$$

E.g. $B^- \to D^0 K^-$ decay: $\mathcal{H} \supset [\bar{b}(x)\gamma^\mu(1-\gamma_5)c(x)][\bar{u}(x)\gamma^\mu(1-\gamma_5)s(x)]$.

5 The canonical formalism

5.1 Canonical variables

We saw in Sec. 4.1 that

$$[\phi(x), \phi(y)] = \Delta(x-y)\,, \tag{5.1}$$

where

$$\Delta(x) = \int \frac{d^3k}{(2\pi)^3 2k^0}\left[e^{-ik\cdot x} - e^{+ik\cdot x}\right]. \tag{5.2}$$

We have $\Delta(0, \boldsymbol{x}) = 0$ and (the dot denotes a derivative w.r.t. time, d/dx^0)

$$\begin{aligned} \dot{\Delta}(x)\Big|_{x^0=0} &= \int \frac{d^3k}{(2\pi)^3 2k^0}\left[-ik^0 e^{-ik\cdot x} - ik^0 e^{+ik\cdot x}\right]\Big|_{x^0=0} \\ &= -\frac{i}{2}\int \frac{d^3k}{(2\pi)^3}\left[e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\right] = -i\delta^3(\boldsymbol{x})\,. \end{aligned} \tag{5.3}$$

Therefore, $q(\boldsymbol{x}, t) \equiv \phi(\boldsymbol{x}, t)$ and $p(\boldsymbol{x}, t) \equiv \dot{\phi}(\boldsymbol{x}, t)$ satisfy the *canonical commutation relations*

$$[q(\boldsymbol{x}, t), p(\boldsymbol{y}, t)] = i\delta^3(\boldsymbol{x}-\boldsymbol{y})\,, \tag{5.4}$$

$$[q(\boldsymbol{x}, t), q(\boldsymbol{y}, t)] = [p(\boldsymbol{x}, t), p(\boldsymbol{y}, t)] = 0\,. \tag{5.5}$$

For the Dirac field we have:

$$\begin{aligned} \{\psi(\boldsymbol{x}, t), \psi^\dagger(\boldsymbol{y}, t)\} &= \int \frac{d^3p d^3p'}{(2\pi)^3}\sum_{\sigma\sigma'}\Big[u(\boldsymbol{p}, \sigma)u^\dagger(\boldsymbol{p}', \sigma')e^{-ip\cdot x + ip'\cdot y'}\overbrace{\{a(\boldsymbol{p}, \sigma), a^\dagger(\boldsymbol{p}', \sigma')\}}^{\delta^3(\boldsymbol{p}-\boldsymbol{p}')\delta_{\sigma\sigma'}} \\ &\qquad + v(\boldsymbol{p}, \sigma)v^\dagger(\boldsymbol{p}', \sigma')e^{ip\cdot x - ip'\cdot y'}\underbrace{\{a^{c\dagger}(\boldsymbol{p}, \sigma), a^c(\boldsymbol{p}', \sigma')\}}_{\delta^3(\boldsymbol{p}-\boldsymbol{p}')\delta_{\sigma\sigma'}}\Big] \\ &= \int \frac{d^3p}{(2\pi)^3}\sum_\sigma\left[u(\boldsymbol{p}, \sigma)u^\dagger(\boldsymbol{p}, \sigma)e^{-i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} + v(\boldsymbol{p}, \sigma)v^\dagger(\boldsymbol{p}, \sigma)e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\right]. \end{aligned} \tag{5.6}$$

Calculate the spin sums:

$$\begin{aligned}
&\sum_\sigma u(\boldsymbol{p},\sigma)u^\dagger(\boldsymbol{p},\sigma)\\
&= \frac{m}{p^0}\sum_\sigma D(L(p))u(0,\sigma)\big(D(L(p))u(0,\sigma)\big)^\dagger\\
&= \frac{m}{p^0}D(L(p))\sum_\sigma u(0,\sigma)u(0,\sigma)^\dagger D(L(p))^\dagger\\
&\overset{(4.106)}{=} \frac{m}{2p^0}D(L(p))(1+\gamma^0)D(L(p))^\dagger\\
&= \frac{m}{2p^0}\big[D(L(p))\gamma^0 D^{-1}(L(p)) + D(L(p))D^{-1}(L(p))\big]\gamma^0\\
&= \frac{1}{2p^0}(\not{p}+m)\gamma^0\,.
\end{aligned} \tag{5.7}$$

(Here, we used $\beta D^\dagger = D^{-1}\beta$ and $1=\gamma^0\gamma^0$ in the second-to-last line, and the same "trick" as in the derivation of the Dirac equation in the last line.) Similarly, one shows

$$\sum_\sigma v(\boldsymbol{p},\sigma)v^\dagger(\boldsymbol{p},\sigma) = \frac{1}{2p^0}(\not{p}-m)\gamma^0\,. \tag{5.8}$$

Insert into Eq. (5.6):

$$\begin{aligned}
\{\psi(\boldsymbol{x},t),\psi^\dagger(\boldsymbol{y},t)\} &= \int\frac{d^3p}{(2\pi)^3}\frac{1}{2p^0}\big[(p^0\gamma^0-\boldsymbol{p}\cdot\boldsymbol{\gamma}+m)e^{-i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\\
&\qquad + (p^0\gamma^0-\boldsymbol{p}\cdot\boldsymbol{\gamma}-m)e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\big]\gamma^0\\
&\overset{\substack{\boldsymbol{p}\to-\boldsymbol{p}\\ \text{in 2nd term}}}{=} \int\frac{d^3p}{(2\pi)^3}\frac{1}{2p^0}\underbrace{\big[(p^0\gamma^0-\boldsymbol{p}\cdot\boldsymbol{\gamma}+m+p^0\gamma^0+\boldsymbol{p}\cdot\boldsymbol{\gamma}-m)\big]\gamma^0}_{2p^0}e^{-i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\\
&= \delta^3(\boldsymbol{x}-\boldsymbol{y})\,,
\end{aligned} \tag{5.9}$$

or, with indices explicit,

$$\{\psi_\ell(\boldsymbol{x},t), i\psi_k^\dagger(\boldsymbol{y},t)\} = i\delta^3(\boldsymbol{x}-\boldsymbol{y})\delta_{\ell k}\,. \tag{5.10}$$

The other anticommutators are (exercise)

$$\{\psi_\ell(\boldsymbol{x},t),\psi_k(\boldsymbol{y},t)\} = \{i\psi_\ell^\dagger(\boldsymbol{x},t), i\psi_k^\dagger(\boldsymbol{y},t)\} = 0\,. \tag{5.11}$$

Hence, we can regard $q=\Psi$, $p=i\Psi^\dagger$ as canonically conjugated variables.

All this carries over to interacting fields. One can show [1] that

$$Q^n(\boldsymbol{x},t) \equiv \exp(iHt)q^n(\boldsymbol{x},0)\exp(-iHt)\,, \tag{5.12}$$

$$P_n(\boldsymbol{x},t) \equiv \exp(iHt)p_n(\boldsymbol{x},0)\exp(-iHt)\,, \tag{5.13}$$

satisfy the same canonical (anti-)commutation relations.

The free Hamiltonian is always given by Eq. (3.15):

$$H_0 \equiv \sum_{n,\sigma}\int d^3k\, a^\dagger(\boldsymbol{k},\sigma,n)a(\boldsymbol{k},\sigma,n)\sqrt{\boldsymbol{k}^2+m_n^2}\,. \tag{5.14}$$

Example: scalar field

$$H_0 = \int d^3x \left[\tfrac{1}{2}p^2 + \tfrac{1}{2}(\nabla q)^2 + \tfrac{1}{2}m^2 q^2\right] \tag{5.15}$$

yields exactly Eq. (5.14) (exercise). Which free Lagrangian gives Eq. (5.15)? Legendre transformation:

$$L_0[q(t), \dot{q}(t)] = \sum_n \int d^3x p_n(\boldsymbol{x}, t)\dot{q}^n(\boldsymbol{x}, t) - H_0 \,. \tag{5.16}$$

Example: scalar field

$$L_0 = \int d^3x \left[p\dot{q} - \tfrac{1}{2}p^2 - \tfrac{1}{2}(\nabla q)^2 - \tfrac{1}{2}m^2 q^2\right] = \int d^3x \left[\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2\right]. \tag{5.17}$$

5.2 The Lagrangian formalism

Goal: formulate field theory with Lagrangians (symmetries!) The Lagrangian $L = L[\Psi(t), \dot{\Psi}(t)]$ is a functional of the fields and their time derivatives. The conjugated fields are defined as

$$\Pi_\ell(\boldsymbol{x}, t) \equiv \frac{\delta L[\Psi(t), \dot{\Psi}(t)]}{\delta\dot{\Psi}^\ell(t)} \,. \tag{5.18}$$

The *equations of motion (e.o.m.)* are

$$\dot{\Pi}_\ell(\boldsymbol{x}, t) = \frac{\delta L[\Psi(t), \dot{\Psi}(t)]}{\delta\Psi^\ell(t)} \,. \tag{5.19}$$

The e.o.m. follow from a *variational principle*: Define the *action* as

$$I[\Psi] \equiv \int_{-\infty}^{\infty} dt\, L[\Psi(t), \dot{\Psi}(t)] \,. \tag{5.20}$$

For an arbitrary variation $\delta\Psi$ with $\delta\Psi(\pm\infty) = 0$ we have

$$\begin{aligned} \delta I[\Psi] &= \int_{-\infty}^{\infty} dt \left[\frac{\delta L}{\delta\Psi^\ell(x)}\delta\Psi^\ell(x) + \frac{\delta L}{\delta\dot{\Psi}^\ell(x)}\delta\dot{\Psi}^\ell(x)\right] \\ &\overset{\text{P.I.}}{=} \int_{-\infty}^{\infty} dt \int d^3x \left[\frac{\delta L}{\delta\Psi^\ell(x)} - \frac{d}{dt}\frac{\delta L}{\delta\dot{\Psi}^\ell(x)}\right]\delta\Psi^\ell(x) \end{aligned} \tag{5.21}$$

(action principle). In Lorentz-invariant field theories, we write L as an integral over a *Lagrangian density* $\mathcal{L}$,

$$L[\Psi(t), \dot{\Psi}(t)] = \int d^3x\, \mathcal{L}\big(\Psi(\boldsymbol{x}, t), \nabla\Psi(\boldsymbol{x}, t), \dot{\Psi}(\boldsymbol{x}, t)\big) \,, \tag{5.22}$$

such that the action becomes ($\partial_\mu \equiv \partial/\partial x^\mu$)

$$I[\Psi] = \int d^4x\, \mathcal{L}\big(\Psi(x), \partial_\mu \Psi(x)\big)\,. \tag{5.23}$$

The variation of L under $\Psi \to \Psi + \delta\Psi$ becomes

$$\begin{aligned} \delta L &= \int d^3x \left[\frac{\partial \mathcal{L}}{\partial \Psi^\ell} \delta \Psi^\ell + \frac{\partial \mathcal{L}}{\partial(\nabla \Psi^\ell)} \nabla \delta \Psi^\ell + \frac{\partial \mathcal{L}}{\partial \dot{\Psi}^\ell} \delta \dot{\Psi}^\ell \right] \\ &= \int d^3x \left[\left(\frac{\partial \mathcal{L}}{\partial \Psi^\ell} \delta \Psi^\ell - \nabla \frac{\partial \mathcal{L}}{\partial(\nabla \Psi^\ell)} \right) \delta \Psi^\ell + \frac{\partial \mathcal{L}}{\partial \dot{\Psi}^\ell} \delta \dot{\Psi}^\ell \right], \end{aligned} \tag{5.24}$$

and comparing with Eq. (5.21) we get

$$\frac{\delta L}{\delta \Psi^\ell} = \frac{\partial \mathcal{L}}{\partial \Psi^\ell} \delta \Psi^\ell - \nabla \frac{\partial \mathcal{L}}{\partial(\nabla \Psi^\ell)} \tag{5.25}$$

$$\frac{\delta L}{\delta \dot{\Psi}^\ell} = \frac{\partial \mathcal{L}}{\partial \dot{\Psi}^\ell}\,. \tag{5.26}$$

The e.o.m. (5.19) are then the *Euler-Lagrange equations*

$$\boxed{\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi^\ell)} = \frac{\partial \mathcal{L}}{\partial \Psi^\ell}\,.} \tag{5.27}$$

The Hamiltonian is given by the Legendre transformation

$$H = \sum_\ell \int d^3x \Pi_\ell(\boldsymbol{x}, t) \dot{\Psi}^\ell(\boldsymbol{x}, t) - L[\Psi(t), \dot{\Psi}(t)]\,. \tag{5.28}$$

5.3 Global symmetries

Assume that the action (5.20) is invariant under the infinitesimal field transformation

$$\Psi^\ell(x) \to \Psi^\ell(x) + i\epsilon \mathcal{F}^\ell(x)\,, \tag{5.29}$$

i.e.

$$0 = \delta I = i\epsilon \int d^4x \frac{\delta I[\Psi]}{\delta \Psi^\ell(x)} \mathcal{F}^\ell(x)\,. \tag{5.30}$$

For ϵ a constant this is a *global symmetry*. (NB: The e.o.m. need not be satisfied, otherwise Eq. (5.30) is trivially satisfied.) Now consider $\epsilon = \epsilon(x)$. In order that $\delta I = 0$ for constant ϵ, we must have

$$\delta I = -\int d^4x \mathcal{J}^\mu(x) \frac{\partial \epsilon(x)}{\partial x^\mu}\,, \tag{5.31}$$

for some current $\mathcal{J}^\mu(x)$. If now the fields satisfy the e.o.m., we have $\delta I = 0$, and integrating by parts yields

$$0 = \frac{\partial \mathcal{J}^\mu(x)}{\partial x^\mu}\,. \tag{5.32}$$

This is *Noether's theorem*: symmetries imply conservation laws. For each conserved current $\mathcal{J}^\mu$, Gauss' theorem implies that

$$F \equiv \int d^3x\, \mathcal{J}^0 \tag{5.33}$$

is conserved:

$$0 = \frac{dF}{dt}\,. \tag{5.34}$$

If the Lagrangian density $\mathcal{L}$ is invariant under the transformation (5.29), we can calculate $\mathcal{J}^\mu$ explicitly.

If only the Lagrangian is invariant, we can obtain an explicit form for F. See Sec. 9.1.

The variation of the action, with $\epsilon = \epsilon(x)$, is

$$\delta I[\Psi] = i \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \Psi^\ell}\mathcal{F}^\ell(x)\epsilon(x) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi^\ell)}\partial_\mu(\mathcal{F}^\ell(x)\epsilon(x))\right]. \tag{5.35}$$

The invariance of $\mathcal{L}$ for constant ϵ requires

$$0 = \frac{\partial \mathcal{L}}{\partial \Psi^\ell}\mathcal{F}^\ell(x) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi^\ell)}\partial_\mu \mathcal{F}^\ell(x)\,, \tag{5.36}$$

hence the variation of I for arbitrary fields is

$$\delta I[\Psi] = i \int d^4x \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi^\ell)}\mathcal{F}^\ell(x)\partial_\mu \epsilon(x)\,. \tag{5.37}$$

Comparison with (5.31) yields

$$\mathcal{J}^\mu = -i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi^\ell)}\mathcal{F}^\ell\,. \tag{5.38}$$

6 Quantum electrodynamics

6.1 Gauge invariance

How can we construct a Lorentz-invariant interaction out of the fields (4.77), (4.124), (4.125)? Recall $A_\mu \to A_\mu + \partial_\mu \Omega$. Require that the action be invariant under the *gauge transformation*

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \epsilon(x)\,. \tag{6.1}$$

To this end, couple $A_\mu(x)$ to a conserved current:

$$I_{\mathrm{M}} \supset -\int d^4x\, \mathcal{J}^\mu A_\mu(x)\,. \tag{6.2}$$

Then under the transformation (6.1)

$$\delta I_{\mathrm{M}} = -\int d^4x\, \mathcal{J}^\mu \partial_\mu \epsilon(x) \overset{\text{P.I.}}{=} \int d^4x\, \partial_\mu \mathcal{J}^\mu \epsilon(x) = 0\,. \tag{6.3}$$

We have seen that the symmetry transformation $\psi(x) \to \psi(x) + \delta\psi(x)$, with

$$\delta\psi(x) = -i\epsilon\psi(x)\,, \tag{6.4}$$

yields a conserved current for constant ϵ, and for $\epsilon = \epsilon(x)$ (see Eq. (5.31))

$$\delta I_{\mathrm{M}} = \int d^4x\, \mathcal{J}^\mu \partial_\mu \epsilon(x)\,, \tag{6.5}$$

so we can couple A_μ to this current $\mathcal{J}^\mu$. In summary, the action must be invariant under the combined transformations

$$\delta A_\mu(x) = \partial_\mu \epsilon(x)\,, \tag{6.6}$$
$$\delta\psi(x) = -i\epsilon(x)e\psi(x)\,, \tag{6.7}$$

where we factored out the *electric charge* e. This is a *local* or *gauge symmetry.*

The antisymmetric tensor field

$$F_{\mu\nu}(x) \equiv \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) \tag{6.8}$$

is invariant under (6.1). We use (6.8) to construct the kinetic term of QED:

$$I_\gamma = -\frac{1}{4}\int d^4x F_{\mu\nu}(x)F^{\mu\nu}(x)\,. \tag{6.9}$$

6.2 Quantization of electrodynamics

The canonical variables are A_μ and

$$\Pi^\mu \equiv \frac{\partial\mathcal{L}}{\partial(\partial_0 A_\mu)}\,. \tag{6.10}$$

Only the kinetic term (6.9) depends on derivatives of A_μ, so

$$\begin{aligned}\Pi^\mu &= -\frac{1}{4}\frac{\partial}{\partial(\partial_0 A_\mu)}F^{\rho\sigma}F_{\rho\sigma} = -\frac{1}{2}F^{\rho\sigma}\frac{\partial}{\partial(\partial_0 A_\mu)}F_{\rho\sigma} \\ &= -\frac{1}{2}F^{\rho\sigma}\frac{\partial}{\partial(\partial_0 A_\mu)}(\partial_\rho A_\sigma - \partial_\sigma A_\rho) = -\frac{1}{2}F^{\rho\sigma}(\delta^0_\rho\delta^\mu_\sigma - \delta^0_\sigma\delta^\mu_\rho) \\ &= -F^{0\mu}\,.\end{aligned} \tag{6.11}$$

The antisymmetry of $F^{\mu\nu}$ implies

$$\Pi^0(x) \equiv 0\,. \tag{6.12}$$

Similarly, one can show

$$\partial_i \Pi^i = \partial_i \frac{\partial \mathcal{L}}{\partial(\partial_0 A_i)} = -\partial_i \frac{\partial \mathcal{L}}{\partial(\partial_i A_0)} \overset{(5.27)}{=} -\frac{\partial \mathcal{L}}{\partial A_0} \overset{(6.2)}{=} \mathcal{J}^0 \,. \tag{6.13}$$

Eq.s (6.12) and (6.13) are not consistent with the canonical commutation relations

$$[A_\mu(\boldsymbol{x}, t), \Pi^\nu(\boldsymbol{y}, t)] = i\delta^\nu_\mu \delta^3(\boldsymbol{x} - \boldsymbol{y}) \,, \qquad [\psi(\boldsymbol{x}, t), \Pi^\nu(\boldsymbol{y}, t)] = 0 \,. \tag{6.14}$$

Possible solution: *quantization in Coulomb gauge*. Choose A_μ such that

$$\nabla \cdot \boldsymbol{A} = 0 \,. \tag{6.15}$$

This is always possible: If $\nabla \cdot \boldsymbol{A} \neq 0$, perform a field transformation $A^\mu \to A'^\mu = A^\mu + \partial^\mu \lambda$, with $\nabla^2 \lambda = -\nabla \cdot \boldsymbol{A}$. It follows that $\nabla \cdot \boldsymbol{A'} = 0$.

Eq. (6.13) implies that

$$\mathcal{J}^0 = \partial_i \Pi^i = \ldots = \partial_i F^{i0} = \partial_i(\partial^i A^0 - \partial^0 A^i) \overset{(6.15)}{=} -\nabla^2 A^0 \,, \tag{6.16}$$

with solution (recall $\nabla^2 1/|\boldsymbol{x} - \boldsymbol{y}| = -4\pi\delta^3(\boldsymbol{x} - \boldsymbol{y})$)

$$A^0(\boldsymbol{x}, t) = \int d^3y \, \frac{\mathcal{J}^0(\boldsymbol{y}, t)}{4\pi|\boldsymbol{x} - \boldsymbol{y}|} \,. \tag{6.17}$$

The other three d.o.f. are subject to the constraint $\nabla \cdot \boldsymbol{A} = 0$. The commutation relations consistent with Eq. (6.15) are [Dirac 1964]

$$[A^i(\boldsymbol{x}, t), \Pi_j(\boldsymbol{y}, t)] = i\delta^\nu_\mu \delta^3(\boldsymbol{x} - \boldsymbol{y}) + i \frac{\partial^2}{\partial x^j \partial x_i} \frac{1}{4\pi|\boldsymbol{x} - \boldsymbol{y}|} \,, \tag{6.18}$$

$$[A^i(\boldsymbol{x}, t), A^j(\boldsymbol{y}, t)] = [\Pi_i(\boldsymbol{x}, t), \Pi_j(\boldsymbol{y}, t)] = 0 \,. \tag{6.19}$$

To solve the problem $[\Psi, \Pi] \neq 0$, $[\Psi^\dagger, \Pi] \neq 0$, we express the Hamiltonian in terms of

$$\boldsymbol{\Pi}_\perp \equiv \boldsymbol{\Pi} - \nabla A^0 = \dot{\boldsymbol{A}} \tag{6.20}$$

(here, we have used $\Pi_j = \partial \mathcal{L}/\partial \dot{A}^j = \dot{A}^j + \partial^j A^0$). It follows that

$$\nabla \cdot \boldsymbol{\Pi}_\perp = 0 \,. \tag{6.21}$$

It is now straightforward to check that

$$[\psi(\boldsymbol{x}, t), \boldsymbol{\Pi}_\perp(\boldsymbol{y}, t)] = [\psi^\dagger(\boldsymbol{x}, t), \boldsymbol{\Pi}_\perp(\boldsymbol{y}, t)] = 0 \,, \tag{6.22}$$

and that $\boldsymbol{\Pi}_\perp$ satisfies the commutation relations (6.18).

6.3 Electrodynamics in the interaction picture

The Lagrangian density for (spinor) electrodynamics is now

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi + \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \mathcal{J}_\mu A^\mu \,, \tag{6.23}$$

with

$$\mathcal{J}^\mu = e\bar{\psi}(x)\gamma^\mu\psi(x) \,. \tag{6.24}$$

A Legendre transformation yields the Hamiltonian:

$$H = \int d^3x \left[\Pi_{\perp i}\dot{A}^i + i\psi^\dagger\dot{\psi} - \mathcal{L}\right] . \tag{6.25}$$

We use Eq. (6.20) to replace $\dot{\boldsymbol{A}}$ by $\boldsymbol{\Pi}_\perp$:

$$H = \int d^3x \left[\boldsymbol{\Pi}_\perp^2 + \tfrac{1}{2}(\nabla\times\boldsymbol{A})^2 - \tfrac{1}{2}(\boldsymbol{\Pi}_\perp - \nabla A^0)^2 - \boldsymbol{\mathcal{J}}\cdot\boldsymbol{A} + \mathcal{J}^0 A^0\right] + H_{\text{M},0} \,, \tag{6.26}$$

where $H_{\text{M},0}$ is the free Dirac Hamiltonian. Now use $\nabla\cdot\boldsymbol{\Pi}_\perp = 0$ in the third term, $(\nabla A^0)^2 = -A^0\nabla^2 A^0$ (integration by parts), and $\mathcal{J}^0 = -\nabla^2 A^0$ (6.16), to find

$$H = \int d^3x \left[\tfrac{1}{2}\boldsymbol{\Pi}_\perp^2 + \tfrac{1}{2}(\nabla\times\boldsymbol{A})^2 - \boldsymbol{\mathcal{J}}\cdot\boldsymbol{A} + \tfrac{1}{2}\mathcal{J}^0 A^0\right] + H_{\text{M},0} \,. \tag{6.27}$$

The last term in brackets is the Coulomb energy,

$$V_{\text{Coul}} = \frac{1}{2}\int d^3x\, \mathcal{J}^0 A^0 \overset{(6.17)}{=} \frac{1}{2}\int d^3x \int d^3y\, \frac{\mathcal{J}^0(\boldsymbol{x},t)\mathcal{J}^0(\boldsymbol{y},t)}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} \,. \tag{6.28}$$

For the transition to the interaction picture, we separate

$$H = H_0 + V \,, \tag{6.29}$$

with

$$H_0 = \int d^3x \left[\tfrac{1}{2}\boldsymbol{\Pi}_\perp^2 + \tfrac{1}{2}(\nabla\times\boldsymbol{A})^2\right] + H_{\text{M},0} \,, \tag{6.30}$$

$$V = -\int d^3x\, \boldsymbol{\mathcal{J}}\cdot\boldsymbol{A} + V_{\text{Coul}} \,. \tag{6.31}$$

Since H is time independent, we can evaluate Eq.s (6.30), (6.31) at $t = 0$. Then we have in the interaction picture

$$V_I(t) = \exp(iH_0 t)V\exp(-iH_0 t) \,. \tag{6.32}$$

We will drop the index I. It is straightforward to see that the fields in the interaction picture satisfy the commutation relations (6.18), as well as

$$\nabla\cdot\boldsymbol{A} = 0 \,, \qquad \nabla\cdot\boldsymbol{\Pi} = 0 \,. \tag{6.33}$$

(I dropped the $\perp$ label here.) Moreover, one can show [1] that

$$\dot{\boldsymbol{A}} = \boldsymbol{\Pi} \tag{6.34}$$

and

$$\Box \boldsymbol{A} \equiv \left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)\boldsymbol{A} = 0\,. \tag{6.35}$$

Since A^0 is not an independent variable, we do not introduce a corresponding operator A^0, and choose $A^0 = 0$ instead (A^0 can always be calculated using Eq. (6.17)). Then we can choose A^μ exactly as the field (4.77). Explicitly, we have now

$$V(t) = \int d^3x \mathcal{J}_\mu(\boldsymbol{x}, t) A^\mu(\boldsymbol{x}, t) + V_{\text{Coul}}\,, \tag{6.36}$$

where

$$\mathcal{J}^\mu(\boldsymbol{x}, t) = \exp(iH_0 t)\mathcal{J}^\mu(\boldsymbol{x}, 0)\exp(-iH_0 t) \tag{6.37}$$

and

$$V_{\text{Coul}}(t) = \frac{1}{2}\int d^3x \int d^3y \,\frac{\mathcal{J}^0(\boldsymbol{x}, t)\mathcal{J}^0(\boldsymbol{y}, t)}{4\pi|\boldsymbol{x} - \boldsymbol{y}|}\,. \tag{6.38}$$

(NB: $\mathcal{J}_\mu A^\mu = -\boldsymbol{\mathcal{J}} \cdot \boldsymbol{A}$ since $A^0 = 0$.)

7 Elementary processes of QED and Feynman rules

7.1 Propagators

As a preparation, we calculate the propagators.

The propagator for the Dirac field

The propagator is defined as the vacuum expectation value of the time-ordered product

$$\begin{aligned} &\langle 0|T\{\psi_l(x)\bar{\psi}_m(y)\}|0\rangle \\ &\quad = \theta(x^0 - y^0)\langle 0|\psi_l(x)\bar{\psi}_m(y)|0\rangle - \theta(y^0 - x^0)\langle 0|\bar{\psi}_m(y)\psi_l(x)|0\rangle\,. \end{aligned} \tag{7.1}$$

Consider the first term:

$$\begin{aligned} &\langle 0|\psi_l(x)\bar{\psi}_m(y)|0\rangle \\ &= \langle 0|\int \frac{d^3p d^3p'}{(2\pi)^3}\sum_{\sigma\sigma'}\Big[\big(u_l(\boldsymbol{p}, \sigma)e^{-ip\cdot x}a(\boldsymbol{p}, \sigma) + v_l(\boldsymbol{p}, \sigma)e^{ip\cdot x}a^{c\dagger}(\boldsymbol{p}, \sigma)\big) \\ &\qquad \times \big(\bar{v}_l(\boldsymbol{p}', \sigma')e^{-ip'\cdot y}a^c(\boldsymbol{p}', \sigma') + \bar{u}_l(\boldsymbol{p}', \sigma')e^{ip'\cdot y}a^\dagger(\boldsymbol{p}', \sigma')\big)\Big]|0\rangle\,. \end{aligned} \tag{7.2}$$

Only the term with $aa^\dagger$ survives. We use

$$\langle 0|a(\boldsymbol{p}, \sigma)a^\dagger(\boldsymbol{p}', \sigma')|0\rangle = -0 + \delta^3(\boldsymbol{p} - \boldsymbol{p}')\delta_{\sigma\sigma'} \tag{7.3}$$

to find

$$\langle 0|\psi_l(x)\bar{\psi}_m(y)|0\rangle = \int \frac{d^3p}{(2\pi)^3} e^{-ip\cdot(x-y)} \sum_\sigma u_l(\boldsymbol{p},\sigma)\bar{u}_m(\boldsymbol{p}',\sigma') \\ \overset{(5.7)}{=} \int \frac{d^3p}{(2\pi)^3} \frac{(\not{p}+m)_{lm}}{2p^0} e^{-ip\cdot(x-y)} \,. \tag{7.4}$$

In complete analogy we find

$$\langle 0|\bar{\psi}_m(y)\psi_l(x)|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{(-\not{p}+m)_{lm}}{2p^0} e^{ip\cdot(x-y)} \,. \tag{7.5}$$

Thus, we can write the propagator as

$$\langle 0|T\{\psi_l(x)\bar{\psi}_m(y)\}|0\rangle \\ = \theta(x^0-y^0)(i\not{\partial}+m)\Delta_+(x-y) + \theta(y^0-x^0)(i\not{\partial}+m)\Delta_+(y-x) \,. \tag{7.6}$$

Now we use

$$\frac{\partial}{\partial x^0}\theta(x^0-y^0) = -\frac{\partial}{\partial x^0}\theta(y^0-x^0) = \delta(x^0-y^0) \tag{7.7}$$

to move the time derivatives in Eq, (7.6) past the θ functions:

$$\langle 0|T\{\psi_l(x)\bar{\psi}_m(y)\}|0\rangle \\ = (i\not{\partial}+m)\big[\theta(x^0-y^0)\Delta_+(x-y) + \theta(y^0-x^0)\Delta_+(y-x)\big] \\ - i\gamma^0\delta(x^0-y^0)\big[\Delta_+(x-y) - \Delta_+(y-x)\big] \,. \tag{7.8}$$

For $x^0 = 0$, $\Delta_+(x)$ is even in x (compensate $\boldsymbol{x} \to -\boldsymbol{x}$ by $\boldsymbol{p} \to -\boldsymbol{p}$), hence the terms in the second line cancel, and we have

$$\langle 0|T\{\psi_l(x)\bar{\psi}_m(y)\}|0\rangle = (i\not{\partial}+m)i\Delta_F(x-y) \,, \tag{7.9}$$

with the *Feynman propagator*

$$i\Delta_F(x) \equiv \theta(x^0)\Delta_+(x) + \theta(-x^0)\Delta_+(-x) \,. \tag{7.10}$$

We can rewrite this as follows, using the Fourier representation of the θ function

$$\theta(t) = \frac{-1}{2\pi i}\int\limits_{-\infty}^{\infty} ds \, \frac{\exp(-ist)}{s+i\epsilon} \,. \tag{7.11}$$

(This can be seen as follows. The integrand has a simple pole at $s = -i\epsilon$. For $t > 0$, the integrand converges for $\mathrm{I}m(s) < 0$, so we can close the contour in the lower half plane. The residue theorem gives $-2\pi i$ for the integral. For $T < 0$ we can close the contour in the upper half plane, and the residue theorem gives zero. See Fig. 1.)

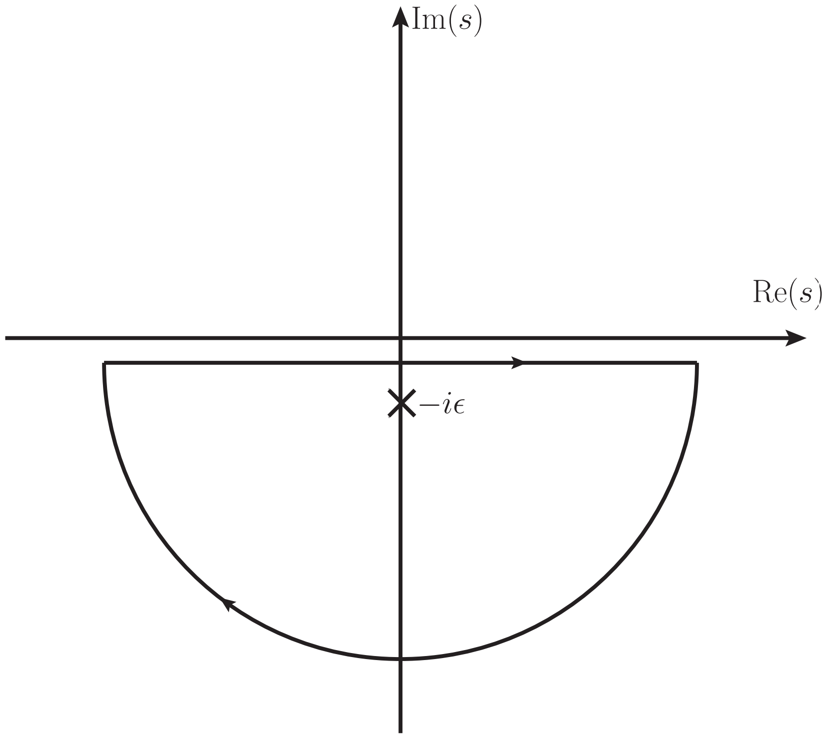


Figure 1: *Fourier representation of the θ function.*

Insert Eq. (7.11) into the Feynman propagator:

$$i\Delta_F(x) = -\frac{1}{2\pi i}\int \frac{d^3p}{(2\pi)^3}\int_{-\infty}^{\infty} ds \frac{1}{2p^0}\left[\frac{\exp(-ip\cdot x - isx^0)}{s+i\epsilon} + \frac{\exp(ip\cdot x + isx^0)}{s+i\epsilon}\right]. \tag{7.12}$$

Now we perform a change of integration variables: $\boldsymbol{q}=\boldsymbol{p}$, $q^0 = p^0 + s$ in the first term, and $\boldsymbol{q} = -\boldsymbol{p}$, $q^0 = -p^0 - s$ in the second term. We then find

$$\begin{aligned} i\Delta_F(x) = &-\frac{1}{2\pi i}\int_{-\infty}^{\infty} dq^0 \int \frac{d^3p}{(2\pi)^3}\frac{\exp(i\boldsymbol{q}\cdot\boldsymbol{x} - iq^0x^0)}{2\sqrt{\boldsymbol{q}^2+m^2}} \\ &\times \underbrace{\left[\frac{1}{q^0 - \sqrt{\boldsymbol{q}^2+m^2}+i\epsilon} + \frac{1}{-q^0-\sqrt{\boldsymbol{q}^2+m^2}+i\epsilon}\right]}_{\frac{-2\sqrt{\boldsymbol{q}^2+m^2}}{-(q^0)^2+\boldsymbol{q}^2+m^2-i\epsilon 2\sqrt{\boldsymbol{q}^2+m^2}+\mathcal{O}(\epsilon^2)}}, \end{aligned} \tag{7.13}$$

and so ($q^2 = (q^0)^2 - \boldsymbol{q}^2$)

$$\Delta_F(x) = \int \frac{d^4q}{(2\pi)^4}\frac{e^{-iq\cdot x}}{q^2 - m^2 + i\epsilon}. \tag{7.14}$$

Hence, the Dirac propagator is

$$\langle 0|T\{\psi(x)\bar{\psi}(y)\}|0\rangle = \int \frac{d^4q}{(2\pi)^4}\frac{i(\not{q}+m)}{q^2-m^2+i\epsilon}e^{-iq\cdot x}\,. \tag{7.15}$$

The photon propagator

Now we want to calculate the *photon propagator*

$$i\Delta_{\mu\nu}(x-y) \equiv \langle 0|T\{A_\mu(x)A_\nu(y)\}|0\rangle\,. \tag{7.16}$$

In analogy to our previous calculation we find

$$i\Delta_{\mu\nu}(x-y) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2|\boldsymbol{p}|}P_{\mu\nu}(\boldsymbol{p})\left[\theta(x^0-y^0)e^{-ip\cdot(x-y)}+\theta(y^0-x^0)e^{ip\cdot(x-y)}\right]. \tag{7.17}$$

Here,

$$P_{\mu\nu}(\boldsymbol{p}) \equiv \sum_{\sigma=\pm 1} e_\mu(\boldsymbol{p},\sigma)e^*_\nu(\boldsymbol{p},\sigma)\,, \tag{7.18}$$

where as usual $p^0 \equiv |\boldsymbol{p}|$, and with Eq.s (4.80), (4.86)

$$\begin{aligned} P_{ij}(\boldsymbol{p}) &= \delta_{ij} - \frac{p_ip_j}{|\boldsymbol{p}|^2}\,,\\ P_{0i}(\boldsymbol{p}) &= P_{i0}(\boldsymbol{p}) = P_{00}(\boldsymbol{p}) = 0\,. \end{aligned} \tag{7.19}$$

Using again Eq. (7.11), we get

$$\Delta_{\mu\nu}(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{P_{\mu\nu}(\boldsymbol{p})}{q^2+i\epsilon}e^{-iq\cdot(x-y)}\,. \tag{7.20}$$

This can be brought into nicer form (later: path integral!). Write

$$P_{\mu\nu}(\boldsymbol{q}) = -\eta_{\mu\nu} + \frac{q_0q_\mu n_\nu + q_0q_\nu n_\mu - q_\mu q_\nu - q^2n_\mu n_\nu}{|\boldsymbol{q}|^2}\,, \tag{7.21}$$

with $n^\mu = (1,0,0,0)$ and $q^2 = (q^0)^2 - |\boldsymbol{q}|^2$. Since we always couple the photon fields to conserved currents, i.e. $q_\mu\mathcal{J}^\mu = 0$, the first three terms in the numerator never contribute to a physical transition amplitude. The fourth term gives a contribution to the S-matrix of the form

$$-\frac{1}{2}\int d^4x\int d^4y\mathcal{J}^0(x)\mathcal{J}^0(y)\int\frac{d^4q}{|\boldsymbol{q}|^2}e^{-iq\cdot(x-y)}\,.$$

Integrating over dq^0 gives $\delta(x^0-y^0)$, and the Fourier integral then yields a correction to the interaction Hamiltonian $V(t)$ of the form

$$-\frac{1}{2}\int d^3x\int d^3y\frac{\mathcal{J}^0(\boldsymbol{x},t)\mathcal{J}^0(\boldsymbol{y},t)}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\,.$$

This exactly cancels the Coulomb interaction (6.38). In summary, we can just drop Eq. (6.38) and use the covariant expression for the photon propagator

$$\Delta_{\mu\nu}(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{-i\eta_{\mu\nu}}{q^2+i\epsilon}e^{-iq\cdot(x-y)}\,. \tag{7.22}$$

7.2 $e^+e^- \to \mu^+\mu^-$

Since V_{Coul} effectively does not contribute, our interaction density is (cf. Eq. (6.36))

$$\mathcal{H}(x) = \mathcal{J}_\mu(x) A^\mu(x)\,, \tag{7.23}$$

with the current

$$\mathcal{J}_\rho(x) = e\,\bar{\psi}_e \gamma_\rho \psi_e(x) + e\,\bar{\psi}_\mu \gamma_\rho \psi_\mu(x)\,. \tag{7.24}$$

We insert this into the S-matrix (2.70). Now consider an e^+e^- initial state, with momenta $\boldsymbol{p}$, $\boldsymbol{p}'$ and spins s, s', and an $\mu^+\mu^-$ final state, with momenta $\boldsymbol{k}$, $\boldsymbol{k}'$ and spins r, r'. I.e. our "in state" is

$$|\boldsymbol{p}, s; \boldsymbol{p}', s'\rangle = a^\dagger(\boldsymbol{p}, s, e^-) a^\dagger(\boldsymbol{p}', s', e^+)|0\rangle\,, \tag{7.25}$$

and the "out state" is

$$\langle \boldsymbol{k}, r; \boldsymbol{k}', r'| = \langle 0| a(\boldsymbol{k}, r, \mu^-) a(\boldsymbol{k}', r', \mu^+)\,. \tag{7.26}$$

By a suitable choice of the vacuum energy we can write all creation operators to the left of all annihilation operators in Eq. (7.24); this is called *normal ordering*, in symbols $:\mathcal{H}(x):$.

It is easy to see that, in our case, the first non-vanishing term in (2.70) is $n = 2$. Therefore, we need to calculate

$$\begin{aligned} S_{\beta\alpha} &= -2\pi i \delta^4(p_\beta - p_\alpha) M_{\beta\alpha} \\ &= \frac{(-i)^2}{2!} \int d^4x \int d^4y \langle 0| a(\boldsymbol{k}, r, \mu^-) a(\boldsymbol{k}', r', \mu^+) \\ &\qquad \times e^2 T\big\{ \big(:\bar{\psi}_e(x) \not{A}(x) \psi_e(x): + :\bar{\psi}_\mu(x) \not{A}(x) \psi_\mu(x): \big) \\ &\qquad\qquad \times \big(:\bar{\psi}_e(y) \not{A}(y) \psi_e(y): + :\bar{\psi}_\mu(y) \not{A}(y) \psi_\mu(y): \big)\big\} \\ &\qquad \times a^\dagger(\boldsymbol{p}, s, e^-) a^\dagger(\boldsymbol{p}', s', e^+)|0\rangle\,. \end{aligned} \tag{7.27}$$

Our strategy is, as usual, to move all annihilation operators to the right, until the annihilate the vacuum state. The only non-zero left-over terms originate from the arising delta functions. Reminder:

$$\begin{aligned} \bar{\psi}_e &\sim a^\dagger(e^-) + a(e^+)\,, \\ \psi_e &\sim a(e^-) + a^\dagger(e^+)\,, \\ \bar{\psi}_\mu &\sim a^\dagger(\mu^-) + a(\mu^+)\,, \\ \psi_\mu &\sim a(\mu^-) + a^\dagger(\mu^+)\,. \end{aligned}$$

We denote the "(anti-)commutation" of an annihilation-creation-pair as a *contraction*; notation: $\ldots \overbrace{a \ldots a^\dagger} \ldots$. The following contractions contribute to the matrix element in (7.27):

$$\langle 0| a(\mu^-) a(\mu^+) T\big\{ \big(\bar{\psi}_e \not{A} \psi_e + \bar{\psi}_\mu \not{A} \psi_\mu \big)(x) \big(\bar{\psi}_e \not{A} \psi_e + \bar{\psi}_\mu \not{A} \psi_\mu \big)(y) \big\} a^\dagger(e^-) a^\dagger(e^+)|0\rangle\,,$$

and

$$\langle 0| a(\mu^-) a(\mu^+) T\big\{ \big(\bar{\psi}_e \not{A} \psi_e + \bar{\psi}_\mu \not{A} \psi_\mu \big)(x) \big(\bar{\psi}_e \not{A} \psi_e + \bar{\psi}_\mu \not{A} \psi_\mu \big)(y) \big\} a^\dagger(e^-) a^\dagger(e^+)|0\rangle\,.$$

(In addition, we need to contract the corresponding photon fields.)

The contraction of the external states with the fields gives (we write only the relevant fields, for simplicity)

$$
\begin{aligned}
&\langle 0|\overline{a(\boldsymbol{k}, r, \mu^-)\bar{\psi}}_\mu(x)\dots|0\rangle \\
&\overset{(4.124)}{=}\langle 0|a(\boldsymbol{k}, r, \mu^-)\int\frac{d^3p}{(2\pi)^{3/2}}\sum_\sigma\left(\bar{u}(\boldsymbol{p},\sigma)e^{ip\cdot x}a^\dagger(\boldsymbol{p},\sigma)+\dots\right)\dots|0\rangle \\
&\overset{(3.10),(3.11)}{=} -0+\langle 0|\int\frac{d^3p}{(2\pi)^{3/2}}\sum_\sigma\bar{u}(\boldsymbol{p},\sigma)e^{ip\cdot x}\delta^3(\boldsymbol{k}-\boldsymbol{p})\delta_{r\sigma}\dots|0\rangle \\
&=\frac{\bar{u}(\boldsymbol{k}, r, \mu^-)}{(2\pi)^{3/2}}e^{ik\cdot x}\langle 0|\dots|0\rangle\,.
\end{aligned}
\tag{7.28}
$$

Similarly, we find

$$
\langle 0|a(\boldsymbol{k}', r', \mu^+)\psi_\mu(x)|0\rangle = \frac{v(\boldsymbol{k}', r', \mu^+)}{(2\pi)^{3/2}}e^{ik'\cdot x}\,, \tag{7.29}
$$

$$
\langle 0|\psi_e(y)a^\dagger(\boldsymbol{p}, s, e^-)|0\rangle = \frac{u(\boldsymbol{p}, s, e^-)}{(2\pi)^{3/2}}e^{-ip\cdot y}\,, \tag{7.30}
$$

$$
\langle 0|\psi_e(y)a(\boldsymbol{p}', s', e^+)|0\rangle = \frac{\bar{v}(\boldsymbol{p}', s', e^+)}{(2\pi)^{3/2}}e^{-ip'\cdot x}\,. \tag{7.31}
$$

In total, we get the two terms

$$
\begin{aligned}
S_{\beta\alpha} &= -\frac{e^2}{2}\int d^4x\int d^4y\frac{1}{(2\pi)^6} \\
&\quad\times(\bar{u}(\boldsymbol{k}, r)\gamma_\mu v(\boldsymbol{k}', r'))(\bar{v}(\boldsymbol{p}', s')\gamma_\nu u(\boldsymbol{p}, s)) \\
&\quad\times\Big[e^{ik\cdot x+ik'\cdot x-ip\cdot y-ip'\cdot y}\,\langle 0|T\{A^\mu(x)A^\nu(y)\psi_e(x)\}|0\rangle \\
&\qquad+e^{ik\cdot y+ik'\cdot y-ip\cdot x-ip'\cdot x}\,\langle 0|T\{A^\mu(y)A^\nu(x)\psi_e(x)\}|0\rangle\Big] \\
&\overset{(7.22)}{=}-\frac{e^2}{(2\pi)^6}(\bar{u}(\boldsymbol{k}, r)\gamma_\mu v(\boldsymbol{k}', r'))(\bar{v}(\boldsymbol{p}', s')\gamma_\nu u(\boldsymbol{p}, s)) \\
&\quad\times\int d^4x\int d^4y\int\frac{d^4q}{(2\pi)^4}\frac{-i\eta_{\mu\nu}}{q^2+i\epsilon}e^{-iq\cdot(x-y)}e^{i(k+k')\cdot x}e^{-i(p+p')\cdot y} \\
&=\frac{ie^2}{(2\pi)^6}(\bar{u}(\boldsymbol{k}, r)\gamma_\mu v(\boldsymbol{k}', r'))(\bar{v}(\boldsymbol{p}', s')\gamma^\mu u(\boldsymbol{p}, s)) \\
&\quad\times\int\frac{d^4q}{(2\pi)^4}\frac{1}{q^2+i\epsilon}(2\pi)^4\delta^4(-q+k+k')(2\pi)^4\delta^4(q-p-p') \\
&=\frac{ie^2}{(2\pi)^2}\frac{(\bar{u}(\boldsymbol{k}, r)\gamma_\mu v(\boldsymbol{k}', r'))(\bar{v}(\boldsymbol{p}', s')\gamma^\mu u(\boldsymbol{p}, s))}{(p+p')^2}\delta^4(k+k'-p-p') \\
&\equiv -2\pi i\delta^4(k+k'-p-p')M_{\beta\alpha}\,.
\end{aligned}
\tag{7.32}
$$

where in the second equality we used that the expression is symmetric under the exchange $x\leftrightarrow y$.

Next, we would like to calculate $|M_{\beta\alpha}|^2$. As a preparation, we need the following relation for the Hermitian conjugate of the Dirac matrices (exercise!):

$$(\gamma^\mu)^\dagger = \beta\gamma^\mu\beta\,, \qquad \mu = 0,1,2,3\,. \tag{7.33}$$

Our matrix element is

$$M_{\beta\alpha} = -\frac{e^2}{(2\pi)^3}\frac{1}{(p+p')^2}(\bar{u}(\boldsymbol{k},r)\gamma_\mu v(\boldsymbol{k}',r'))(\bar{v}(\boldsymbol{p}',s')\gamma^\mu u(\boldsymbol{p},s))\,. \tag{7.34}$$

We have

$$(\bar{u}\gamma_\mu v)^* = (u^\dagger\beta\gamma_\mu v)^\dagger = v^\dagger\gamma_\mu^\dagger\beta^\dagger u = v^\dagger\beta\gamma_\mu\beta\beta u = \bar{v}\gamma_\mu u\,, \tag{7.35}$$

and similarly

$$(\bar{v}\gamma_\mu u)^* = \bar{u}\gamma_\mu v\,. \tag{7.36}$$

Therefore, we find

$$\begin{aligned}|M_{\beta\alpha}|^2 = \frac{e^4}{(2\pi)^6}\frac{1}{((p+p')^2)^2} \quad &(\bar{u}(\boldsymbol{k},r)\gamma_\mu v(\boldsymbol{k}',r'))(\bar{v}(\boldsymbol{k}',r')\gamma_\nu u(\boldsymbol{k},r)) \\ &\times (\bar{v}(\boldsymbol{p}',s')\gamma^\mu u(\boldsymbol{p},s))(\bar{u}(\boldsymbol{p},s)\gamma^\nu v(\boldsymbol{p}',s'))\,.\end{aligned} \tag{7.37}$$

In realistic experiments one frequently does not know the polarization of the electrons in the initial state. If we are not interested in the polarization of the muons in the final state, we average the cross section over the spin states of the initial-state particles and sum over the spin states in the final state. Hence, we need to calculate

$$\tfrac{1}{2}\sum_s \tfrac{1}{2}\sum_{s'}\sum_{rr'}|M_{\beta\alpha}|^2\,.$$

We insert (7.37) and use the spin sums (5.7), (5.8). The first factor in Eq. (7.37) gives (writing the spinor indices explicitly):

$$\begin{aligned}&\sum_{rr'}\sum_{abcd}(\bar{u}_a(\boldsymbol{k},r)(\gamma_\mu)_{ab}v_b(\boldsymbol{k}',r')\bar{v}_c(\boldsymbol{k}',r')(\gamma_\nu)_{cd}u_d(\boldsymbol{k},r)) \\ =&\sum_{abcd}\left[(\not{k}+m_\mu)_{da}(\gamma_\mu)_{ab}(\not{k}'-m_\mu)_{bc}(\gamma_\nu)_{cd}\right]\frac{1}{2k^0 2k'^0} \\ =&\frac{1}{4k^0k'^0}\mathrm{Tr}\left\{(\not{k}+m_\mu)\gamma_\mu(\not{k}'-m_\mu)\gamma_\nu\right\}\,.\end{aligned} \tag{7.38}$$

Similar for the second factor in (7.37). In total, we have for the "summed and averaged" absolute value squared,

$$\overline{|M|^2} \equiv \tfrac{1}{4}\sum_{\text{spins}}|M|^2\,, \tag{7.39}$$

writing $p + p' = q$,

$$\overline{|M|^2} = \frac{1}{4}\frac{e^4}{(2\pi)^6}\frac{1}{q^4}\frac{1}{16k^0k'^0p^0p'^0}\mathrm{Tr}\left\{(\not{k}+m_\mu)\gamma_\mu(\not{k}'-m_\mu)\gamma_\nu\right\}\mathrm{Tr}\left\{\not{p}'\gamma^\mu\not{p}\gamma^\nu\right\} \tag{7.40}$$

where we neglected the electron mass (the electrons in this process are always ultra-relativistic, since $m_\mu \gg m_e$). Next we need to evaluate the traces. We have

$$\mathrm{Tr}\{\gamma^\mu\} = \mathrm{Tr}\{\gamma^\mu\gamma_5\gamma_5\} = -\mathrm{Tr}\{\gamma_5\gamma^\mu\gamma_5\} = -\mathrm{Tr}\{\gamma^\mu\gamma_5\gamma_5\} = -\mathrm{Tr}\{\gamma^\mu\} \tag{7.41}$$

and so $\mathrm{Tr}\{\gamma^\mu\} = 0$. In general, the trace of a product of an odd number of Dirac matrices vanishes. Furthermore,

$$\mathrm{Tr}\{\gamma^\mu\gamma^\nu\} = \tfrac{1}{2}\mathrm{Tr}\{\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\} = \tfrac{1}{2}\cdot 2\cdot \eta^{\mu\nu}\mathrm{Tr}\{\mathbb{1}\} = 4\eta^{\mu\nu}\,. \tag{7.42}$$

Similarly, one can show that

$$\mathrm{Tr}\{\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\} = 4\big(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}\big)\,. \tag{7.43}$$

Using these results, we get

$$\mathrm{Tr}\big\{\not{p}'\gamma^\mu\not{p}\gamma^\nu\big\} = p'_\rho p_\sigma \mathrm{Tr}\big\{\gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\nu\big\} = 4(p'^\mu p^\nu - (p'\cdot p)\,\eta^{\mu\nu} + p'^\nu p^\mu) \tag{7.44}$$

and

$$\mathrm{Tr}\big\{(\not{k} + m_\mu)\gamma_\mu(\not{k}' - m_\mu)\gamma_\nu\big\} = -4m_\mu^2\eta_{\mu\nu} + 4(k'_\mu k_\nu - (k'\cdot k)\,\eta_{\mu\nu} + k'_\nu k_\mu)\,. \tag{7.45}$$

Inserting this into (7.40) yields

$$\begin{aligned}\overline{|M|^2} = &\frac{1}{4}\frac{e^4}{(2\pi)^6}\frac{1}{q^4}\frac{1}{16k^0k'^0p^0p'^0}\\ &\times 16\left[2m_\mu^2(p'\cdot p) + 2(p'\cdot k')(p\cdot k) + 2(p'\cdot k)(p\cdot k')\right].\end{aligned} \tag{7.46}$$

To simplify this further, consider kinematics in center-of-mass system (see Fig. 2). Here, $|\boldsymbol{k}| = \sqrt{E^2 - m_\mu^2}$ and $k_z = |\boldsymbol{k}|\cos\theta$. We have $q^2 \equiv (p+p')^2 = 4E^2$; $p\cdot p' = 2E^2$; $p\cdot k = p'\cdot k' = E^2 - E|\boldsymbol{k}|\cos\theta$; $p\cdot k' = p'\cdot k = E^2 + E|\boldsymbol{k}|\cos\theta$. With this we get

$$\begin{aligned}\overline{|M|^2} = &\frac{1}{4}\frac{2e^4}{(2\pi)^6}\frac{1}{16E^4}\frac{1}{E_eE'_eE_\mu E'_\mu}\\ &\times\big[2m_\mu^2E^2 + \underbrace{E^2(E - |\boldsymbol{k}|\cos\theta)^2 + E^2(E + |\boldsymbol{k}|\cos\theta)^2}_{2E^4 + 2E^2(E^2 - m_\mu^2)\cos^2\theta}\big]\\ = &\frac{1}{4}\frac{2e^4}{(2\pi)^6}\frac{1}{16E^4}\frac{1}{E^4}2E^4\left[1 + \frac{m_\mu^2}{E^2} + \left(1 - \frac{m_\mu^2}{E^2}\right)\cos^2\theta\right].\end{aligned} \tag{7.47}$$

Inserting into (2.55) gives

$$\begin{aligned}&\frac{d\sigma(e^+e^- \to \mu^+\mu^-)}{d\Omega} = \frac{(2\pi)^4|\boldsymbol{k}|E^4}{4E^2\cdot E}\,\overline{|M|^2}\\ &= \frac{1}{4}\frac{e^4}{(2\pi)^2}\frac{1}{4E^4}\left[1 + \frac{m_\mu^2}{E^2} + \left(1 - \frac{m_\mu^2}{E^2}\right)\cos^2\theta\right]\frac{E(E^2 - m_\mu^2)^{1/2}}{4}\\ &= \frac{\alpha^2}{4s}\sqrt{1 - \frac{m_\mu^2}{E^2}}\left[1 + \frac{m_\mu^2}{E^2} + \left(1 - \frac{m_\mu^2}{E^2}\right)\cos^2\theta\right],\end{aligned} \tag{7.48}$$

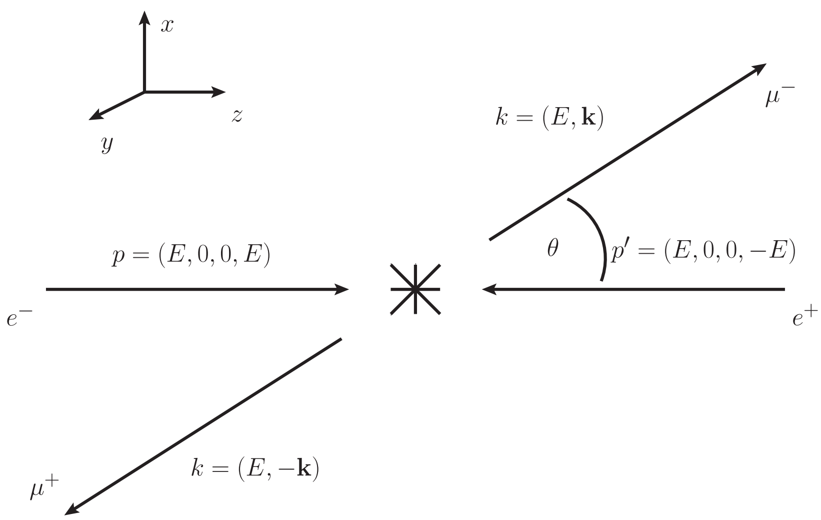


Figure 2: *Kinematics for $e^+e^- \to \mu^+\mu^-$ in the center-of-mass system.*

with $s \equiv (2E)^2$ the square of the center-of-mass energy and $\alpha = e^2/(4\pi) \approx 1/137$ the fine structure constant. Integrating over $d\Omega = 2\pi d\cos\theta$ we obtain the *total cross section*

$$\sigma(e^+e^- \to \mu^+\mu^-) = \frac{4\pi\alpha^2}{3s}\sqrt{1 - \frac{m_\mu^2}{E^2}}\left(1 + \frac{1}{2}\frac{m_\mu^2}{E^2}\right). \tag{7.49}$$

In the high-energy limit ($E \gg m_\mu$) we obtain

$$\frac{d\sigma}{d\Omega} \xrightarrow{E \gg m_\mu} \frac{\alpha^2}{4s}(1 + \cos^2\theta)\,, \tag{7.50}$$

$$\sigma \xrightarrow{E \gg m_\mu} \frac{4\pi\alpha^2}{3s} + \mathcal{O}\left(\frac{m_\mu^4}{E^4}\right). \tag{7.51}$$

For $E \gg m_\mu$, the energy is the only dimensionful quantity in the process, so Eq. (7.51) follows (up to the constant factor) from "naive dimensional analysis" (NDA).

7.3 The Feynman rules for QED

Our strategy to calculate the S-matrix element by bringing all annihilation operators to the right generalizes ("Wicks theorem", C. G. Wick 1950 [9]). Formally, one transforms a time-ordered into a normal-ordered product. The appearing contractions can be represented graphically. As usual, we classify external states by their three-momentum $\boldsymbol{p}$ and spin-z component (or helicity) σ. Then we have for an

incoming fermion: 

: $$\frac{u_\ell(\boldsymbol{p}, \sigma)}{(2\pi)^{3/2}}\,, \tag{7.52}$$

incoming antifermion: 

: $$\frac{\bar{v}_\ell(\boldsymbol{p}, \sigma)}{(2\pi)^{3/2}}\,, \tag{7.53}$$

outgoing fermion: 

$$: \quad \frac{\bar{u}_\ell(\boldsymbol{p},\sigma)}{(2\pi)^{3/2}}\,, \tag{7.54}$$

outgoing antifermion: 

$$: \quad \frac{v_\ell(\boldsymbol{p},\sigma)}{(2\pi)^{3/2}}\,, \tag{7.55}$$

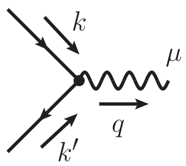
incoming photon: 

$$: \quad \frac{e_\mu(\boldsymbol{p},\sigma)}{(2\pi)^{3/2}\sqrt{2p^0}}\,, \tag{7.56}$$

outgoing photon: 

$$: \quad \frac{e^*_\mu(\boldsymbol{p},\sigma)}{(2\pi)^{3/2}\sqrt{2p^0}}\,. \tag{7.57}$$

The interactions are symbolized by vertices. The integration over $x_1, x_2, \ldots$ in (2.70) effectively yields a momentum-conservation delta function for each x_i. Hence, each vertex gives a factor



$$: \; -iQ_f(\gamma^\mu)_{\ell m}(2\pi)^4\delta^4(k+k'-q)\,. \tag{7.58}$$

Here, Q_f denotes the charge of fermion f. The contraction of two internal fields yields a factor

$$: \frac{1}{(2\pi)^4}\frac{i(\not{q}+m)}{q^2-m^2+i\epsilon} \tag{7.59}$$

for fermions, and

$$: \frac{1}{(2\pi)^4}\frac{-i\eta^{\mu\nu}}{q^2+i\epsilon} \tag{7.60}$$

for photons. The *Feynman rules* to obtain the transition matrix elements are:

1. Draw all Feynman diagrams contributing to a given process, with a given maximal number of vertices, using the building blocks (7.52)-(7.60).

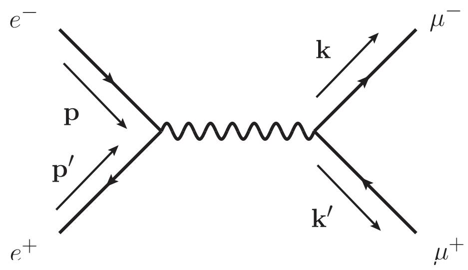
2. Replace each building block by its mathematical expression.

3. Integrate over the momenta of all internal lines, and sum over contracted Lorentz and Dirac index pairs.

4. Add the contributions of all diagrams.

5. Each closed fermion line yields a factor (-1). If two Feynman diagrams differ by an odd number of permutations of fermionic annihilation or creation operators, they get a relative minus sign.

Example: Muon pair production.



NB: We symbolize all contributions obtained by a mere renumbering of internal vertices by the *same* Feynman diagram! This cancels the factor $1/n!$ in (2.70).

7.4 $e^+e^- \to \mu^+\mu^-$: High-energy limit and helicity structure

In this section we will treat electrons and muons as massless (high-energy limit). Where does the angular dependence $(1+\cos\theta)$ in Eq. (7.50) come from? As an exercise, we construct the transition amplitude using the Feynman rules:

$$\begin{aligned}
&-2\pi i\delta^4(k+k'-p-p')M \\
&\quad= \int d^4q \left[\frac{\bar{u}(\boldsymbol{k},r)}{(2\pi)^{3/2}}(-ie)\gamma^\mu \frac{v(\boldsymbol{k}',r')}{(2\pi)^{3/2}}(2\pi)^4\delta^4(k+k'-q)\right. \\
&\qquad\qquad \times \frac{1}{(2\pi)^4}\frac{-i\eta_{\mu\nu}}{q^2+i\epsilon} \\
&\qquad\qquad \left.\times \frac{\bar{v}(\boldsymbol{p}',s')}{(2\pi)^{3/2}}(-ie)\gamma^\nu \frac{u(\boldsymbol{p},s)}{(2\pi)^{3/2}}(2\pi)^4\delta^4(q-p-p')\right] \\
&\quad= \frac{ie^2}{(2\pi)^2}\frac{(\bar{u}(\boldsymbol{k},r)\gamma^\mu v(\boldsymbol{k}',r'))(\bar{v}(\boldsymbol{p}',s')\gamma_\mu u(\boldsymbol{p},s))}{(p+p')^2}\delta^4(k+k'-p-p')\,.
\end{aligned} \tag{7.61}$$

NB: When not writing explicit spinor indices, all fermion lines must be "evaluated" *against* the direction of arrows.

We will evaluate the amplitude separately for all helicities (massless fermions!). First, we decompose all spinors into their LH and RH components:

$$\psi = P_L\psi + P_R\psi = \psi_L + \psi_R = \tfrac{1}{2}(1-\gamma_5)\psi + \tfrac{1}{2}(1+\gamma_5)\psi\,. \tag{7.62}$$

We have seen that the massless LH and RH spinors, $u(\boldsymbol{p},-1/2) = P_L u(\boldsymbol{p},-1/2)$ and $u(\boldsymbol{p},1/2) = P_R u(\boldsymbol{p},1/2)$, are eigenstates of the helicity operator with eigenvalues $-1/2$ and $+1/2$, respectively. Similarly, $v(\boldsymbol{p},-1/2) = P_L v(\boldsymbol{p},-1/2)$ and $v(\boldsymbol{p},1/2) = P_R v(\boldsymbol{p},1/2)$, are eigenstates with eigenvalues $+1/2$ and $-1/2$, respectively. Using this, we can project onto the different spin states. Consider, for instance, the second factor in (7.86). We replace

$$\bar{v}(\boldsymbol{p}',s')\gamma^\mu u(\boldsymbol{p},s) \to \bar{v}(\boldsymbol{p}',s')\gamma^\mu \tfrac{1}{2}(1+\gamma_5)u(\boldsymbol{p},s)\,. \tag{7.63}$$

Then the amplitude vanishes for a LH polarized electron ($h = -1/2$), while it is unchanged for a RH electron ($h = +1/2$). We have

$$\bar{v}\gamma_\mu \tfrac{1}{2}(1+\gamma_5)u = v^\dagger \beta\gamma^\mu \tfrac{1}{2}(1+\gamma_5)u = v^\dagger \tfrac{1}{2}(1+\gamma_5)\beta\gamma^\mu u = (\tfrac{1}{2}(1+\gamma_5)v)^\dagger \beta\gamma^\mu u\,, \tag{7.64}$$

so the positron must be RH polarized! In general, the amplitude vanishes unless electron and positron have opposite helicity.

Let's calculate the squared matrix element. The "electron factor" yields now

$$\begin{aligned}\sum_{\text{spins}} \left|\bar{v}(\boldsymbol{p}', s')\gamma^\mu \tfrac{1}{2}(1+\gamma_5)u(\boldsymbol{p}, s)\right|^2 &= \operatorname{Tr}\left\{\not{p}'\gamma^\mu \tfrac{1}{2}(1+\gamma_5)\not{p}\gamma^\nu \tfrac{1}{2}(1+\gamma_5)\right\} \\ &= \operatorname{Tr}\left\{\not{p}'\gamma^\mu \not{p}\gamma^\nu \tfrac{1}{2}(1+\gamma_5)\right\}.\end{aligned} \tag{7.65}$$

Now we use

$$\operatorname{Tr}\left\{\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_5\right\} = -4i\epsilon^{\mu\nu\rho\sigma} \tag{7.66}$$

and obtain

$$\sum_{\text{spins}} \left|\bar{v}(\boldsymbol{p}', s')\gamma^\mu \tfrac{1}{2}(1+\gamma_5)u(\boldsymbol{p}, s)\right|^2 = \frac{2}{4p^0 p'^0}\left(p'^\mu p^\nu + p'^\nu p^\mu - \eta^{\mu\nu} p\cdot p' - i\epsilon^{\alpha\mu\beta\nu} p'_\alpha p_\beta\right). \tag{7.67}$$

The analogous calculation for a RH μ^- and a LH μ^+ yields

$$\sum_{\text{spins}} \left|\bar{u}(\boldsymbol{k}, r)\gamma_\mu \tfrac{1}{2}(1+\gamma_5)v(\boldsymbol{k}', r')\right|^2 = \frac{2}{4k^0 k'^0}\left(k'_\mu k_\nu + k'_\nu k_\mu - \eta_{\mu\nu} k\cdot k' - i\epsilon_{\rho\mu\sigma\nu} k^\rho k'^\sigma\right). \tag{7.68}$$

Therefore, the squared matrix element for $e^-_R e^+_L \to \mu^-_R \mu^+_L$ is

$$\begin{aligned}\overline{|M|^2} &= \frac{e^4}{(2\pi)^6}\frac{1}{q^4}\frac{4}{16p^0 p'^0 k^0 k'^0} \\ &\quad\times\left[2(p\cdot k)(p'\cdot k') + 2(p'\cdot k)(p\cdot k') - \epsilon^{\alpha\mu\beta\nu}\epsilon_{\rho\mu\sigma\nu} p'_\alpha p_\beta k^\rho k'^\sigma\right] \\ &= \frac{e^4}{(2\pi)^6}\frac{1}{q^4}\frac{16}{16p^0 p'^0 k^0 k'^0}(p'\cdot k)(p\cdot k') \\ &\overset{\text{c.m.s.}}{=} \frac{e^4}{(2\pi)^6}\frac{1}{16E^4}(1+\cos\theta)^2\,,\end{aligned} \tag{7.69}$$

with $p^0 p'^0 k^0 k'^0 = E$; $q^2 = 4E^2$, $(p'\cdot k) = (p\cdot k') = E^2(1+\cos\theta)$. In the first line, we used $\epsilon^{\alpha\mu\beta\nu}\epsilon_{\rho\mu\sigma\nu} = 2(\delta^\alpha_\sigma\delta^\beta_\rho - \delta^\alpha_\rho\delta^\beta_\sigma)$. Inserting into (2.55) yields the cross section

$$\frac{d\sigma(e^-_R e^+_L \to \mu^-_R \mu^+_L)}{d\Omega} = \frac{\alpha^2}{4s}(1+\cos\theta)^2\,. \tag{7.70}$$

To calculate the remaining three non-vanishing amplitudes, we just need to reverse the sign of γ_5 in (7.67) and / or (7.68). That just changes the signs of the terms with the Levi-Civita tensor, and we get

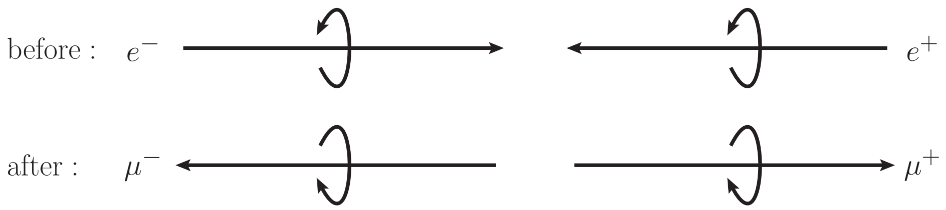
$$\frac{d\sigma(e^-_R e^+_L \to \mu^-_L \mu^+_R)}{d\Omega} = \frac{\alpha^2}{4s}(1-\cos\theta)^2\,, \tag{7.71}$$

$$\frac{d\sigma(e_L^- e_R^+ \to \mu_R^- \mu_L^+)}{d\Omega} = \frac{\alpha^2}{4s}(1-\cos\theta)^2\,, \tag{7.72}$$

$$\frac{d\sigma(e_L^- e_R^+ \to \mu_L^- \mu_R^+)}{d\Omega} = \frac{\alpha^2}{4s}(1+\cos\theta)^2\,. \tag{7.73}$$

Summing the four terms and dividing by 4 for spin averaging, we reproduce (7.50).

The physical meaning of, for instance, Eq. (7.70) can be understood as follows. For $\theta = \pi$ the cross section vanishes. This is nothing but conservation of angular momentum:



Since the total angular momentum is conserved, the amplitude must vanish (see Ref. [4], ch. 5.2 for more details). Helicity is conserved in the high-energy limit.

7.5 $e^+e^- \to \mu^+\mu^-$: Nonrelativistic limit

For $E \approx m_\mu$, the unpolarized cross section (7.48) becomes

$$\frac{d\sigma(e^+e^- \to \mu^+\mu^-)}{d\Omega} = \frac{\alpha^2}{4s}\underbrace{\sqrt{1-\frac{m_\mu^2}{E^2}}}_{|\boldsymbol{k}|/E}\left[\underbrace{1+\frac{m_\mu^2}{E^2}}_{\approx 2}+\Bigg(\underbrace{1-\frac{m_\mu^2}{E^2}}_{\approx 0}\Bigg)\cos^2\theta\right] \xrightarrow{E\approx m_\mu} \frac{\alpha^2}{2s}\frac{|\boldsymbol{k}|}{E}\,. \tag{7.74}$$

How can we understand the absence of the angular dependence? Let's calculate (7.74) explicitly in the NR limit.

Consider again the "electron factor" in (7.86). Since e^+ and e^- must be ultrarelativistic ($m_\mu \gg m_e$!), we choose helicity eigenstates, e.g. RH e^- in z direction, LH e^+ in $-z$ direction. The corresponding spinors are (cf. Eq. (4.114))

$$u(\boldsymbol{p},1/2) = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}\;; \qquad v(\boldsymbol{p},1/2) = \begin{pmatrix}0\\0\\0\\-1\end{pmatrix}\,. \tag{7.75}$$

Using (4.102) we obtain

$$\bar{v}(\boldsymbol{p},1/2)\gamma^\mu u(\boldsymbol{p},1/2) = (0,-1)\sigma^\mu\begin{pmatrix}1\\0\end{pmatrix}\,, \tag{7.76}$$

with

$$\sigma^\mu \equiv (\mathbb{1},\boldsymbol{\sigma})\,. \tag{7.77}$$

A simple calculation gives

$$\bar{v}(\boldsymbol{p},1/2)\gamma^\mu u(\boldsymbol{p},1/2) = (0,-1,-i,0)\,. \tag{7.78}$$

For the "muon factor" we use the general basis in the NR limit, Eq. (4.106). We write

$$u(0,\sigma) = \frac{1}{\sqrt{2}} \begin{pmatrix} \chi \\ \chi \end{pmatrix} ; \qquad v(0,\sigma) = \frac{1}{\sqrt{2}} \begin{pmatrix} \chi' \\ -\chi' \end{pmatrix} , \tag{7.79}$$

where (see Eq. (4.106))

$$\begin{aligned} \chi &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \quad \chi' = \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \qquad \text{for} \quad \sigma = +\tfrac{1}{2} ; \\ \chi &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \quad \chi' = \begin{pmatrix} -1 \\ 0 \end{pmatrix} , \qquad \text{for} \quad \sigma = -\tfrac{1}{2} . \end{aligned} \tag{7.80}$$

Defining

$$\bar{\sigma}^\mu \equiv (\mathbb{1}, -\boldsymbol{\sigma}) , \tag{7.81}$$

Eq. (4.102) becomes

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} , \tag{7.82}$$

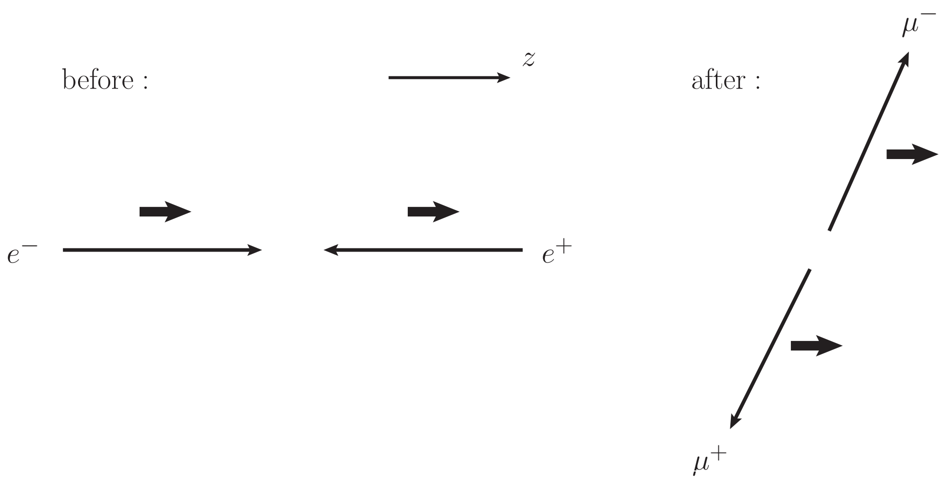
and we get

$$\begin{aligned} \bar{u}(\boldsymbol{k}, r)\gamma^\mu v(\boldsymbol{k}', r') &\overset{\boldsymbol{k},\boldsymbol{k}'\to 0}{=} \frac{1}{2}(\chi^\dagger, \chi^\dagger) \begin{pmatrix} \sigma^\mu & 0 \\ 0 & \bar{\sigma}^\mu \end{pmatrix} \begin{pmatrix} \chi' & -\chi' \end{pmatrix} \\ &= \begin{cases} 0 , & \mu = 0 , \\ -\chi^\dagger \sigma^i \chi' , & \mu = 1, 2, 3 . \end{cases} \end{aligned} \tag{7.83}$$

The Lorentz scalar product of (7.78) and (7.83) finally gives the scattering amplitude ($q^2 = 4m_\mu^2$)

$$M(e_R^- e_L^+ \to \mu^+ \mu^-) = -\frac{e^2}{(2\pi)^3} \frac{1}{4m_\mu^2} \chi^\dagger \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \chi' . \tag{7.84}$$

This expression is independent of the scattering angle; the orbital angular momentum of the $\mu^+\mu^-$ pair is zero ("s wave"). We see from (7.80), (7.84) that we need to choose $\sigma = +1/2$ for both μ^+ and μ^- to get a non-vanishing scattering amplitude. This is again conservation of angular momentum!



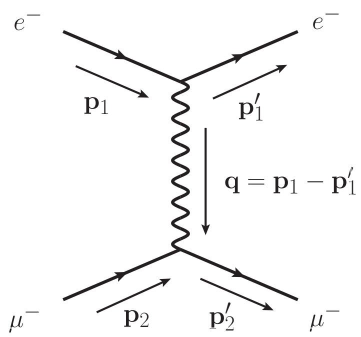
We find the scattering cross section for this process by summing over the muon spins (only one terms contributes); this gives (cf. (2.55))

$$\frac{d\sigma(e_R^- e_L^+ \to \mu^+\mu^-)}{d\Omega} = \frac{\alpha^2}{s}\frac{|\boldsymbol{k}|}{E}\,. \tag{7.85}$$

The amplitude for the process $e_L^- e_R^+ \to \mu^+\mu^-$ yields the same cross section. Summing the two terms and dividing by 4 (spin average) yields again (7.74).

7.6 $e^-\mu^- \to e^-\mu^-$ and "crossing symmetry"

We now consider the process $e^-\mu^- \to e^-\mu^-$. To leading order in QED we have



The corresponding scattering amplitude i

$$-2\pi i M = \frac{ie^2}{(2\pi)^2}\frac{(\bar{u}(\boldsymbol{p}_1', s')\gamma^\mu u(\boldsymbol{p}_1, s))(\bar{v}(\boldsymbol{p}_2', r')\gamma_\mu v(\boldsymbol{p}_2, r))}{(p_1 - p_1')^2}\,. \tag{7.86}$$

The summed and spin-averaged square is then (taking $m_e = 0$ as before)

$$\begin{aligned}\overline{|M|^2} = &\frac{1}{4}\frac{e^4}{(2\pi)^6}\frac{1}{((p_1 - p_1')^2)^2}\frac{1}{16p_1^0 p_1'^0 p_2^0 p_2'^0}\\ &\times \mathrm{Tr}\big\{\not{p}_1'\gamma^\mu \not{p}_1\gamma^\nu\big\}\mathrm{Tr}\big\{(\not{p}_2' + m_\mu)\gamma_\mu(\not{p}_2 + m_\mu)\gamma_\nu\big\}\,.\end{aligned} \tag{7.87}$$

This is the same expression as (7.40) after the replacements

$$p \to p_1\,, \qquad p' \to -p_1'\,, \qquad k \to p_2'\,, \qquad k' \to -p_2\,. \tag{7.88}$$

We can do the same replacements in (7.89), so we do not need to calculate the traces again. That gives

$$\begin{aligned}\overline{|M|^2} = &\frac{1}{4}\frac{e^4}{(2\pi)^6}\frac{1}{(p_1 - p_1')^4}\frac{1}{16p_1^0 p_1'^0 p_2^0 p_2'^0}\\ &\times 16\left[-2m_\mu^2(p_1\cdot p_1') + 2(p_1'\cdot p_2')(p_1\cdot p_2) + 2(p_1'\cdot p_2)(p_1\cdot p_2')\right].\end{aligned} \tag{7.89}$$

The kinematics for this process is, however, totally different. We work again in the c.m.s. (see Fig. 3), where $E^2 = k^2 + m_\mu^2$, $\boldsymbol{k}_z = k\cos\theta$, $\sqrt{s} = E + k$. With this we get $p_1\cdot p_2 = p_1'\cdot p_2' =$

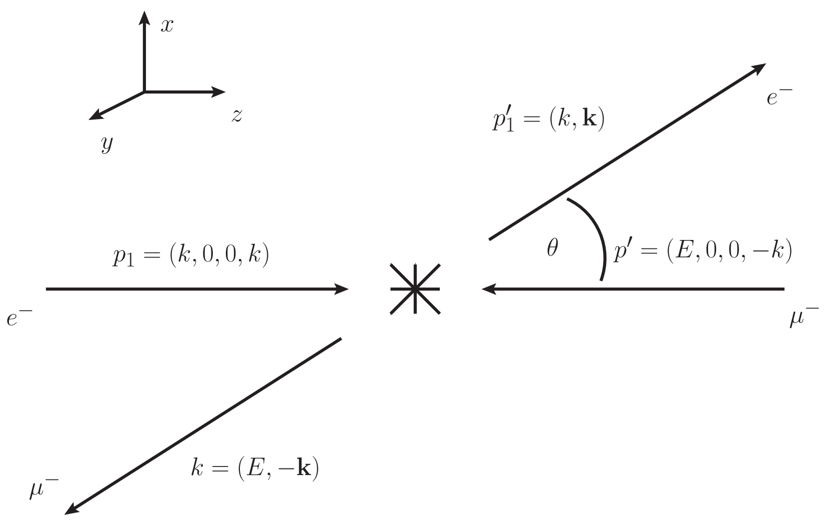


Figure 3: *Kinematics for* $e^-\mu^- \to e^-\mu^-$ *in the center-of-mass system.*

$k(E+k)$; $p_1' \cdot p_2 = p_1 \cdot p_2' = k(E + k\cos\theta)$; $p_1 \cdot p_1' = k^2(1-\cos\theta)$; $q^2 = (p_1 - p_1')^2 = -2p_1 \cdot p_1' = -2k^2(1-\cos\theta)$, and obtain

$$\begin{aligned}\overline{|M|^2} = & \frac{1}{4}\frac{e^4}{(2\pi)^6}\frac{1}{(p_1-p_1')^4}\frac{1}{16p_1^0 p_1'^0 p_2^0 p_2'^0} \\ & \times \frac{1}{4k^2(1-\cos\theta)^2}\left[(E+k)^2 + (E+k\cos\theta)^2 - m_\mu^2(1-\cos\theta)\right].\end{aligned} \tag{7.90}$$

Inserting into (2.55) gives the differential cross section

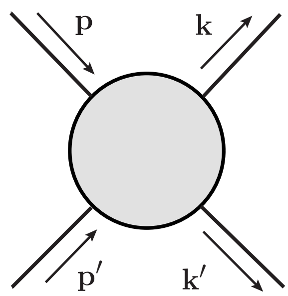
$$\begin{aligned}\frac{d\sigma(e^-\mu^- \to e^-\mu^-)}{d\Omega} = & \frac{\alpha^2}{2k^2 s(1-\cos\theta)^2} \\ & \times \left[(E+k)^2 + (E+k\cos\theta)^2 - m_\mu^2(1-\cos\theta)\right],\end{aligned} \tag{7.91}$$

and in the high-energy limit ($m_\mu \to 0$, $E \approx k$)

$$\frac{d\sigma(e^-\mu^- \to e^-\mu^-)}{d\Omega} = \frac{\alpha^2}{2s(1-\cos\theta)^2}\left[4 + (1+\cos\theta)^2\right]. \tag{7.92}$$

Remark: The differential cross section diverges like $1/\theta^4$ for $\theta \to 0$ ("Rutherford peak").

The *crossing relation* becomes more transparent if we use *Mandelstam variables*:

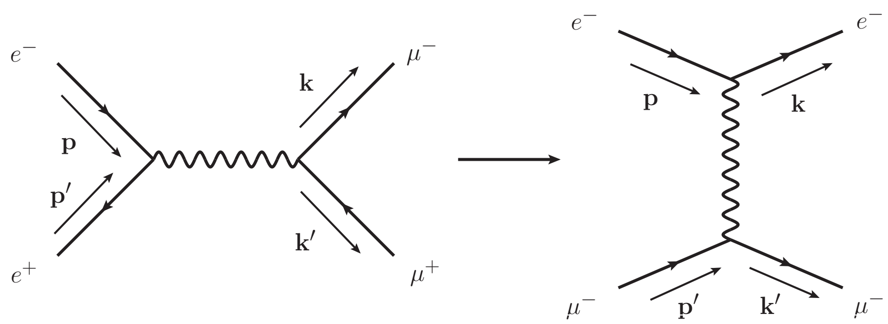


$$s = (p+p')^2 = (k+k')^2\,, \tag{7.93}$$
$$t = (k-p)^2 = (k'-p')^2\,, \tag{7.94}$$
$$u = (k'-p)^2 = (k-p')^2\,. \tag{7.95}$$

Example: $e^+e^- \to \mu^+\mu^-$ in the high-energy limit
Here, $t = -2p\cdot k = -2p'\cdot k'$, $u = -2p'\cdot k = -2p\cdot k'$, and the squared amplitude becomes (cf. (7.89))

$$\overline{|M|^2} = \frac{1}{4}\frac{e^4}{(2\pi)^6}\frac{1}{p_1^0 p_1'^0 p_2^0 p_2'^0}\;\frac{1}{s^2}\left[\frac{t^2}{2}+\frac{u^2}{2}\right]. \tag{7.96}$$

Crossing:



Hence, we have to perform the following replacements (particles in initial state → antiparticles in final state and vice versa; momentum with opposite sign)

$$p' \to -k\,, \qquad k' \to -p'\,, \qquad p \to p\,, \qquad k \to k'\,, \tag{7.97}$$

and hence

$$s = (p+p')^2 \to (p-k)^2 = t\,, \tag{7.98}$$
$$t = (k-p)^2 \to (k'-p)^2 = u\,, \tag{7.99}$$
$$u = (k-p')^2 \to (k'+k)^2 = s\,. \tag{7.100}$$

Therefore,

$$\overline{|M|^2} \to \frac{1}{4}\frac{e^4}{(2\pi)^6}\frac{1}{p_1^0 p_1'^0 p_2^0 p_2'^0}\;\frac{1}{t}\left[\frac{u^2}{2}+\frac{s^2}{2}\right]. \tag{7.101}$$

This agrees with (7.89).

If only a single diagram contributes to a given process, once refers to "s-channel, t-channel, u-channel". They lead to a characteristic angular dependence of the cross section:

s-channel: $\sim \frac{1}{s} = \frac{1}{E^2_{\text{c.m.s.}}}$;

t-channel: $\sim \frac{1}{t} = \frac{1}{1-\cos\theta}$;

u-channel: $\sim \frac{1}{u} = \frac{1}{1+\cos\theta}$.

8 Renormalization

8.1 Basic ideas

1. Self interactions generate dynamical contributions to mass and potential. This "renormalizes" measured values w.r.t. parameters in the Lagrangian.

2. Frequently, these "renormalizations" are divergent (infinite). In *renormalizable* theories, these are the only infinities.

3. One can cancel these infinities by introducing "counterterms" for fields, massses, and couplings , such that the resulting interaction is finite. (It follows that the theory cannot predict these masses and couplings.)

4. To be quantitative, we need to *regularize* the theory in intermediate steps of the calculation (lattice, cut-off, dimensional, …).

Example: Scalar field with ϕ^3 interaction

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_0\partial^\mu\phi_0 - \frac{1}{2}m_0^2\phi_0^2 - \frac{g_0}{3!}\phi_0^3\,. \tag{8.1}$$

We rescale the field: $\phi_0 = Z^{1/2}\phi$, and write

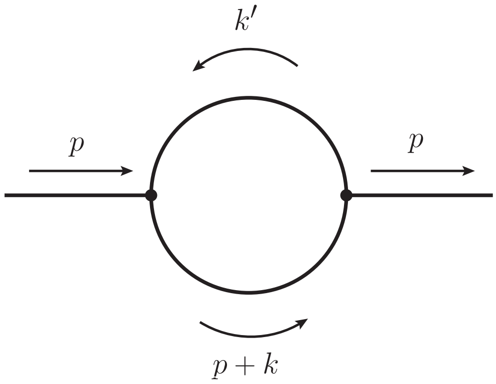
$$\mathcal{L} = \frac{1}{2}Z\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m_0^2 Z\phi^2 - \frac{g_0}{3!}Z^{3/2}\phi^3 \tag{8.2}$$

$$= \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{g}{3!}\phi^3 \tag{8.3}$$

$$+ \frac{1}{2}\delta Z\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\delta m^2\phi^2 - \frac{\delta g}{3!}\phi^3\,, \tag{8.4}$$

where $\delta Z \equiv Z - 1$, $\delta m^2 \equiv Zm_0^2 - m^2$, and $\delta g \equiv Z^{3/2}g_0 - g$. We choose δZ, δm^2, δg such that the divergences in the regularization parameter are exactly canceled. We call (8.3) the *basic Lagrangian* and (8.4) the *counterterm Lagrangian*; we regard the latter as part of the interaction. We expand the counterterms in the renormalized coupling g such that the divergences of all graphs are canceled.

8.2 One-loop self energy in ϕ^3 theory



$$\equiv i(2\pi)^4\Sigma_1(p^2)$$

$$\begin{aligned} i(2\pi)^4\Sigma_1(p^2) &= \frac{1}{2}\frac{g^2}{(2\pi)^3}\frac{(2\pi)^8}{(2\pi)^8}\int d^4k d^4k' \, \frac{\delta^4(p+k'-p-k)\delta^4(p+k-p-k')}{[(k+p)^2-m^2+i\epsilon][(k')^2-m^2+i\epsilon]} \\ &= \frac{1}{2}\frac{g^2}{(2\pi)^3}\int d^4k \, \frac{1}{[k^2-m^2+i\epsilon][(k+p)^2-m^2+i\epsilon]} \, . \end{aligned} \tag{8.5}$$

(For the "symmetry factor" $\frac{1}{2}$ consider the contractions

$$\langle 0|aT\{\phi\phi\phi(x) \quad \phi\phi\phi(y)\}a^\dagger|0\rangle \equiv \langle 0|aT\{\phi\phi\phi(x) \quad \phi\phi\phi(y)\}a^\dagger|0\rangle \, . \tag{8.6}$$

Each ϕ^3 has $3!$ contractions, cancels the factor $1/3!$ in Lagrangian, but two contractions are identical!)

For $k^\mu \to \infty$ this integral is logarithmically divergent (to see this, write $d^4k = k^3 dk d\Omega$); *UV divergence*. The UV structure is most transparent in Euclidean metric (otherwise one can have situations like "$k^2 = (k^0)^2 - \boldsymbol{k}^2$").

Wick rotation

Substitute $k^0 \to ik^4$, $dk^0 \to idk^4$; write $k_\mathrm{E} = (k^1, k^2, k^3, k^4)$, $k_\mathrm{E}^2 = \boldsymbol{k}^2 + (k^4)^2$. The pole structure is complicated, but the UV divergence is independent of these complications. Simplest example: scalar propagator (see Fig. 4)

$$\int_{-\infty}^{\infty} \frac{dk^0}{k^2-m^2+i\epsilon} = \int_{-i\infty}^{i\infty} \frac{dk^0}{k^2-m^2+i\epsilon} = -i\int_{-\infty}^{\infty} \frac{dk^4}{k_\mathrm{E}^2+m^2-i\epsilon} \, . \tag{8.7}$$

The poles of the integrand lie at $k^0 = \pm\sqrt{\boldsymbol{k}^2+m^2} - i\epsilon$.

For large k, the integrand in the self energy (8.5) behaves like $1/(k^2)^2$. Therefore, we write

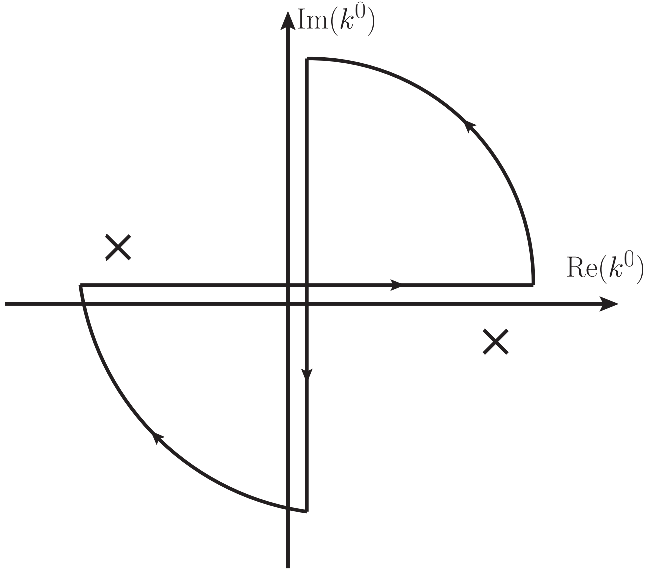


Figure 4: *Wick rotation. It is easy to verify by explicit calculation that the contributions of the two arc sections cancel each other. Since the contour does not enclose any of the poles, the integration over the real axis give the same contribution as the one over the imaginary axis.*

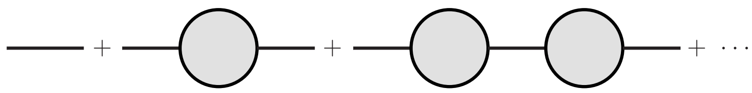
(μ^2 because of IR divergence!)

$$\begin{aligned}(2\pi)^4\Sigma_1(p^2)\\ &= \frac{-ig^2}{2(2\pi)^3}\int d^4k\left\{\left[\frac{1}{(k^2-m^2+i\epsilon)[(k+p)^2-m^2+i\epsilon]}-\frac{1}{(k^2-\mu^2+i\epsilon)^2}\right]\right.\\ &\qquad\left.+\frac{1}{(k^2-\mu^2+i\epsilon)^2}\right\}\\ &\equiv (2\pi)^4\big(\Sigma_{1,\text{finite}}(p^2,m^2,\mu^2)+\Sigma_{1,\text{div}}(m^2,\mu^2)\big)\,.\end{aligned} \tag{8.8}$$

This exact decomposition has the following propoerties: the term in square brackets $[\dots]$ is finite; Σ_1 is independent of μ^2; the last term is independent of p^2. In order to calculate this divergent last term, we change to Euclidean metric and introduce a UV cut-off Λ:

$$\begin{aligned}(2\pi)^4\Sigma_1(p^2) &= \frac{-ig^2}{16\pi^3}\int d^4k\,\frac{1}{(k^2-\mu^2+i\epsilon)^2}+\Sigma_{1,\text{finite}}\\ &= \frac{g^2}{16\pi^3}\int d^4k_\text{E}\,\frac{1}{(k_\text{E}^2+\mu^2-i\epsilon)^2}+\Sigma_{1,\text{finite}}\\ &\to \frac{g^2}{16\pi^3}\underbrace{\int d\Omega}_{2\pi^2}\int_0^\Lambda \frac{|k_\text{E}|^3d|k_\text{E}|}{(k_\text{E}^2+\mu^2-i\epsilon)^2}+\Sigma_{1,\text{finite}}\\ &= \frac{g^2}{8\pi}\left[\log\left(1+\frac{\Lambda^2}{\mu^2}\right)+1-\frac{1}{1-\Lambda^2/\mu^2}\right]+\Sigma_{1,\text{finite}}\,.\end{aligned} \tag{8.9}$$

Interpretation: Let $i(2\pi)^4\Sigma$ be the sum of all one-particle irreducible (1PI) self-energy graphs. Then the "full" propagator is

$G_2(p) =$ 

$$
\begin{aligned}
&= \frac{i}{(2\pi)^4}\frac{1}{p^2 - m_0^2 + i\epsilon} + \frac{i}{(2\pi)^4}\frac{1}{p^2 - m_0^2 + i\epsilon}\left[i(2\pi)^4\Sigma\right]\frac{i}{(2\pi)^4}\frac{1}{p^2 - m_0^2 + i\epsilon} + \dots \\
&= \frac{i}{(2\pi)^4}\frac{1}{p^2 - m_0^2 + i\epsilon} + \frac{1}{1 + \frac{\Sigma}{p^2 - m_0^2 + i\epsilon}} \\
&= \frac{i}{(2\pi)^4}\frac{1}{p^2 - m_0^2 + \Sigma(p^2) + i\epsilon} \,. \qquad (8.10)
\end{aligned}
$$

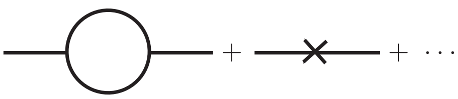
The "physical mass" of the particle is given by the pole of the full propagator, $p^2 = m_{\text{ph}}^2$, where

$$m_{\text{ph}}^2 = m_0^2 - \Sigma(p^2 = m_{\text{ph}}^2, m_0^2) \,. \qquad (8.11)$$

Traditionally, one parameterizes the theory in terms of m_{ph}^2 instead of m_0 which is unobservable:

$$-\frac{1}{2}m_0^2\phi^2 = \underbrace{-\frac{1}{2}m_{\text{ph}}^2\phi^2}_{\supset\text{free Lagrangian}} \qquad \underbrace{-\frac{1}{2}\delta m^2\phi^2}_{\supset\text{interaction Lagrangian}} \,. \qquad (8.12)$$

The free propagator then becomes $\frac{i}{(2\pi)^4}\frac{1}{p^2 - m_{\text{ph}}^2 + i\epsilon}$, and δm^2 is chosen such that m_{ph} is the "physical mass". In perturbation theory, δm^2 is determined as a power series in g. The renormalized self energy is given by

$i(2\pi)^4\Sigma_{1,\text{R}} =$ 

$$
\begin{aligned}
&= i(2\pi)^4\{\Sigma_1(p^2, m_{\text{ph}}^2, \Lambda) + \delta m^2\}\Big|_{\Lambda\to\infty} \\
&= i(2\pi)^4\{\Sigma_{1,\text{finite}}(p^2, m_{\text{ph}}^2, \mu^2, \Lambda = \infty) + [\delta m^2 + \Sigma_{1,\text{div}}(m_{\text{ph}}^2, \mu^2, \Lambda = \infty)]\} \,.
\end{aligned}
$$

(Recall that $\Sigma_{1,\text{div}}$ does not depend on p!) First, we choose δm^2 such that the the divergent part in $\Sigma_{1,\text{div}}$ is canceled. Then, we choose the finite part of δm^2 such that

$$\delta m^2 + \Sigma_{1,\text{div}}(m_{\text{ph}}^2, \mu^2, \Lambda = \infty) = -\Sigma_{1,\text{finite}}(p^2 = m_{\text{ph}}^2, m_{\text{ph}}^2, \mu^2, \Lambda = \infty) \,. \qquad (8.13)$$

Then, to order g^2, the self energy becomes

$$
\begin{aligned}
&(2\pi)^4\Sigma_{1,\text{R}}(p^2) \\
&\quad = \frac{-ig^2}{2(2\pi)^3}\int d^4k\left[\frac{1}{(k^2 - m_{\text{ph}}^2 + i\epsilon)[(k+p)^2 - m_{\text{ph}}^2 + i\epsilon]} - \frac{1}{(k^2 - \mu^2 + i\epsilon)^2}\right] \qquad (8.14) \\
&\qquad - \text{value at } p^2 = m_{\text{ph}}^2 \,.
\end{aligned}
$$

The μ^2 dependence obviously cancels.

Calculation of $\Sigma_{1,R}$

Differentiate $\Sigma_{1,\mathrm{R}}$ w.r.t. p^2. Integration of the result yields back $\Sigma_{1,\mathrm{R}}$; the integration constant is fixed by the "on-shell" condition $\Sigma_{1,\mathrm{R}}(p^2 = m_{\mathrm{ph}}^2) = 0$.

$$\begin{aligned}(2\pi)^4 \frac{\partial}{\partial p^2}\Sigma_{1,\mathrm{R}}(p^2) &= (2\pi)^4 \frac{p^\mu}{2p^2}\frac{\partial}{\partial p^\mu}\Sigma_{1,\mathrm{R}}(p^2) \\ &= \frac{ig^2}{16\pi^3 p^2}\int d^4k\, \frac{p\cdot(p+k)}{(k^2 - m_{\mathrm{ph}}^2 + i\epsilon)[(k+p)^2 - m_{\mathrm{ph}}^2 + i\epsilon]^2}\,.\end{aligned} \tag{8.15}$$

To calculate this, we use *Feynman parameters:*

$$\frac{1}{A^\alpha B^\beta} = \int_0^1 dx\, \frac{x^{\alpha-1}(1-x)^{\beta-1}}{[xA + (1-x)B]^{\alpha+\beta}}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\,, \tag{8.16}$$

where $\Gamma(z)$ is Euler's Γ function.

The Γ function is the unique extrapolation of the factorial with $\Gamma(z+1) = z\Gamma(z)$ and $\Gamma(0) = 1$. A useful integral representation, valid for $\mathrm{R}e(z) > 0$, is

$$\Gamma(z) = \int_0^\infty dt\, t^{z-1}e^{-t}\,. \tag{8.17}$$

Using this, we get

$$\begin{aligned}(2\pi)^4 \frac{\partial}{\partial p^2}\Sigma_{1,\mathrm{R}}(p^2) &= \frac{ig^2}{16\pi^3 p^2}\int d^4k \int_0^1 dx\, \frac{p\cdot(p+k)x}{[\underbrace{x(k+p)^2 + (1-x)k^2}_{xp^2+k^2+2xp\cdot k=(k+xp)^2+x(1-x)p^2} - m_{\mathrm{ph}}^2 + i\epsilon]^3}\frac{\Gamma(3)}{\Gamma(1)\Gamma(2)} \\ &\overset{k\to k-xp}{=} \frac{2ig^2}{16\pi^3 p^2}\int_0^1 dx \int d^4k\, \frac{p\cdot[k + p(1-x)]x}{[k^2 - m_{\mathrm{ph}}^2 + x(1-x)p^2 + i\epsilon]^3}\,.\end{aligned} \tag{8.18}$$

Let's calculate a more general integral first, for future convenience:

$$
\begin{aligned}
I_n(A) &\equiv \int d^d k \, (k^2 - A + i\epsilon)^{-n} \\
&= (-1)^n i \int d^d k_{\mathrm{E}} \, (k_{\mathrm{E}}^2 + A - i\epsilon)^{-n} \\
&= (-1)^n i \int d\Omega_d \int\limits_0^\infty dk_{\mathrm{E}} \, k_{\mathrm{E}}^{d-1} (k_{\mathrm{E}}^2 + A - i\epsilon)^{-n} \\
&= (-1)^n \frac{i}{2} \frac{2\pi^{d/2}}{\Gamma(d/2)} \int\limits_0^\infty dk_{\mathrm{E}}^2 \, (k_{\mathrm{E}}^2)^{d/2-1} (k_{\mathrm{E}}^2 + A - i\epsilon)^{-n} \\
&= (-1)^n \frac{i\pi^{d/2}}{\Gamma(d/2)} (A - i\epsilon)^{d/2-n} \underbrace{\int\limits_0^\infty dy \, y^{d/2-1} (1+y)^{-n}}_{B(d/2, n-d/2)} \\
&= (-1)^n i \pi^{d/2} \frac{\Gamma(n - d/2)}{\Gamma(n)} (A - i\epsilon)^{d/2-n} \, ,
\end{aligned}
\tag{8.19}
$$

with *Euler's Beta function*

$$
B(x, y) \equiv \int\limits_0^\infty dt \, t^{x-1} (1+t)^{-x-y} = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \, . \tag{8.20}
$$

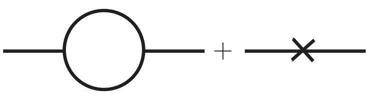
Using this result for $n = 3$, $d = 4$ we get

$$
\begin{aligned}
(2\pi)^4 \frac{\partial}{\partial p^2} \Sigma_{1,\mathrm{R}}(p^2) &= \frac{g^2}{16\pi} \int\limits_0^1 dx \, \frac{x(1-x)}{m_{\mathrm{ph}}^2 - p^2 x(1-x)} \\
&= -\frac{g^2}{16\pi} \frac{\partial}{\partial p^2} \int\limits_0^1 dx \, \log[m_{\mathrm{ph}}^2 - p^2 x(1-x)] \, .
\end{aligned}
\tag{8.21}
$$

Using the condition $\Sigma_{1,\mathrm{R}}(p^2 = m_{\mathrm{ph}}^2) = 0$ yields finally

$$
(2\pi)^4 \Sigma_{1,\mathrm{R}}(p^2) = -\frac{g^2}{16\pi} \int\limits_0^1 dx \, \log\left[\frac{m_{\mathrm{ph}}^2 - p^2 x(1-x)}{m_{\mathrm{ph}}^2 (1 - x(1-x))} \right] . \tag{8.22}
$$

8.3 Higher order

We introduced a mass counterterm such that  is finite. This counterterm can appear at arbitrary places in general diagrams.

8.4 Degree of divergence

UV divergences are a "purely Euclidean" problem – can use power counting. Consider, for instance, the self energy in d space-time dimensions:

$$(2\pi)^d \Sigma_1(p^2, m^2, d) = \frac{-ig^2}{2(2\pi)^{d-1}} \int d^d k \, \frac{1}{[k^2 - m^2 + i\epsilon][(k+p)^2 - m^2 + i\epsilon]} \,. \tag{8.23}$$

We call the leading power of k in the integral ($d-4$ in our example) the *overall* or *superficial degree of divergence* δ. The diagram is finite for $\delta < 0$, and divergent for $\delta \geq 0$.

Strictly speaking, at higher orders the last statement is only correct after subtraction of subdivergences.

Differentiating (8.23) once w.r.t. p^μ gives convergence for $d = 4$:

$$(2\pi)^d \frac{\partial \Sigma_1}{\partial p^\mu} = \frac{ig^2}{(2\pi)^{d-1}} \int d^d k \, \frac{(p+k)_\mu}{[k^2 - m^2 + i\epsilon][(k+p)^2 - m^2 + i\epsilon]^2} \,. \tag{8.24}$$

This expression would be logarithmically divergent in $d = 5$. However, from symmetry considerations, we expect the divergence to vanish, see below.

Differentiating again gives a result with $\delta = d - 6$:

$$(2\pi)^d \frac{\partial^2 \Sigma_1}{\partial p^\mu \partial p^\nu} = \frac{-ig^2}{(2\pi)^{d-1}} \int d^d k \, \frac{4(p+k)_\mu (p+k)_\nu - \eta_{\mu\nu}[(p+k)^2] - m^2}{[k^2 - m^2 + i\epsilon][(k+p)^2 - m^2 + i\epsilon]^3} \,. \tag{8.25}$$

Integrating twice gives constants of integration $A + B_\mu p^\mu$. Lorentz invariance of $\Sigma_{1,\mathrm{R}}$ implies that $B_\mu = 0$. A mass counterterm yields a finite $\Sigma_{1,\mathrm{R}}$ in $d = 5$.

In $d = 6$ we need to differentiate Σ_1 three times, this gives a finite result with $\delta = d - 7$. Integration gives terms $A - B\,p^2$. Now we need two counterterms: $\delta m^2 + \delta Z p^2$, generated by

$$\mathcal{L} \supset -\frac{1}{2}\delta m^2 \phi^2 + \frac{1}{2}\delta Z \partial_\mu \phi \partial^\mu \phi \,. \tag{8.26}$$

The degree of divergence (of the original, undifferentiated graph) corresponds to the maximal number of derivatives in the counterterms.

Interpretation: The complete propagator for unrenormalized fields (in momentum space) is

$$\begin{aligned} G_{2,(0)} &= \frac{i}{(2\pi)^6} \frac{Z}{p^2 - m_{\mathrm{ph}}^2 + \Sigma_{1,(0)}(p^2) - \delta m^2 + \delta Z p^2 + i\epsilon} \\ &= \frac{i}{(2\pi)^6} \frac{Z}{p^2 - m_{\mathrm{ph}}^2 + \Sigma_{1,\mathrm{R}}(p^2) + i\epsilon} \,. \end{aligned} \tag{8.27}$$

In general, the residue of the propagator is not equal to unity:

$$G_{2(0)} \overset{p^2\to m_{\text{ph}}^2}{=} \frac{i}{(2\pi)^6}\frac{R_{(0)}}{p^2 - m_{\text{ph}}^2 + i\epsilon} + \text{finite}\,. \tag{8.28}$$

We now choose $Z = R(0)$; then the renormalized propagator is ($\phi = Z^{-1/2}\phi_0$)

$$G_2 \overset{p^2\to m_{\text{ph}}^2}{=} \frac{i}{(2\pi)^6}\frac{1}{p^2 - m_{\text{ph}}^2 + i\epsilon} + \text{finite}\,. \tag{8.29}$$

We chose δm^2 such that $\Sigma_{1,\text{R}}(p^2 = m_{\text{ph}}^2) = 0$. With this choice

$$\begin{aligned} \text{res}_{p^2=m_{\text{ph}}^2} G_2 &= \frac{i}{(2\pi)^6}\lim_{p^2\to m_{\text{ph}}^2}(p^2 - m_{\text{ph}}^2)\frac{1}{p^2 - m_{\text{ph}}^2 + \Sigma_{1,\text{R}}(p^2)} \\ &= \frac{i}{(2\pi)^6}\lim_{p^2\to m_{\text{ph}}^2}\frac{1}{1 + \frac{\Sigma_{1,\text{R}}(p^2) - \Sigma_{1,\text{R}}(m_{\text{ph}}^2)}{p^2 - m_{\text{ph}}^2}} \overset{!}{=} \frac{i}{(2\pi)^6}\,, \end{aligned} \tag{8.30}$$

and so

$$\frac{\partial}{\partial p^2}\Sigma_{1,\text{R}}(p^2)\Big|_{p^2=m_{\text{ph}}^2} = 0\,. \tag{8.31}$$

This fixes both integration constants.

8.5 (Non-)Renormalizability

In $d = 8$, the self energy has degree of divergence $\delta = 4$; the requisite counterterms are

$$\mathcal{L}_c = +\frac{1}{2}\delta Z\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\delta m^2\phi^2 + \frac{1}{2}E(\Box\phi)^2\,. \tag{8.32}$$

The last term cannot be interpreted as a “renormalization” of the basic Lagrangian, hence the theory is non-renormalizable.

More generally, let Γ be a 1PI graph, with degree of divergence $\delta(\Gamma)$ and mass dimension $d(\Gamma)$. Then

$$d(\Gamma) = \delta(\Gamma) + \Delta(\Gamma)\,, \tag{8.33}$$

with $\Delta(\Gamma)$ the dimension of the couplings in Γ. The counterterms to Γ are a polynomial of degree $\delta(\Gamma)$ in the external momenta. For each counterterm, let $\delta(C)$ be the number of derivatives and $\Delta(C)$ the mass dimension of the coefficient, such that

$$\delta(C) + \Delta(C) = d(\Gamma)\,. \tag{8.34}$$

The maximal number of derivatives in the counterterms is

$$\delta(\Gamma) = d(\Gamma) - \Delta(\Gamma)\,. \tag{8.35}$$

1. For couplings with negative mass dimension $\delta(\Gamma)$ becomes arbitrarily large, hence the theory is not expected to be renormalizable without miraculous cancelations.

2. For couplings with mass dimension ≥ 0 we have $-\Delta(\Gamma) \leq 0$; $d(\Gamma)$ decreases with number of external lines. We have

$$\Delta(C) = d(\Gamma) - \delta(C) \geq d(\Gamma) - \delta(\Gamma) = \Delta(\Gamma)\,, \tag{8.36}$$

hence also the counterterm coefficients have mass dimension ≥ 0. The theory is expected to be renormalizable.

Is renormalizability necessary?

- Renormalizable theories have finitely many terms in the Lagrangian
- E.g. QED: No terms like $\bar{\psi}\sigma_{\mu\nu}\psi F^{\mu\nu}$
- Frequently, “non-renormalizable” theories are in fact renormalizable if one includes all counterterms allowed by symmetries (infinitely many)
- Couplings g with mass dimension $\Delta < 0$: expect $g \propto M^{\Delta}$, hint towards a new scale (e.g. Fermi theory, weak scale)
- Sensible description of physics by “effective field theories” – expansion in ratio E/M (e.g. chiral Lagrangian)

8.6 Renormalization scheme

Traditionally: Renormalization $\to$ replace unobservable “bare” coupling parameters by measured “physical” parameters (e.g. m_{ph}). This view is too restrictive! E.g. QCD: quarks do not exist as asymptotic states.

Example:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{g}{3!}\phi^3 + \frac{1}{2}\delta Z\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\delta m^2\phi^2 - \frac{\delta g}{3!}\phi^3\,, \tag{8.37}$$

with $m \neq m_{\text{ph}}$ etc. The self energy (8.13) in $d = 4$ (i.e. $\delta Z = 0$) becomes

$$\Sigma_{1,\text{R}} = \Sigma_{1,\text{fin}} + \left(\delta m^2 + \Sigma_{1,\text{div}}\right). \tag{8.38}$$

The divergent part of δm^2 is fixed (it has to subtract the divergence of the graph); the finite part is *arbitrary*! Does this ruin the predictiviy of the theory? Consider the propagator with “on-shell” condition $\Sigma_{1,\text{R}}(p^2 = m_{\text{ph}}^2) = 0$:

$$G_2(p) = \frac{i}{(2\pi)^4}\frac{1}{p^2 - m_{\text{ph}}^2 + \Sigma_{1,\text{fin}}(p^2, m_{\text{ph}}^2) - \Sigma_{1,\text{fin}}(m_{\text{ph}}^2, m_{\text{ph}}^2) + \mathcal{O}(g^4)}\,, \tag{8.39a}$$

$$m_0^2 = m_{\text{ph}}^2 - \Sigma_{1,\text{div}} - \Sigma_{1,\text{fin}}(m_{\text{ph}}^2) + \mathcal{O}(g^4)\,. \tag{8.39b}$$

Now with a different finite part (here, C is an arbitrary real number):

$$G_2(p) = \frac{i}{(2\pi)^4} \frac{1}{p^2 - m^2 + \Sigma_{1,\text{fin}}(p^2, m^2) + g^2 C + \mathcal{O}(g^4)}\,, \tag{8.40a}$$

$$m_0^2 = m^2 - \Sigma_{1,\text{div}} + g^2 C + \mathcal{O}(g^4)\,. \tag{8.40b}$$

Obviously, both prescriptions (8.39) and (8.40) give the same result, since the bare mass m_0^2 is the same in both cases. The splitting of the fundamental parameter into $m^2 + \delta m^2$ is arbitrary and without physical meaning. We have

$$m^2 = m_{\text{ph}}^2 - \Sigma_{1,\text{fin}}(m_{\text{ph}}^2) - g^2 C + \mathcal{O}(g^4)\,. \tag{8.41}$$

It follows that the difference $m^2 - m_{\text{ph}}^2$ is of $\mathcal{O}(g^2)$. A shift in C can be compensated by a shift in m. The choice of C is called a *renormalization prescription* or *renormalization scheme.* A change of scheme corresponds to a reshuffling of terms from the free into the interaction Lagrangian (reordering of the perturbation series).

8.7 Dimensional regularization

Observation: Graphs become convergent for sufficiently small space-time dimension d. Consider d as a continous variable.

Example: ϕ^3 self energy around $d = 4$ (drop external leg factor)

$$\begin{aligned}
i(2\pi)^d \Sigma_1(p^2, m^2, d) &= \frac{g^2}{2} \int d^d k \, \frac{1}{(k^2 - m^2)[(k+p)^2 - m^2]} \\
&= \frac{g^2}{2} \int_0^1 dx \int d^d k \, \frac{1}{[x(p+k)^2 + (1-x)k - m^2]^2} \frac{\Gamma(2)}{\Gamma(1)\Gamma(1)} \\
&= \frac{g^2}{2} \int_0^1 dx \int d^d k \, \frac{1}{[k^2 + x(1-x)p^2 - m^2]^2} \\
&\overset{(8.19)}{=} \frac{i\pi^{d/2} g^2}{2} \frac{\Gamma(2 - d/2)}{\Gamma(2)} \int_0^1 dx \, [m^2 - x(1-x)p^2]^{d/2-2}\,.
\end{aligned} \tag{8.42}$$

$\Gamma(z)$ has a pole at $z = 0$:

$$\Gamma(z) = \frac{1}{z} - \gamma_{\text{E}} + \mathcal{O}(z)\,, \tag{8.43}$$

with the *Euler constant* $\gamma_{\text{E}} = 0.5772\ldots$. Using the relation $\Gamma(z+1) = z\Gamma(z)$, we see that the divergences show up as poles for $d = 4, 6, 8, \ldots$.

On-shell renormalization:

$$\Sigma_{1,\mathrm{R}}^{(\mathrm{OS})}(p^2, m^2, d) = \Sigma_1(p^2, m_{\mathrm{ph}}^2, d) - \Sigma_1(p^2 = m_{\mathrm{ph}}^2, m_{\mathrm{ph}}^2, d)$$
$$= \frac{\pi^{d/2} g^2}{2}\Gamma(2 - d/2) \int_0^1 dx \left\{ [m_{\mathrm{ph}}^2 - x(1-x)p^2]^{d/2-2} - [m_{\mathrm{ph}}^2(1 - x - x^2)]^{d/2-2} \right\} . \tag{8.44}$$

We write $d = 4 - 2\epsilon$. For $d \to 4$ ($\epsilon \to 0$) use $A^\epsilon \equiv \exp(\epsilon \log A) = 1 + \epsilon \log A + \mathcal{O}(\epsilon^2)$ to find

$$\Sigma_{1,\mathrm{R}}^{(\mathrm{OS})}(p^2, m^2, d) = \frac{\pi^{2-\epsilon} g^2}{2(2\pi)^{4-2\epsilon}} \left(\tfrac{1}{\epsilon} - \gamma_{\mathrm{E}} + \dots\right)$$
$$\times \int_0^1 dx \left\{ 1 - \epsilon \log[m_{\mathrm{ph}}^2 - x(1-x)p^2] - 1 + \epsilon \log[m_{\mathrm{ph}}^2(1 - x - x^2)] \right\} + \mathcal{O}(\epsilon) \tag{8.45}$$
$$= \frac{-g^2}{32\pi^2} \int_0^1 dx \, \log\left[\frac{m_{\mathrm{ph}}^2 - x(1-x)p^2}{m_{\mathrm{ph}}^2(1 - x - x^2)} \right] + \mathcal{O}(\epsilon) .$$

This is exactly (8.22)$\times(2\pi)^3$.

8.8 Minimal subtraction

The divergence of the self energy corresponds to a simple pole for $d \to 4$. We can choose δm^2 such that exactly this pole is subtracted:

$$\delta m^2 = \frac{g^2}{32\pi^2} \cdot \frac{1}{\epsilon} . \tag{8.46}$$

Using

$$\frac{\pi^{2-\epsilon}}{2(2\pi)^{4-2\epsilon}} = \frac{1}{32\pi^2}(4\pi)^\epsilon = \frac{1}{32\pi^2}(1 + \epsilon \log(4\pi) + \dots)$$

we then get

$$\Sigma_{1,\mathrm{R}} = \frac{-g^2}{32\pi^2} \int_0^1 dx \left\{ \log\left[\frac{m^2 - x(1-x)p^2}{4\pi} \right] + \gamma_{\mathrm{E}} \right\} . \tag{8.47}$$

What about $\log(m^2)$?? Notice that $g = g(d)$.

From the kinetic terms in the Lagrangian, we read off the mass dimension of the field: $[\phi] = d/2 - 1 = (d-2)/2$. Then from the interaction term $[g] = d - 3(d-2)/2 = 3 - d/2$.

We write

$$g \to \mu^{2-d/2} g , \tag{8.48}$$

such that the rescaled coupling has mass dimension 1 (as in $d = 4$). Then our result reads

$$\Sigma_{1,\mathrm{R}}^{(\mathrm{MS})} = \frac{-g^2}{32\pi^2} \int_0^1 dx \left\{ \log\left[\frac{m^2 - x(1-x)p^2}{4\pi\mu^2}\right] + \gamma_{\mathrm{E}} \right\}. \tag{8.49}$$

The "'t Hooft mass scale" μ is completely arbitrary. A choice of μ defines a renormalization prescription; a change in μ can (must) be compensated by a change in m (i.e. $m = m(\mu)$).

Frequently one performs the change of scheme

$$\mu \to \mu \left(\frac{e_{\mathrm{E}}^{\gamma}}{4\pi}\right)^{1/2}; \tag{8.50}$$

this defines the $\overline{MS}$ *scheme*, modified minimal subtraction. In this scheme,

$$\Sigma_{1,\mathrm{R}}^{(\overline{\mathrm{MS}})} = \frac{-g^2}{32\pi^2} \int_0^1 dx \left\{ \log\left[\frac{m^2 - x(1-x)p^2}{\mu^2}\right] \right\}. \tag{8.51}$$

The self energies behave like $\Sigma_{1,\mathrm{R}}^{(\mathrm{MS})} \sim \log(p^2/\mu^2)$, which implies large perturbative corrections for $p^2 \to \infty$, unless one chooses $\mu^2 \approx p^2$.

8.9 The renormalization group equations

Physical observables (e.g. the S-matrix) are invariant under the combined change

$$(\mu, g(\mu), m(\mu)) \to (\mu', g(\mu'), m(\mu')). \tag{8.52}$$

(The fundamental "bare" parameters are invariant.) Infinitesimally:

$$\mu \frac{dS}{d\mu} = 0. \tag{8.53}$$

The total derivative is

$$\mu \frac{d}{d\mu} = \mu \frac{\partial}{\partial\mu} + \beta \frac{\partial}{\partial g} - \gamma_m m^2 \frac{\partial}{\partial m^2}, \tag{8.54}$$

with

$$\beta = \mu \frac{dg(\mu)}{d\mu}, \qquad \gamma_m = -\frac{\mu}{m^2} \frac{\partial m^2(\mu)}{\partial\mu}. \tag{8.55}$$

In the MS scheme, β and γ_m are mass independent, and it is easy to solve the differential equations (8.55). β and γ_m can be determined from the conditions

$$\mu \frac{dg_0}{d\mu} = 0, \qquad \mu \frac{\partial m_0^2}{\partial\mu} = 0. \tag{8.56}$$

As an example, consider ϕ^3 theory in $d = 6$ (such that g is dimensionless):

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\partial_\mu\phi_0\partial^\mu\phi_0 - \frac{1}{2}m_0^2\phi_0^2 - \frac{g_0}{3!}\phi_0^3 \\ &= \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \mu^{3-d/2}\frac{g}{3!}\phi^3 \\ &\quad + \frac{1}{2}\delta Z\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\delta m^2\phi^2 - \mu^{3-d/2}\frac{\delta g}{3!}\phi^3 \,.\end{aligned} \tag{8.57}$$

The one-loop self energy and vertex correction are (exercise!)

$$\Sigma_1 = -\frac{g^2}{128\pi^3}\frac{1}{\epsilon}\Big[m^2 - \tfrac{1}{6}p^2\Big] + \text{finite}\,, \tag{8.58}$$

$$V_1 = -\frac{g^3}{128\pi^3}\frac{1}{\epsilon} + \text{finite}\,, \tag{8.59}$$

where $d \equiv 6 - 2\epsilon$. This implies

$$\delta m^2 = -\frac{g^2}{128\pi^3}\frac{m^2}{\epsilon}\,, \tag{8.60}$$

$$\delta Z = -\frac{g^2}{768\pi^3}\frac{1}{\epsilon}\,, \tag{8.61}$$

$$\delta g = -\frac{g^3}{128\pi^3}\frac{1}{\epsilon}\,. \tag{8.62}$$

We had defined $\delta Z = Z - 1$, $\delta m^2 = Zm_0^2 - m^2$, $\delta g = Z^{3/2}g_0 - g$. Solving these relations for m_0^2 we find

$$\begin{aligned}m_0^2 &= Z^{-1}(m^2 + \delta m^2) = (1 - \delta Z)(m^2 + \delta m^2) = m^2 - m^2\delta Z + \delta m^2 \\ &= m^2\left[1 - \frac{5g^2}{768\pi^3\epsilon} + \mathcal{O}(g^4)\right],\end{aligned} \tag{8.63}$$

$$g_0 = (1 - \tfrac{3}{2}\delta Z)\mu^{3-d/2}(g + \delta g) = \mu^\epsilon g\left[1 - \frac{3g^2}{512\pi^3\epsilon} + \mathcal{O}(g^4)\right]. \tag{8.64}$$

Using these results, we find

$$0 = \mu\frac{dg_0}{d\mu} = \mu^\epsilon\left\{\epsilon g - \frac{3g^3}{512\pi^3} + \mathcal{O}(g^5) + \beta(g)\left[1 - \frac{9g^2}{512\pi^3\epsilon} + \mathcal{O}(g^4)\right]\right\}. \tag{8.65}$$

From this we can extract the β function

$$\beta(g) = -\epsilon g - \frac{3g^3}{256\pi^3} + \mathcal{O}(g^5) \xrightarrow{d\to 6} -\frac{3g^3}{256\pi^3} + \mathcal{O}(g^5)\,. \tag{8.66}$$

Similarly, from

$$0 = \mu\frac{dm_0^2}{d\mu} = -\gamma_m(g)m^2\Big[1 + \mathcal{O}(g^2)\Big] + m^2\beta(g)\left[-\frac{5g}{384\pi^3\epsilon} + \mathcal{O}(g^3)\right] \tag{8.67}$$

we obtain

$$\gamma_m(g) = \frac{5g^2}{384\pi^3} + \mathcal{O}(g^4)\,. \tag{8.68}$$

The ϵ terms in $\beta(g)$ are important! $\beta(g)$ and $\gamma_m(g)$ are mass independent. The solutions of (8.55) are

$$\log\frac{\mu'}{\mu} = \int\limits_{g(\mu)}^{g(\mu')} \frac{dg'}{\beta(g')}\,, \tag{8.69}$$

$$m^2(\mu') = m^2(\mu)\exp\left[-\int\limits_{\mu}^{\mu'} \frac{d\bar\mu}{\bar\mu}\gamma_m(g(\bar\mu))\right] \overset{(8.69)}{=} m^2(\mu)\exp\left[-\int\limits_{g(\mu)}^{g(\mu')} dg'\,\frac{\gamma_m(g')}{\beta(g')}\right]. \tag{8.70}$$

Example: Running coupling to leading order:

$$\begin{aligned}\log\frac{\mu'}{\mu} &= -\frac{256\pi^3}{3}\int\limits_{g(\mu)}^{g(\mu')} \frac{dg'}{g'^3}\left[1+\mathcal{O}(g^2)\right]\\ &= \frac{128\pi^3}{3}\left[\frac{1}{g(\mu')^2} - \frac{1}{g(\mu)^2}\right] + \mathcal{O}\left(\log\frac{g(\mu')}{g(\mu)}\right),\end{aligned} \tag{8.71}$$

and so

$$g(\mu')^2 = \frac{g(\mu)^2}{1+\frac{3g(\mu)^2}{128\pi^3}\log\frac{\mu'}{\mu}} \overset{\mu'\to\infty}{\longrightarrow} 0\,. \tag{8.72}$$

"Asymptotic freedom"

Varieties of asymptotic behaviour:

1. Singularity at finite energy. If $\beta(g)$ grows sufficiently fast, such that $\int^\infty dg/\beta(g) < \infty$, Eq. (8.55) implies

$$\mu_\infty = \mu\exp\left(\int\limits_{g(\mu)}^{\infty}\frac{dg'}{\beta(g')}\right) \tag{8.73}$$

   E.g. $\mu_\infty = e^{646.6}m_e$ in QED.

2. Continued growth

3. Fixed point at finite $g = g^*$ (e.g. phase transitions)

4. Asymptotic freedom (e.g. QCD)

9 One-loop radiative corrections in QED

9.1 Electromagnetic form factors

We are interested in the general form of matrix elements of the electromagnetic current

$$\mathcal{J}^\mu(x) = \sum_f Q_f \bar{\psi}_f \gamma^\mu \psi_f \,. \tag{9.1}$$

Translational invariance implies

$$\langle \boldsymbol{p}', \sigma' | \mathcal{J}^\mu(x) | \boldsymbol{p}, \sigma \rangle = e^{-i(p-p')\cdot x} \langle \boldsymbol{p}', \sigma' | \mathcal{J}^\mu(0) | \boldsymbol{p}, \sigma \rangle \,. \tag{9.2}$$

This can be derived as follows. First, we recognize that spatial translation leave the Lagrangian (though not the Lagrangian density!) invariant. We can use the methods in Sec. 5.3 to find the QM generators for the translations. (This procedure works in general to find the QM generators of any symmetry that leaves the Lagrangian invariant; see Ref. [1] for the details.)

Consider a field variation $\Psi^\ell(x) \to \Psi^\ell(x) + i\epsilon(t)\mathcal{F}^\ell(x)$, where ϵ depends only on time. The variation of the action is then

$$\delta I = i \int dt \int d^3x \left[\frac{\delta \mathcal{L}}{\delta \Psi^\ell(\boldsymbol{x},t)} \epsilon(t) \mathcal{F}^\ell(\boldsymbol{x},t) + \frac{\delta \mathcal{L}}{\delta \dot{\Psi}^\ell(\boldsymbol{x},t)} \frac{d}{dt} \big(\epsilon(t) \mathcal{F}^\ell(\boldsymbol{x},t) \big) \right] . \tag{9.3}$$

For the Lagrangian to be invariant for constant ϵ we need

$$0 = \int d^3x \left[\frac{\delta \mathcal{L}}{\delta \Psi^\ell(\boldsymbol{x},t)} \mathcal{F}^\ell(\boldsymbol{x},t) + \frac{\delta \mathcal{L}}{\delta \dot{\Psi}^\ell(\boldsymbol{x},t)} \frac{d}{dt} \mathcal{F}^\ell(\boldsymbol{x},t) \right] , \tag{9.4}$$

so for general fields (that may not satisfy the e.o.m.) we have

$$\delta I = i \int dt \int d^3x \frac{\delta \mathcal{L}}{\delta \dot{\Psi}^\ell(\boldsymbol{x},t)} \dot{\epsilon}(t) \mathcal{F}^\ell(\boldsymbol{x},t) \,. \tag{9.5}$$

Comparison with (5.31) yields

$$F = -i \int d^3x \frac{\delta \mathcal{L}}{\delta \dot{\Psi}^\ell(\boldsymbol{x},t)} \mathcal{F}^\ell(\boldsymbol{x},t) \,. \tag{9.6}$$

Now we use the definition of the canonically conjugated variable,

$$P_n(\boldsymbol{x},t) = \frac{\delta \mathcal{L}[Q(t), \dot{Q}(t)]}{\delta \dot{Q}^n(\boldsymbol{x},t)} \,, \tag{9.7}$$

to rewrite Eq. (9.7) as

$$F = -i \int d^3x P_n(\boldsymbol{x},t) \mathcal{F}^n(\boldsymbol{x},t) \,. \tag{9.8}$$

Since $dF/dt = 0$, we can use the equal-time commutation relations to calculate

$$[F, Q^n(\boldsymbol{x},t)] = -\mathcal{F}^n(\boldsymbol{x},t) \,. \tag{9.9}$$

For an infinitesimal translation, we have $\mathcal{F}^n(\boldsymbol{x},t) = -i\nabla Q^n(\boldsymbol{x},t)$ and the generator of translations is, therefore,

$$\boldsymbol{P} = -\int d^3x P_n(\boldsymbol{x},t) \nabla Q^n(\boldsymbol{x},t) \,. \tag{9.10}$$

It has the following commutation relations:

$$[\boldsymbol{P}, Q^n(\boldsymbol{x}, t)] = i\nabla Q^n(\boldsymbol{x}, t)\,, \tag{9.11}$$

$$[\boldsymbol{P}, P_n(\boldsymbol{x}, t)] = i\nabla P_n(\boldsymbol{x}, t)\,, \tag{9.12}$$

so for any function $G[Q, P]$ that does not explicitly depend on $\boldsymbol{x}$, we have

$$[\boldsymbol{P}, G(x)] = i\nabla G(x)\,. \tag{9.13}$$

Furthermore, we have seen in Sec. 2.1 that the Hamiltonian H acts as the generator of time translations:

$$[H, G(x)] = -i\dot{G}(x)\,. \tag{9.14}$$

Hence, we have in total

$$[P_\mu, O(x)] = -i\frac{\partial}{\partial x^\mu}O(x)\,, \tag{9.15}$$

for any local field operator $O(x)$. It follows

$$\langle\beta|[P_\mu, O(x)]|\alpha\rangle = (p_\beta - p_\alpha)_\mu\langle\beta|O(x)|\alpha\rangle = -i\frac{\partial}{\partial x^\mu}\langle\beta|O(x)|\alpha\rangle\,, \tag{9.16}$$

and so

$$\langle\beta|O(x)|\alpha\rangle = \exp[i(p_\beta - p_\alpha)\cdot x]\langle\beta|O(0)|\alpha\rangle\,. \tag{9.17}$$

Current conservation then requires

$$0 = (p - p')_\mu\langle\boldsymbol{p}', \sigma'|\mathcal{J}^\mu(0)|\boldsymbol{p}, \sigma\rangle\,. \tag{9.18}$$

Setting $\mu = 0$ in Eq. (9.2), integrating over $\boldsymbol{x}$, and recalling Eq. (5.33) gives

$$\langle\boldsymbol{p}', \sigma'|Q|\boldsymbol{p}, \sigma\rangle = (2\pi)^3\delta^3(\boldsymbol{p}' - \boldsymbol{p})\langle\boldsymbol{p}', \sigma'|\mathcal{J}^0(0)|\boldsymbol{p}, \sigma\rangle\,. \tag{9.19}$$

Assuming that the charge of the state $|\boldsymbol{p}, \sigma\rangle$ is q, we find

$$\langle\boldsymbol{p}, \sigma'|\mathcal{J}^0(0)|\boldsymbol{p}, \sigma\rangle = \frac{q}{(2\pi)^3}\delta_{\sigma'\sigma}\,. \tag{9.20}$$

For on-shell, spin-$1/2$ external states Lorentz covariance requires

$$\langle\boldsymbol{p}', \sigma'|\mathcal{J}^\mu(0)|\boldsymbol{p}, \sigma\rangle = \frac{q}{(2\pi)^3}\bar{u}(\boldsymbol{p}', \sigma')\Gamma^\mu(p', p)u(\boldsymbol{p}, \sigma)\,, \tag{9.21}$$

with $\Gamma^\mu(p', p)$ a 4×4 matrix. One can show [1] that

$$\begin{aligned}&\bar{u}(\boldsymbol{p}', \sigma')\Gamma^\mu(p', p)u(\boldsymbol{p}, \sigma)\\ =&\bar{u}(\boldsymbol{p}', \sigma')\left[\gamma^\mu F(q^2) + \frac{(p+p')^\mu}{2m}G(q^2) + \frac{i(p-p')^\mu}{2m}H(q^2)\right]u(\boldsymbol{p}, \sigma)\,,\end{aligned} \tag{9.22}$$

with F, G, H real functions of $q^2 \equiv (p - p')^2 = 2m^2 - 2p\cdot p'$. Current conservation (9.18) implies $H(q^2) \equiv 0$. One can use Gordon's identity

$$\bar{u}(\boldsymbol{p}', \sigma')\gamma^\mu u(\boldsymbol{p}, \sigma) = u(\boldsymbol{p}', \sigma')\left[\frac{(p+p')^\mu}{2m} + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\right]u(\boldsymbol{p}, \sigma)\,, \tag{9.23}$$

with $\sigma^{\mu\nu} = i[\gamma^\mu, \gamma^\nu]/2$, to write (9.22) as

$$\bar{u}(\boldsymbol{p}', \sigma')\Gamma^\mu(p', p)u(\boldsymbol{p}, \sigma) = u(\boldsymbol{p}', \sigma')\left[\gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2(q^2)\right]u(\boldsymbol{p}, \sigma)\,, \tag{9.24}$$

where

$$F_1(q^2) = F(q^2) + G(q^2)\,, \tag{9.25}$$
$$F_2(q^2) = -G(q^2)\,. \tag{9.26}$$

For $\boldsymbol{p}' \to \boldsymbol{p}$ we find

$$\langle \boldsymbol{p}, \sigma' | \mathcal{J}^\mu(0) | \boldsymbol{p}, \sigma\rangle = \frac{q}{(2\pi)^3}\bar{u}(\boldsymbol{p}, \sigma')\gamma^\mu F_1(0) u(\boldsymbol{p}, \sigma)\,. \tag{9.27}$$

Using

$$\bar{u}(\boldsymbol{p}, \sigma')\gamma^0 u(\boldsymbol{p}, \sigma) = u^\dagger(\boldsymbol{p}, \sigma')u(\boldsymbol{p}, \sigma) \overset{(4.113)}{=} \delta_{\sigma'\sigma} \tag{9.28}$$

we get

$$\langle \boldsymbol{p}, \sigma' | \mathcal{J}^0(0) | \boldsymbol{p}, \sigma\rangle = \frac{q}{(2\pi)^3}\delta_{\sigma'\sigma}F_1(0)\,, \tag{9.29}$$

and comparing with (9.20) gives the normalization condition

$$F_1(0) = 1\,. \tag{9.30}$$

Let's rewrite (9.22) in a third way:

$$\bar{u}(\boldsymbol{p}', \sigma')\Gamma^\mu(p', p)u(\boldsymbol{p}, \sigma) = u(\boldsymbol{p}', \sigma')\left[\frac{(p+p')_\mu}{2m}\gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\big(F_1(q^2) + F_2(q^2)\big)\right]u(\boldsymbol{p}, \sigma)\,. \tag{9.31}$$

Consider the nonrelativistic limit $|\boldsymbol{p}|, |\boldsymbol{p}|' \ll m$. The first term in the brackets in Eq. (9.31) is spin independent and leads to Coulomb scattering. The second term is spin dependent and leads to the magnetic moment of the fermion.

For zero external momenta, we have using Eq.s (4.102), (4.103), (4.105), (4.106)

$$\bar{u}(0, \sigma')[\gamma^i, \gamma^j]u(0, \sigma) = -4i\epsilon^{ijk}\big(\mathcal{J}^k_{1/2}\big)_{\sigma'\sigma}\,, \tag{9.32}$$
$$\bar{u}(0, \sigma')[\gamma^i, \gamma^0]u(0, \sigma) = 0\,, \tag{9.33}$$

where $\boldsymbol{\mathcal{J}}_{1/2} \equiv \boldsymbol{\sigma}/2$ is the spin operator for spin $1/2$. So, to first order in $\boldsymbol{p}, \boldsymbol{p}'$ we have

$$\bar{u}(\boldsymbol{p}', \sigma')\boldsymbol{\Gamma}(p', p)u(\boldsymbol{p}, \sigma) \to \frac{\boldsymbol{p} + \boldsymbol{p}'}{2m}\delta_{\sigma'\sigma} + i\frac{[(\boldsymbol{p} - \boldsymbol{p}') \times \boldsymbol{\mathcal{J}}_{1/2}]_{\sigma'\sigma}}{m}\big(1 + F_2(0)\big)\,. \tag{9.34}$$

Now we couple this current to a weak, time-independent external vector potential $\boldsymbol{A}(\boldsymbol{x})$ with interaction Hamiltonian

$$H = -\int d^3x\, \boldsymbol{\mathcal{J}}_{1/2}(\boldsymbol{x}) \cdot \boldsymbol{A}(\boldsymbol{x})\,. \tag{9.35}$$

We obtain

$$\begin{aligned}\langle \boldsymbol{p}', \sigma'|H|\boldsymbol{p}, \sigma\rangle &= \frac{-iq(1+F_2(0))}{(2\pi)^3 m}\int d^3x\, e^{-i(\boldsymbol{p}-\boldsymbol{p}')\cdot\boldsymbol{x}}\boldsymbol{A}(\boldsymbol{x})\cdot[(\boldsymbol{p}-\boldsymbol{p}')\times\boldsymbol{\mathcal{J}}_{1/2}]_{\sigma'\sigma} \\ &= \frac{-q(1+F_2(0))}{(2\pi)^3 m}\int d^3x\, e^{-i(\boldsymbol{p}-\boldsymbol{p}')\cdot\boldsymbol{x}}\boldsymbol{B}(\boldsymbol{x})\cdot(\boldsymbol{\mathcal{J}}_{1/2})_{\sigma'\sigma}\,,\end{aligned} \tag{9.36}$$

where $\boldsymbol{B} = \nabla\times\boldsymbol{A}$. If $\boldsymbol{B}$ is slowly varying, we can take it out of the integral and get

$$\langle \boldsymbol{p}', \sigma'|H|\boldsymbol{p}, \sigma\rangle = \frac{-q(1+F_2(0))}{m}\boldsymbol{B}(\boldsymbol{x})\cdot(\boldsymbol{\mathcal{J}}_{1/2})_{\sigma'\sigma}\delta(\boldsymbol{p}'-\boldsymbol{p})\,. \tag{9.37}$$

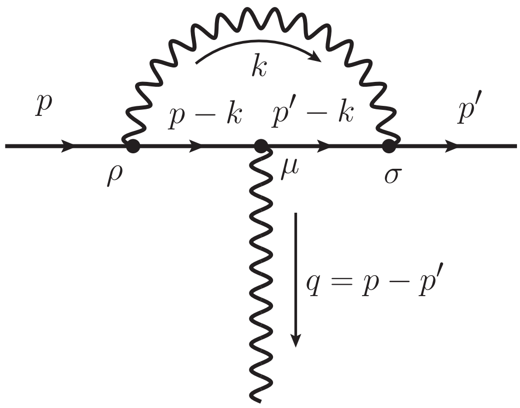
By definition,

$$\mu = \frac{q}{2m}(1+F_2(0)) \tag{9.38}$$

is the magnetic moment of a particle with charge q, mass m, and spin $1/2$. Without radiative corrections, we obtain Dirac's result $\mu = q/(2m)$ for an elementary particle.

9.2 Anomalous magnetic moment

The shift at leading-order QED in the magnetic moment is given by the one-loop diagram:



Using the Feynman rules we obtain (I'm dropping the external-line factors for simplicity)

$$\begin{aligned}&-ie(2\pi)^4\Gamma^\mu_{\text{1loop}}(p', p) \\ &\quad\equiv \int d^4k(-ie)(2\pi)^4\gamma^\sigma\frac{i}{(2\pi)^4}\frac{\not p' - \not k + m}{(p'-k)^2 - m^2 + i\epsilon}(-ie)(2\pi)^4\gamma^\mu \\ &\qquad\times\frac{i}{(2\pi)^4}\frac{\not p - \not k + m}{(p-k)^2 - m^2 + i\epsilon}(-ie)(2\pi)^4\gamma^\rho\frac{1}{(2\pi)^4}\frac{-i\eta_{\rho\sigma}}{k^2 + i\epsilon}\,.\end{aligned} \tag{9.39}$$

This gives

$$\Gamma^\mu_{\text{1loop}} = -ie^2\int\frac{d^4k}{(2\pi)^4}\frac{\gamma_\rho(\not p' - \not k + m)\gamma^\mu(\not p - \not k + m)\gamma^\rho}{[(p'-k)^2 - m^2][(p-k)^2 - m^2]k^2}\,. \tag{9.40}$$

Now we use (exercise!)

$$\frac{1}{ABC} = 2\int_0^1 dx \int_0^x dy \, \frac{1}{[Ay + B(x-y) + C(1-x)]^3} \,, \tag{9.41}$$

so the denominator becomes (note that $p^2 = p'^2 = m^2$ and $q = p - p'$)

$$\begin{aligned} &\frac{1}{[(p'-k)^2 - m^2][(p-k)^2 - m^2]k^2} \\ &= 2\int_0^1 dx \int_0^x dy \left\{[(p'-k)^2 - m^2]y + [(p-k)^2 - m^2](x-y) + k^2(1-x)\right\}^{-3} \\ &= 2\int_0^1 dx \int_0^x dy \left\{k^2 - 2k\cdot[p'y + p(x-y)] - m^2 x + p'^2 y + p^2(x-y)\right\}^{-3} \\ &= 2\int_0^1 dx \int_0^x dy \left\{[k - p'y - p(x-y)]^2 + q^2 y(x-y) - m^2x^2\right\}^{-3} , \end{aligned} \tag{9.42}$$

so we can shift $k \to k + p'y + p(x-y)$ and get

$$\begin{aligned} \Gamma^\mu_{\text{1loop}} = -2ie^2 \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx \int_0^x dy \, \frac{1}{[k^2 + q^2 y(x-y) - m^2x^2]^3} \\ \times \gamma_\rho[\not{p}'(1-y) - \not{k} - \not{p}(x-y) + m]\gamma^\mu[\not{p}(1-x+y) - \not{k} - \not{p}'y + m]\gamma^\rho \,. \end{aligned} \tag{9.43}$$

After Wick rotation, the denominator is even in k, so we can use $k^\mu k^\nu = \eta^{\mu\nu}k^2/4$, and drop terms odd in k. This gives

$$\begin{aligned} \Gamma^\mu_{\text{1loop}} = -\frac{4\pi^2 e^2}{(2\pi)^4} \int_0^1 dx \int_0^x dy \int_0^\infty \kappa^3 d\kappa \, \frac{1}{[\kappa^2 - q^2 y(x-y) + m^2x^2]^3} \\ \times \Big\{\gamma_\rho[\not{p}'(1-y) - \not{p}(x-y) + m]\gamma^\mu[\not{p}(1-x+y) - \not{p}'y + m]\gamma^\rho \\ - \frac{\kappa^2}{4}\gamma_\rho\gamma_\sigma\gamma^\mu\gamma^\sigma\gamma^\rho\Big\} \,. \end{aligned} \tag{9.44}$$

For on-shell external states, we can use the Dirac equation

$$\bar{u}'(\not{p}' - m) = 0 \,, \qquad (\not{p} - m)u = 0 \,. \tag{9.45}$$

We use the Clifford relation (4.87) to bring all $\not{p}'$ to the left and all $\not{p}$ to the right, and find the coefficients of

$$(1-y)(1-x+y): \qquad 2\not{p}'\gamma^\mu\not{p} + 4p'\cdot p\gamma^\mu - 4p^\mu\not{p}' - 4p'^\mu\not{p} \,, \tag{9.46}$$

$$(1-y)(-y): \qquad 2m^2\gamma^\mu - 4p'^\mu \not{p}' \,, \tag{9.47}$$
$$-(x-y)(1-x+y): \qquad 2m^2\gamma^\mu - 4p^\mu \not{p} \,, \tag{9.48}$$
$$-(x-y)(-y): \qquad -2\not{p}'\gamma^\mu\not{p} \,, \tag{9.49}$$
$$(1-y)m: \qquad 4p'^\mu \,, \tag{9.50}$$
$$-(x-y)m: \qquad 4p^\mu \,, \tag{9.51}$$
$$(1-x+y)m: \qquad 4p^\mu \,, \tag{9.52}$$
$$(-y)m: \qquad 4p'^\mu \,, \tag{9.53}$$
$$m^2: \qquad -2\gamma^\mu \,. \tag{9.54}$$

Here, we used $\gamma_\rho\gamma^\mu\gamma^\rho = -2\gamma^\mu$, $\gamma_\rho\gamma^\mu\gamma^\nu\gamma^\rho = 4\eta^{\mu\nu}$, and $\gamma_\rho\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho = -2\gamma^\sigma\gamma^\nu\gamma^\mu$. Putting terms together, we find

$$\begin{aligned}\bar{u}'\big[\gamma_\rho \ldots \gamma^\rho\big]u = \bar{u}'\big[&4p^\mu(y+xy-x^2)m + 4p'^\mu(x-y-xy)m \\ &+ 4\gamma^\mu p\cdot p'(1-y^2-x+xy) \\ &+ 4m^2\gamma^\mu(y^2-x-xy+\tfrac{1}{2}x^2)\big]u\,.\end{aligned} \tag{9.55}$$

Using now $4p\cdot p' = -2(p-p')^2 + 4m^2$, we obtain

$$\begin{aligned}\bar{u}'\Gamma^\mu_{\text{1loop}}u = -\frac{4\pi^2e^2}{(2\pi)^4}\int_0^1 dx\int_0^x dy\int_0^\infty \kappa^3 d\kappa\,\frac{1}{[\kappa^2 - q^2y(x-y) + m^2x^2]^3} \\ \times\,\bar{u}'\Big\{\gamma^\mu\big[-\kappa^2 + 2m^2(x^2-4x+2) - 2q^2(y(x-y)+1-x)\big] \\ + 4mp^\mu(y+xy-x^2) - 4mp'^\mu(x-y-xy)\Big\}u\,.\end{aligned} \tag{9.56}$$

Note now that the denominator as well as the integration over y are symmetric under $y \to x-y$. Moreover, under this reflection,

$$\begin{aligned}y+xy-x^2 &\to x-y+x^2-xy-x^2 = x-y-xy\,,\\ x-y-xy &\to x+y-x-x^2+xy = y+xy-x^2\,,\end{aligned}$$

so we can replace these factors by their average,

$$\tfrac{1}{2}(y+xy-x^2) + \tfrac{1}{2}(x-y-xy) = \tfrac{1}{2}x(1-x)\,,$$

and we have, finally,

$$\begin{aligned}\bar{u}'\Gamma^\mu_{\text{1loop}}u = -\frac{4\pi^2e^2}{(2\pi)^4}\int_0^1 dx\int_0^x dy\int_0^\infty \frac{\kappa^3 d\kappa}{[\kappa^2 - q^2y(x-y) + m^2x^2]^3} \\ \times\,\bar{u}'\Big\{\gamma^\mu\big[-\kappa^2 + 2m^2(x^2-4x+2) - 2q^2(y(x-y)+1-x)\big] \\ + 2m(p+p')^\mu x(1-x)\Big\}u\,.\end{aligned} \tag{9.57}$$

This is now in the form (9.22), and we can identify the form factor $G(q^2)$:

$$G(q^2) = -\frac{16m^2\pi^2e^2}{(2\pi)^4}\int_0^1 dx\int_0^x dy\int_0^\infty \frac{\kappa^3 d\kappa\, x(1-x)}{[\kappa^2 - q^2y(x-y) + m^2x^2]^3}\,, \tag{9.58}$$

and, therefore,

$$\begin{aligned} G(0) &= -\frac{m^2e^2}{\pi^2}\int_0^1 dx\int_0^x dy\int_0^\infty \frac{\kappa^3 d\kappa\, x(1-x)}{[\kappa^2 + m^2x^2]^3} \\ &= -\frac{m^2e^2}{\pi^2}\int_0^1 dx\int_0^x dy\frac{1}{2}\int_{m^2x^2}^\infty \frac{(u-m^2x^2)du}{u^3}x(1-x) \\ &= -\frac{m^2e^2}{4\pi^2}\int_0^1 dx\int_0^x dy\,\frac{x(1-x)}{m^2x^2} = -\frac{e^2}{4\pi^2}\int_0^1 dx\,\frac{x^2(1-x)}{x^2} = -\frac{e^2}{8\pi^2} = -\frac{\alpha}{2\pi}\,. \end{aligned} \tag{9.59}$$

The magnetic moment of the electron was defined as (see Eq.s (9.38), (9.26))

$$\mu_e = \frac{e}{2m_e}(1 - G(0)) = \frac{e}{2m_e}\left(1 + \frac{\alpha}{2\pi}\right) = \frac{e}{2m_e}(1 + 0.001162)\,. \tag{9.60}$$

[Schwinger 1948]

9.3 Electric charge and the Ward identity

Gauge invariance (6.7) and Noether's theorem imply the existence of a conserved current for spinor electrodynamics

$$\mathcal{J}^\mu = -i\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Psi^\ell)}q_\ell\Psi_\ell\,, \tag{9.61}$$

with

$$\partial_\mu\mathcal{J}^\mu = 0 \tag{9.62}$$

and

$$i\frac{d}{dt}Q = [Q, H] = 0\,, \tag{9.63}$$

where

$$Q \equiv \int d^3x\mathcal{J}^0\,. \tag{9.64}$$

Obviously,

$$[Q, \boldsymbol{P}] = 0\,, \tag{9.65}$$

and, since $\mathcal{J}^\mu$ is a four-vector,

$$[Q, \mathcal{J}^{\mu\nu}] = 0 \tag{9.66}$$

(see, e.g., Ref. [10]). It follows that

$$Q|0\rangle = 0\,, \tag{9.67}$$

and

$$Q|\boldsymbol{p},\sigma,n\rangle = q(n)|\boldsymbol{p},\sigma,n\rangle\,. \tag{9.68}$$

What is the relation of the "charge" $q(n)$ to the parameters q_ℓ in th Lagrangian? The canonical commutation relations give

$$\begin{aligned}[]
[\mathcal{J}^0(\boldsymbol{x},t),\psi_\ell(\boldsymbol{y},t)] &= \left[-i\frac{\partial\mathcal{L}}{\partial\dot{\psi}_{\bar{\ell}}},\psi_\ell(\boldsymbol{y},t)\right] \\
&= -iq_\ell(-i\delta^3(\boldsymbol{x}-\boldsymbol{y})\delta_{\bar{\ell}\ell})\psi_{\bar{\ell}}(\boldsymbol{x},t) = -q_\ell\psi_\ell(\boldsymbol{x},t)\delta^3(\boldsymbol{x}-\boldsymbol{y})\,,
\end{aligned} \tag{9.69}$$

and integrating over $\boldsymbol{x}$ we find

$$[Q,\psi_\ell(\boldsymbol{y},t)] = -q_\ell\psi_\ell(\boldsymbol{y},t)\,. \tag{9.70}$$

Taking the matrix element of (9.70) between the vacuum and a one-particle state, and using (9.67) and (9.68), we have

$$0 = \langle 0|[Q,\psi_\ell(\boldsymbol{y},t)] + q_\ell\psi_\ell(\boldsymbol{y},t)|\boldsymbol{p},\sigma,n\rangle = \langle 0|\psi_\ell(\boldsymbol{y},t)|\boldsymbol{p},\sigma,n\rangle(q_\ell - q(n))\,. \tag{9.71}$$

It follows that $q_\ell = q(n)$. However, the absolute scale of q_ℓ in a gauge transformation $\psi_\ell \to \exp(iq_\ell\epsilon)\psi_\ell$ is not fixed. Writing the QED interaction in terms of the *covariant derivative*

$$D_\mu \equiv \partial_\mu - iq_\ell A_\mu \tag{9.72}$$

as

$$\bar{\psi}_\ell(i\not{D} - m)\psi_\ell\,, \tag{9.73}$$

we define the physical electric charge as the parameter appearing in the covariant derivative of the *renormalized* photon field. Writing, as usual, $A^\mu_{\text{bare}} = Z_A^{1/2}A^\mu$, the covariant derivative becomes

$$\partial_\mu - iq_\ell^{\text{bare}}A_\mu^{\text{bare}} = \partial_\mu - iq_\ell^{\text{bare}}Z_A^{1/2}A_\mu \equiv \partial_\mu - iq_\ell A_\mu\,, \tag{9.74}$$

and thus

$$q_\ell = Z_A^{1/2}q_\ell^{\text{bare}}\,. \tag{9.75}$$

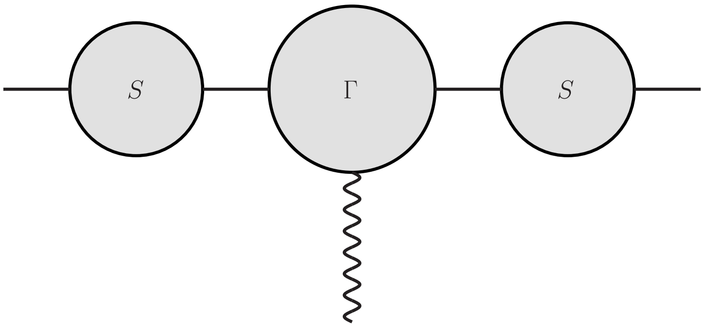
Why is the photon field renormalization sufficient to cancel all divergences? The cancelation of all other divergences is guaranteed by the *Ward identity*. To derive the Ward identity, we define the electromagnetic vertex function by

$$\begin{aligned}
&\int d^4x\,d^4y\,d^4z\,e^{iq\cdot x}e^{ip\cdot y}e^{-ip'\cdot z}\langle 0|T\{\mathcal{J}^\mu(x)\psi_k(y)\bar{\psi}_l(z)\}|0\rangle \\
&\qquad \equiv i(2\pi)^4 eS_{kk'}(p)\Gamma_{k'l'}(p,p')S_{ll'}(p')\delta^4(q+p-p')\,,
\end{aligned} \tag{9.76}$$

where

$$i(2\pi)^4S_{kl}(p)\delta^4(p-p') \equiv \int d^4y\,d^4z\,e^{ip\cdot y}e^{-ip'\cdot z}\langle 0|T\{\psi_k(y)\bar{\psi}_l(z)\}|0\rangle \tag{9.77}$$

is the fermion propagator function.



Using (7.15), it is easy to see that in the limit of no interactions

$$S(p) \to \frac{1}{\not{p} - m + i\epsilon}\,, \qquad \Gamma^\mu(p,p') \to \gamma^\mu\,. \tag{9.78}$$

Now we calculate

$$\begin{aligned}
&\frac{\partial}{\partial x^\mu} T\{\mathcal{J}^\mu(x)\psi_k(y)\bar{\psi}_l(z)\} \\
&= \frac{\partial}{\partial x^\mu}\big[\quad \theta(x^0-y^0)\theta(y^0-z^0)\mathcal{J}^\mu(x)\psi_k(y)\bar{\psi}_l(z) \\
&\qquad + \theta(y^0-x^0)\theta(x^0-z^0)\psi_k(y)\mathcal{J}^\mu(x)\bar{\psi}_l(z) \\
&\qquad - \theta(x^0-z^0)\theta(z^0-y^0)\mathcal{J}^\mu(x)\bar{\psi}_l(z)\psi_k(y) \\
&\qquad - \theta(z^0-x^0)\theta(x^0-y^0)\bar{\psi}_l(z)\mathcal{J}^\mu(x)\psi_k(y) \\
&\qquad - \theta(z^0-y^0)\theta(y^0-x^0)\bar{\psi}_l(z)\psi_k(y)\mathcal{J}^\mu(x) \\
&\qquad + \theta(y^0-z^0)\theta(z^0-x^0)\psi_k(y)\bar{\psi}_l(z)\mathcal{J}^\mu(x)\big] \\
&= \ldots = T\{\partial_\mu \mathcal{J}^\mu(x)\psi_k(y)\bar{\psi}_l(z)\} \\
&\qquad + \delta(x^0-y^0)T\{[\mathcal{J}^0(x),\psi_k(y)]\bar{\psi}_l(z)\} \\
&\qquad + \delta(x^0-z^0)T\{\psi_k(y)[\mathcal{J}^0(x),\bar{\psi}_l(z)]\}\,.
\end{aligned} \tag{9.79}$$

Using $\partial_\mu \mathcal{J}^\mu = 0$ and (9.69), as well as

$$[\mathcal{J}^0(\boldsymbol{x},t), \bar{\psi}_\ell(\boldsymbol{y},t)] = q_\ell \bar{\psi}_\ell(\boldsymbol{x},t)\delta^3(\boldsymbol{x}-\boldsymbol{y}) \tag{9.80}$$

(here, $q_\ell \equiv e$), we get

$$\begin{aligned}
&\frac{\partial}{\partial x^\mu} T\{\mathcal{J}^\mu(x)\psi_k(y)\bar{\psi}_l(z)\} \\
&\qquad = -e\delta^4(x-y)T\{\psi_k(y)\bar{\psi}_l(z)\} + e\delta^4(x-z)T\{\psi_k(y)\bar{\psi}_l(z)\}\,.
\end{aligned} \tag{9.81}$$

Using this in (9.76), (9.77) gives

$$(p'-p)_\mu S(p)\Gamma^\mu(p,p')S(p') = S(p) - S(p')\,, \tag{9.82}$$

or

$$\boxed{(p'-p)_\mu \Gamma^\mu(p,p') = S^{-1}(p') - S^{-1}(p)\,.} \tag{9.83}$$

This is the *Ward-Takahashi identity*. Taking the limit $p' \to p$ we get

$$\Gamma^\mu(p,p') = \frac{\partial}{\partial p^\mu} S^{-1}(p)\,. \tag{9.84}$$

Writing the full (inverse) propagator in terms of 1PI self-energy diagrams $\Sigma(p)$ as

$$S^{-1}(p) = \not{p} - m + \Sigma(p)\,, \tag{9.85}$$

Eq. (9.84) becomes

$$\Gamma^\mu(p,p') = \gamma^\mu + \frac{\partial}{\partial p^\mu}\Sigma(p)\,. \tag{9.86}$$

Hence, for a properly (on-shell) renormalized Dirac field, we find that on the mass shell

$$\bar{u}'\Gamma^\mu(p,p')u = \bar{u}'\gamma^\mu u\,. \tag{9.87}$$

The vertex corrections cancel for the interaction of an on-shell fermion in the limit of vanishing momentum transfer.

9.4 Gauge invariance and photon mass

Consider the matrix element

$$\begin{aligned} M^{\mu_1\mu_2\cdots}_{\beta\alpha}(q_1,q_2,\ldots) \equiv \int & d^4x_1\, d^4x_2\, \ldots\, e^{iq_1\cdot x_1}e^{iq_2\cdot x_2}\cdots \\ & \times \langle\beta,\text{out}|T\{\mathcal{J}^{\mu_1}(x_1)\mathcal{J}^{\mu_2}(x_2)\ldots\}|\alpha,\text{in}\rangle\,. \end{aligned} \tag{9.88}$$

We want to show that $q_{\mu_1}M^{\mu_1\mu_2\cdots}_{\beta\alpha} = q_{\mu_2}M^{\mu_1\mu_2\cdots}_{\beta\alpha} = \ldots = 0$. Integration by parts gives

$$\begin{aligned} q_{\mu_1}M^{\mu_1\mu_2\cdots}_{\beta\alpha}(q_1,q_2,\ldots) = i\int & d^4x_1\, d^4x_2\, \ldots\, e^{iq_1\cdot x_1}e^{iq_2\cdot x_2}\cdots \\ & \times \langle\beta,\text{out}|\frac{\partial}{\partial x_1^\mu}T\{\mathcal{J}^{\mu_1}(x_1)\mathcal{J}^{\mu_2}(x_2)\ldots\}|\alpha,\text{in}\rangle\,. \end{aligned} \tag{9.89}$$

Only the contact terms contribute; e.g., for two currents

$$\begin{aligned} & \frac{\partial}{\partial x^\mu}T\{\mathcal{J}^\mu(x)\mathcal{J}^\nu(y)\} \\ &= \frac{\partial}{\partial x^\mu}\left[\theta(x^0-y^0)\mathcal{J}^\mu(x)\mathcal{J}^\nu(y) + \theta(y^0-x^0)\mathcal{J}^\nu(y)\mathcal{J}^\mu(x)\right] \\ &= \delta(x^0-y^0)\mathcal{J}^0(x)\mathcal{J}^\nu(y) + \delta(y^0-x^0)\mathcal{J}^\nu(y)\mathcal{J}^0(x) \\ &= \delta(x^0-y^0)[\mathcal{J}^0(x),\mathcal{J}^\nu(y)]\,. \end{aligned} \tag{9.90}$$

Since the current is electrically neutral (or using (9.69) and (9.80)), we have

$$[\mathcal{J}^0(\boldsymbol{x},t),\mathcal{J}^\nu(\boldsymbol{y},t)] = 0 \tag{9.91}$$

and, therefore,

$$q_\mu M^{\mu\mu_1\mu_2\cdots}_{\beta\alpha}(q, q_1, q_2, \ldots) = 0\,. \tag{9.92}$$

It follows that S-matrix elements are unaffected if we change any photon propagator by

$$\Delta_{\mu\nu}(q) \to \Delta_{\mu\nu}(q) + \alpha_\mu q_\nu + q_\mu \beta_\nu\,, \tag{9.93}$$

or any polarization vector by

$$e_\rho(\boldsymbol{k}, \lambda) \to e_\rho(\boldsymbol{k}, \lambda) + c\, k_\rho\,, \tag{9.94}$$

where $k^0 \equiv |\boldsymbol{k}|$ and α_μ, β_ν, and c are entirely arbitrary.

As an application, consider the "full" photon propagator

$$\Delta'_{\mu\nu}(q) = \Delta_{\mu\nu}(q) + \Delta_{\mu\rho}(q) M^{\rho\sigma} \Delta_{\sigma\nu}(q)\,, \tag{9.95}$$

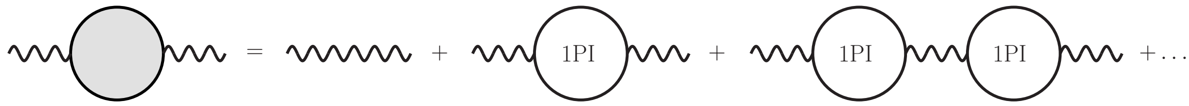
where $M^{\rho\sigma}$ is (9.88) with $\beta = \langle 0|$, $\alpha = |0\rangle$. Here, the bare photon propagator in R_ξ gauge is given by[2]

$$\Delta_{\mu\nu}(q) = \frac{-i}{q^2}\left[\eta_{\mu\nu} - (1-\xi)\frac{q_\mu q_\nu}{q^2}\right]. \tag{9.96}$$

Eq. (9.92) implies $q^\mu M_{\mu\nu} = 0$, such that

$$\begin{aligned} q^\mu \Delta'_{\mu\nu}(q) &= q^\mu \frac{-i}{q^2}\left[\eta_{\mu\nu} - (1-\xi)\frac{q_\mu q_\nu}{q^2}\right] + q^\mu \eta_{\mu\rho} M^{\rho\sigma} \eta_{\sigma\nu} \\ &= q^\mu \Delta_{\mu\nu}(q) = \frac{-i}{q^2}\Big[q_\nu - (1-\xi) q_\nu\Big] = \frac{-i\xi q_\nu}{q^2}\,. \end{aligned} \tag{9.97}$$

We can express the photon propagator in terms of 1PI diagrams $\Pi^*_{\mu\nu}(q)$,



$$\begin{aligned} \Delta'_{\mu\nu}(q) &= \Delta_{\mu\nu}(q) + \big(\Delta(q)\Pi^*(q)\Delta(q)\big)_{\mu\nu} + \big(\Delta(q)\Pi^*(q)\Delta(q)\Pi^*(q)\Delta(q)\big)_{\mu\nu} + \ldots \\ &= \Big[\Delta(q)\big(1 - \Pi^*(q)\Delta(q)\big)^{-1}\Big]_{\mu\nu} = \big(\Delta^{-1}(q) - \Pi^*(q)\big)^{-1}_{\mu\nu} \\ &= \Delta_{\mu\nu}(q) + \big(\Delta(q)\Pi^*(q)\Delta'(q)\big)_{\mu\nu}\,. \end{aligned} \tag{9.98}$$

Eq. (9.97) then implies

$$q^\rho \Pi^*_{\rho\sigma}(q) = 0\,. \tag{9.99}$$

Using Lorentz covariance, we see that

$$-i\Pi^*_{\rho\sigma}(q) = (q^2 \eta_{\rho\sigma} - q_\rho q_\sigma)\Pi(q^2)\,. \tag{9.100}$$

[2] This result will be derived, using path-integral methods, in QFT2.

Using $\Delta^{-1}_{\mu\nu}(q) = i\eta_{\mu\nu}q^2 - i(1-1/\xi)q_\mu q_\nu$, Eq. (9.98) gives the complete propagator

$$\Delta'_{\mu\nu}(q) = \frac{-i}{q^2[1-\Pi(q^2)]}\left(\eta_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\big(1-\xi[1-\Pi(q^2)]\big)\right). \tag{9.101}$$

Since $\Pi^*_{\mu\nu}(q)$ is 1PI, it has no pole at $q^2 = 0$. Hence, $\Pi(q^2)$ has no pole at $q^2 = 0$. It follows that radiative corrections do not give the photon a mass.

For an (on-shell) renormalized photon field, the residue of the gauge-independent part of the photon propagator should be $-i\eta_{\mu\nu}$, so we require

$$\Pi(q^2 = 0) = 0\,. \tag{9.102}$$

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