PHYS 8015 – Particle Physics I

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Abstract

"It doesn't matter what we cover. It matters what you discover."
[Attributed to Viktor Weisskopf, theoretical physicist, 1908 – 2002]

Contents

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particle	spin/helicity	baryon number	lepton number	electric charge
up, charm, top	1/2	1/3	0	2/3
down, strange, bottom	1/2	1/3	0	-1/3
ν_e, ν_μ, ν_τ	1/2	0	1	0
electron, μ, τ	1/2	0	1	-1
photon	1	0	0	0
gluon	1	0	0	0
$W^\pm$, Z	1	0	0	$\pm$ 1, 0
h	0	0	0	0

Table 1: *SM particle content*

symmetry	conserved quantity
Spatial translations	Momentum
Time translations	Energy
Rotations	Angular momentum
Global inner symmetries	Charge (electric, color, ...)
"Accidental" symmetries	baryon number, lepton number

Table 2: *Symmetries*

1 Introductory Remarks

Elementary particle physics is the search for the fundamental laws of nature. The theoretical framework is given by relativistic quantum field theories (QFT – a consistent combination of quantum mechanics and special relativity). While QFT gives the general framework, it is brought to life by assigning a specific particle content (a "model").

"Elementary" means roughly "no substructure". More precisely, a QM state that is characterized by its energy, its momentum, and further *discrete* quantum numbers, such as charge, spin, etc. I.e. we label each free one-particle state as $|E\boldsymbol{p}n\rangle$, where the discrete index n denotes particle type, spin, charge, etc. The most important model of particle physics is the *Standard Model*. Its particle content is summarized in Tab. 1.

What are suitable observables? According to the principles of QM, we cannot exactly observe particle trajectories.

Time-independent: bound states (energy spectra, ...)

Time-dependent: scattering processes, particle decays, ...

Example: muon decay $\mu^- \to e^- \bar{\nu}_e \nu_\mu$

The number of possible processes is restriced by symmetries: (Non-)examples: $\mu^- \to e^- \nu_e$, $p \to e^+$, $p \to e^+ \gamma$, $e^- \to \mu^- \bar{\nu}_\mu \nu_e$.

2 Relativistic Kinematics

See this PDG article for a good review. In general, the PDG has excellent review articles on the topics of this lecture.

2.1 Lorentz transformations

The laws of nature are invariant under *Lorentz transformations* – space-time transformations of all four-vectors

$$x^\mu \equiv (x^0, \boldsymbol{x}) \tag{2.1}$$

that leave the Lorentz scalar product

$$x \cdot y \equiv x^0 y^0 - \boldsymbol{x} \cdot \boldsymbol{y} \tag{2.2}$$

invariant. Defining the *metric tensor*

$$\eta_{\mu\nu} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} , \tag{2.3}$$

Eq. (2.2) can be written

$$x \cdot y = \sum_{\mu\nu} x^\mu y^\nu \eta_{\mu\nu} \equiv \sum_\mu x^\mu y_\mu \tag{2.4}$$

(the summation sign is often dropped by convention). It follows that Lorentz transformations

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \tag{2.5}$$

must satisfy

$$\eta_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = \eta_{\rho\sigma} , \tag{2.6}$$

because then

$$x' \cdot y' = x'^\mu y'^\nu \eta_{\mu\nu} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma x^\rho y^\sigma \eta_{\mu\nu} = x^\rho y^\sigma \eta_{\rho\sigma} = x \cdot y \, . \tag{2.7}$$

Application: Four-momentum. For a particle with mass m in the rest system, we define $p^\mu = (m, \mathbf{0})$, and $p^2 = m^2$ is invariant.

2.2 Kinematics of scattering processes

We can deduce several properties of $p_a p_b \to p_c p_d$ scattering without knowledge of the detailed dynamics of the process. Conservation of energy and momentum implies

$$p_a + p_b = p_c + p_d \, . \tag{2.8}$$

In the center-of-mass system (c.m.s.), denoted here by the asterisk, we have

$$\boldsymbol{p}_a^* + \boldsymbol{p}_b^* = \boldsymbol{p}_c^* + \boldsymbol{p}_d^* = 0 \, . \tag{2.9}$$

As an example, we would like to express energy and momentum of particle a in the rest frame of particle b ("lab frame") in terms of Lorentz-invariant quantities. We define

$$\begin{aligned} s \equiv (p_a + p_b)^2 &= \underbrace{(E_a^* + E_b^*)^2}_{\text{c.m.s.}} \\ &= p_a^2 + p_b^2 + 2p_a \cdot p_b \\ &= \underbrace{m_a^2 + m_b^2 + 2E_a m_b}_{\text{lab frame}} \, . \end{aligned} \tag{2.10}$$

From this expression we obtain the energy

$$E_a = \frac{s - m_a^2 - m_b^2}{2m_b} \tag{2.11}$$

and the absolute value of the momentum

$$\begin{aligned} |\boldsymbol{p}_a| &= \sqrt{E_a^2 - m_a^2} \\ &= \frac{1}{2m_b}\sqrt{s^2 + m_a^4 + m_b^4 - 2sm_a^2 - 2sm_b^2 + 2m_a^2 m_b^2 - 4m_a^2 m_b^2} \\ &\equiv \frac{1}{2m_b}\lambda(s, m_a^2, m_b^2) \, , \end{aligned} \tag{2.12}$$

with the symmetric *phase-space function*

$$\lambda(x, y, z) \equiv \sqrt{x^2 + y^2 + z^2 - 2xy - 2xz - 2yz} \, . \tag{2.13}$$

More generally, we can express *all* kinematic variables in $2 \to 2$ scattering in terms of the Lorentz-invariant *Mandelstam variables*

$$s \equiv (p_a + p_b)^2 \, , \tag{2.14}$$

$$t \equiv (p_a - p_c)^2 \, , \tag{2.15}$$

$$u \equiv (p_a - p_d)^2 \, . \tag{2.16}$$

They are not linearly independent:

$$\begin{aligned} s + t + u &= (p_a + p_b)^2 + (p_a - p_c)^2 + (p_a - p_d)^2 \\ &= m_a^2 + m_b^2 + m_c^2 + m_d^2 + 2p_a \cdot (p_a + p_b - p_c - p_d) = \sum_i m_i^2 \, . \end{aligned} \tag{2.17}$$

(There are only two independent kinematic variables for $2 \to 2$ scattering: of the 8 components of the momenta p_c and p_d, the energies are fixed by $E_i = \sqrt{m_i^2 + |\boldsymbol{p}_i|^2}$. Conservation of energy and momentum yields four more conditions, so we have 8-2-3-1 = 2 independent variables.) Every choice of two variables (energies, scattering angles, …) can be expressed in terms of two Mandelstam variables.

Integration on the mass shell

The Lorentz-invariant integral of an arbitrary (Lorentz-)scalar function $f(p)$ of the four-momentum p^μ with $p^2 = m^2 > 0$ and $p^0 > 0$ can be written as

$$\begin{aligned}
&\int d^4p\, \delta(p^2 - m^2)\theta(p^0) f(p) \\
&= \int d^3\boldsymbol{p} dp^0 \delta\big((p^0)^2 - \boldsymbol{p}^2 - m^2\big)\theta(p^0) f(\boldsymbol{p}, p^0) \\
&= \int d^3\boldsymbol{p} \frac{f(\boldsymbol{p}, \sqrt{\boldsymbol{p}^2 + m^2})}{2\sqrt{\boldsymbol{p}^2 + m^2}} \,.
\end{aligned} \tag{2.18}$$

It follows that the Lorentz-invariant integration measure for integration "on the mass shell" is $d^3\boldsymbol{p}/\sqrt{\boldsymbol{p}^2 + m^2}$. The Dirac delta function is defined by

$$F(\boldsymbol{p}) = \int F(\boldsymbol{p}')\delta\big(\boldsymbol{p} - \boldsymbol{p}'\big) d^3\boldsymbol{p} = \int F(\boldsymbol{p}') \Big[\sqrt{\boldsymbol{p}'^2 + m^2}\delta\big(\boldsymbol{p} - \boldsymbol{p}'\big)\Big] \frac{d^3\boldsymbol{p}}{\sqrt{\boldsymbol{p}'^2 + m^2}} \,, \tag{2.19}$$

so the invariant delta function is given by

$$\sqrt{\boldsymbol{p}'^2 + m^2}\delta\big(\boldsymbol{p} - \boldsymbol{p}'\big) = p^0 \delta\big(\boldsymbol{p} - \boldsymbol{p}'\big) \,. \tag{2.20}$$

3 Scattering theory

3.1 The S-matrix

We are interested in scattering experiments. For $t \to -\infty$: particles far apart, not interacting; for finite t: interaction; for $t \to \infty$: particles far apart, not interacting.

Assume we can split the Hamiltonian H of a system into a free part H_0 and an interaction term V,

$$H = H_0 + V \,, \tag{3.1}$$

such that H_0 describes an arbitrary number of non-interacting particles, and V is Hermitian, describes the interaction, and vanishes if all particles are far apart from each other.

We will use the Heisenberg picture and define incoming and outgoing states, $|\alpha, +\rangle$ and $|\alpha, -\rangle$, as eigenstates of the full Hamiltonian,

$$H|\alpha, \pm\rangle = E_\alpha |\alpha, \pm\rangle \,, \tag{3.2}$$

such that they appear, for measurements at $t \to \mp\infty$, like eigenstates $|\alpha\rangle_0$ of the free Hamiltonian,

$$H_0|\alpha\rangle_0 = E_\alpha |\alpha\rangle_0 \,. \tag{3.3}$$

Our normalization convention for the states is

$${}_0\langle\alpha|\beta\rangle_0 = \delta(\alpha - \beta) = \delta^3(\boldsymbol{p}_1' - \boldsymbol{p}_1)\delta_{\sigma_1'\sigma_1}\delta_{n_1'n_1} \cdots \,. \tag{3.4}$$

The states $|\alpha, +\rangle$ and $|\alpha, -\rangle$ are elements of the *same* Hilbert space, so we can express, e.g., the $|\alpha, +\rangle$ in terms of the $|\alpha, -\rangle$:

$$|\alpha, +\rangle = \int d\beta S_{\beta\alpha} |\beta, -\rangle \,. \tag{3.5}$$

This defines the *S-matrix* $S_{\beta\alpha}$. The incoming state $|\alpha, +\rangle$ looks, for $t \to -\infty$, like a state of free particles, $|\alpha\rangle_0$, and for $t \to +\infty$ like the superposition $\int d\beta S_{\beta\alpha} |\beta\rangle_0$. Therefore, $S_{\beta\alpha}$ contains the full information about the scattering process.

One can show that the states $|\alpha, +\rangle$ and $|\alpha, -\rangle$ are orthonormal, so

$$\langle\beta, -|\alpha, +\rangle = \int d\gamma S_{\gamma\alpha} \langle\beta, -|\gamma, -\rangle = \int d\gamma S_{\gamma\alpha} \delta(\beta - \gamma) = S_{\beta\alpha} \,. \tag{3.6}$$

We see that $S_{\beta\alpha}$ is the transition amplitude for the process $|\alpha, +\rangle \to |\beta, -\rangle$. As the incoming and outgoing states are orthonormal, the S-matrix is *unitary*.

3.2 Decay rates and cross sections

Time and translation invariance imply conservation of total energy and momentum, respectively; hence, we can write

$$S_{\beta\alpha} = \delta(\beta - \alpha) - 2\pi i \delta(E_\beta - E_\alpha) \delta^3(\boldsymbol{p}_\beta - \boldsymbol{p}_\alpha) M_{\beta\alpha} \,. \tag{3.7}$$

What is the transition probability $|S_{\beta\alpha}|^2$? Imagine the system in a finite box (Volume V) for a finite time T (later $V, T \to \infty$). Then

$$\boldsymbol{p} = \frac{2\pi}{L}(n_1, n_2, n_3) \,, \tag{3.8}$$

with $n_i \in \mathbb{N}$ and $L^3 = V$, and the delta functions become

$$\delta^3_V(\boldsymbol{p}' - \boldsymbol{p}) \equiv \frac{1}{(2\pi)^3} \int\limits_V d^3x e^{i(\boldsymbol{p}' - \boldsymbol{p})\cdot \boldsymbol{x}} = \frac{V}{(2\pi)^3} \delta_{\boldsymbol{p}', \boldsymbol{p}} \,. \tag{3.9}$$

Hence, the "box states" have a factor $(V/(2\pi)^3)^N$ in the scalar product (N is the number of particles in the box state). Therefore, define normalized states

$$\Psi^{\text{BOX}}_\alpha \equiv \left(\frac{(2\pi)^3}{V}\right)^{N_\alpha/2} \Psi_\alpha \,, \tag{3.10}$$

with

$$(\Psi^{\text{BOX}}_\alpha, \Psi^{\text{BOX}}_\beta) = \delta_{\alpha\beta} \,. \tag{3.11}$$

The time delta function becomes

$$\delta_T(E_\alpha - E_\beta) \equiv \frac{1}{2\pi} \int\limits_{-T/2}^{T/2} dt \, e^{i(E_\alpha - E_\beta)t} \,. \tag{3.12}$$

The transition probability into a specific final state is

$$P(\alpha \to \beta) = \left|S^{\text{BOX}}_{\beta\alpha}\right|^2 = \left(\frac{(2\pi)^3}{V}\right)^{N_\alpha+N_\beta} \left|S_{\beta\alpha}\right|^2 . \tag{3.13}$$

The number of one-particle box states in momentum volume element d^3p is $V d^3p/(2\pi)^3$ (see Eq. (3.9)). We define $d\beta = d^3p'_1 \cdots d^3p'_{N_\beta}$, such that the number of states in $d\beta$ is

$$d\mathcal{N}_\beta = \left(\frac{V}{(2\pi)^3}\right)^{N_\beta} d\beta . \tag{3.14}$$

Hence the total probability for the transition into the range $d\beta$ is

$$dP(\alpha \to \beta) = P(\alpha \to \beta) d\mathcal{N}_\beta = \left(\frac{(2\pi)^3}{V}\right)^{N_\alpha} \left|S_{\beta\alpha}\right|^2 d\beta . \tag{3.15}$$

Interpretation of the squares of delta functions:

$$\left[\delta^3_V(\boldsymbol{p}_\beta - \boldsymbol{p}_\alpha)\right]^2 = \delta^3_V(\boldsymbol{p}_\beta - \boldsymbol{p}_\alpha)\delta^3_V(0) = \delta^3_V(\boldsymbol{p}_\beta - \boldsymbol{p}_\alpha)\frac{V}{(2\pi)^3} , \tag{3.16}$$

$$\left[\delta_T(E_\beta - E_\alpha)\right]^2 = \delta_T(E_\beta - E_\alpha)\delta_T(0) = \delta_T(E_\beta - E_\alpha)\frac{T}{2\pi} , \tag{3.17}$$

$$\tag{3.18}$$

and Eq. (3.15) becomes

$$dP(\alpha \to \beta) \overset{(3.7)}{=} (2\pi)^2 \left(\frac{(2\pi)^3}{V}\right)^{N_\alpha - 1} \frac{T}{2\pi} \left|M_{\beta\alpha}\right|^2 \delta^3_V(\boldsymbol{p}_\beta - \boldsymbol{p}_\alpha)\delta_T(E_\beta - E_\alpha) d\beta . \tag{3.19}$$

The transition probability is proportional to T; the coefficient is the *differential transition rate* $d\Gamma$. For $V, T \to \infty$

$$d\Gamma(\alpha \to \beta) \equiv dP(\alpha \to \beta)/T = (2\pi)^{3N_\alpha - 2} V^{1-N_\alpha} \left|M_{\beta\alpha}\right|^2 \delta^4(p_\beta - p_\alpha) d\beta , \tag{3.20}$$

where (for $\alpha \neq \beta$)

$$S_{\beta\alpha} \equiv -2\pi i \delta^4(p_\beta - p_\alpha) M_{\beta\alpha} . \tag{3.21}$$

$N_\alpha = 1$: Decay rate

Here, the volume cancels and

$$d\Gamma(\alpha \to \beta) = 2\pi \left|M_{\beta\alpha}\right|^2 \delta^4(p_\beta - p_\alpha) d\beta . \tag{3.22}$$

$N_\alpha = 2$: Collision of two particles

Rate is proportional to $1/V$, density of one particle at the position of the other particle. Usually, one measures the rate per *flux* Φ_α of incoming particles,

$$\Phi_\alpha = \frac{u_\alpha}{V}\,, \tag{3.23}$$

where u_α is the relative velocity between the two particles. This is called the *differential cross section*

$$d\sigma(\alpha \to \beta) \equiv \frac{d\Gamma(\alpha \to \beta)}{\Phi_\alpha} = (2\pi)^4 u_\alpha^{-1} \left|M_{\beta\alpha}\right|^2 \delta^4(p_\beta - p_\alpha) d\beta\,. \tag{3.24}$$

The cases $N_\alpha \geq 3$ play a role in astrophysics, cosmology, and chemistry, but rarely in particle physics.

What is u_α? One can show that

$$\sum_{\text{spins}} \left|M_{\beta\alpha}\right|^2 \prod_\beta E \prod_\alpha E \equiv R_{\beta\alpha} \tag{3.25}$$

is a Lorentz scalar function of all four-momenta. Hence the decay rate (3.22), summed over particle spins, can be written as

$$\sum_{\text{spins}} d\Gamma(\alpha \to \beta) = 2\pi E_\alpha^{-1} R_{\beta\alpha} \delta^4(p_\beta - p_\alpha) \frac{d\beta}{\prod_\beta E}\,. \tag{3.26}$$

This is Lorentz invariant apart from factor E_α^{-1} – the faster the particle moves, the slower it decays (time dilation).

Similarly, we write the spin-summed differential cross section as

$$\sum_{\text{spins}} d\sigma(\alpha \to \beta) = (2\pi)^4 u_\alpha^{-1} E_1^{-1} E_2^{-1} R_{\beta\alpha} \delta^4(p_\beta - p_\alpha) \frac{d\beta}{\prod_\beta E}\,. \tag{3.27}$$

To make this a Lorentz-invariant function, one usually defines u_α such that $u_\alpha E_1 E_2$ is a Lorentz scalar. Moreover, in rest frame of one particle, u_α is the velocity of the other particle. Hence

$$u_\alpha = \frac{\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}}{E_1 E_2}\,. \tag{3.28}$$

(If particle 1 is at rest, $\boldsymbol{p}_1 = 0$, $E_1 = m_1$, and $p_1 \cdot p_2 = m_1 E_2$. Therefore,

$$u_\alpha = \frac{\sqrt{E_2^2 - m_2^2}}{E_2} = \frac{|\boldsymbol{p}_2|}{E_2}\,, \tag{3.29}$$

the velocity of particle 2.)

Phase space To calculate the *phase space factor* $\delta^4(p_\beta - p_\alpha)d\beta$, we work in the center-of-mass frame, such that

$$\boldsymbol{p}_\alpha = 0\,. \tag{3.30}$$

For a final state with momenta $p'_1, p'_2, \ldots$ we have

$$\delta^4(p_\beta - p_\alpha)d\beta = \delta^3(\boldsymbol{p}'_1 + \boldsymbol{p}'_2 + \ldots)\delta(E'_1 + E'_2 + \ldots - E)d^3\boldsymbol{p}'_1 d^3\boldsymbol{p}'_2 \ldots\,, \tag{3.31}$$

where E is the total energy of the initial state. The integral over, e.g., $\boldsymbol{p}'_1$ can be easily solved using the momentum delta function:

$$\delta^4(p_\beta - p_\alpha)d\beta \to \delta(E'_1 + E'_2 + \ldots - E)d^3\boldsymbol{p}'_2 \ldots\,, \tag{3.32}$$

where now everywhere

$$\boldsymbol{p}'_1 = -\boldsymbol{p}'_2 - \boldsymbol{p}'_3 - \ldots\,. \tag{3.33}$$

What about $\delta(E)$? The easiest case is $N_\beta = 2$:

$$\begin{aligned}\delta^4(p_\beta - p_\alpha)d\beta &\to \delta(E'_1 + E'_2 - E\ldots)d^3\boldsymbol{p}'_2 \\ &= \delta(\sqrt{|\boldsymbol{p}'_1|^2 + m'^2_1} + \sqrt{|\boldsymbol{p}'_1|^2 + m'^2_2} - E)|\boldsymbol{p}'_1|^2 d|\boldsymbol{p}'_1|d\Omega\,,\end{aligned} \tag{3.34}$$

where $\boldsymbol{p}'_2 = -\boldsymbol{p}'_1$ and the solid angle is $d\Omega = \sin\theta d\theta d\phi$. Now we use

$$\delta(f(x)) = \frac{\delta(x - x_0)}{|f'(x_0)|}\,, \tag{3.35}$$

where x_0 is a simple zero of f. In our case, the zero of the argument of the energy delta function is $k' \equiv |\boldsymbol{p}'_1|$, where

$$k' = \frac{\sqrt{(E^2 - m'^2_1 - m'^2_2)^2 - 4m'^2_1 m'^2_2}}{2E} \equiv \frac{\lambda(E^2, m'^2_1, m'^2_2)}{2E}\,, \tag{3.36}$$

and, therefore,

$$E'_1 = \sqrt{k'^2 + m'^2_1} = \frac{E^2 - m'^2_2 + m'^2_1}{2E}\,, \tag{3.37}$$

$$E'_2 = \sqrt{k'^2 + m'^2_2} = \frac{E^2 - m'^2_1 + m'^2_2}{2E}\,, \tag{3.38}$$

and the derivative

$$\frac{d}{d|\boldsymbol{p}'_1|}\left(\sqrt{|\boldsymbol{p}'_1|^2 + m'^2_1} + \sqrt{|\boldsymbol{p}'_1|^2 + m'^2_2} - E\right)\Big|_{|\boldsymbol{p}'_1| = k'} = \frac{k'}{E'_1} + \frac{k'}{E'_2} = \frac{k'E}{E'_1 E'_2}\,. \tag{3.39}$$

Finally, we have

$$\delta^4(p_\beta - p_\alpha)d\beta \to \frac{k'E'_1E'_2}{E}d\Omega\,. \tag{3.40}$$

In particular, the differential decay rate of a particle at rest with energy E into two particles is

$$\boxed{\frac{d\Gamma(\alpha \to \beta)}{d\Omega} = \frac{2\pi k' E_1' E_2'}{E} |M_{\beta\alpha}|^2 \,,} \tag{3.41}$$

and the differential cross section for $2 \to 2$ scattering (in the c.m.s.) is

$$\boxed{\frac{d\sigma(\alpha \to \beta)}{d\Omega} = \frac{(2\pi)^4 k' E_1' E_2'}{E u_\alpha} |M_{\beta\alpha}|^2 = \frac{(2\pi)^4 k' E_1' E_2' E_1 E_2}{E^2 k} |M_{\beta\alpha}|^2 \,,} \tag{3.42}$$

where[1] $k \equiv |\boldsymbol{p}_1| = |\boldsymbol{p}_2|$.

For $N_\beta = 3$ we have

$$\delta^4(p_\beta - p_\alpha) d\beta \to \delta^4\Big(\sqrt{(\boldsymbol{p}_2' + \boldsymbol{p}_3')^2 + m_1'^2} + \sqrt{\boldsymbol{p}_2'^2 + m_2'^2} + \sqrt{\boldsymbol{p}_3'^2 + m_3'^2} - E\Big) d^3\boldsymbol{p}_2' d^3\boldsymbol{p}_3' \,. \tag{3.46}$$

We define the polar and azimuthal angles of $\boldsymbol{p}_2'$ with respect to the $\boldsymbol{p}_3'$ direction as θ_{23} and ϕ_{23}, and write

$$d^3\boldsymbol{p}_2' d^3\boldsymbol{p}_3' = |\boldsymbol{p}_2'|^2 d|\boldsymbol{p}_2'| |\boldsymbol{p}_3'|^2 d|\boldsymbol{p}_3'| d\Omega_3 d\cos\theta_{23} d\phi_{23} \,, \tag{3.47}$$

with the differential solid angle $d\Omega$ for $\boldsymbol{p}_3'$. The angle θ_{23} is fixed by energy conservation,

$$\sqrt{(|\boldsymbol{p}_2'|^2 + 2|\boldsymbol{p}_2'||\boldsymbol{p}_3'|\cos\theta_{23} + |\boldsymbol{p}_3'|^2 + m_1'^2} + \sqrt{|\boldsymbol{p}_2'|^2 + m_2'^2} + \sqrt{|\boldsymbol{p}_3'|^2 + m_3'^2} = E \,. \tag{3.48}$$

The derivative with respect to $\cos\theta_{23}$ of the argument of the delta function is

$$\frac{\partial E_1'}{\partial \cos\theta_{23}} = \frac{|\boldsymbol{p}_2'||\boldsymbol{p}_3'|}{E_1'} \,, \tag{3.49}$$

and so

$$\delta^4(p_\beta - p_\alpha) d\beta \to |\boldsymbol{p}_2'| d|\boldsymbol{p}_2'| |\boldsymbol{p}_3'| d|\boldsymbol{p}_3'| E_1' d\Omega_3 d\phi_{23} \,. \tag{3.50}$$

Using $dE/dp = d\sqrt{p^2 + m^2}/dp = p/E$ we finally obtain

$$\delta^4(p_\beta - p_\alpha) d\beta \to E_1' E_2' E_3' dE_2' dE_3' d\Omega_3 d\phi_{23} \,. \tag{3.51}$$

Eq. (3.25) shows that $\sum_{\text{spins}} |M_{\beta\alpha}|^2 \prod_\alpha E_\alpha \prod_\beta E_\beta$ is a scalar of four-momenta. If we assume this function is constant, Eq. (3.51) tells us that the final states are uniformly distributed in the $E_2' - E_3'$ plane. Any departure from a uniform distribution of events gives useful clues about resonances or asymmetries in the decay process ("Dalitz plot" [1]).

[1] Here, we have used

$$\begin{aligned} u_\alpha &= \frac{\sqrt{(E_1 E_2 + k^2)^2 - (E_1^2 - k^2)(E_2 - k^2)}}{E_1 E_2} && (3.43)\\ &= \frac{\sqrt{E_1^2 E_2^2 + 2k E_1 E_2 + k^4 - E_1^2 E_2^2 - k^4 + k^2(E_1^2 + k^2)}}{E_1 E_2} && (3.44)\\ &= \frac{kE}{E_1 E_2} \,. && (3.45) \end{aligned}$$

3.3 Perturbation Theory

We will calculate the S-matrix as a power series in the interaction term V (see Eq. (3.1)).

As a preparation, we need to find a slightly different form of the S-matrix. We have defined in and out states via the condition

$$\lim_{t\to\mp\infty} \exp(-iHt)|\alpha,\pm\rangle = \lim_{t\to\mp\infty} \exp(-iH_0 t)|\alpha\rangle_0 \,. \tag{3.52}$$

Then we have

$$|\alpha,\pm\rangle = \Omega(\mp\infty)|\alpha\rangle_0 \,, \tag{3.53}$$

with

$$\Omega(\tau) = \exp(+iH\tau)\exp(-iH_0\tau) \,. \tag{3.54}$$

Now we can write the S-matrix as

$$S_{\beta\alpha} = \langle\beta,-|\alpha,+\rangle = {}_0\langle\beta|\Omega^\dagger(+\infty)\Omega(-\infty)|\alpha\rangle_0 \equiv {}_0\langle\beta|U(+\infty,-\infty)|\alpha\rangle_0 \,, \tag{3.55}$$

where

$$U(t,t') \equiv \Omega^\dagger(t)\Omega(t') = \exp(iH_0 t)\exp[-iH(t-t')]\exp(-iH_0 t') \,. \tag{3.56}$$

To derive a perturbative expansion of the S-matrix, we differentiate Eq. (3.56) with respect to t:

$$\begin{aligned}
i\frac{d}{dt}U(t,t') &= i\frac{d}{dt}\Big[\exp(iH_0t)\exp(-iH(t-t'))\exp(-iH_0t')\Big] \\
&= i\Big[\exp(iH_0t)(iH_0 - iH)\exp(-iH(t-t'))\exp(-iH_0t')\Big] \\
&= \Big[\exp(iH_0t)V\exp(-iH_0t)\Big]\Big[\exp(iH_0t)\exp(-iH(t-t'))\exp(-iH_0t')\Big] \\
&\equiv V_I(t)U(t,t') \,.
\end{aligned} \tag{3.57}$$

Here, the index "I" denotes the *interaction picture* (time dependence of operators given by free Hamiltonian H_0). The initial condition for U is

$$U(t,t) = \mathbb{1} \,. \tag{3.58}$$

Eq.s (3.57) and (3.58) are equivalent to the integral equation

$$U(t,t') = \mathbb{1} - i\int_{t'}^{t} d\tau\, V_I(\tau)U(\tau,t') \,. \tag{3.59}$$

Solve through iteration:

$$\begin{aligned}
U(t,t') = \mathbb{1} &- i\int_{t'}^{t} d\tau_1\, V_I(\tau_1) + (-i)^2\int_{t'}^{t} d\tau_1\int_{t'}^{\tau_1} d\tau_2\, V_I(\tau_1)V_I(\tau_2) \\
&+ (-i)^3\int_{t'}^{t} d\tau_1\int_{t'}^{\tau_1} d\tau_2\int_{t'}^{\tau_2} d\tau_3\, V_I(\tau_1)V_I(\tau_2)V_I(\tau_2) + \ldots \,.
\end{aligned} \tag{3.60}$$

This may be rewritten using the *time-ordered product*, s.th. operators with larger (later) time argument appear to the left of the earlier ones:

$$T\{V(\tau)\} = V(\tau)\,, \tag{3.61}$$
$$T\{V(\tau_1), V(\tau_2)\} = \theta(\tau_1-\tau_2)V(\tau_1)V(\tau_2) + \theta(\tau_2-\tau_1)V(\tau_2)V(\tau_1)\,, \tag{3.62}$$

etc., with $n!$ terms in the time-ordered product of n operators V. Each term, integrated between t' and t, gives the same integral as the n-th term in Eq. (3.60). Hence we have

$$U(t,t') = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{t'}^{t} d\tau_1 \ldots d\tau_n T\{V(\tau_1)\cdots V(\tau_n)\}\,. \tag{3.63}$$

From Eq. (3.63) we obtain the *Dyson series* for the S-matrix (F. Dyson 1949 [2]):

$$S = \mathbb{1} + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} d\tau_1 \ldots d\tau_n T\{V(\tau_1)\cdots V(\tau_n)\}\,. \tag{3.64}$$

Short-hand notation:

$$S = T\exp\left(-i \int_{-\infty}^{\infty} dt\, V(t)\right)\,. \tag{3.65}$$

However, in general the series does not converge; it may be regarded as an asymptotic series in some coupling constants.

What about Lorentz invariance? One can show [3] that S commutes with the generators of the Lorentz group if

$$V(t) = \int d^3x \mathcal{H}(\boldsymbol{x}, t)\,, \tag{3.66}$$

where $\mathcal{H}(x)$ is a scalar "Hamilton density", in the sense that

$$U(\Lambda, a)\mathcal{H}(x)U^{-1}(\Lambda, a) = \mathcal{H}(\Lambda x + a)\,. \tag{3.67}$$

We can then write

$$\boxed{S = \mathbb{1} + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int d^4x_1 \ldots d^4x_n T\{\mathcal{H}(x_1)\cdots \mathcal{H}(x_n)\}\,.} \tag{3.68}$$

The time ordering of two events x_1, x_2 is Lorentz invariant unless their difference is space-like, $(x_1-x_2)^2 < 0$. If all $\mathcal{H}(x)$ commute at space-like distances, no special inertial system is introduced:

$$\boxed{[\mathcal{H}(x), \mathcal{H}(x')] = 0 \quad \text{for} \quad (x-x') \leq 0\,.} \tag{3.69}$$

3.4 Unitarity and optical theorem

Using

$$S_{\beta\alpha} = \delta(\beta - \alpha) - 2\pi i \delta^4(p_\beta - p_\alpha) M_{\beta\alpha} \,, \tag{3.70}$$

the unitarity condition $S^\dagger S = 1$ can be written as

$$\begin{aligned} \delta(\gamma - \alpha) &= \int d\beta S^*_{\beta\gamma} S_{\beta\alpha} \\ &= \delta(\gamma - \alpha) - 2\pi i \delta^4(p_\gamma - p_\alpha) M_{\gamma\alpha} + 2\pi i \delta^4(p_\gamma - p_\alpha) M^*_{\alpha\gamma} \\ &\quad + 4\pi^2 \int d\beta \delta^4(p_\beta - p_\gamma) \delta^4(p_\beta - p_\alpha) M^*_{\beta\gamma} M_{\beta\alpha} \,. \end{aligned} \tag{3.71}$$

For $p_\alpha = p_\gamma$ this gives

$$0 = -iM_{\gamma\alpha} + iM^*_{\alpha\gamma} + 2\pi \int d\beta \delta^4(p_\beta - p_\alpha) M^*_{\beta\gamma} M_{\beta\alpha} \,. \tag{3.72}$$

For the special case $\alpha = \gamma$, this is

$$\text{Im}(M_{\alpha\alpha}) = -\pi \int d\beta \delta^4(p_\beta - p_\alpha) |M_{\beta\alpha}|^2 \,. \tag{3.73}$$

We can use this result to calculate the total rate for a given initial state (see Eq. (3.21))

$$\begin{aligned} \Gamma_\alpha &\equiv \int d\beta \frac{d\Gamma(\alpha \to \beta)}{d\beta} = (2\pi)^{3N_\alpha - 2} V^{1-N_\alpha} \int \left|M_{\beta\alpha}\right|^2 \delta^4(p_\beta - p_\alpha) d\beta \\ &= -\frac{1}{\pi} (2\pi)^{3N_\alpha - 2} V^{1-N_\alpha} \text{Im}(M_{\alpha\alpha}) \,. \end{aligned} \tag{3.74}$$

If α is a two-particle state and σ_α the total cross section,

$$\sigma_\alpha \equiv \int d\beta \frac{d\sigma(\alpha \to \beta)}{d\beta} = (2\pi)^4 u_\alpha^{-1} \int \left|M_{\beta\alpha}\right|^2 \delta^4(p_\beta - p_\alpha) d\beta \,, \tag{3.75}$$

we can write Eq. (3.74) as

$$\text{Im}(M_{\alpha\alpha}) = -\frac{u_\alpha \sigma_\alpha}{16\pi^3} \,. \tag{3.76}$$

Introducing the *scattering amplitude*

$$f(\alpha \to \beta) \equiv -\frac{4\pi^2}{E} \sqrt{E_1' E_2' E_1 E_2} \sqrt{\frac{k'}{k}} M_{\beta\alpha} \,, \tag{3.77}$$

we can write Eq. (3.42) as

$$\frac{d\sigma(\alpha \to \beta)}{d\Omega} = |f(\alpha \to \beta)|^2 \,. \tag{3.78}$$

For elastic scattering, we have $k' = k$, $E_i' = E_i$, and so (using Eq. (3.28))

$$\text{Im}[f(\alpha \to \alpha)] = \frac{k}{4\pi} \sigma_\alpha \,. \tag{3.79}$$

This is the *optical theorem.*

3.5 Partial-wave expansions*

Frequently, it is useful to choose a basis of states where all variables, other than total energy and momentum, are discrete. (For instance, for two particles in the c.m.s., the components of the particle momenta lie on a two-dimensional spherical surface and can be expanded in spherical harmonics. More generally, for fixed total momentum and energy, the components of the individual particle momenta always form a compact space.) Therefore, we label the free-particle states as $|E\boldsymbol{p}N\rangle$, normalized as

$$\langle E'\boldsymbol{p}'N'|E\boldsymbol{p}N\rangle = \delta(E'-E)\delta^3(\boldsymbol{p}'-\boldsymbol{p})\delta_{N'N}\,. \tag{3.80}$$

The S-matrix elements are then

$$\langle E'\boldsymbol{p}'N'|S|E\boldsymbol{p}N\rangle = \delta(E'-E)\delta^3(\boldsymbol{p}'-\boldsymbol{p})S_{N'N}(E,\boldsymbol{p})\,, \tag{3.81}$$

where $S_{N'N}$ is a finite unitary matrix. The transition matrix elements are now defined by

$$S_{N'N}(E,\boldsymbol{p}) = \delta_{N'N} - 2i\pi M_{N'N}(E,\boldsymbol{p})\,. \tag{3.82}$$

As an example, consider a state of two non-identical particles n_1, n_2, with non-zero masses m_1, m_2, and spins s_1, s_2. We can label the state by the total momentum $\boldsymbol{p} = \boldsymbol{p}_1 + \boldsymbol{p}_2$, the energy E, the species labels n_1, n_2, and use Clebsch-Gordan coefficients (see Sec. B) to combine the two spins into a total spin s with z-component μ, and then combine the total spin with the orbital angular momentum ℓ with z-component m into a total angular momentum j with z-component σ. I.e. we label the states as $|E\boldsymbol{p}j\sigma\ell sn\rangle$. These states have scalar products with states of definite individual momenta and spins

$$\begin{aligned}\langle \boldsymbol{p}_1\sigma_1\boldsymbol{p}_2\sigma_2 n'|E\boldsymbol{p}j\sigma\ell sn\rangle = {} & \frac{\delta^3(\boldsymbol{p}-\boldsymbol{p}_1-\boldsymbol{p}_2)\delta\Big(E-\sqrt{|\boldsymbol{p}_1|^2+m_1^2}-\sqrt{|\boldsymbol{p}_2|^2+m_2^2}\Big)\delta_{n'n}}{\sqrt{|\boldsymbol{p}_1|E_1E_2/E}} \\ & \times\sum_{m,\mu} C_{s_1s_2}(s,\mu;\sigma_1,\sigma_2)C_{\ell,s}(j,\sigma;m,\mu)Y_\ell^m(\hat{\boldsymbol{p}}_1)\,.\end{aligned} \tag{3.83}$$

Here, Y_ℓ^m are the usual spherical harmonics. With this definition, the states are properly normalized in the c.m.s.:

$$\langle E'\boldsymbol{p}'j'\sigma'\ell's'n'|E\mathbf{0}j\sigma\ell sn\rangle = \delta^3(\boldsymbol{p}')\delta(E'-E)\delta_{j'j}\delta_{\sigma'\sigma}\delta_{\ell'\ell}\delta_{s's}\delta_{n'n}\,. \tag{3.84}$$

If the transition operator M is translationally and rotationally invariant, its matrix elements in the c.m.s. must take the form

$$\langle E'\boldsymbol{p}'j'\sigma'\ell's'n'|M|E\mathbf{0}j\sigma\ell sn\rangle = \delta^3(\boldsymbol{p}')\delta(E'-E)M^j_{\ell's'n',\ell sn}(E)\delta_{j'j}\delta_{\sigma'\sigma}\,. \tag{3.85}$$

It follows that the scattering amplitude in the c.m.s. is given by

$$
\begin{aligned}
&f\big(\boldsymbol{k}\sigma_1, -\boldsymbol{k}\sigma_2, n \to \boldsymbol{k}'\sigma_1', -\boldsymbol{k}'\sigma_2', n'\big) \\
&\equiv -4\pi^2\sqrt{\frac{k' E_1' E_2' E_1 E_2}{k E^2}}\langle \boldsymbol{k}'\sigma_1' - \boldsymbol{k}'\sigma_2' n' | M | \boldsymbol{k}\sigma_1 - \boldsymbol{k}\sigma_2 n\rangle \\
&= -\frac{4\pi^2}{k} \sum_{j\sigma\ell' m' s'\mu' \ell m s \mu} C_{s_1 s_2}(s,\mu;\sigma_1,\sigma_2) C_{\ell,s}(j,\sigma;m,\mu) \\
&\quad \times C_{s_1' s_2'}(s',\mu';\sigma_1',\sigma_2') C_{\ell',s'}(j,\sigma;m',\mu') Y_\ell^{m*}(\hat{\boldsymbol{k}}) Y_{\ell'}^{m'}(\hat{\boldsymbol{k}}') M_{\ell' s' n', \ell s n}(E)\,.
\end{aligned}
\tag{3.86}
$$

where we inserted two completeness relations, used Eq. (3.83), and note that the M amplitude is defined with a total energy and momentum conservation delta function factored out.

We will now take the z-axis in direction of the initial momentum $\boldsymbol{k}$, such that

$$
Y_\ell^m(\hat{\boldsymbol{k}}) = \delta_{m0}\sqrt{\frac{2\ell+1}{4\pi}}\,. \tag{3.87}
$$

The differential scattering cross section is given by $|f|^2$. Integrating this over the direction of the final momentum $\boldsymbol{k}'$ and summing and averaging over final and initial spins, we obtain the total cross section for the transition $n \to n'$,

$$
\sigma(n \to n'; E) = \frac{\pi}{k^2(2s_1+1)(2s_2+1)} \sum_{j\ell s \ell' s'} (2j+1)\left|\delta_{\ell'\ell}\delta_{s's}\delta_{n'n} - S^j_{\ell' s' n', \ell s n}(E)\right|^2. \tag{3.88}
$$

(Here, we used several standard sum rules for the Clebsch-Gordan coefficients, see Eq. (B.3)-(B.5).) Summing over all two-body channels then gives the total cross section for all two-body reactions,

$$
\begin{aligned}
\sum_{n'} \sigma(n \to n'; E) = {}& \frac{\pi}{k^2(2s_1+1)(2s_2+1)} \sum_{j\ell s}(2j+1) \\
& \times \left[(1 - S^j(E))^\dagger (1 - S^j(E))\right]_{\ell s n, \ell s n}.
\end{aligned}
\tag{3.89}
$$

We can compare this to the total cross. Using Eqs. (3.82), (3.86), and (3.87), the spin-averaged forward scattering amplitude is

$$
f(n; E) = \frac{i}{2k(2s_1+1)(2s_2+1)} \sum_{j\ell s}(2j+1)\left[1 - S^j(E)\right]_{\ell s n, \ell s n}. \tag{3.90}
$$

The optical theorem then gives the total cross section

$$
\sigma_{\text{total}}(n; E) = \frac{2\pi}{k^2(2s_1+1)(2s_2+1)} \sum_{j\ell s}(2j+1)\mathrm{Re}\left[1 - S^j(E)\right]_{\ell s n, \ell s n}. \tag{3.91}
$$

This agrees with Eq. (3.89) if only two-body final states can be reached at energy E, because then the S-matrix (or at least the submatrix relevant for 2-to-2 scattering) is unitary, and so

$$
\left[(1 - S^j(E))^\dagger (1 - S^j(E))\right]_{\ell s n, \ell s n} = 2\mathrm{Re}\left[1 - S^j(E)\right]_{\ell s n, \ell s n}. \tag{3.92}
$$

If final states with more than two particles are accessible, then the difference between Eqs. (3.91) and (3.89) gives the total cross section for producing extra particles,

$$\sigma_{\text{production}}(n;E) = \frac{\pi}{k^2(2s_1+1)(2s_2+1)} \sum_{j\ell s} (2j+1) \left[1 - S^j(E)^\dagger S^j(E)\right]_{\ell sn,\ell sn} . \quad (3.93)$$

If the relevant part of the S-matrix is diagonal (e.g. for $\pi^+ - \pi^+$ scattering below the threshold for producing additional pions), then unitarity requires

$$S^j_{\ell' s' n', \ell s n} = \exp[2i\delta_{j\ell sn}(E)]\delta_{\ell'\ell}\delta_{s's}\delta_{n'n} , \quad (3.94)$$

where real phase $\delta_{j\ell sn}(E)$ is called the *phase shift*. The elastic and total cross sections are then given by

$$\sigma(n \to n; E) = \sigma_{\text{total}}(n;E) = \frac{4\pi}{k^2(2s_1+1)(2s_2+1)} \sum_{j\ell s} (2j+1) \sin^2 \delta_{j\ell sn}(E) . \quad (3.95)$$

Threshold behaviour of cross sections

We expect the matrix element $\langle \boldsymbol{k}'\sigma_1' - \boldsymbol{k}'\sigma_2' n' | M | \boldsymbol{k}\sigma_1 - \boldsymbol{k}\sigma_2 n\rangle$ to be an analytic function of the momenta $\boldsymbol{k}$ and $\boldsymbol{k}'$ near $k = 0$ and / or $k' = 0$ (if the interaction falls off sufficiently fast in position space). This allows us to obtain some information about the threshold behaviour of the scattering amplitudes. First, we note that $k^\ell Y_\ell^m(\hat{\boldsymbol{k}})$ is a polynomial function of $\boldsymbol{k}$, so looking at the partial-wave expansion Eq. (3.86), we see that $M^j_{\ell' s' n', \ell s n}$ must go as $k^{\ell+1/2} k'^{\ell'+1/2}$ as $k, k' \to 0$, and in this limit the scattering amplitude is dominated by the lowest partial waves. There are three different cases:

Exothermic reactions

In this case, $k' \to$ const. for $k \to 0$, so $M^j_{\ell' s' n', \ell s n} \to k^{\ell_{\text{min}}+1/2}$, and the cross section goes as $k^{2\ell_{\text{min}}-1}$. Here, ℓ_{min} is the lowest contributing orbital momentum; typically, $\ell_{\text{min}} = 0$. Note that the reaction rate is given by the cross section multiplied by the flux, so for $\ell_{\text{min}} = 0$ the reaction rate approaches a constant. However, the probability of absorption for a beam crossing a target is proportional to the scattering cross section. An example is the absorption of slow neutrons in a nuclear reactor.

Endothermic reactions

Here, the reaction is forbidden until k reaches a threshold, at which $k' = 0$. Just above threshold $M^j_{\ell' s' n', \ell s n} \to (k')^{\ell'_{\text{min}}+1/2}$, where ℓ'_{min} is the lowest orbital momentum that can be produced, typically $\ell'_{\text{min}} = 0$. The scattering cross section goes as $(k')^{2\ell'_{\text{min}}+1}$; for $\ell'_{\text{min}} = 0$, this is[2] $k' \sim \sqrt{E - E_{\text{threshold}}}$. Example: associate production of strange particles, e.g. $p + \pi^- \to \Lambda^0 + K^0$.

Elastic reactions

Here, $k' = k$, and the lowest orbital momenta are $\ell = \ell' = 0$. Hence, the scattering cross section approaches a constant value. This is conventionally written in terms of a *scattering length*, defined by

$$M^s_{0sn',0sn} \to -\frac{k}{\pi} a_s(n \to n') \,. \tag{3.97}$$

The scattering amplitude becomes

$$\begin{aligned} &f\big(\boldsymbol{k}\sigma_1, -\boldsymbol{k}\sigma_2, n \to \boldsymbol{k}'\sigma_1', -\boldsymbol{k}'\sigma_2', n'\big) \\ &\qquad \to \sum_{s\sigma} C_{s_1 s_2}(s,\sigma;\sigma_1,\sigma_2) C_{s_1 s_2}(s,\sigma;\sigma_1',\sigma_2') a_s(n \to n') \,. \end{aligned} \tag{3.98}$$

The total cross section is $4\pi|f|^2$; summing and averaging over final and initial spins gives

$$\begin{aligned} &\sigma(n \to n'; k = 0) \\ &= \frac{4\pi}{(2s_1+1)(2s_2+1)} \sum_{s\sigma\sigma_1\sigma_2 s'\sigma'\sigma_1'\sigma_2'} C_{s_1 s_2}(s,\sigma;\sigma_1,\sigma_2) C_{s_1 s_2}(s',\sigma';\sigma_1,\sigma_2) \\ &\qquad \times C_{s_1 s_2}(s,\sigma;\sigma_1',\sigma_2') C_{s_1 s_2}(s',\sigma';\sigma_1',\sigma_2') a_s(n \to n') a_{s'}(n \to n') \\ &= \frac{4\pi}{(2s_1+1)(2s_2+1)} \sum_{s\sigma s'\sigma'} \big(\delta_{s's}\big)^2 \big(\delta_{\sigma'\sigma}\big)^2 a_s(n \to n') a_{s'}(n \to n') \\ &= \frac{4\pi}{(2s_1+1)(2s_2+1)} \sum_s (2s+1) a_s^2(n \to n') \,. \end{aligned} \tag{3.99}$$

Example: neutron - proton scattering, with $a_0 \gg a_1$.

[2]At threshold, we must have $E_{\text{threshold}} = m_1' + m_2'$, hence (for $E \approx E_{\text{thr.}}$)

$$\begin{aligned} 2k' &= \frac{\sqrt{(E^2 - m_1'^2 - m_2'^2)^2 - 4m_1'^2 m_2'^2}}{E} \\ &= \frac{\sqrt{(E^2 - E_{\text{thr.}}^2 + 2m_1' m_2')^2 - 4m_1'^2 m_2'^2}}{E} \\ &= \frac{\sqrt{(E^2 - E_{\text{thr.}}^2)^2 + 2m_1' m_2'(E^2 - E_{\text{thr.}}^2)}}{E} \\ &= \sqrt{(E - E_{\text{thr.}})^2 (E + E_{\text{thr.}})^2 / E^2 + 2m_1' m_2' (E - E_{\text{thr.}})(E + E_{\text{thr.}})/E^2} \\ &= \sqrt{2(E - E_{\text{thr.}})^2 + 4m_1' m_2' (E - E_{\text{thr.}})/E_{\text{thr.}}} \\ &\sim \sqrt{E - E_{\text{thr.}}} \,. \end{aligned} \tag{3.96}$$

3.6 Resonances*

4 Quantum fields

Goal: construction of a Lorentz-invariant interaction Lagrangian

4.1 Fock space of multi-particle states

Denote the state containing N particles with momenta $\boldsymbol{p}_i$, spins σ_i, of type n_i, by $|\boldsymbol{p}_1, \sigma_1, n_1; \ldots; \boldsymbol{p}_N, \sigma_N, n_N\rangle$. All known particles are either *bosons* or *fermions*; i.e., for identical particles we have

$$|\ldots \boldsymbol{p}\sigma n \ldots \boldsymbol{p}'\sigma' n \ldots\rangle = \pm |\ldots \boldsymbol{p}'\sigma' n \ldots \boldsymbol{p}\sigma n \ldots\rangle \,. \tag{4.1}$$

We choose the normalization accordingly:

$$N = 0: \qquad \langle 0|0\rangle = 1\,, \tag{4.2}$$

$$N = 1: \qquad \langle q'|q\rangle \equiv \langle \boldsymbol{p}'\sigma' n'|\boldsymbol{p}\sigma n\rangle = \delta^3(\boldsymbol{p}' - \boldsymbol{p})\delta_{\sigma'\sigma}\delta_{n'n} \equiv \delta(q' - q)\,, \tag{4.3}$$

$$N = 2: \qquad \langle q_1' q_2'|q_1 q_2\rangle = \delta(q_1' - q_1)\delta(q_2' - q_2) \pm \delta(q_2' - q_1)\delta(q_1' - q_2)\,, \tag{4.4}$$

and in general

$$\langle q_1' q_2' \ldots q_M'|q_1 q_2 \ldots q_N\rangle = \delta_{NM} \sum_{\mathcal{P}} \delta_{\mathcal{P}} \prod_i \delta(q_i' - q_{\mathcal{P}i})\,, \tag{4.5}$$

where $\mathcal{P}$ is a permutation operator. We define the *creation operator* $a^\dagger(\boldsymbol{p}\sigma n)$ by

$$a^\dagger(q)|q_1 q_2 \ldots q_N\rangle = |q q_1 q_2 \ldots q_N\rangle \,. \tag{4.6}$$

Thus, we can obtain the N-particle state from the vacuum state $|0\rangle$ as

$$|q_1 q_2 \ldots q_N\rangle = a^\dagger(q_1) a^\dagger(q_2) \cdots a^\dagger(q_N)|0\rangle \,. \tag{4.7}$$

The adjoint *annihilation operator* $a(\boldsymbol{p}\sigma n)$ removes a particle from the state. In particular,

$$a(q)|q\rangle = |0\rangle\,, \tag{4.8}$$

$$a(q)|0\rangle = 0\,. \tag{4.9}$$

One can show [3] that these operators satisfy the following *(anti-)commutation relations*

$$[a(q), a^\dagger(q')]_\mp \equiv a(q)a^\dagger(q') \mp a^\dagger(q')a(q) = \delta(q' - q)\,, \tag{4.10}$$

$$[a(q), a(q')]_\mp = [a^\dagger(q), a^\dagger(q')]_\mp = 0\,. \tag{4.11}$$

All operators acting on the Hilbert space of multi-particle states can be written as a sum of products of creation and annihilation operators (see Ref. [3] for a proof). In particular, this applies to the interaction Hamiltonian (density) $\mathcal{H}(x)$. The interaction Hamiltonian must be a Lorentz scalar and satisfy the condition (3.69). This is achieved in the simplest way of constructing the Hamiltonian out of *quantum fields*.

4.2 Dirac algebra and spinors

Reminder: Lie algebra of the Lorentz group. The generator of the rotations, $\boldsymbol{J}$, satisfy the commutation relations

$$[J^i, J^j] = i\epsilon^{ijk} J^k \,. \tag{4.12}$$

Together with the generators of the Lorentz boosts, $\boldsymbol{K}$, we have in addition

$$[J^i, K^j] = i\epsilon^{ijk} K^k \,, \tag{4.13}$$

$$[K^i, K^j] = -i\epsilon^{ijk} J^k \,. \tag{4.14}$$

Defining $\boldsymbol{J} \equiv (J^{23}, J^{31}, J^{12})$ and $\boldsymbol{K} \equiv (J^{01}, J^{02}, J^{03})$, this can be written in a compact way as

$$\left[J^{\mu\nu}, J^{\rho\sigma}\right] = i\left(\eta^{\nu\rho} J^{\mu\sigma} - \eta^{\mu\rho} J^{\nu\sigma} - \eta^{\sigma\mu} J^{\rho\nu} + \eta^{\sigma\nu} J^{\rho\mu}\right) , \tag{4.15}$$

where $J^{\mu\nu} = -J^{\nu\mu}$ are the generators of the Lorentz group, i.e.

$$D(\Lambda) = \exp\left(-\tfrac{i}{2}\omega_{\mu\nu} J^{\mu\nu}\right) , \tag{4.16}$$

with $\omega_{\mu\nu} = -\omega_{\nu\mu}$. For instance, for four-vectors we have $D(\Lambda) = \Lambda$; for a boost in the z direction we find the familiar expression

$$\Lambda^{\alpha}{}_{\beta} = \exp\left(-i\omega_{03} J^{03}\right)^{\alpha}{}_{\beta} = \exp\begin{pmatrix} 0 & 0 & 0 & \eta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \eta & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \cosh\eta & 0 & 0 & \sinh\eta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sinh\eta & 0 & 0 & \cosh\eta \end{pmatrix} . \tag{4.17}$$

Goal: construction of the spin-1/2 representation of the Lorentz group ("spinor representation").

Start with a set of four 4×4 matrices γ^μ satisfying the *Clifford relation*

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu} \,. \tag{4.18}$$

We then define

$$\mathcal{J}^{\mu\nu} \equiv \tfrac{i}{4}[\gamma^\mu, \gamma^\nu] \,. \tag{4.19}$$

It is tedious but straightforward to show that these $\mathcal{J}^{\mu\nu}$ satisfy Eq. (4.15). One can also show

$$D(\Lambda)\mathbb{1}D^{-1}(\Lambda) = \mathbb{1} \,, \tag{4.20}$$

$$D(\Lambda)\gamma^\rho D^{-1}(\Lambda) = \Lambda_\sigma{}^\rho \gamma^\sigma \,, \tag{4.21}$$

$$D(\Lambda)\mathcal{J}^{\rho\sigma} D^{-1}(\Lambda) = \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma \mathcal{J}^{\mu\nu} \,. \tag{4.22}$$

We now define

$$\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 \,. \tag{4.23}$$

It is easy to show that

$$\{\gamma_5, \gamma^\mu\} = 0 \,, \tag{4.24}$$

and

$$\mathbb{1}\,; \qquad \gamma^\mu\,; \qquad \mathcal{J}^{\mu\nu}\,; \qquad \gamma^\mu\gamma_5\,; \qquad \gamma_5 \tag{4.25}$$

form a basis of the *Clifford algebra*.

We can introduce a “parity transformation” using

$$\beta \equiv \gamma^0 = \beta^{-1}\,. \tag{4.26}$$

We have

$$\beta\mathbb{1}\beta = \mathbb{1} \qquad \ldots\text{scalar} \tag{4.27}$$

$$\beta\gamma^i\beta = -\gamma^i\,; \quad \beta\gamma^0\beta = \gamma^0 \qquad \ldots\text{vector} \tag{4.28}$$

$$\beta\mathcal{J}^{ij}\beta = \mathcal{J}^{ij}\,; \quad \beta\mathcal{J}^{i0}\beta = -\mathcal{J}^{i0} \qquad \ldots\text{tensor} \tag{4.29}$$

$$\beta\gamma^i\gamma_5\beta = \gamma^i\gamma_5\,; \quad \beta\gamma^0\gamma_5\beta = -\gamma^0\gamma_5 \qquad \ldots\text{axial vector} \tag{4.30}$$

$$\beta\gamma_5\beta = -\gamma_5 \qquad \ldots\text{pseudoscalar} \tag{4.31}$$

A useful explicit representation of the Dirac matrices is the so-called *chiral representation*:

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}\,, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\,, \tag{4.32}$$

with the *Pauli matrices*

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\,, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,. \tag{4.33}$$

In this representation,

$$\gamma_5 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}\,. \tag{4.34}$$

The matrices $D(\Lambda)$ in this representation act on four-component *Dirac spinors* (not on Lorentz four-vectors!). The generators are (in the chiral representation):

$$\mathcal{J}^{ij} = \frac{1}{2}\epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}\,, \qquad \mathcal{J}^{j0} = -\frac{i}{2}\begin{pmatrix} \sigma^j & 0 \\ 0 & -\sigma^j \end{pmatrix}\,. \tag{4.35}$$

Now we choose a basis for the Dirac spinors:

$$\begin{aligned} u(0,\tfrac{1}{2}) &= \frac{1}{\sqrt{2}}\begin{pmatrix} \xi \\ \xi \end{pmatrix}\,, \qquad & u(0,-\tfrac{1}{2}) &= \frac{1}{\sqrt{2}}\begin{pmatrix} \eta \\ \eta \end{pmatrix}\,, \\ v(0,\tfrac{1}{2}) &= \frac{1}{\sqrt{2}}\begin{pmatrix} \eta \\ -\eta \end{pmatrix}\,, \qquad & v(0,-\tfrac{1}{2}) &= \frac{1}{\sqrt{2}}\begin{pmatrix} -\xi \\ \xi \end{pmatrix}\,, \end{aligned} \tag{4.36}$$

where

$$\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\,, \qquad \eta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\,. \tag{4.37}$$

These are eigenvectors of the spin-z operator

$$\mathcal{J}^3 = \mathcal{J}^{12} = \frac{1}{2}\begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}, \tag{4.38}$$

with eigenvalues $\frac{1}{2}$, $-\frac{1}{2}$, $-\frac{1}{2}$, $\frac{1}{2}$. How do the spinors transform under a L.T.?

Reminder: four-momentum

$$\begin{pmatrix} E \\ 0 \\ 0 \\ p^3 \end{pmatrix} = \exp\bigl(-i\theta K^3\bigr)\begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix} = \exp\begin{pmatrix} 0 & 0 & 0 & \theta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \theta & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} m\cosh\theta \\ 0 \\ 0 \\ m\sinh\theta \end{pmatrix}. \tag{4.39}$$

We have $E = m\cosh\theta = \frac{m}{2}(e^\theta + e^{-\theta})$, $p^3 = m\sinh\theta = \frac{m}{2}(e^\theta - e^{-\theta})$, and therefore

$$e^\theta = \frac{E + p^3}{m}, \qquad e^{-\theta} = \frac{E - p^3}{m}. \tag{4.40}$$

Now let us perform the same boost for spinors. We write $u(0,\sigma) = (\chi,\chi)^T/\sqrt{2}$, where $\chi = \xi, \eta$; and $\boldsymbol{p} = (0, 0, p^3)$:[3]

$$\begin{aligned} u(\boldsymbol{p},\sigma) &= \sqrt{\frac{m}{2E}}\exp\left[-\frac{\theta}{2}\begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}\right]\begin{pmatrix} \chi \\ \chi \end{pmatrix} \\ &= \sqrt{\frac{m}{2E}}\begin{pmatrix} e^{-\theta/2} & & & \\ & e^{\theta/2} & & \\ & & e^{\theta/2} & \\ & & & e^{-\theta/2} \end{pmatrix}\begin{pmatrix} \chi \\ \chi \end{pmatrix} \\ &= \sqrt{\frac{m}{2E}}\begin{pmatrix} e^{\theta/2}\left(\frac{1-\sigma^3}{2}\right) + e^{-\theta/2}\left(\frac{1+\sigma^3}{2}\right) & 0 \\ 0 & e^{\theta/2}\left(\frac{1+\sigma^3}{2}\right) + e^{-\theta/2}\left(\frac{1-\sigma^3}{2}\right) \end{pmatrix}\begin{pmatrix} \chi \\ \chi \end{pmatrix} \end{aligned} \tag{4.41}$$

Now we use $\sqrt{m}e^{\theta/2} = \sqrt{E + p^3}$ and $\sqrt{m}e^{-\theta/2} = \sqrt{E - p^3}$, and find

$$u(\boldsymbol{p},\sigma) = \frac{1}{\sqrt{2E}}\begin{pmatrix} \left[\sqrt{E+p^3}\left(\frac{1-\sigma^3}{2}\right) + \sqrt{E-p^3}\left(\frac{1+\sigma^3}{2}\right)\right]\chi \\ \left[\sqrt{E+p^3}\left(\frac{1+\sigma^3}{2}\right) + \sqrt{E-p^3}\left(\frac{1-\sigma^3}{2}\right)\right]\chi \end{pmatrix}. \tag{4.42}$$

One can easily show, by an explicit calculation, that

$$u^\dagger(\boldsymbol{p},\sigma)u(\boldsymbol{p},\sigma) = 1. \tag{4.43}$$

For $\chi = \xi = (1,0)^T$ and a large boost ($E \approx p^3$), Eq. (4.42) becomes

$$u(\boldsymbol{p},1/2) = \frac{1}{\sqrt{2E}}\begin{pmatrix} \sqrt{E-p^3}\xi \\ \sqrt{E+p^3}\xi \end{pmatrix} \to \begin{pmatrix} 0 \\ \xi \end{pmatrix}. \tag{4.44}$$

[3] The presence of the normalization factor $\sqrt{m/E}$ is required such that the quantum fields (to be defined later) have the desired Lorentz transformation properties. See Ref. [3] for details.

For $\chi = \eta = (0,1)^T$

$$u(\boldsymbol{p}, -1/2) = \frac{1}{\sqrt{2E}} \begin{pmatrix} \sqrt{E+p^3}\eta \\ \sqrt{E-p^3}\eta \end{pmatrix} \to \begin{pmatrix} \eta \\ 0 \end{pmatrix} . \tag{4.45}$$

The spinors (4.44), (4.45) are eigenstates of the *helicity operator*

$$h = \frac{\boldsymbol{p} \cdot \boldsymbol{\mathcal{J}}}{|\boldsymbol{p}|} = \frac{1}{2}\hat{p}^k \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} , \tag{4.46}$$

with eigenvalues $+\frac{1}{2}$ and $-\frac{1}{2}$, respectively. Generally, using

$$P_R \equiv \tfrac{1}{2}\big(1+\gamma_5\big) \overset{(4.23)}{=} \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix} , \qquad P_L \equiv \tfrac{1}{2}\big(1-\gamma_5\big) \overset{(4.23)}{=} \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix} , \tag{4.47}$$

we can decompose every spinor Ψ into a "left-handed" and a "right-handed" part:

$$\Psi = (P_R + P_L)\Psi \equiv \Psi_L + \Psi_R . \tag{4.48}$$

Lorentz-covariant bilinears

Not all generators in Eq. (4.35) are Hermitian: $(\mathcal{J}^{ij})^\dagger = \mathcal{J}^{ij}$, but $(\mathcal{J}^{0j})^\dagger = -\mathcal{J}^{0j}$. Hence, in general $D(\Lambda)^\dagger \neq D(\Lambda)^{-1}$, and $u^\dagger u$ is not a Lorentz scalar. We can use a "trick" to construct a Lorentz scalar, as follows: Define $\bar{u} \equiv u^\dagger \beta = u^\dagger \gamma^0$. Then

$$\begin{aligned} \bar{u}u = u^\dagger \beta u &\overset{\text{L.T.}}{\to} (D(\Lambda)u)^\dagger \beta D(\Lambda) u \\ &= u^\dagger D(\Lambda)^\dagger \beta D(\Lambda) u \\ &= u^\dagger \exp\big(\tfrac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu\dagger}\big) \beta \exp\big(-\tfrac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}\big) u \\ &\overset{(4.29)}{=} u^\dagger \beta \exp\big(\tfrac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}\big) \exp\big(-\tfrac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}\big) u \\ &= \bar{u}u . \end{aligned} \tag{4.49}$$

In analogy, we can construct Lorentz four-vectors $\bar{u}\gamma^\mu u$, etc.

The Dirac spinors $u(\boldsymbol{p}, \sigma)$, $u(\boldsymbol{p}, \sigma)$ satisfy the *Dirac equation* (here, we introduce Feynman's short-hand notation $p_\mu \gamma^\mu = \not{p}$):

$$(\not{p} - m) u(\boldsymbol{p}, \sigma) = 0 , \tag{4.50}$$

$$(\not{p} + m) v(\boldsymbol{p}, \sigma) = 0 . \tag{4.51}$$

Consider, e.g., Eq. (4.50) in the rest frame:

$$(\not{p} - m) u(\boldsymbol{p}, \sigma) = (m\gamma^0 - m) u(0, \sigma) \overset{(4.32),(4.36)}{=} \frac{m}{\sqrt{2}} \begin{pmatrix} -\mathbb{1} & \mathbb{1} \\ \mathbb{1} & -\mathbb{1} \end{pmatrix} \begin{pmatrix} \xi \\ \xi \end{pmatrix} = 0 . \tag{4.52}$$

Perform a Lorentz boost (drop global factor $\sqrt{m/E}$):

$$\begin{aligned} 0 &= D(\Lambda)(m\gamma^0 - m) u(0,\sigma) = [mD(\Lambda)\gamma^0 D^{-1}(\Lambda) - m] D(\Lambda) u(0,\sigma) \\ &\overset{(4.21)}{=} [m{\Lambda_\mu}^0 \gamma^\mu - m] u(\boldsymbol{p}, \sigma) \overset{k=(m,\boldsymbol{0})}{=} [k_\nu {\Lambda_\mu}^\nu \gamma^\mu - m] u(\boldsymbol{p}, \sigma) \\ &= (\not{p} - m) u(\boldsymbol{p}, \sigma) . \end{aligned} \tag{4.53}$$

Similarly for $v(\boldsymbol{p}, \sigma)$.

4.3 The Dirac field

Using the Dirac spinors, we can write the *Dirac field* as

$$\boxed{\psi_\ell(x) = \sum_\sigma \int \frac{d^3p}{(2\pi)^{3/2}} \left[u_\ell(\boldsymbol{p},\sigma) a(\boldsymbol{p},\sigma) e^{-ip\cdot x} + v_\ell(\boldsymbol{p},\sigma) a^{c\dagger}(\boldsymbol{p},\sigma) e^{ip\cdot x}\right].} \tag{4.54}$$

Here, the u_ℓ, v_ℓ are given by Eq. (4.36). One can show [3] that $a(\boldsymbol{p},\sigma)$ annihilates a spin-1/2 *fermion* with momentum $\boldsymbol{p}$, spin-z component σ, and mass m. $a^{c\dagger}(\boldsymbol{p},\sigma)$ creates the corresponding antiparticle with opposite charge. The four-momenta are on-shell, i.e. $p^0 = \sqrt{|\boldsymbol{p}|^2 + m^2}$. We also have

$$\boxed{\bar\psi_\ell(x) \equiv \psi_\ell \beta = \sum_\sigma \int \frac{d^3p}{(2\pi)^{3/2}} \left[\bar v_\ell(\boldsymbol{p},\sigma) a^{c}(\boldsymbol{p},\sigma) e^{-ip\cdot x} + \bar u_\ell(\boldsymbol{p},\sigma) a^{\dagger}(\boldsymbol{p},\sigma) e^{ip\cdot x}\right].} \tag{4.55}$$

Eq.s (4.50), (4.51) immediately imply that the Dirac fields satisfy the *Dirac equation*

$$\boxed{(i\partial\!\!\!/ - m)\psi_\ell(x) = 0\,.} \tag{4.56}$$

Let's calculate the equal-time commutators for the Dirac field:

$$\begin{aligned}
\{\psi(\boldsymbol{x},t), \psi^\dagger(\boldsymbol{y},t)\} &= \int \frac{d^3p d^3p'}{(2\pi)^3} \sum_{\sigma\sigma'} \Big[u(\boldsymbol{p},\sigma) u^\dagger(\boldsymbol{p}',\sigma') e^{-ip\cdot x + ip'\cdot y'} \overbrace{\{a(\boldsymbol{p},\sigma), a^\dagger(\boldsymbol{p}',\sigma')\}}^{\delta^3(\boldsymbol{p}-\boldsymbol{p}')\delta_{\sigma\sigma'}} \\
&\qquad + v(\boldsymbol{p},\sigma) v^\dagger(\boldsymbol{p}',\sigma') e^{ip\cdot x - ip'\cdot y'} \underbrace{\{a^{c\dagger}(\boldsymbol{p},\sigma), a^c(\boldsymbol{p}',\sigma')\}}_{\delta^3(\boldsymbol{p}-\boldsymbol{p}')\delta_{\sigma\sigma'}}\Big] \\
&= \int \frac{d^3p}{(2\pi)^3} \sum_\sigma \left[u(\boldsymbol{p},\sigma) u^\dagger(\boldsymbol{p},\sigma) e^{-i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} + v(\boldsymbol{p},\sigma) v^\dagger(\boldsymbol{p},\sigma) e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\right].
\end{aligned} \tag{4.57}$$

Calculate the spin sums:

$$\begin{aligned}
&\sum_\sigma u(\boldsymbol{p},\sigma) u^\dagger(\boldsymbol{p},\sigma) \\
&= \frac{m}{p^0} \sum_\sigma D(L(p)) u(0,\sigma) \big(D(L(p)) u(0,\sigma)\big)^\dagger \\
&= \frac{m}{p^0} D(L(p)) \sum_\sigma u(0,\sigma) u(0,\sigma)^\dagger D(L(p))^\dagger \\
&\overset{(4.36)}{=} \frac{m}{2p^0} D(L(p)) (1+\gamma^0) D(L(p))^\dagger \\
&= \frac{m}{2p^0} \big[D(L(p))\gamma^0 D^{-1}(L(p)) + D(L(p)) D^{-1}(L(p))\big]\gamma^0 \\
&= \frac{1}{2p^0} (p\!\!\!/ + m)\gamma^0\,.
\end{aligned} \tag{4.58}$$

(Here, we used $\beta D^\dagger = D^{-1}\beta$ and $1 = \gamma^0\gamma^0$ in the second-to-last line, and the same "trick" as in the derivation of the Dirac equation in the last line.) Similarly, one shows

$$\sum_\sigma v(\boldsymbol{p},\sigma)v^\dagger(\boldsymbol{p},\sigma) = \frac{1}{2p^0}(\not{p} - m)\gamma^0 \,. \tag{4.59}$$

Insert into Eq. (4.57):

$$\begin{aligned}\{\psi(\boldsymbol{x},t),\psi^\dagger(\boldsymbol{y},t)\} &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2p^0}\big[(p^0\gamma^0 - \boldsymbol{p}\cdot\boldsymbol{\gamma} + m)e^{-i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} \\ &\qquad\qquad + (p^0\gamma^0 - \boldsymbol{p}\cdot\boldsymbol{\gamma} - m)e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\big]\gamma^0 \\ &\overset{\substack{\boldsymbol{p}\to-\boldsymbol{p}\\ \text{in 2nd term}}}{=} \int \frac{d^3p}{(2\pi)^3}\frac{1}{2p^0}\underbrace{\big[(p^0\gamma^0 - \boldsymbol{p}\cdot\boldsymbol{\gamma} + m + p^0\gamma^0 + \boldsymbol{p}\cdot\boldsymbol{\gamma} - m)\big]\gamma^0}_{2p^0} e^{-i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} \\ &= \delta^3(\boldsymbol{x}-\boldsymbol{y})\,,\end{aligned} \tag{4.60}$$

or, with explicit indices,

$$\{\psi_\ell(\boldsymbol{x},t), i\psi_k^\dagger(\boldsymbol{y},t)\} = i\delta^3(\boldsymbol{x}-\boldsymbol{y})\delta_{\ell k}\,. \tag{4.61}$$

The other anticommutators are (exercise)

$$\{\psi_\ell(\boldsymbol{x},t),\psi_k(\boldsymbol{y},t)\} = \{i\psi_\ell^\dagger(\boldsymbol{x},t), i\psi_k^\dagger(\boldsymbol{y},t)\} = 0\,. \tag{4.62}$$

4.4 Canonical formalism for the Dirac field

Eqs. (4.61) and (4.61) show that we can regard $q = \Psi$, $p = i\Psi^\dagger$ as canonically conjugated variables.

The Lagrangian $L = L[\Psi(t),\dot\Psi(t)]$ is a functional of the fields and their time derivatives. The conjugated fields are defined as

$$\Pi_\ell(\boldsymbol{x},t) \equiv \frac{\delta L[\Psi(t),\dot\Psi(t)]}{\delta\dot\Psi^\ell(t)}\,. \tag{4.63}$$

The *equations of motion (e.o.m.)* are

$$\dot\Pi_\ell(\boldsymbol{x},t) = \frac{\delta L[\Psi(t),\dot\Psi(t)]}{\delta\Psi^\ell(t)}\,. \tag{4.64}$$

The e.o.m. follow from a *variational principle*: Define the *action* as

$$I[\Psi] \equiv \int_{-\infty}^{\infty} dt\, L[\Psi(t),\dot\Psi(t)]\,. \tag{4.65}$$

For an arbitrary variation $\delta\Psi$ with $\delta\Psi(\pm\infty) = 0$ we have

$$\begin{aligned}\delta I[\Psi] &= \int_{-\infty}^{\infty} dt \left[\frac{\delta L}{\delta\Psi^\ell(x)}\delta\Psi^\ell(x) + \frac{\delta L}{\delta\dot{\Psi}^\ell(x)}\delta\dot{\Psi}^\ell(x)\right] \\ &\overset{\text{P.I.}}{=} \int_{-\infty}^{\infty} dt \int d^3x \left[\frac{\delta L}{\delta\Psi^\ell(x)} - \frac{d}{dt}\frac{\delta L}{\delta\dot{\Psi}^\ell(x)}\right]\delta\Psi^\ell(x)\end{aligned} \tag{4.66}$$

(action principle). In Lorentz-invariant field theories, we write L as an integral over a *Lagrangian density* $\mathcal{L}$,

$$L[\Psi(t), \dot{\Psi}(t)] = \int d^3x\, \mathcal{L}\big(\Psi(\boldsymbol{x}, t), \nabla\Psi(\boldsymbol{x}, t), \dot{\Psi}(\boldsymbol{x}, t)\big)\,, \tag{4.67}$$

such that the action becomes ($\partial_\mu \equiv \partial/\partial x^\mu$)

$$I[\Psi] = \int d^4x\, \mathcal{L}\big(\Psi(x), \partial_\mu\Psi(x)\big)\,. \tag{4.68}$$

The variation of L under $\Psi \to \Psi + \delta\Psi$ becomes

$$\begin{aligned}\delta L &= \int d^3x \left[\frac{\partial\mathcal{L}}{\partial\Psi^\ell}\delta\Psi^\ell + \frac{\partial\mathcal{L}}{\partial(\nabla\Psi^\ell)}\nabla\delta\Psi^\ell + \frac{\partial\mathcal{L}}{\partial\dot{\Psi}^\ell}\delta\dot{\Psi}^\ell\right] \\ &= \int d^3x \left[\left(\frac{\partial\mathcal{L}}{\partial\Psi^\ell} - \nabla\frac{\partial\mathcal{L}}{\partial(\nabla\Psi^\ell)}\right)\delta\Psi^\ell + \frac{\partial\mathcal{L}}{\partial\dot{\Psi}^\ell}\delta\dot{\Psi}^\ell\right],\end{aligned} \tag{4.69}$$

and comparing with Eq. (4.66) we get

$$\frac{\delta L}{\delta\Psi^\ell} = \frac{\partial\mathcal{L}}{\partial\Psi^\ell} - \nabla\frac{\partial\mathcal{L}}{\partial(\nabla\Psi^\ell)} \tag{4.70}$$

$$\frac{\delta L}{\delta\dot{\Psi}^\ell} = \frac{\partial\mathcal{L}}{\partial\dot{\Psi}^\ell}\,. \tag{4.71}$$

The e.o.m. (4.64) are then the *Euler-Lagrange equations*

$$\boxed{\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Psi^\ell)} = \frac{\partial\mathcal{L}}{\partial\Psi^\ell}\,.} \tag{4.72}$$

The Hamiltonian is given by the Legendre transformation

$$H = \sum_\ell \int d^3x \Pi_\ell(\boldsymbol{x}, t)\dot{\Psi}^\ell(\boldsymbol{x}, t) - L[\Psi(t), \dot{\Psi}(t)]\,. \tag{4.73}$$

4.5 Global symmetries

Assume that the action (4.65) is invariant under the infinitesimal field transformation

$$\Psi^\ell(x) \to \Psi^\ell(x) + i\epsilon \mathcal{F}^\ell(x)\,, \tag{4.74}$$

i.e.

$$0 = \delta I = i\epsilon \int d^4x \frac{\delta I[\Psi]}{\delta \Psi^\ell(x)} \mathcal{F}^\ell(x)\,. \tag{4.75}$$

For ϵ a constant this is a *global symmetry*. (NB: The e.o.m. need not be satisfied, otherwise Eq. (4.75) is trivially satisfied.) Now consider $\epsilon = \epsilon(x)$. In order that $\delta I = 0$ for constant ϵ, we must have

$$\delta I = -\int d^4x \mathcal{J}^\mu(x) \frac{\partial \epsilon(x)}{\partial x^\mu}\,, \tag{4.76}$$

for some current $\mathcal{J}^\mu(x)$. If now the fields satisfy the e.o.m., we have $\delta I = 0$, and integrating by parts yields

$$0 = \frac{\partial \mathcal{J}^\mu(x)}{\partial x^\mu}\,. \tag{4.77}$$

This is *Noether's theorem*: symmetries imply conservation laws. For each conserved current $\mathcal{J}^\mu$, Gauss' theorem implies that

$$F \equiv \int d^3x\, \mathcal{J}^0 \tag{4.78}$$

is conserved:

$$0 = \frac{dF}{dt}\,. \tag{4.79}$$

If the Lagrangian density $\mathcal{L}$ is invariant under the transformation (4.74), we can calculate $\mathcal{J}^\mu$ explicitly. The variation of the action, with $\epsilon = \epsilon(x)$, is

$$\delta I[\Psi] = i\int d^4x \left[\frac{\partial \mathcal{L}}{\partial \Psi^\ell} \mathcal{F}^\ell(x)\epsilon(x) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi^\ell)} \partial_\mu(\mathcal{F}^\ell(x)\epsilon(x))\right]. \tag{4.80}$$

The invariance of $\mathcal{L}$ for constant ϵ requires

$$0 = \frac{\partial \mathcal{L}}{\partial \Psi^\ell} \mathcal{F}^\ell(x) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi^\ell)} \partial_\mu \mathcal{F}^\ell(x)\,, \tag{4.81}$$

hence the variation of I for arbitrary fields is

$$\delta I[\Psi] = i\int d^4x \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi^\ell)} \mathcal{F}^\ell(x) \partial_\mu \epsilon(x)\,. \tag{4.82}$$

Comparison with (4.76) yields

$$\mathcal{J}^\mu = -i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi^\ell)} \mathcal{F}^\ell\,. \tag{4.83}$$

Application to the Dirac field

The Lagrangian density for the free Dirac field is

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi \,. \tag{4.84}$$

The Euler-Lagrange equations give

$$0 = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} - \frac{\partial \mathcal{L}}{\partial \psi} = \partial_\mu i\bar{\psi}\gamma^\mu + m\bar{\psi} \,, \tag{4.85}$$

and, after multiplying by β and Hermitian conjugation,

$$(i\partial\!\!\!/ - m)\psi = 0 \,. \tag{4.86}$$

(This is the Dirac equation again.)

Invanriance of Eq. (4.84) under $\psi \to e^{i\epsilon}\psi = \psi + i\epsilon\psi$ gives the corresponding Noether current

$$J^\mu = -i\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = \bar{\psi}\gamma^\mu\psi \,. \tag{4.87}$$

Spin*

4.6 Vector fields

In analogy to Eq. (4.54) we now define a *quantum vector field* (for now, we only consider massive particles), i.e. here we have $D(\Lambda)^\mu{}_\nu = \Lambda^\mu{}_\nu$. We have

$$\boxed{V^\mu(x) = \sum_\sigma \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2p^0}} \left[e^\mu(\boldsymbol{p},\sigma) a(\boldsymbol{p},\sigma) e^{-ip\cdot x} + e^{\mu *}(\boldsymbol{p},\sigma) a^{c\dagger}(\boldsymbol{p},\sigma) e^{ip\cdot x} \right] .} \tag{4.88}$$

Here, $a(\boldsymbol{p},\sigma)$ annihilates a massive spin-1 particle with momentum $\boldsymbol{p}$ and spin-z component σ, and $a^{c\dagger}(\boldsymbol{p},\sigma)$ creates the corresponding antiparticle. We can choose the *polarization vectors as follows.* For $\boldsymbol{p} = 0$ we define

$$e^\mu(0,0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} , \quad e^\mu(0,+1) = -\frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ i \\ 0 \end{pmatrix} , \quad e^\mu(0,0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -i \\ 0 \end{pmatrix} . \tag{4.89}$$

The polarization vectors for non-zero momenta $\boldsymbol{p} \neq 0$ can then be obtained by a Lorentz boost. We denote by $L(\boldsymbol{p})$ the L.T. that transforms $(M, \boldsymbol{0})$ into $(E, \boldsymbol{p})$. Then we have

$$e^\mu(\boldsymbol{p},\sigma) \equiv L(p)^\mu{}_\nu e^\nu(0,\sigma) \,, \tag{4.90}$$

For the polarization sum we obtain (exercise!)

$$\Pi^{\mu\nu}(\boldsymbol{p}) \equiv \sum_\sigma e^\mu(\boldsymbol{p},\sigma) e^{\nu *}(\boldsymbol{p},\sigma) = -\eta^{\mu\nu} + \frac{p^\mu p^\nu}{M^2} \,. \tag{4.91}$$

The field $V(x)$ satisfies an important field equation. Defining $k^\mu \equiv (M, \mathbf{0})$ and using Eq. (4.90), we see

$$p_\mu e^\mu(\boldsymbol{p}, \sigma) = \eta_{\mu\rho} p^\rho e^\mu(\boldsymbol{p}, \sigma) = \eta_{\mu\rho} L^\rho{}_\sigma(p) L^\mu{}_\nu(p) k^\sigma e^\mu(0, \sigma) = \eta_{\sigma\nu} k^\sigma e^\nu(0, \sigma) = 0\,, \quad (4.92)$$

and therefore[4]

$$\boxed{\partial_\mu V^\mu(x) = 0\,.} \quad (4.95)$$

4.7 Massless vector fields?

Consider the "object"

$$A^\mu(x) = \sum_{\sigma=\pm1} \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2p^0}} \left[e^\mu(\boldsymbol{p}, \sigma) a(\boldsymbol{p}, \sigma) e^{-ip\cdot x} + e^{\mu *}(\boldsymbol{p}, \sigma) a^\dagger(\boldsymbol{p}, \sigma) e^{ip\cdot x}\right]. \quad (4.96)$$

Here, a, $a^\dagger$ annihilate / create massless spin-one particles. We can choose polarization vectors as follows. Let $k^\mu = (\kappa, 0, 0, \kappa)$, $k^2 = 0$, and define

$$e^\mu(\boldsymbol{k}, \pm1) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ \pm i \\ 0 \end{pmatrix}. \quad (4.97)$$

Denote by $R(\hat{p})$ the rotation that brings the z-axis into the direction of $\boldsymbol{p}$. Then the e^μ for general momenta are (note that a boost in z direction does not affect $e^\mu(\boldsymbol{k}, \pm1)$)

$$e^\mu(\boldsymbol{p}, \pm1) = R(\hat{p})^\mu{}_\nu e^\nu(\boldsymbol{k}, \pm1)\,. \quad (4.98)$$

In particular, $e^0(\boldsymbol{k}, \pm1) = 0$ and $\boldsymbol{k} \cdot \boldsymbol{e}(\boldsymbol{k}, \pm1) = 0$, so

$$e^0(\boldsymbol{p}, \pm1) = 0 \quad (4.99)$$

and

$$\boldsymbol{p} \cdot \boldsymbol{e}(\boldsymbol{p}, \pm1) = 0\,. \quad (4.100)$$

[4] Interestingly, these would be the equations of electrodynamics for $M \to 0$ in Lorentz gauge. Is it allowed to take this limit? Assume a Hamilton density of the form $\mathcal{H} = J_\mu V^\mu$, where J_μ is some four-vector current. The squared transition matrix element then has the form

$$\sum_\sigma \left|\langle J_\mu \rangle e^{\mu *}(\boldsymbol{p}, \sigma)\right|^2 = \langle J_\mu \rangle \langle J_\nu \rangle^* \Big(-\eta^{\mu\nu} + \underbrace{\frac{p^\mu p^\nu}{M^2}}_{\longrightarrow\infty \text{ as } M\to0} \Big)\,. \quad (4.93)$$

We see that $\langle J_\mu \rangle p^\mu$ has to vanish, or, equivalently, J_μ has to be conserved:

$$\partial_\mu J^\mu = 0\,. \quad (4.94)$$

Therefore, $A^\mu(x)$ satiesfies the field equations

$$A^0(x) = 0 \tag{4.101}$$

and

$$\nabla \cdot \boldsymbol{A}(x) = 0\,. \tag{4.102}$$

This shows immediately that $A^\mu(x)$ cannot be a Lorentz four-vector. In fact, it can be shown [3] that under a L.T. A^μ transforms like a four-vector only up to a *gauge transformation*,

$$A_\mu(x) \to \Lambda^\nu{}_\mu A_\nu(\Lambda x) + \partial_\nu \Omega(x)\,, \tag{4.103}$$

with a scalar function $\Omega(x)$. One has to construct $\mathcal{H}(x)$ in terms of

$$F_{\mu\nu} = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) \tag{4.104}$$

(this is a Lorentz tensor, due to antisymmetry), or with terms of the form $A_\mu J^\mu$, with $\partial_\mu J^\mu = 0$. *This is the origin of gauge invariance.*

From Eq.s (4.97), (4.98) we obtain the polarization sum

$$\sum_{\sigma=\pm} e^i(\boldsymbol{p},\sigma) e^{j*}(\boldsymbol{p},\sigma) = \delta^{ij} - \frac{p^i p^j}{|\boldsymbol{p}|^2}\,. \tag{4.105}$$

5 Quantum electrodynamics

5.1 Gauge invariance

How can we construct a Lorentz-invariant interaction out of the fields (4.96), (4.54), (4.55)? Recall $A_\mu \to A_\mu + \partial_\mu \Omega$. Require that the action be invariant under the *gauge transformation*

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \epsilon(x)\,. \tag{5.1}$$

To this end, couple $A_\mu(x)$ to a conserved current:

$$I_{\mathrm{M}} \supset -\int d^4x\, \mathcal{J}^\mu A_\mu(x)\,. \tag{5.2}$$

Then under the transformation (5.1)

$$\delta I_{\mathrm{M}} = -\int d^4x\, \mathcal{J}^\mu \partial_\mu \epsilon(x) \overset{\text{P.I.}}{=} \int d^4x\, \partial_\mu \mathcal{J}^\mu \epsilon(x) = 0\,. \tag{5.3}$$

We have seen that the symmetry transformation $\psi(x) \to \psi(x) + \delta\psi(x)$, with

$$\delta\psi(x) = -i\epsilon\psi(x)\,, \tag{5.4}$$

yields a conserved current for constant ϵ, and for $\epsilon = \epsilon(x)$ (see Eq. (4.76))

$$\delta I_{\mathrm{M}} = \int d^4x\, \mathcal{J}^\mu \partial_\mu \epsilon(x)\,, \tag{5.5}$$

so we can couple A_μ to this current $\mathcal{J}^\mu$. In summary, the action must be invariant under the combined transformations

$$\delta A_\mu(x) = \partial_\mu \epsilon(x) \,, \tag{5.6}$$
$$\delta\psi(x) = -i\epsilon(x)e\psi(x) \,, \tag{5.7}$$

where we factored out the *electric charge* e. This is a *local* or *gauge symmetry.*

The antisymmetric tensor field

$$F_{\mu\nu}(x) \equiv \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) \tag{5.8}$$

is invariant under (5.1). We use (5.8) to construct the kinetic term of QED:

$$I_\gamma = -\frac{1}{4}\int d^4x F_{\mu\nu}(x)F^{\mu\nu}(x) \,. \tag{5.9}$$

5.2 Quantization of electrodynamics

Quantizing electrodynamics is a complicated business, due to gauge invariance and several constraints (such as $A^0 \equiv 0$). In this lecture, I will not attempt to explain it and refer to Ref. [3] or my lecture notes on QFT. Instead, I will just give the result of the analysis in a few lines. For the rest of the lecture, we will just use these results to calculate physical processes.

The QED Lagrangian in the interaction picture is given by

$$\mathcal{L} = \bar\psi(i\partial\!\!\!/ - m)\psi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \mathcal{J}_\mu A^\mu \,, \tag{5.10}$$

with

$$\mathcal{J}^\mu = e\bar\psi(x)\gamma^\mu\psi(x) \,. \tag{5.11}$$

The interaction Hamiltonian can be obtained by a Legendre transformation. In particular, the interaction term is given by

$$V(t) = \int d^3x \mathcal{J}_\mu(\boldsymbol{x},t)A^\mu(\boldsymbol{x},t) \,, \tag{5.12}$$

where

$$\mathcal{J}^\mu(\boldsymbol{x},t) = \exp(iH_0t)\mathcal{J}^\mu(\boldsymbol{x},0)\exp(-iH_0t) \,. \tag{5.13}$$

5.3 Propagators

As a preparation for later, we will calculate the propagators.

5.3.1 The propagator for the Dirac field

The propagator is defined as the vacuum expectation value of the time-ordered product

$$\begin{aligned} &\langle 0|T\{\psi_l(x)\bar\psi_m(y)\}|0\rangle \\ &\quad = \theta(x^0-y^0)\langle 0|\psi_l(x)\bar\psi_m(y)|0\rangle - \theta(y^0-x^0)\langle 0|\bar\psi_m(y)\psi_l(x)|0\rangle \,. \end{aligned} \tag{5.14}$$

Consider the first term:

$$
\begin{aligned}
&\langle 0|\psi_l(x)\bar{\psi}_m(y)|0\rangle \\
&= \langle 0| \int \frac{d^3p d^3p'}{(2\pi)^3} \sum_{\sigma\sigma'} \Big[\big(u_l(\boldsymbol{p},\sigma) e^{-ip\cdot x} a(\boldsymbol{p},\sigma) + v_l(\boldsymbol{p},\sigma) e^{ip\cdot x} a^{c\dagger}(\boldsymbol{p},\sigma) \big) \\
&\qquad \times \big(\bar{v}_m(\boldsymbol{p}',\sigma') e^{-ip'\cdot y} a^c(\boldsymbol{p}',\sigma') + \bar{u}_m(\boldsymbol{p}',\sigma') e^{ip'\cdot y} a^\dagger(\boldsymbol{p}',\sigma') \big) \Big] |0\rangle \,.
\end{aligned} \tag{5.15}
$$

Only the term with $aa^\dagger$ survives. We use

$$
\langle 0|a(\boldsymbol{p},\sigma)a^\dagger(\boldsymbol{p}',\sigma')|0\rangle = -0 + \delta^3(\boldsymbol{p}-\boldsymbol{p}')\delta_{\sigma\sigma'} \tag{5.16}
$$

to find

$$
\begin{aligned}
\langle 0|\psi_l(x)\bar{\psi}_m(y)|0\rangle &= \int \frac{d^3p}{(2\pi)^3} e^{-ip\cdot(x-y)} \sum_\sigma u_l(\boldsymbol{p},\sigma)\bar{u}_m(\boldsymbol{p}',\sigma') \\
&\overset{(4.58)}{=} \int \frac{d^3p}{(2\pi)^3} \frac{(\not{p}+m)_{lm}}{2p^0} e^{-ip\cdot(x-y)} \,.
\end{aligned} \tag{5.17}
$$

In complete analogy we find

$$
\langle 0|\bar{\psi}_m(y)\psi_l(x)|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{(-\not{p}+m)_{lm}}{2p^0} e^{ip\cdot(x-y)} \,. \tag{5.18}
$$

Thus, we can write the propagator as

$$
\begin{aligned}
&\langle 0|T\{\psi_l(x)\bar{\psi}_m(y)\}|0\rangle \\
&\quad = \theta(x^0-y^0)(i\not{\partial}+m)\Delta_+(x-y) + \theta(y^0-x^0)(i\not{\partial}+m)\Delta_+(y-x) \,.
\end{aligned} \tag{5.19}
$$

Now we use

$$
\frac{\partial}{\partial x^0}\theta(x^0-y^0) = -\frac{\partial}{\partial x^0}\theta(y^0-x^0) = \delta(x^0-y^0) \tag{5.20}
$$

to move the time derivatives in Eq, (5.19) past the θ functions:

$$
\begin{aligned}
&\langle 0|T\{\psi_l(x)\bar{\psi}_m(y)\}|0\rangle \\
&\quad = (i\not{\partial}+m)\big[\theta(x^0-y^0)\Delta_+(x-y) + \theta(y^0-x^0)\Delta_+(y-x)\big] \\
&\qquad - i\gamma^0\delta(x^0-y^0)\big[\Delta_+(x-y) - \Delta_+(y-x)\big] \,.
\end{aligned} \tag{5.21}
$$

For $x^0 = 0$, $\Delta_+(x)$ is even in x (compensate $\boldsymbol{x} \to -\boldsymbol{x}$ by shifting the integration variables $\boldsymbol{p} \to -\boldsymbol{p}$), hence the terms in the second line cancel, and we have

$$
\langle 0|T\{\psi_l(x)\bar{\psi}_m(y)\}|0\rangle = (i\not{\partial}+m)_{lm}\, i\Delta_F(x-y) \,, \tag{5.22}
$$

with the *Feynman propagator*

$$
i\Delta_F(x) \equiv \theta(x^0)\Delta_+(x) + \theta(-x^0)\Delta_+(-x) \,. \tag{5.23}
$$

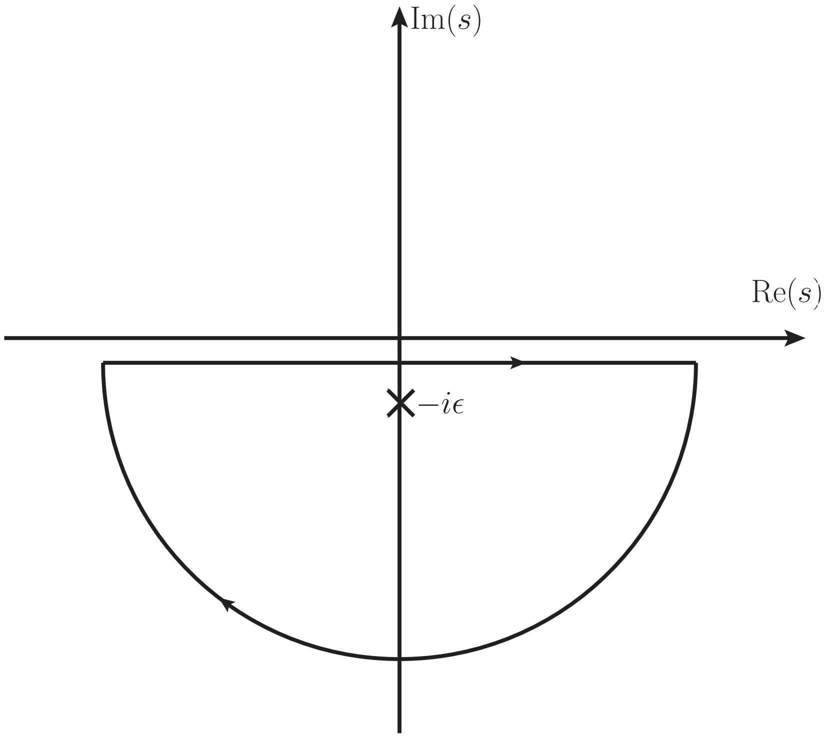


***Figure 1:** Fourier representation of the θ function.*

We can rewrite this as follows, using the Fourier representation of the θ function

$$\theta(t) = \frac{-1}{2\pi i} \int_{-\infty}^{\infty} ds \, \frac{\exp(-ist)}{s + i\epsilon} \,. \tag{5.24}$$

(This can be seen as follows. The integrand has a simple pole at $s = -i\epsilon$. For $t > 0$, the integrand converges for $\mathrm{I}m(s) < 0$, so we can close the contour in the lower half plane. The residue theorem gives $-2\pi i$ for the integral. For $T < 0$ we can close the contour in the upper half plane, and the residue theorem gives zero. See Fig. 1.)

Insert Eq. (5.24) into the Feynman propagator:

$$i\Delta_F(x) = -\frac{1}{2\pi i} \int \frac{d^3p}{(2\pi)^3} \int_{-\infty}^{\infty} ds \frac{1}{2p^0} \left[\frac{\exp(-ip \cdot x - isx^0)}{s + i\epsilon} + \frac{\exp(ip \cdot x + isx^0)}{s + i\epsilon} \right] . \tag{5.25}$$

Now we perform a change of integration variables: $\boldsymbol{q} = \boldsymbol{p}$, $q^0 = p^0 + s$ in the first term, and

$\boldsymbol{q} = -\boldsymbol{p}$, $q^0 = -p^0 - s$ in the second term. We then find

$$
\begin{aligned}
i\Delta_F(x) = &-\frac{1}{2\pi i}\int\limits_{-\infty}^{\infty} dq^0 \int \frac{d^3q}{(2\pi)^3}\frac{\exp(i\boldsymbol{q}\cdot\boldsymbol{x} - iq^0x^0)}{2\sqrt{\boldsymbol{q}^2+m^2}} \\
&\times \underbrace{\left[\frac{1}{q^0-\sqrt{\boldsymbol{q}^2+m^2}+i\epsilon}+\frac{1}{-q^0-\sqrt{\boldsymbol{q}^2+m^2}+i\epsilon}\right]}_{\frac{-2\sqrt{\boldsymbol{q}^2+m^2}}{-(q^0)^2+\boldsymbol{q}^2+m^2-i\epsilon 2\sqrt{\boldsymbol{q}^2+m^2}+\mathcal{O}(\epsilon^2)}},
\end{aligned}
\tag{5.26}
$$

and so ($q^2 = (q^0)^2 - \boldsymbol{q}^2$)

$$
\Delta_F(x) = \int \frac{d^4q}{(2\pi)^4}\frac{e^{-iq\cdot x}}{q^2-m^2+i\epsilon}\,. \tag{5.27}
$$

Hence, the Dirac propagator is

$$
\langle 0|T\{\psi(x)\bar{\psi}(y)\}|0\rangle = \int \frac{d^4q}{(2\pi)^4}\frac{i(\not{q}+m)}{q^2-m^2+i\epsilon}e^{-iq\cdot(x-y)}\,. \tag{5.28}
$$

5.3.2 The propagator for the photon field

The calculation of the photon propagator proceeds along similar lines, but is slightly more involved. It can really be understood only using path integrals. Therefore, we just quote the result. The photon propagator is defined as

$$
i\Delta_{\mu\nu}(x-y) \equiv \langle 0|T\{A_\mu(x)A_\nu(y)\}|0\rangle\,. \tag{5.29}
$$

The explicit calculation gives the result

$$
i\Delta_{\mu\nu}(x-y) = \int \frac{d^4q}{(2\pi)^4}\frac{-i\eta_{\mu\nu}}{q^2+i\epsilon}e^{-iq\cdot(x-y)}\,. \tag{5.30}
$$

6 Elementary processes of QED

6.1 $e^+e^- \to \mu^+\mu^-$

Our interaction density for QED is (cf. Eq. (5.12))

$$
\mathcal{H}(x) = \mathcal{J}_\mu(x)A^\mu(x)\,, \tag{6.1}
$$

with the current

$$
\mathcal{J}_\rho(x) = e\,\bar{\psi}_e\gamma_\rho\psi_e(x) + e\,\bar{\psi}_\mu\gamma_\rho\psi_\mu(x)\,. \tag{6.2}
$$

We insert this into the S-matrix (3.68). Now consider an e^+e^- initial state, with momenta $\boldsymbol{p}$, $\boldsymbol{p}'$ and spins s, s', and an $\mu^+\mu^-$ final state, with momenta $\boldsymbol{k}$, $\boldsymbol{k}'$ and spins r, r'. I.e. our "in state" is

$$
|\boldsymbol{p}, s; \boldsymbol{p}', s'\rangle = a^\dagger(\boldsymbol{p}, s, e^-)a^\dagger(\boldsymbol{p}', s', e^+)|0\rangle\,, \tag{6.3}
$$

and the "out state" is

$$\langle \boldsymbol{k}, r; \boldsymbol{k}', r'| = \langle 0|a(\boldsymbol{k}, r, \mu^-)a(\boldsymbol{k}', r', \mu^+)\,. \tag{6.4}$$

By a suitable choice of the vacuum energy we can write all creation operators to the left of all annihilation operators in Eq. (6.2); this is called *normal ordering*, in symbols $:\mathcal{H}(x):$.

It is easy to see that, in our case, the first non-vanishing term in (3.68) is $n = 2$. Therefore, we need to calculate

$$\begin{aligned} S_{\beta\alpha} &= -2\pi i\delta^4(p_\beta - p_\alpha)M_{\beta\alpha} \\ &= \frac{(-i)^2}{2!}\int d^4x \int d^4y \langle 0|a(\boldsymbol{k}, r, \mu^-)a(\boldsymbol{k}', r', \mu^+) \\ &\quad \times e^2 T\big\{\big(:\bar{\psi}_e(x)\not{A}(x)\psi_e(x): + :\bar{\psi}_\mu(x)\not{A}(x)\psi_\mu(x):\big) \\ &\qquad \times \big(:\bar{\psi}_e(y)\not{A}(y)\psi_e(y): + :\bar{\psi}_\mu(y)\not{A}(y)\psi_\mu(y):\big)\big\} \\ &\quad \times a^\dagger(\boldsymbol{p}, s, e^-)a^\dagger(\boldsymbol{p}', s', e^+)|0\rangle\,. \end{aligned} \tag{6.5}$$

Our strategy is, as usual, to move all annihilation operators to the right, until the annihilate the vacuum state. The only non-zero left-over terms originate from the arising delta functions. Reminder:

$$\begin{aligned} \bar{\psi}_e &\sim a^\dagger(e^-) + a(e^+)\,, \\ \psi_e &\sim a(e^-) + a^\dagger(e^+)\,, \\ \bar{\psi}_\mu &\sim a^\dagger(\mu^-) + a(\mu^+)\,, \\ \psi_\mu &\sim a(\mu^-) + a^\dagger(\mu^+)\,. \end{aligned}$$

We denote the "(anti-)commutation" of an annihilation-creation-pair as a *contraction*; notation: $\ldots a \ldots a^\dagger \ldots$. The following contractions contribute to the matrix element in (6.5):

$$\langle 0|a(\mu^-)a(\mu^+)T\big\{\big(\bar{\psi}_e\not{A}\psi_e + \bar{\psi}_\mu\not{A}\psi_\mu\big)(x)\big(\bar{\psi}_e\not{A}\psi_e + \bar{\psi}_\mu\not{A}\psi_\mu\big)(y)\big\}a^\dagger(e^-)a^\dagger(e^+)|0\rangle\,,$$

and

$$\langle 0|a(\mu^-)a(\mu^+)T\big\{\big(\bar{\psi}_e\not{A}\psi_e + \bar{\psi}_\mu\not{A}\psi_\mu\big)(x)\big(\bar{\psi}_e\not{A}\psi_e + \bar{\psi}_\mu\not{A}\psi_\mu\big)(y)\big\}a^\dagger(e^-)a^\dagger(e^+)|0\rangle\,.$$

(In addition, we need to contract the corresponding photon fields.)

The contraction of the external states with the fields gives (we write only the relevant fields, for simplicity)

$$\begin{aligned} &\langle 0|a(\boldsymbol{k}, r, \mu^-)\bar{\psi}_\mu(x)\ldots|0\rangle \\ &\overset{(4.54)}{=}\langle 0|a(\boldsymbol{k}, r, \mu^-)\int\frac{d^3p}{(2\pi)^{3/2}}\sum_\sigma\big(\bar{u}(\boldsymbol{p},\sigma)e^{ip\cdot x}a^\dagger(\boldsymbol{p},\sigma) + \ldots\big)\ldots|0\rangle \\ &\overset{(4.10),(4.11)}{=} -0 + \langle 0|\int\frac{d^3p}{(2\pi)^{3/2}}\sum_\sigma \bar{u}(\boldsymbol{p},\sigma)e^{ip\cdot x}\delta^3(\boldsymbol{k}-\boldsymbol{p})\delta_{r\sigma}\ldots|0\rangle \\ &=\frac{\bar{u}(\boldsymbol{k}, r, \mu^-)}{(2\pi)^{3/2}}e^{ik\cdot x}\langle 0|\ldots|0\rangle\,. \end{aligned} \tag{6.6}$$

Similarly, we find

$$\langle 0|a(\boldsymbol{k}',r',\mu^+)\psi_\mu(x)|0\rangle = \frac{v(\boldsymbol{k}',r',\mu^+)}{(2\pi)^{3/2}}e^{ik'\cdot x}\,, \tag{6.7}$$

$$\langle 0|\psi_e(y)a^\dagger(\boldsymbol{p},s,e^-)|0\rangle = \frac{u(\boldsymbol{p},s,e^-)}{(2\pi)^{3/2}}e^{-ip\cdot y}\,, \tag{6.8}$$

$$\langle 0|\psi_e(y)a(\boldsymbol{p}',s',e^+)|0\rangle = \frac{\bar{v}(\boldsymbol{p}',s',e^+)}{(2\pi)^{3/2}}e^{-ip'\cdot y}\,. \tag{6.9}$$

In total, we get the two terms

$$\begin{aligned}
S_{\beta\alpha} &= -\frac{e^2}{2}\int d^4x\int d^4y\frac{1}{(2\pi)^6}\\
&\quad\times(\bar{u}(\boldsymbol{k},r)\gamma_\mu v(\boldsymbol{k}',r'))(\bar{v}(\boldsymbol{p}',s')\gamma_\nu u(\boldsymbol{p},s))\\
&\quad\times\big[e^{ik\cdot x+ik'\cdot x-ip\cdot y-ip'\cdot y}\,\langle 0|T\{A^\mu(x)A^\nu(y)\psi_e(x)\}|0\rangle\\
&\qquad+e^{ik\cdot y+ik'\cdot y-ip\cdot x-ip'\cdot x}\,\langle 0|T\{A^\mu(y)A^\nu(x)\psi_e(x)\}|0\rangle\big]\\
&\overset{(5.30)}{=}-\frac{e^2}{(2\pi)^6}(\bar{u}(\boldsymbol{k},r)\gamma_\mu v(\boldsymbol{k}',r'))(\bar{v}(\boldsymbol{p}',s')\gamma_\nu u(\boldsymbol{p},s))\\
&\quad\times\int d^4x\int d^4y\int\frac{d^4q}{(2\pi)^4}\frac{-i\eta_{\mu\nu}}{q^2+i\epsilon}e^{-iq\cdot(x-y)}e^{i(k+k')\cdot x}e^{-i(p+p')\cdot y}\\
&=\frac{ie^2}{(2\pi)^6}(\bar{u}(\boldsymbol{k},r)\gamma_\mu v(\boldsymbol{k}',r'))(\bar{v}(\boldsymbol{p}',s')\gamma^\mu u(\boldsymbol{p},s))\\
&\quad\times\int\frac{d^4q}{(2\pi)^4}\frac{1}{q^2+i\epsilon}(2\pi)^4\delta^4(-q+k+k')(2\pi)^4\delta^4(q-p-p')\\
&=\frac{ie^2}{(2\pi)^2}\frac{(\bar{u}(\boldsymbol{k},r)\gamma_\mu v(\boldsymbol{k}',r'))(\bar{v}(\boldsymbol{p}',s')\gamma^\mu u(\boldsymbol{p},s))}{(p+p')^2}\delta^4(k+k'-p-p')\\
&\equiv-2\pi i\delta^4(k+k'-p-p')M_{\beta\alpha}\,.
\end{aligned} \tag{6.10}$$

where in the second equality we used that the expression is symmetric under the exchange $x\leftrightarrow y$.

Next, we would like to calculate $|M_{\beta\alpha}|^2$. As a preparation, we need the following relation for the Hermitian conjugate of the Dirac matrices (exercise!):

$$(\gamma^\mu)^\dagger=\beta\gamma^\mu\beta\,,\qquad \mu=0,1,2,3\,. \tag{6.11}$$

Our matrix element is

$$M_{\beta\alpha}=-\frac{e^2}{(2\pi)^3}\frac{1}{(p+p')^2}(\bar{u}(\boldsymbol{k},r)\gamma_\mu v(\boldsymbol{k}',r'))(\bar{v}(\boldsymbol{p}',s')\gamma^\mu u(\boldsymbol{p},s))\,. \tag{6.12}$$

We have

$$(\bar{u}\gamma_\mu v)^*=(u^\dagger\beta\gamma_\mu v)^\dagger=v^\dagger\gamma_\mu^\dagger\beta^\dagger u=v^\dagger\beta\gamma_\mu\beta\beta u=\bar{v}\gamma_\mu u\,, \tag{6.13}$$

and similarly

$$(\bar{v}\gamma_\mu u)^*=\bar{u}\gamma_\mu v\,. \tag{6.14}$$

Therefore, we find

$$|M_{\beta\alpha}|^2 = \frac{e^4}{(2\pi)^6}\frac{1}{((p+p')^2)^2} \quad \begin{aligned}&(\bar{u}(\boldsymbol{k},r)\gamma_\mu v(\boldsymbol{k}',r'))(\bar{v}(\boldsymbol{k}',r')\gamma_\nu u(\boldsymbol{k},r)) \\ &\times (\bar{v}(\boldsymbol{p}',s')\gamma^\mu u(\boldsymbol{p},s))(\bar{u}(\boldsymbol{p},s)\gamma^\nu v(\boldsymbol{p}',s'))\,.\end{aligned} \tag{6.15}$$

In realistic experiments one frequently does not know the polarization of the electrons in the initial state. If we are not interested in the polarization of the muons in the final state, we average the cross section over the spin states of the initial-state particles and sum over the spin states in the final state. Hence, we need to calculate

$$\tfrac{1}{2}\sum_s \tfrac{1}{2}\sum_{s'}\sum_{rr'}|M_{\beta\alpha}|^2\,.$$

We insert (6.15) and use the spin sums (4.58), (4.59). The first factor in Eq. (6.15) gives (writing the spinor indices explicitly):

$$\begin{aligned}&\sum_{rr'}\sum_{abcd}(\bar{u}_a(\boldsymbol{k},r)(\gamma_\mu)_{ab}v_b(\boldsymbol{k}',r')\bar{v}_c(\boldsymbol{k}',r')(\gamma_\nu)_{cd}u_d(\boldsymbol{k},r)) \\ =&\sum_{abcd}\left[(\not{k}+m_\mu)_{da}(\gamma_\mu)_{ab}(\not{k}'-m_\mu)_{bc}(\gamma_\nu)_{cd}\right]\frac{1}{2k^0 2k'^0} \\ =&\frac{1}{4k^0k'^0}\mathrm{Tr}\big\{(\not{k}+m_\mu)\gamma_\mu(\not{k}'-m_\mu)\gamma_\nu\big\}\,.\end{aligned} \tag{6.16}$$

Similar for the second factor in (6.15). In total, we have for the "summed and averaged" absolute value squared,

$$\overline{|M|^2} \equiv \tfrac{1}{4}\sum_{\text{spins}}|M|^2\,, \tag{6.17}$$

writing $p + p' = q$,

$$\overline{|M|^2} = \frac{1}{4}\frac{e^4}{(2\pi)^6}\frac{1}{q^4}\frac{1}{16k^0k'^0p^0p'^0}\mathrm{Tr}\big\{(\not{k}+m_\mu)\gamma_\mu(\not{k}'-m_\mu)\gamma_\nu\big\}\mathrm{Tr}\big\{\not{p}'\gamma^\mu\not{p}\gamma^\nu\big\} \tag{6.18}$$

where we neglected the electron mass (the electrons in this process are always ultra-relativistic, since $m_\mu \gg m_e$). Next we need to evaluate the traces. We have

$$\mathrm{Tr}\{\gamma^\mu\} = \mathrm{Tr}\{\gamma^\mu\gamma_5\gamma_5\} = -\mathrm{Tr}\{\gamma_5\gamma^\mu\gamma_5\} = -\mathrm{Tr}\{\gamma^\mu\gamma_5\gamma_5\} = -\mathrm{Tr}\{\gamma^\mu\} \tag{6.19}$$

and so $\mathrm{Tr}\{\gamma^\mu\} = 0$. In general, the trace of a product of an odd number of Dirac matrices vanishes. Furthermore,

$$\mathrm{Tr}\{\gamma^\mu\gamma^\nu\} = \tfrac{1}{2}\mathrm{Tr}\{\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu\} = \tfrac{1}{2}\cdot 2\cdot\eta^{\mu\nu}\mathrm{Tr}\{\mathbb{1}\} = 4\eta^{\mu\nu}\,. \tag{6.20}$$

Similarly, one can show that

$$\mathrm{Tr}\{\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\} = 4\big(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}\big)\,. \tag{6.21}$$

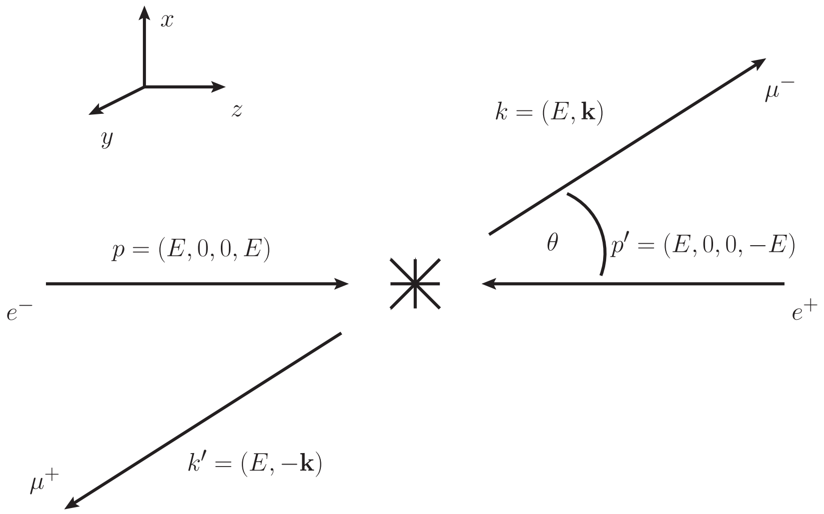


Figure 2: *Kinematics for* $e^+e^- \to \mu^+\mu^-$ *in the center-of-mass system.*

Using these results, we get

$$\mathrm{Tr}\left\{\not{p}'\gamma^\mu\not{p}\gamma^\nu\right\} = p'_\rho p_\sigma \mathrm{Tr}\left\{\gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\nu\right\} = 4(p'^\mu p^\nu - (p'\cdot p)\,\eta^{\mu\nu} + p'^\nu p^\mu) \tag{6.22}$$

and

$$\mathrm{Tr}\left\{(\not{k} + m_\mu)\gamma_\mu(\not{k}' - m_\mu)\gamma_\nu\right\} = -4m_\mu^2\eta_{\mu\nu} + 4(k'_\mu k_\nu - (k'\cdot k)\,\eta_{\mu\nu} + k'_\nu k_\mu)\,. \tag{6.23}$$

Inserting this into (6.18) yields

$$\begin{aligned}\overline{|M|^2} = {}& \frac{1}{4}\frac{e^4}{(2\pi)^6}\frac{1}{q^4}\frac{1}{16k^0k'^0p^0p'^0} \\ & \times 16\left[2m_\mu^2(p'\cdot p) + 2(p'\cdot k')(p\cdot k) + 2(p'\cdot k)(p\cdot k')\right].\end{aligned} \tag{6.24}$$

To simplify this further, consider kinematics in center-of-mass system (see Fig. 2). Here, $|\boldsymbol{k}| = \sqrt{E^2 - m_\mu^2}$ and $k_z = |\boldsymbol{k}|\cos\theta$. We have $q^2 \equiv (p+p')^2 = 4E^2$; $p\cdot p' = 2E^2$; $p\cdot k = p'\cdot k' = E^2 - E|\boldsymbol{k}|\cos\theta$; $p\cdot k' = p'\cdot k = E^2 + E|\boldsymbol{k}|\cos\theta$. With this we get

$$\begin{aligned}\overline{|M|^2} &= \frac{1}{4}\frac{2e^4}{(2\pi)^6}\frac{1}{16E^4}\frac{1}{E_eE'_eE_\mu E'_\mu} \\ &\quad\times\left[2m_\mu^2E^2 + \underbrace{E^2(E - |\boldsymbol{k}|\cos\theta)^2 + E^2(E + |\boldsymbol{k}|\cos\theta)^2}_{2E^4 + 2E^2(E^2 - m_\mu^2)\cos^2\theta}\right] \\ &= \frac{1}{4}\frac{2e^4}{(2\pi)^6}\frac{1}{16E^4}\frac{1}{E^4}2E^4\left[1 + \frac{m_\mu^2}{E^2} + \left(1 - \frac{m_\mu^2}{E^2}\right)\cos^2\theta\right].\end{aligned} \tag{6.25}$$

Inserting into (3.42) gives

$$
\begin{aligned}
\frac{d\sigma(e^+e^- \to \mu^+\mu^-)}{d\Omega} &= \frac{(2\pi)^4 |\boldsymbol{k}| E^4}{4E^2 \cdot E}\, \overline{|M|^2} \\
&= \frac{1}{4}\frac{e^4}{(2\pi)^2}\frac{1}{4E^4}\left[1 + \frac{m_\mu^2}{E^2} + \left(1 - \frac{m_\mu^2}{E^2}\right)\cos^2\theta\right]\frac{E(E^2 - m_\mu^2)^{1/2}}{4} \\
&= \frac{\alpha^2}{4s}\sqrt{1 - \frac{m_\mu^2}{E^2}}\left[1 + \frac{m_\mu^2}{E^2} + \left(1 - \frac{m_\mu^2}{E^2}\right)\cos^2\theta\right],
\end{aligned}
\tag{6.26}
$$

with $s \equiv (2E)^2$ the square of the center-of-mass energy and $\alpha = e^2/(4\pi) \approx 1/137$ the fine structure constant. Integrating over $d\Omega = 2\pi d\cos\theta$ we obtain the *total cross section*

$$
\sigma(e^+e^- \to \mu^+\mu^-) = \frac{4\pi\alpha^2}{3s}\sqrt{1 - \frac{m_\mu^2}{E^2}}\left(1 + \frac{1}{2}\frac{m_\mu^2}{E^2}\right). \tag{6.27}
$$

In the high-energy limit ($E \gg m_\mu$) we obtain

$$
\frac{d\sigma}{d\Omega} \xrightarrow{E \gg m_\mu} \frac{\alpha^2}{4s}(1 + \cos^2\theta)\,, \tag{6.28}
$$

$$
\sigma \xrightarrow{E \gg m_\mu} \frac{4\pi\alpha^2}{3s} + \mathcal{O}\left(\frac{m_\mu^4}{E^4}\right). \tag{6.29}
$$

For $E \gg m_\mu$, the energy is the only dimensionful quantity in the process, so Eq. (6.29) follows (up to the constant factor) from "naive dimensional analysis" (NDA).

6.2 The Feynman rules for QED

Our strategy to calculate the S-matrix element by bringing all annihilation operators to the right generalizes ("Wicks theorem", C. G. Wick 1950 [4]). Formally, one transforms a time-ordered into a normal-ordered product. The appearing contractions can be represented graphically. As usual, we classify external states by their three-momentum $\boldsymbol{p}$ and spin-z component (or helicity) σ. Then we have for an

incoming fermion: 

: $$\frac{u_\ell(\boldsymbol{p},\sigma)}{(2\pi)^{3/2}}\,, \tag{6.30}$$

incoming antifermion: 

: $$\frac{\bar{v}_\ell(\boldsymbol{p},\sigma)}{(2\pi)^{3/2}}\,, \tag{6.31}$$

outgoing fermion: 

: $$\frac{\bar{u}_\ell(\boldsymbol{p},\sigma)}{(2\pi)^{3/2}}\,, \tag{6.32}$$

outgoing antifermion: 

: $$\frac{v_\ell(\boldsymbol{p},\sigma)}{(2\pi)^{3/2}}\,, \tag{6.33}$$

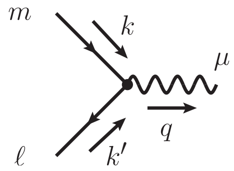
incoming photon: 

: $$\frac{e_\mu(\boldsymbol{p},\sigma)}{(2\pi)^{3/2}\sqrt{2p^0}}\,, \tag{6.34}$$

outgoing photon: 

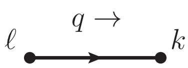
$$: \quad \frac{e^*_\mu(\boldsymbol{p},\sigma)}{(2\pi)^{3/2}\sqrt{2p^0}}\,. \qquad (6.35)$$

The interactions are symbolized by vertices. The integration over $x_1, x_2, \ldots$ in (3.68) effectively yields a momentum-conservation delta function for each x_i. Hence, each vertex gives a factor



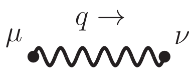
$$: \; -iQ_f(\gamma^\mu)_{\ell m}(2\pi)^4\delta^4(k+k'-q)\,. \qquad (6.36)$$

Here, Q_f denotes the charge of fermion f. The contraction of two internal fields yields a factor



$$: \frac{1}{(2\pi)^4}\frac{i(\not{q}+m)}{q^2-m^2+i\epsilon} \qquad (6.37)$$

for fermions, and



$$: \frac{1}{(2\pi)^4}\frac{-i\eta_{\mu\nu}}{q^2+i\epsilon} \qquad (6.38)$$

for photons. The *Feynman rules* to obtain the transition matrix elements are:

1. Draw all Feynman diagrams contributing to a given process, with a given maximal number of vertices, using the building blocks (6.30)-(7.43).

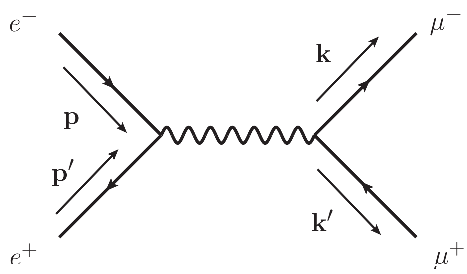
2. Replace each building block by its mathematical expression.

3. Integrate over the momenta of all internal lines, and sum over contracted Lorentz and Dirac index pairs.

4. Add the contributions of all diagrams.

5. Each closed fermion line yields a factor (-1). If two Feynman diagrams differ by an odd number of permutations of fermionic annihilation or creation operators, they get a relative minus sign.

Example: Muon pair production.



NB: We symbolize all contributions obtained by a mere renumbering of internal vertices by the *same* Feynman diagram! This cancels the factor $1/n!$ in (3.68).

6.3 $e^+e^- \to \mu^+\mu^-$: High-energy limit and helicity structure

In this section we will treat electrons and muons as massless (high-energy limit). Where does the angular dependence $(1+\cos\theta)$ in Eq. (6.28) come from? As an exercise, we construct the transition amplitude using the Feynman rules:

$$\begin{aligned}
&-2\pi i\delta^4(k+k'-p-p')M \\
&\quad = \int d^4q \left[\frac{\bar{u}(\boldsymbol{k},r)}{(2\pi)^{3/2}}(-ie)\gamma^\mu \frac{v(\boldsymbol{k}',r')}{(2\pi)^{3/2}}(2\pi)^4\delta^4(k+k'-q)\right. \\
&\qquad\qquad \times \frac{1}{(2\pi)^4}\frac{-i\eta_{\mu\nu}}{q^2+i\epsilon} \\
&\qquad\qquad \left.\times \frac{\bar{v}(\boldsymbol{p}',s')}{(2\pi)^{3/2}}(-ie)\gamma^\nu \frac{u(\boldsymbol{p},s)}{(2\pi)^{3/2}}(2\pi)^4\delta^4(q-p-p')\right] \\
&\quad = \frac{ie^2}{(2\pi)^2}\frac{(\bar{u}(\boldsymbol{k},r)\gamma^\mu v(\boldsymbol{k}',r'))(\bar{v}(\boldsymbol{p}',s')\gamma_\mu u(\boldsymbol{p},s))}{(p+p')^2}\delta^4(k+k'-p-p')\,.
\end{aligned} \tag{6.39}$$

NB: When not writing explicit spinor indices, all fermion lines must be "evaluated" *against* the direction of arrows.

We will evaluate the amplitude separately for all helicities (massless fermions!). First, we decompose all spinors into their LH and RH components:

$$\psi = P_L\psi + P_R\psi = \psi_L + \psi_R = \tfrac{1}{2}(1-\gamma_5)\psi + \tfrac{1}{2}(1+\gamma_5)\psi\,. \tag{6.40}$$

We have seen that the massless LH and RH spinors, $u(\boldsymbol{p},-1/2) = P_L u(\boldsymbol{p},-1/2)$ and $u(\boldsymbol{p},1/2) = P_R u(\boldsymbol{p},1/2)$, are eigenstates of the helicity operator with eigenvalues $-1/2$ and $+1/2$, respectively. Similarly, $v(\boldsymbol{p},-1/2) = P_L v(\boldsymbol{p},-1/2)$ and $v(\boldsymbol{p},1/2) = P_R v(\boldsymbol{p},1/2)$, are eigenstates with eigenvalues $+1/2$ and $-1/2$, respectively. Using this, we can project onto the different spin states. Consider, for instance, the second factor in (6.39). We replace

$$\bar{v}(\boldsymbol{p}',s')\gamma^\mu u(\boldsymbol{p},s) \to \bar{v}(\boldsymbol{p}',s')\gamma^\mu \tfrac{1}{2}(1+\gamma_5)u(\boldsymbol{p},s)\,. \tag{6.41}$$

Then the amplitude vanishes for a LH polarized electron ($h=-1/2$), while it is unchanged for a RH electron ($h=+1/2$). We have

$$\bar{v}\gamma_\mu\tfrac{1}{2}(1+\gamma_5)u = v^\dagger\beta\gamma^\mu\tfrac{1}{2}(1+\gamma_5)u = v^\dagger\tfrac{1}{2}(1+\gamma_5)\beta\gamma^\mu u = (\tfrac{1}{2}(1+\gamma_5)v)^\dagger\beta\gamma^\mu u\,, \tag{6.42}$$

so the positron must be RH polarized! In general, the amplitude vanishes unless electron and positron have opposite helicity.

Let's calculate the squared matrix element. The "electron factor" yields now

$$\begin{aligned}
\sum_{\text{spins}}\left|\bar{v}(\boldsymbol{p}',s')\gamma^\mu\tfrac{1}{2}(1+\gamma_5)u(\boldsymbol{p},s)\right|^2 &= \mathrm{Tr}\big\{\not{p}'\gamma^\mu\tfrac{1}{2}(1+\gamma_5)\not{p}\gamma^\nu\tfrac{1}{2}(1+\gamma_5)\big\} \\
&= \mathrm{Tr}\big\{\not{p}'\gamma^\mu\not{p}\gamma^\nu\tfrac{1}{2}(1+\gamma_5)\big\}\,.
\end{aligned} \tag{6.43}$$

Now we use

$$\mathrm{Tr}\big\{\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_5\big\} = -4i\epsilon^{\mu\nu\rho\sigma} \tag{6.44}$$

and obtain

$$\sum_{\text{spins}} \left|\bar{v}(\boldsymbol{p}', s')\gamma^\mu \tfrac{1}{2}(1+\gamma_5)u(\boldsymbol{p}, s)\right|^2 = \frac{2}{4p^0 p'^0}\left(p'^\mu p^\nu + p'^\nu p^\mu - \eta^{\mu\nu} p\cdot p' - i\epsilon^{\alpha\mu\beta\nu} p'_\alpha p_\beta\right). \quad (6.45)$$

The analogous calculation for a RH μ^- and a LH μ^+ yields

$$\sum_{\text{spins}} \left|\bar{u}(\boldsymbol{k}, r)\gamma_\mu \tfrac{1}{2}(1+\gamma_5)v(\boldsymbol{k}', r')\right|^2 = \frac{2}{4k^0 k'^0}\left(k'_\mu k_\nu + k'_\nu k_\mu - \eta_{\mu\nu} k\cdot k' - i\epsilon_{\rho\mu\sigma\nu} k^\rho k'^\sigma\right). \quad (6.46)$$

Therefore, the squared matrix element for $e^-_R e^+_L \to \mu^-_R \mu^+_L$ is

$$\begin{aligned}
\overline{|M|^2} &= \frac{e^4}{(2\pi)^6}\frac{1}{q^4}\frac{4}{16p^0 p'^0 k^0 k'^0} \\
&\quad\times\left[2(p\cdot k)(p'\cdot k') + 2(p'\cdot k)(p\cdot k') - \epsilon^{\alpha\mu\beta\nu}\epsilon_{\rho\mu\sigma\nu} p'_\alpha p_\beta k^\rho k'^\sigma\right] \\
&= \frac{e^4}{(2\pi)^6}\frac{1}{q^4}\frac{16}{16p^0 p'^0 k^0 k'^0}(p'\cdot k)(p\cdot k') \\
&\overset{\text{c.m.s.}}{=} \frac{e^4}{(2\pi)^6}\frac{1}{16E^4}(1+\cos\theta)^2,
\end{aligned} \quad (6.47)$$

with $p^0 p'^0 k^0 k'^0 = E$; $q^2 = 4E^2$, $(p'\cdot k) = (p\cdot k') = E^2(1+\cos\theta)$. In the first line, we used $\epsilon^{\alpha\mu\beta\nu}\epsilon_{\rho\mu\sigma\nu} = 2(\delta^\alpha_\sigma\delta^\beta_\rho - \delta^\alpha_\rho\delta^\beta_\sigma)$. Inserting into (3.42) yields the cross section

$$\frac{d\sigma(e^-_R e^+_L \to \mu^-_R \mu^+_L)}{d\Omega} = \frac{\alpha^2}{4s}(1+\cos\theta)^2. \quad (6.48)$$

To calculate the remaining three non-vanishing amplitudes, we just need to reverse the sign of γ_5 in (6.45) and / or (6.46). That just changes the signs of the terms with the Levi-Civita tensor, and we get

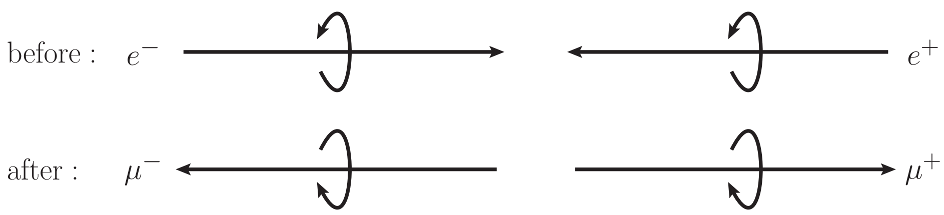
$$\frac{d\sigma(e^-_R e^+_L \to \mu^-_L \mu^+_R)}{d\Omega} = \frac{\alpha^2}{4s}(1-\cos\theta)^2, \quad (6.49)$$

$$\frac{d\sigma(e^-_L e^+_R \to \mu^-_R \mu^+_L)}{d\Omega} = \frac{\alpha^2}{4s}(1-\cos\theta)^2, \quad (6.50)$$

$$\frac{d\sigma(e^-_L e^+_R \to \mu^-_L \mu^+_R)}{d\Omega} = \frac{\alpha^2}{4s}(1+\cos\theta)^2. \quad (6.51)$$

Summing the four terms and dividing by 4 for spin averaging, we reproduce (6.28).

The physical meaning of, for instance, Eq. (6.48) can be understood as follows. For $\theta = \pi$ the cross section vanishes. This is nothing but conservation of angular momentum:



Since the total angular momentum is conserved, the amplitude must vanish (see Ref. [5], ch. 5.2 for more details). Helicity is conserved in the high-energy limit.

6.4 $e^+e^- \to \mu^+\mu^-$: Nonrelativistic limit

For $E \approx m_\mu$, the unpolarized cross section (6.26) becomes

$$\frac{d\sigma(e^+e^- \to \mu^+\mu^-)}{d\Omega} = \frac{\alpha^2}{4s} \underbrace{\sqrt{1 - \frac{m_\mu^2}{E^2}}}_{|\boldsymbol{k}|/E} \left[\underbrace{1 + \frac{m_\mu^2}{E^2}}_{\approx 2} + \Bigg(\underbrace{1 - \frac{m_\mu^2}{E^2}}_{\approx 0} \Bigg) \cos^2\theta \right] \xrightarrow{E \approx m_\mu} \frac{\alpha^2}{2s} \frac{|\boldsymbol{k}|}{E} . \quad (6.52)$$

How can we understand the absence of the angular dependence? Let's calculate (6.52) explicitly in the NR limit.

Consider again the "electron factor" in (6.39). Since e^+ and e^- must be ultrarelativistic ($m_\mu \gg m_e$!), we choose helicity eigenstates, e.g. RH e^- in z direction, LH e^+ in $-z$ direction. The corresponding spinors are (cf. Eq. (4.44))

$$u(\boldsymbol{p}, 1/2) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} ; \qquad v(\boldsymbol{p}, 1/2) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} . \quad (6.53)$$

Using (4.32) we obtain

$$\bar{v}(\boldsymbol{p}, 1/2)\gamma^\mu u(\boldsymbol{p}, 1/2) = (0, -1)\sigma^\mu \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \quad (6.54)$$

with

$$\sigma^\mu \equiv (\mathbb{1}, \boldsymbol{\sigma}) . \quad (6.55)$$

A simple calculation gives

$$\bar{v}(\boldsymbol{p}, 1/2)\gamma^\mu u(\boldsymbol{p}, 1/2) = (0, -1, -i, 0) . \quad (6.56)$$

For the "muon factor" we use the general basis in the NR limit, Eq. (4.36). We write

$$u(0, \sigma) = \frac{1}{\sqrt{2}} \begin{pmatrix} \chi \\ \chi \end{pmatrix} ; \qquad v(0, \sigma) = \frac{1}{\sqrt{2}} \begin{pmatrix} \chi' \\ -\chi' \end{pmatrix} , \quad (6.57)$$

where (see Eq. (4.36))

$$\begin{aligned} \chi &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \quad \chi' = \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \qquad \text{for} \quad \sigma = +\tfrac{1}{2} ; \\ \chi &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \quad \chi' = \begin{pmatrix} -1 \\ 0 \end{pmatrix} , \qquad \text{for} \quad \sigma = -\tfrac{1}{2} . \end{aligned} \quad (6.58)$$

Defining

$$\bar{\sigma}^\mu \equiv (\mathbb{1}, -\boldsymbol{\sigma}) , \quad (6.59)$$

Eq. (4.32) becomes

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} , \quad (6.60)$$

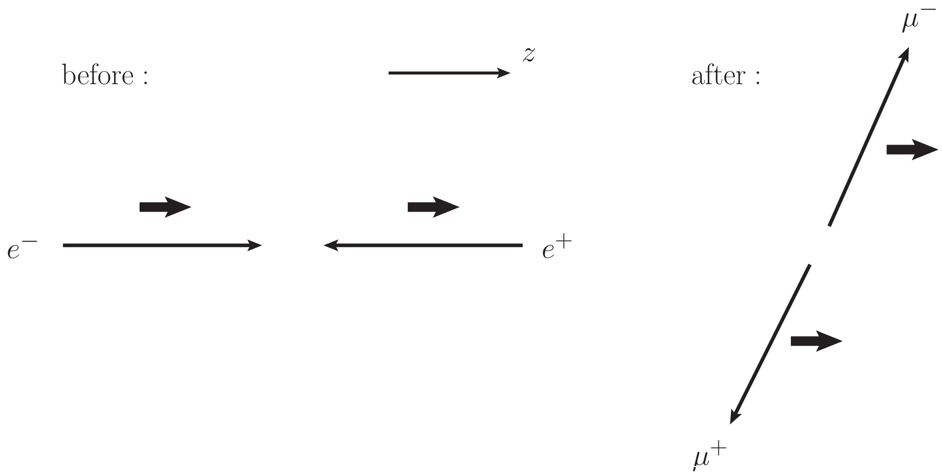
and we get

$$\bar{u}(\boldsymbol{k},r)\gamma^\mu v(\boldsymbol{k}',r') \overset{\boldsymbol{k},\boldsymbol{k}'\to 0}{=} \frac{1}{2}(\chi^\dagger,\chi^\dagger)\begin{pmatrix}\sigma^\mu & 0\\ 0 & \bar{\sigma}^\mu\end{pmatrix}\begin{pmatrix}\chi' & -\chi'\end{pmatrix}$$
$$= \begin{cases} 0\,, & \mu = 0\,,\\ -\chi^\dagger\sigma^i\chi'\,, & \mu = 1,2,3\,.\end{cases} \tag{6.61}$$

The Lorentz scalar product of (6.56) and (6.61) finally gives the scattering amplitude ($q^2 = 4m_\mu^2$)

$$M(e_R^- e_L^+ \to \mu^+\mu^-) = -\frac{e^2}{(2\pi)^3}\frac{1}{4m_\mu^2}\,\chi^\dagger\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\chi'\,. \tag{6.62}$$

This expression is independent of the scattering angle; the orbital angular momentum of the $\mu^+\mu^-$ pair is zero ("s wave"). We see from (6.58), (6.62) that we need to choose $\sigma = +1/2$ for both μ^+ and μ^- to get a non-vanishing scattering amplitude. This is again conservation of angular momentum!



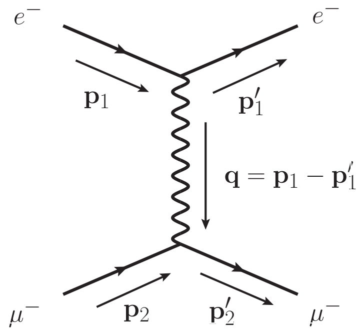
We find the scattering cross section for this process by summing over the muon spins (only one terms contributes); this gives (cf. (3.42))

$$\frac{d\sigma(e_R^- e_L^+ \to \mu^+\mu^-)}{d\Omega} = \frac{\alpha^2}{s}\frac{|\boldsymbol{k}|}{E}\,. \tag{6.63}$$

The amplitude for the process $e_L^- e_R^+ \to \mu^+\mu^-$ yields the same cross section. Summing the two terms and dividing by 4 (spin average) yields again (6.52).

6.5 $e^-\mu^- \to e^-\mu^-$ and "crossing symmetry"

We now consider the process $e^-\mu^- \to e^-\mu^-$. To leading order in QED we have



The corresponding scattering amplitude is

$$-2\pi i M = \frac{ie^2}{(2\pi)^2} \frac{(\bar{u}(\boldsymbol{p}_1', s')\gamma^\mu u(\boldsymbol{p}_1, s))(\bar{u}(\boldsymbol{p}_2', r')\gamma_\mu u(\boldsymbol{p}_2, r))}{(p_1 - p_1')^2} . \tag{6.64}$$

The summed and spin-averaged square is then (taking $m_e = 0$ as before)

$$\begin{aligned} \overline{|M|^2} = \frac{1}{4} \frac{e^4}{(2\pi)^6} & \frac{1}{((p_1 - p_1')^2)^2} \frac{1}{16 p_1^0 p_1'^0 p_2^0 p_2'^0} \\ & \times \operatorname{Tr}\{\not{p}_1' \gamma^\mu \not{p}_1 \gamma^\nu\} \operatorname{Tr}\{(\not{p}_2' + m_\mu)\gamma_\mu(\not{p}_2 + m_\mu)\gamma_\nu\} . \end{aligned} \tag{6.65}$$

This is the same expression as (6.18) after the replacements

$$p \to p_1 , \qquad p' \to -p_1' , \qquad k \to p_2' , \qquad k' \to -p_2 . \tag{6.66}$$

We can do the same replacements in (6.24), so we do not need to calculate the traces again. That gives

$$\begin{aligned} \overline{|M|^2} = & \frac{1}{4} \frac{e^4}{(2\pi)^6} \frac{1}{(p_1 - p_1')^4} \frac{1}{16 p_1^0 p_1'^0 p_2^0 p_2'^0} \\ & \times 16 \left[-2m_\mu^2 (p_1 \cdot p_1') + 2(p_1' \cdot p_2')(p_1 \cdot p_2) + 2(p_1' \cdot p_2)(p_1 \cdot p_2') \right] . \end{aligned} \tag{6.67}$$

The kinematics for this process is, however, totally different. We work again in the c.m.s. (see Fig. 3), where $E^2 = k^2 + m_\mu^2$, $\boldsymbol{k}_z = k \cos\theta$, $\sqrt{s} = E + k$. With this we get $p_1 \cdot p_2 = p_1' \cdot p_2' = k(E + k)$; $p_1' \cdot p_2 = p_1 \cdot p_2' = k(E + k\cos\theta)$; $p_1 \cdot p_1' = k^2(1 - \cos\theta)$; $q^2 = (p_1 - p_1')^2 = -2p_1 \cdot p_1' = -2k^2(1 - \cos\theta)$, and obtain

$$\begin{aligned} \overline{|M|^2} = & \frac{1}{4} \frac{e^4}{(2\pi)^6} \frac{1}{16 p_1^0 p_1'^0 p_2^0 p_2'^0} \\ & \times \frac{1}{4k^2(1 - \cos\theta)^2} \left[(E + k)^2 + (E + k\cos\theta)^2 - m_\mu^2 (1 - \cos\theta) \right] . \end{aligned} \tag{6.68}$$

Inserting into (3.42) gives the differential cross section

$$\begin{aligned} \frac{d\sigma(e^-\mu^- \to e^-\mu^-)}{d\Omega} = & \frac{\alpha^2}{2k^2 s (1 - \cos\theta)^2} \\ & \times \left[(E + k)^2 + (E + k\cos\theta)^2 - m_\mu^2 (1 - \cos\theta) \right] , \end{aligned} \tag{6.69}$$

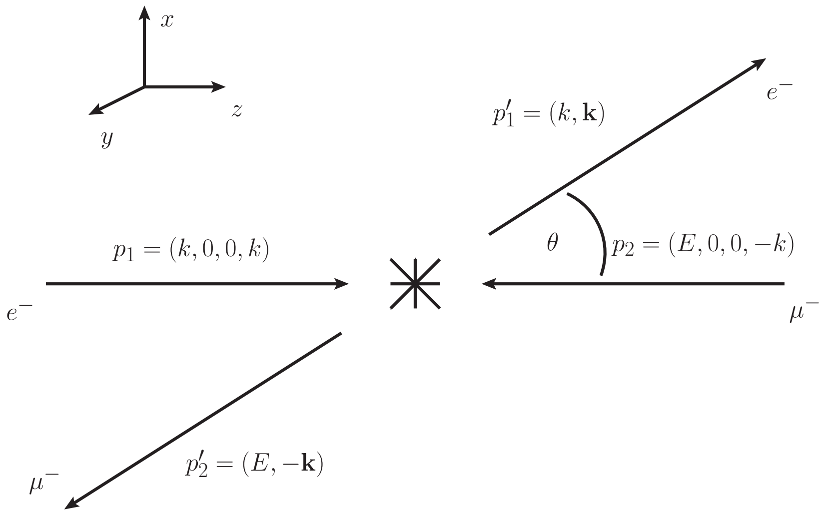


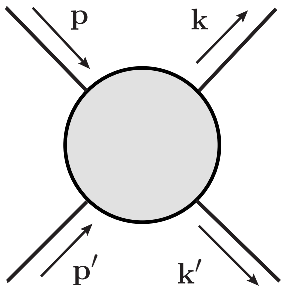
Figure 3: *Kinematics for* $e^-\mu^- \to e^-\mu^-$ *in the center-of-mass system.*

and in the high-energy limit ($m_\mu \to 0$, $E \approx k$)

$$\frac{d\sigma(e^-\mu^- \to e^-\mu^-)}{d\Omega} = \frac{\alpha^2}{2s(1-\cos\theta)^2}\left[4 + (1+\cos\theta)^2\right]. \tag{6.70}$$

Remark: The differential cross section diverges like $1/\theta^4$ for $\theta \to 0$ ("Rutherford peak").

The *crossing relation* becomes more transparent if we use *Mandelstam variables*:

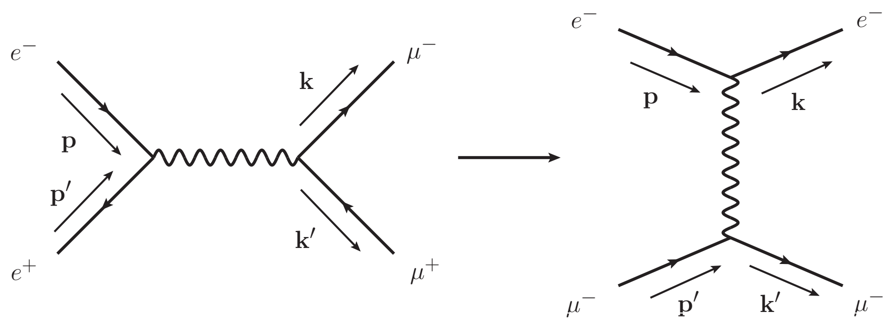


$$s = (p+p')^2 = (k+k')^2\,, \tag{6.71}$$
$$t = (k-p)^2 = (k'-p')^2\,, \tag{6.72}$$
$$u = (k'-p)^2 = (k-p')^2\,. \tag{6.73}$$

Example: $e^+e^- \to \mu^+\mu^-$ in the high-energy limit
Here, $t = -2p\cdot k = -2p'\cdot k'$, $u = -2p'\cdot k = -2p\cdot k'$, and the squared amplitude becomes (cf. (6.24))

$$\overline{|M|^2} = \frac{1}{4}\frac{e^4}{(2\pi)^6}\frac{1}{k^0k'^0p^0p'^0}\;\frac{1}{s^2}\left[\frac{t^2}{2}+\frac{u^2}{2}\right]. \tag{6.74}$$

Crossing:



Hence, we have to perform the following replacements (particles in initial state $\to$ antiparticles in final state and vice versa; momentum with opposite sign)

$$p' \to -k\,, \qquad k' \to -p'\,, \qquad p \to p\,, \qquad k \to k'\,, \tag{6.75}$$

and hence

$$s = (p+p')^2 \to (p-k)^2 = t\,, \tag{6.76}$$
$$t = (k-p)^2 \to (k'-p)^2 = u\,, \tag{6.77}$$
$$u = (k-p')^2 \to (k'+k)^2 = s\,. \tag{6.78}$$

Therefore,

$$\overline{|M|^2} \to \frac{1}{4}\frac{e^4}{(2\pi)^6}\frac{1}{p_1^0 p_1'^0 p_2^0 p_2'^0}\,\frac{1}{t}\left[\frac{u^2}{2}+\frac{s^2}{2}\right]. \tag{6.79}$$

This agrees with (6.67).

If only a single diagram contributes to a given process, once refers to "s-channel, t-channel, u-channel". They lead to a characteristic angular dependence of the scattering amplitude:

s-channel: $\sim \frac{1}{s} = \frac{1}{E^2_{\text{c.m.s.}}}$;

t-channel: $\sim \frac{1}{t} = \frac{1}{1-\cos\theta}$;

u-channel: $\sim \frac{1}{u} = \frac{1}{1+\cos\theta}$.

6.6 Compton scattering*

Compton scattering plays a large role in cosmology and astrophysics, so we discuss it here in detail. The two lowest-order Feynman diagrams are shown in Fig. 4. They give the amplitude

$$\begin{aligned}
&-2\pi i\delta^4(k+p-k'-p')M = \frac{(-ie)^2}{(2\pi)^2\sqrt{2k^0}\sqrt{2k'^0}}e_\mu(k,\sigma)e^*_\nu(k',\sigma') \\
&\times\left[\frac{\bar u(\boldsymbol{p}',s')(\gamma^\nu(\not p+\not k+m_e)\gamma^\mu)u(\boldsymbol{p},s)}{(p+k)^2-m_e^2}+\frac{\bar u(\boldsymbol{p}',s')(\gamma^\mu(\not p-\not k'+m_e)\gamma^\nu)u(\boldsymbol{p},s)}{(p-k')^2-m_e^2}\right] \\
&\times\delta^4(k+p-k'-p')\,.
\end{aligned} \tag{6.80}$$

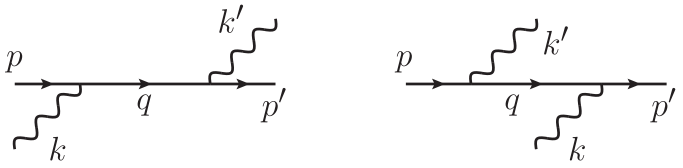


Figure 4: *Leading-order diagrams for Compton scattering.*

Simplifying the denominators, this gives

$$\begin{aligned} M = {} & \frac{i(-ie)^2}{4(2\pi)^3\sqrt{k^0k'^0}} e_\mu(k,\sigma)e^*_\nu(k',\sigma') \\ & \times \left[\frac{\bar{u}(\boldsymbol{p}',s')(\gamma^\nu(\not{p}+\not{k}+m_e)\gamma^\mu)u(\boldsymbol{p},s)}{p\cdot k} - \frac{\bar{u}(\boldsymbol{p}',s')(\gamma^\mu(\not{p}-\not{k}'+m_e)\gamma^\nu)u(\boldsymbol{p},s)}{p\cdot k'} \right] . \end{aligned} \tag{6.81}$$

Now we simplify the numerators by using the Clifford relations and the Dirac equation:

$$(\not{p}+m_e)\gamma^\mu u(\boldsymbol{p},s) = [2p^\mu - \gamma^\mu(\not{p}-m_e)]u(\boldsymbol{p},s) = 2p^\mu u(\boldsymbol{p},s)\,, \tag{6.82}$$

and similarly for $(\not{p}+m_e)\gamma^\nu u(\boldsymbol{p},s)$. This gives

$$\begin{aligned} M &= -\frac{ie^2}{4(2\pi)^3\sqrt{k^0k'^0}} e_\mu(k,\sigma)e^*_\nu(k',\sigma') \\ &\quad \times \bar{u}(\boldsymbol{p}',s')\left[\frac{2\gamma^\nu p^\mu + \gamma^\nu\not{k}\gamma^\mu}{p\cdot k} - \frac{2\gamma^\mu p^\nu - \gamma^\mu\not{k}'\gamma^\nu}{p\cdot k'} \right] u(\boldsymbol{p},s) \\ &= -\frac{ie^2}{4(2\pi)^3\sqrt{k^0k'^0}} \bar{u}(\boldsymbol{p}',s')\left[\frac{2\not{e}'^*(p\cdot e) + \not{e}'^*\not{k}\not{e}}{p\cdot k} - \frac{2\not{e}(p\cdot e^*) - \not{e}\not{k}'\not{e}'^*}{p\cdot k'} \right] u(\boldsymbol{p},s)\,. \end{aligned} \tag{6.83}$$

Next, we calculate the squared matrix element, summing over the final and averaging over the initial electron spins. For this, we need to evaluate the trace

$$\begin{aligned} \mathrm{Tr}\Bigg\{ & (\not{p}'+m_e)\left[\frac{2\not{e}'^*(p\cdot e) + \not{e}'^*\not{k}\not{e}}{p\cdot k} - \frac{2\not{e}(p\cdot e^*) - \not{e}\not{k}'\not{e}'^*}{p\cdot k'} \right] \\ & \times (\not{p}+m_e)\left[\frac{2\not{e}'(p\cdot e^*) + \not{e}^*\not{k}\not{e}'}{p\cdot k} - \frac{2\not{e}^*(p\cdot e) - \not{e}'\not{k}'\not{e}^*}{p\cdot k'} \right] \Bigg\} . \end{aligned} \tag{6.84}$$

This can be further simplified observing that $(p\cdot e) = (p\cdot e') = (p\cdot e^*) = (p\cdot e'^*) = 0$, which can be seen to be true in the rest frame of the initial-state electron and in Coulomb gauge. The trace simplifies to

$$\mathrm{Tr}\Bigg\{ (\not{p}'+m_e)\left[\frac{\not{e}'^*\not{k}\not{e}}{p\cdot k} + \frac{\not{e}\not{k}'\not{e}'^*}{p\cdot k'} \right] (\not{p}+m_e)\left[\frac{\not{e}^*\not{k}\not{e}'}{p\cdot k} + \frac{\not{e}'\not{k}'\not{e}^*}{p\cdot k'} \right] \Bigg\} . \tag{6.85}$$

Before we proceed, we need to say a few words about the photon polarization. (We do not want to immediately sum over photon polarizations, as the polarized cross section plays a role in both astrophysics and cosmology.)

6.6.1 Photon polarization

In Sec. 4.7, we have defined the polarization vectors for photons of helicity ± 1 as

$$e^\mu(\boldsymbol{p}, \pm 1) = R(\hat{p}) \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ \pm i/\sqrt{2} \\ 0 \end{pmatrix} . \tag{6.86}$$

The matrix element for absorbing or emitting a photon in a general superposition of helicity eigenstates $\alpha_+|\boldsymbol{p}, +1\rangle + \alpha_-|\boldsymbol{p}, -1\rangle$, with $|\alpha_+|^2 + |\alpha_-|^2 = 1$, can be obtained by simply using the corresponding linear combination of polarization vectors

$$e^\mu(\boldsymbol{p}) = \alpha_+ e^\mu(\boldsymbol{p}, +1) + \alpha_- e^\mu(\boldsymbol{p}, -1) \tag{6.87}$$

in the Feynman rules. Eq. (6.86) shows that the polarization vectors for definite helicity are normalized as

$$e^*_\mu(\boldsymbol{p}, \sigma') e^\mu(\boldsymbol{p}, \sigma) = \delta_{\sigma'\sigma} , \tag{6.88}$$

and therefore the general polarization vectors satisfy

$$e^*_\mu(\boldsymbol{p}) e^\mu(\boldsymbol{p}) = 1 . \tag{6.89}$$

The limiting cases are *circular polarization*, with $\alpha_+ = 0$ or $\alpha_- = 0$, and *linear polarization* with $|\alpha_+| = |\alpha_-| = 1/\sqrt{2}$. By adjusting an overall phase, we can choose the coefficients for linear polarization as $\alpha_\pm = e^{\mp i\phi}/\sqrt{2}$, such that the polarization vector is

$$e^\mu(\boldsymbol{p}) = R(\hat{p}) \begin{pmatrix} 0 \\ \cos\phi \\ \sin\phi \\ 0 \end{pmatrix} , \tag{6.90}$$

with ϕ the angle of photon polarization in a plane perpendicular to the photon momentum $\boldsymbol{p}$. In this case, the polarization vector is real.

In general, the initial photon state may be in a statistical mixture of polarization states $e^{(r)}_\mu(\boldsymbol{p})$ with probabilities P_r, normalized as $\sum_r P_r = 1$. The rate for absorbing such a photon is then proportional to

$$\sum_r P_r |e^{(r)}_\mu(\boldsymbol{p}) M^\mu|^2 = M^{\mu *} M^\nu \rho_{\nu\mu}(\boldsymbol{p}) , \tag{6.91}$$

which can be expressed in terms of a density matrix

$$\rho_{\nu\mu}(\boldsymbol{p}) = \sum_r P_r e^{(r)}_\nu(\boldsymbol{p}) e^{(r)*}_\mu(\boldsymbol{p}) . \tag{6.92}$$

It is straightforward to see that the matrix $\rho(\boldsymbol{p})$ is Hermitian, positive definite, and has unit trace, and further that $\rho_{0\mu}(\boldsymbol{p}) = \rho_{\nu 0}(\boldsymbol{p}) = 0$ and $\rho_{\nu\mu}(\boldsymbol{p}) p^\mu = \rho_{\nu\mu}(\boldsymbol{p}) p^\nu = 0$. Therefore, it can be diagonalized with two real, positive eigenvalues λ_1, λ_2 satisfying $\lambda_1 + \lambda_2 = 1$, i.e.

$$\rho_{\nu\mu}(\boldsymbol{p}) = \sum_s \lambda_s e_\nu(\boldsymbol{p}, s) e^*_\mu(\boldsymbol{p}, s) , \tag{6.93}$$

where $e_\mu(\boldsymbol{p}, s)$ are the two normalized eigenvectors coresponding to λ_s, satisfying

$$e_0(\boldsymbol{p}, s) = e_\mu(\boldsymbol{p}, s) p^\mu = 0 \,. \tag{6.94}$$

It follows that the rate for absorbing the photon is proportional to

$$\sum_s \lambda_s |e^{(r)}_\mu(\boldsymbol{p}, s) M^\mu|^2 \,, \tag{6.95}$$

showing that any statistical mixture of initial photon polarization states is equivalent to two orthonormal polarization states with probabilities λ_s. In particular, if the initial photon polarization is unknown, we have $\lambda_1 = \lambda_2 = 1/2$, and

$$\rho_{ij}(\boldsymbol{p}) = \frac{1}{2} \sum_s e_i(\boldsymbol{p}, s) e^*_j(\boldsymbol{p}, s) = \frac{1}{2}(\delta_{ij} - \hat{p}_i \hat{p}_j) \tag{6.96}$$

(see Eq. (4.105)). This result does not depend on which particular orthonormal pairs of polarization vectors we average. Similarly, if we do not measure the final photon polarization, we may sum over any pair of orthonormal final photon polarization vectors.

6.6.2 Polarized cross section

We now proceed to evaluate the trace (6.97). A straightforward calculation leads to a lengthy result that can, however, be simplified by using the on-shell conditions $k^2 = (k')^2 = 0$, the normalization conditions $|e|^2 = |e'|^2 = 1$ and $e^* \cdot e' = 0$, as well as the relations $p' \cdot k = p \cdot k'$, $p' \cdot k' = p \cdot k$, $p \cdot p' = k \cdot k' + m_e^2$, $e' \cdot p' = e' \cdot k$, and $e \cdot p' = -e \cdot k'$ that follow from the conservation of total four-momentum. We obtain

$$\begin{aligned} &\mathrm{Tr}\left\{ (\not{p}' + m_e) \left[\frac{\not{e}'^* \not{k} \not{e}}{p \cdot k} + \frac{\not{e} \not{k}' \not{e}'^*}{p \cdot k'} \right] (\not{p} + m_e) \left[\frac{\not{e}^* \not{k} \not{e}'}{p \cdot k} + \frac{\not{e}' \not{k}' \not{e}^*}{p \cdot k'} \right] \right\} \\ &= 32 (e \cdot e')^2 + \frac{8 (k \cdot k')^2}{(p \cdot k)(p \cdot k')} \,, \end{aligned} \tag{6.97}$$

and the squared matrix element, summed over initial and final electron spins, becomes

$$\sum_{s,s'} |M|^2 = \frac{e^4}{64 (2\pi)^6 p^0 p'^0 k^0 k'^0} \left[32 (e \cdot e')^2 + \frac{8 (k \cdot k')^2}{(p \cdot k)(p \cdot k')} \right] . \tag{6.98}$$

Evaluating this in the laboratory frame, with $k = (\omega, 0, 0, \omega)$, $k' = (\omega', \boldsymbol{k}')$, $p = (m_e, \boldsymbol{0})$, we find $k \cdot k' = \omega\omega'(1 - \cos\theta)$, $p \cdot k = m_e \omega$, and $p \cdot k' = m_e \omega'$, and so

$$\sum_{s,s'} |M|^2 = \frac{e^4}{64 (2\pi)^6 m_e p'^0 \omega \omega'} \left[32 (e \cdot e')^2 + \frac{8 \omega \omega' (1 - \cos\theta)^2}{m_e^2} \right] . \tag{6.99}$$

According to problem set 1, the scattering cross section in the laboratory frame is given by

$$\frac{d\sigma(e^- \gamma \to e^- \gamma)}{d\Omega} = \frac{(2\pi)^4 p'^0 \omega' \omega}{m_e + \omega - \omega \cos\theta} \frac{\omega'}{\omega} |M|^2 \,. \tag{6.100}$$

This can be written more compactly by using the relation between ω and ω' that is given by the kinematics of the process; conservation of energy gives

$$m_e + \omega = \sqrt{m_e^2 + (\boldsymbol{p}')^2} + \omega' = \sqrt{m_e^2 + \omega^2 + (\omega')^2 - 2\omega\omega'\cos\theta} + \omega' \,. \tag{6.101}$$

It follows that

$$(m_e + \omega - \omega')^2 = m_e^2 + \omega^2 + (\omega')^2 - 2\omega\omega'\cos\theta \,, \tag{6.102}$$

or

$$\omega' = \omega \frac{m_e}{m_e + \omega(1-\cos\theta)} \,. \tag{6.103}$$

Alternatively, this can be written as a shift in wavelength,

$$\frac{1}{\omega'} - \frac{1}{\omega} = \frac{1-\cos\theta}{m_e} \,. \tag{6.104}$$

Using Eq. (6.103), the cross section formula (6.100) becomes

$$\frac{d\sigma(e^-\gamma \to e^-\gamma)}{d\Omega} = \frac{(2\pi)^4 p'^0 (\omega')^3}{m_e \omega} |M|^2 \,. \tag{6.105}$$

Inserting the matrix element, using Eq. (6.103) again, and including a factor $1/2$ for electron spin averaging gives

$$\frac{d\sigma(e^- + \gamma, e \to e^- + \gamma, e')}{d\Omega} = \frac{\alpha^2}{4m_e^2}\frac{\omega'^2}{\omega^2}\left[\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - 2 + 4(e\cdot e')^2\right] . \tag{6.106}$$

This is the *Klein-Nishina formula.*

Frequently, the polarization of the initial-state photon is not known, while the polarization of the final-state photon can be measured (for instance, in measuring the polarization of the cosmic microwave background). In this case, we should average over the helicities of the photons in the initial state, as explained in Sec. 6.6.1. Using Eq. (6.96) then gives

$$\frac{d\sigma(e^- + \gamma \to e^- + \gamma, e)}{d\Omega} = \frac{\alpha^2}{4m_e^2}\frac{\omega'^2}{\omega^2}\left[\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - 2(\hat{\boldsymbol{k}}\cdot\boldsymbol{e}')^2\right] . \tag{6.107}$$

The scattered photon tends to be polarized perpendicular not only to the outgoing, but also the incoming photon momentum, i.e. perpendicular to plane in which the scattering takes place.

6.6.3 Unpolarized cross section and Thomson scattering

If the final photon polarization is not measured either, we sum over its possible polarizations. Applying Eq. (6.96) (without the factor $1/2$) for the final polarizations, we find

$$\frac{d\sigma(e^- + \gamma \to e^- + \gamma)}{d\Omega} = \frac{\alpha^2}{2m_e^2}\frac{\omega'^2}{\omega^2}\left[\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2\theta\right] . \tag{6.108}$$

where $\cos\theta = \hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{k}}'$.

In the non-relativistic limit, i.e. $\omega \ll m_e$, Eq. (6.103) shows that $\omega = \omega'$, and the differential cross section reduces to

$$\frac{d\sigma(e^- + \gamma \to e^- + \gamma)}{d\Omega} = \frac{\alpha^2}{2m_e^2}\left(1 + \cos^2\theta\right) . \tag{6.109}$$

Performing the integration over $d\Omega$, we obtain the *Thomson cross section*

$$\sigma_T = \frac{8\pi\alpha^2}{3m_e^2} . \tag{6.110}$$

We conclude this section by pointing out a shortcut to the calculation of the unpolarized cross section. As a consequence of gauge invariance, in calculating the squared matrix element summed over initial and / or final photon helicities, it is permissible to replace the photon polarization sum (4.105) by an "effective" polarization sum

$$\sum_{\sigma=\pm} e^\mu(\boldsymbol{p},\sigma) e^{\nu *}(\boldsymbol{p},\sigma) = -\eta^{\mu\nu} . \tag{6.111}$$

Instead of proving this statement, we will just use it to calculate the unpolarized cross section and verify that we obtain the same result. We start with the amplitude

$$\begin{aligned} M = &-\frac{ie^2}{4(2\pi)^3\sqrt{k^0 k'^0}} e_\mu(k,\sigma) e_\nu^*(k',\sigma') \\ &\times \bar{u}(\boldsymbol{p}',s')\left[\frac{2\gamma^\nu p^\mu + \gamma^\nu \not{k}\gamma^\mu}{p\cdot k} - \frac{2\gamma^\mu p^\nu - \gamma^\mu \not{k}'\gamma^\nu}{p\cdot k'}\right] u(\boldsymbol{p},s) . \end{aligned} \tag{6.112}$$

Taking the absolute value squared and using the relation (6.111), we obtain

$$\begin{aligned} \sum_{\sigma\sigma'} |M|^2 = &\frac{e^4}{16(2\pi)^6 k^0 k'^0} \bar{u}(\boldsymbol{p}',s')\left[\frac{2\gamma^\nu p^\mu + \gamma^\nu \not{k}\gamma^\mu}{p\cdot k} - \frac{2\gamma^\mu p^\nu - \gamma^\mu \not{k}'\gamma^\nu}{p\cdot k'}\right] u(\boldsymbol{p},s) \\ &\times \bar{u}(\boldsymbol{p},s)\left[\frac{2\gamma_\nu p_\mu + \gamma_\mu \not{k}\gamma_\nu}{p\cdot k} - \frac{2\gamma_\mu p_\nu - \gamma_\nu \not{k}'\gamma_\mu}{p\cdot k'}\right] u(\boldsymbol{p}',s') . \end{aligned} \tag{6.113}$$

Summing now also over initial and final electron spins leads to the trace

$$\begin{aligned} &\mathrm{Tr}\bigg\{ (\not{p}' + m_e)\left[\frac{2\gamma^\nu p^\mu + \gamma^\nu \not{k}\gamma^\mu}{p\cdot k} - \frac{2\gamma^\mu p^\nu - \gamma^\mu \not{k}'\gamma^\nu}{p\cdot k'}\right] \\ &\qquad \times (\not{p} + m_e)\left[\frac{2\gamma_\nu p_\mu + \gamma_\mu \not{k}\gamma_\nu}{p\cdot k} - \frac{2\gamma_\mu p_\nu - \gamma_\nu \not{k}'\gamma_\mu}{p\cdot k'}\right]\bigg\} \\ &= \frac{32}{4p^0 p'^0}\left[\frac{p\cdot k'}{p\cdot k} + \frac{p\cdot k}{p\cdot k'} + 2m_e^2\left(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'}\right) + m_e^4\left(\frac{1}{p\cdot k'} - \frac{1}{p\cdot k}\right)^2\right] . \end{aligned} \tag{6.114}$$

Evaluating the dot products in the laboratory frame gives, as before, $p \cdot k = m_e\omega$ and $p \cdot k' = m_e\omega'$, so the spin averaged squared matrix element becomes (in the frame where the electron is initially at rest)

$$\frac{1}{4}\sum_{\sigma\sigma'}|M|^2 = \frac{32e^4}{64(2\pi)^6 4\omega\omega' m_e p'^0}\left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} + 2m_e\left(\frac{1}{\omega} - \frac{1}{\omega'}\right) + m_e^2\left(\frac{1}{\omega'} - \frac{1}{\omega}\right)^2\right], \quad (6.115)$$

or, using Eq. (6.104),

$$\frac{1}{4}\sum_{\sigma\sigma'}|M|^2 = \frac{2\pi^2\alpha^2}{(2\pi)^6\omega\omega' m_e p'^0}\left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2\theta\right]. \quad (6.116)$$

Inserting this into Eq. (6.105), we recover the cross section (6.108).

7 Quantum chromodynamics

Short reminder: QED as gauge theory

We have seen in Sec. (4.6) that the description of massless spin-one particles (such as the photon) requires the invariance of the action under local gauge transformations:

$$A_\mu(x) \to A_\mu(x) + \partial_\mu\alpha(x)\,, \quad (7.1)$$

$$\psi(x) \to e^{-ie\alpha(x)}\psi(x) = \psi(x) - ie\alpha(x)\psi(x) + \dots\,. \quad (7.2)$$

The photon field must couple to a conserved fermion current. In total, the QED Lagrangian is

$$\mathcal{L} = \bar\psi\big(i\partial\!\!\!/ - eA\!\!\!/ - m\big)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,. \quad (7.3)$$

We can write this more neatly by introducing the *covariant derivative*

$$D_\mu \equiv \partial_\mu + ieA_\mu\,. \quad (7.4)$$

Then we can write Eq. (7.3) as

$$\mathcal{L} = \bar\psi\big(iD\!\!\!\!/ - m\big)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,. \quad (7.5)$$

The combination $D_\mu\psi(x)$ transforms linearly under local transformations, in the sense that

$$\begin{aligned} D_\mu\psi(x) &= \big(\partial_\mu + ieA_\mu(x)\big)\psi(x) \\ &\to \big[\partial_\mu + ieA_\mu(x) + ie(\partial_\mu\alpha(x))\big]e^{-ie\alpha(x)}\psi(x) \\ &= e^{-ie\alpha(x)}\big[\partial_\mu + ieA_\mu(x) + ie(\partial_\mu\alpha(x)) - ie(\partial_\mu\alpha(x))\big]\psi(x) \\ &= e^{-ie\alpha(x)}D_\mu\psi(x)\,, \end{aligned} \quad (7.6)$$

and so $\bar\psi D_\mu\psi$ (and hence the Lagrangian (7.5)) is gauge invariant. More generally we can construct gauge-invariant Lagrangians by replacing ordinary by covariant derivatives.

7.1 Lie algebras and gauge invariance

Eqs. (7.1) and (7.2) tell us that the QED Lagrangian in invariant under a local phase transformation of the matter fields. The *gauge group* of QED is $U(1)$. The standard model of particle physics uses a generalization of this construction.

We consider a number of fermion fields $\psi_\ell(x)$, with $\ell = 1, \ldots, N$, and require that the Lagrangian be invariant under the local transformation

$$\psi_\ell(x) \to U_\ell{}^m(x)\psi_m(x)\,, \tag{7.7}$$

where U is an element of the special unitary group $SU(N)$ (these are complex $N \times N$ matrices that satisfy the conditions $U^\dagger U = 1$ and $\det(U) = 1$). Any such matrix can be written as

$$U(x) = \exp\left[i\alpha^a(x)T^a\right], \tag{7.8}$$

where the *generators* T^a satisfy the conditions $(T^a)^\dagger = T^a$ and $\mathrm{Tr}\{T^a\} = 0$. One can show that the generators T^a form a *Lie algebra* with the commutation relations

$$\left[T^a, T^b\right] \equiv T^aT^b - T^bT^a = iC^{abc}T^c\,. \tag{7.9}$$

The *structure constants* C^{abc} can be chosen completely antisymmetric and uniquely determine the group structure.

By explicit calculation, it is straighforward to verify the *Jacobi identity*

$$0 = \left[\left[T^a, T^b\right], T^c\right] + \left[\left[T^c, T^a\right], T^b\right] + \left[\left[T^b, T^c\right], T^a\right]. \tag{7.10}$$

Inserting Eq. (7.9) yields a further condition on the structure constants:

$$0 = C^{dab}C^{edc} + C^{dca}C^{edb} + C^{dbc}C^{eda}\,. \tag{7.11}$$

This allows us to define matrices

$$\left(t_A^a\right)_{bc} \equiv -iC^{abc} \tag{7.12}$$

that satisfy the conditions (7.9):

$$\begin{aligned}
\left[t_A^a, t_A^b\right]_{cd} &= \left(t_A^a\right)_{ce}\left(t_A^b\right)_{ed} - \left(t_A^b\right)_{ce}\left(t_A^a\right)_{ed} \\
&= -C^{ace}C^{bed} + C^{bce}C^{aed} \\
&= -C^{eac}C^{deb} + C^{ecb}C^{dea} \\
&= -C^{eba}C^{dec} = C^{abe}C^{ecd} \\
&= iC^{abe}\left(-iC^{ecd}\right) = iC^{abe}\left(t_A^e\right)_{cd}\,.
\end{aligned} \tag{7.13}$$

This is called the *adjoint representation* of the Lie algebra.

Example: Quarks and color $SU(3)$

Each quark appears in one of three indistinguishable "color" states (r, g, b). Hence, for each quark we introduce a triplet of fields,

$$q(x) \equiv \begin{pmatrix} q^r(x) \\ q^g(x) \\ q^b(x) \end{pmatrix} , \tag{7.14}$$

and require that the QCD Lagrangian be invariant under $SU(3)$ rotation of the quark fields

$$q(x) \to U \cdot q(x) , \tag{7.15}$$

with $U \in SU(3)$. If we further require that these rotations be a local ("gauge") symmetry, we can couple the quarks to massless spin-one particle (*gluons*), in analogy to QED.

7.2 The Lagrangian for QCD

The kinetic term for the quark fields,

$$\mathcal{L}_{textQCD,kin} = \bar{q}(x)\big(i\partial\!\!\!/ - m_q\big)q(x) , \tag{7.16}$$

contains a derivative. We want to replace this by a covariant derivative constructed such that the Lagrangian is invariant under local $SU(3)$ transformations.

How many generators does $SU(3)$ have? T^a has N^2 complex, hence $2N^2$ real parameters. $T^\dagger = T$ gives N^2 conditions, and $\mathrm{Tr}\{T\} = 0$ gives one further condition, so we have $2N^2 - N^2 - 1 = N^2 - 1$ independent generators.

We can choose the eight *Gell-Mann matrices* as generators for $SU(3)$: $T^a = \frac{1}{2}\lambda^a$, with

$$\begin{aligned}
\lambda^1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad
\lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad
\lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \\
\lambda^4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} , \quad
\lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} , \\
\lambda^6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} , \quad
\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} , \quad
\lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} .
\end{aligned} \tag{7.17}$$

The normalization is chosen such that $\mathrm{Tr}\{T^a T^b\} = \frac{1}{2}\delta^{ab}$.

Let's first consider an infinitesimal gauge transformation:

$$q_i(x) \to q_i(x) + i\epsilon^a(x) T^a_{ij} q_j(x) . \tag{7.18}$$

Then

$$\partial q_i(x) \to \partial q_i(x) + i\epsilon^a(x) T^a_{ij} \partial q_j(x) + i\partial\epsilon^a(x) T^a_{ij} q_j(x) . \tag{7.19}$$

To construct an invariant Lagrangian we need to introduce a field whose transformation property allows to cancel the last term. Since $\epsilon^a(x)$ carries an adjoint index a, we suspect that the field transforms with t_A^a. We write tentatively $G_\mu^a(x) \to G_\mu^a(x) + \delta G_\mu^a(x)$, with

$$\delta G_\mu^a(x) = \partial_\mu \epsilon^a(x) + i\epsilon^b(x)\big(t_A^b\big)^{ac} G_\mu^c(x) = \partial_\mu \epsilon^a(x) + i\epsilon^b(x) f^{bac} G_\mu^c(x) \,. \tag{7.20}$$

Using this, we can again define a covariant derivative, acting on quark fields:

$$\big(D_\mu q(x)\big)_i \equiv \partial_\mu q_i(x) - iG_\mu^b(x) T_{ij}^b q_j(x) \,. \tag{7.21}$$

It transforms as

$$\begin{aligned} \delta\big(D_\mu q(x)\big)_i \equiv \partial_\mu\big(i\epsilon^b(x)T_{ij}^b q_j(x)\big) & \\ - i\big[\partial_\mu \epsilon^b(x) + f^{dbc}\epsilon^d(x) G_\mu^c(x)\big] T_{ij}^b q_j(x) & \\ - iG_\mu^b(x) T_{ik}^b T_{kj}^d i\epsilon^d(x) q_j(x) \,. & \end{aligned} \tag{7.22}$$

Now we use

$$-if^{dbc}T_{ij}^b = if^{dcb}T_{ij}^b = T_{ik}^d T_{kj}^c - T_{ik}^c T_{kj}^d \,, \tag{7.23}$$

and renaming the indices, we obtain

$$\begin{aligned} \delta\big(D_\mu q(x)\big)_i &= i\epsilon^b(x) T_{ij}^b \partial_\mu q_j(x) + \epsilon^b(x) T_{ik}^b T_{kj}^c G_\mu^c(x) q_j(x) \\ &= i\epsilon^b(x) T_{ik}^b \big[\delta_{kj}\partial_\mu q_j(x) - iT_{kj}^c G_\mu^c(x) q_j(x)\big] \\ &= i\epsilon^b(x) T_{ik}^b \big[D_\mu q(x)\big]_k \,, \end{aligned} \tag{7.24}$$

i.e. $D_\mu q(x)$ transforms like $q(x)$. The finite gauge transformations are

$$q(x) \to U(x) q(x) \,, \tag{7.25}$$

$$D_\mu q(x) \to U(x) D_\mu q(x) \,, \tag{7.26}$$

$$G_\mu^a(x) T^a \to U(x) G_\mu^a(x) T^a U^\dagger(x) - i\big(\partial_\mu U(x)\big) U^\dagger(x) \,, \tag{7.27}$$

where, as always, $U(x) = \exp\big[i\alpha^b(x)T^b\big]$.

Unfortunately we are not yet finished, as the terms with derivatives acting on the gluon fields are still missing. The term $\partial_\mu\partial_\nu\epsilon^a(x)$ cancels in the transformation of the antisymmetric combination $\partial_\mu G_\nu^a - \partial_\nu G_\mu^a$; however, the second terms in Eq. (7.20) still contribute.

We start with the commutator of two covariant derivatives acting of a quark field,

$$\begin{aligned} \big([D_\mu, D_\nu] q(x)\big)_i &= \big(\partial_\mu \delta_{ij} - iG_\mu^a T_{ij}^a\big)\big(\partial_\nu \delta_{jk} - iG_\nu^b T_{jk}^b\big) q_k(x) - (\mu \leftrightarrow \nu) \\ &= \partial_\mu\partial_\nu q_i - i(\partial_\mu G_\nu^b) T_{ik}^b q_k - iG_\nu^b T_{ik}^b \partial_\mu q_k - iG_\mu^b T_{ik}^b \partial_\nu q_k \\ &\qquad - G_\mu^a G_\nu^b T_{ij}^a T_{jk}^b q_k - (\mu \leftrightarrow \nu) \\ &= -iT_{ik}^a\big(\partial_\mu G_\nu^a - \partial_\nu G_\mu^a\big) q_k - if^{abc} G_\mu^a G_\nu^b q_k \\ &\equiv -iT_{ik}^a G_{\mu\nu}^a q_k \,, \end{aligned} \tag{7.28}$$

with the *gluon field strength tensor*

$$G_{\mu\nu}^a = \partial_\mu G_\nu^a - \partial_\nu G_\mu^a + f^{abc} G_\mu^b G_\nu^c \,. \tag{7.29}$$

The transformation properties $q \to Uq$ and $[D_\mu, D_\nu]q \to U[D_\mu, D_\nu]q = U[D_\mu, D_\nu]U^\dagger U q$ finally imply

$$T^a G^a_\mu \to U T^a G^a_\mu U^\dagger \,. \tag{7.30}$$

Even though $G^a_{\mu\nu}$ is not gauge invariant itself, we can now easily construct a Lorentz- and gauge-invariant term:

$$\mathrm{Tr}\{T^a G^a_{\mu\nu} T^b G^{b\,\mu\nu}\} = \frac{1}{2} G^a_{\mu\nu} G^{a\,\mu\nu} \,. \tag{7.31}$$

Conventially one makes the gauge coupling explicit by rescaling $T^a \to g_s T^a$, $f^{abc} \to g_s f^{abc}$. In this way, we have now found all ingredients of the QCD Lagrangian,

$$\mathcal{L}_{\mathrm{QCD}} = \bar{q}(x)\big(i\not{D} - m_q\big)q(x) - \frac{1}{4} G^a_{\mu\nu}(x) G^{a\,\mu\nu}(x) \,, \tag{7.32}$$

with the covariant derivative

$$D_\mu = \partial_\mu - i g_s T^a G^a_\mu(x) \,, \tag{7.33}$$

and the field strength tensor

$$G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g_s f^{abc} G^b_\mu G^c_\nu \,. \tag{7.34}$$

7.3 The Feynman rules for QCD

Most of the QCD Feynman rules are a simple generalization of the rules for QED. Dropping the spinor index and displaying the color index instead, we have the following factors for external lines

incoming quark (color index i): 

$$: \quad \frac{u^i(\boldsymbol{p}, \sigma)}{(2\pi)^{3/2}} \,, \tag{7.35}$$

incoming antiquark: 

$$: \quad \frac{\bar{v}^i(\boldsymbol{p}, \sigma)}{(2\pi)^{3/2}} \,, \tag{7.36}$$

outgoing quark: 

$$: \quad \frac{\bar{u}^i(\boldsymbol{p}, \sigma)}{(2\pi)^{3/2}} \,, \tag{7.37}$$

outgoing antiquark: 

$$: \quad \frac{v^i(\boldsymbol{p}, \sigma)}{(2\pi)^{3/2}} \,, \tag{7.38}$$

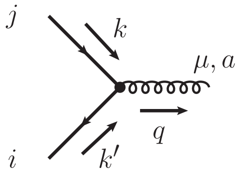
incoming gluon: 

$$: \quad \frac{e_\mu(\boldsymbol{p}, \sigma, a)}{(2\pi)^{3/2}\sqrt{2p^0}} \,, \tag{7.39}$$

outgoing gluon: 

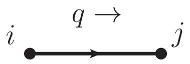
$$: \quad \frac{e^*_\mu(\boldsymbol{p}, \sigma, a)}{(2\pi)^{3/2}\sqrt{2p^0}} \,, \tag{7.40}$$

as well as the quark-gluon vertex



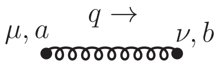
$$: \; +i g_s \gamma^\mu T^a_{ij} (2\pi)^4 \delta^4(k + k' - q) \,. \tag{7.41}$$

Quark propagator:



$$: \frac{1}{(2\pi)^4} \frac{i\delta_{ij}(\not{q} + m)}{q^2 - m^2 + i\epsilon} \tag{7.42}$$

Gluon propagator:



$$: \frac{1}{(2\pi)^4} \frac{-i\eta_{\mu\nu}\delta^{ab}}{q^2 + i\epsilon} \tag{7.43}$$

In QCD, there are additional interaction terms involving three and four gluon fields, respectively. Their Feynman rules can be most conveniently derived using the path integral formalism. As an example, we will show the "conventional" derivation of the rule for the three-gluon vertex. The requisite term in the QCD Lagrangian is

$$\begin{aligned} \mathcal{L}_{\text{QCD}} &\supset -\tfrac{1}{4} G^a_{\mu\nu}(x) G^{a\,\mu\nu}(x) \\ &\supset -\tfrac{1}{4}\big(\partial_\mu G^a_\nu - \partial_\nu G^a_\mu\big) g_s f^{abc} G^{\mu b} G^{\nu c} - \tfrac{1}{4} g_s f^{abc} G^b_\mu G^c_\nu \big(\partial^\mu G^{\nu a} - \partial^\nu G^{\mu a}\big) \\ &= -g_s f^{abc} \big(\partial_\mu G^a_\nu\big) G^{\mu b} G^{\nu c} \,. \end{aligned} \tag{7.44}$$

We consider the following, unphysical "scattering process"

The corresponding scattering amplitude is

$$\begin{aligned} &- 2i\pi \mathcal{M} \delta^4(k + p + q) \\ &= +i \int d^4x \langle 0| - g_s f^{def} \big(\partial_\lambda G^d_\kappa\big) G^{\lambda e} G^{\kappa f} a^\dagger(\boldsymbol{k}, r, a) a^\dagger(\boldsymbol{p}, s, b) a^\dagger(\boldsymbol{q}, t, c) |0\rangle \,. \end{aligned} \tag{7.45}$$

Only the terms in the gluon fields containing annihilation operators give a non-zero contribution; they supply a factor $e^{-i\ell\cdot x}$. Therefore, the derivative yields a factor $-i\ell_\lambda$. Next, we need to calculate the contractions; there are 3! different ones. The contraction indicated above gives

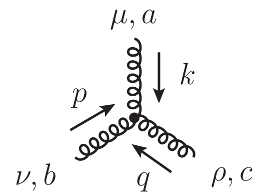
$$\begin{aligned} &- 2i\pi \mathcal{M} \delta^4(k + p + q) \\ &= -ig_s \int d^4x f^{def} (-ik_\lambda) \frac{e_\kappa(\boldsymbol{k}, r, d)}{(2\pi)^{3/2}\sqrt{2k^0}} \delta^{ad} \frac{e^\lambda(\boldsymbol{p}, s, e)}{(2\pi)^{3/2}\sqrt{2p^0}} \delta^{eb} \\ &\qquad \times \frac{e^\kappa(\boldsymbol{q}, t, f)}{(2\pi)^{3/2}\sqrt{2q^0}} \delta^{fc} e^{-i(k+p+q)\cdot x} \\ &= -g_s f^{abc} k_\lambda \eta_{\mu\kappa} \delta^\lambda_\nu \delta^\kappa_\rho \underbrace{\frac{e^\mu(\boldsymbol{k}, r, a) e^\nu(\boldsymbol{p}, s, b) e^\rho(\boldsymbol{q}, t, c)}{(2\pi)^{3/2}\sqrt{2k^0}(2\pi)^{3/2}\sqrt{2p^0}(2\pi)^{3/2}\sqrt{2q^0}}}_{(*)} (2\pi)^4 \delta(k + p + q) \,. \end{aligned} \tag{7.46}$$

The factor $(*)$ belongs to the Feynman rules for the external lines. In total, this contraction then gives a contribution to the Feynman rule of the three-vertex of

$$\ldots = -g_s f^{abc} k_\nu \eta_{\mu\rho} (2\pi)^4 \delta(k + p + q) \,. \tag{7.47}$$

The remaining five contractions yield similar contributions, with permuted Lorentz and color indices. Note that f^{abc} is antisymmetric – sign changes!

By convention one represents by the *same* Feynman diagram all contributions that differ only by a permutation of indices. The Feynman rule for the three-gluon vertex is therefore



$$: \; g_s f^{abc}\big[\eta_{\mu\nu}(k-p)_\rho + \eta_{\nu\rho}(p-q)_\mu + \eta_{\rho\mu}(q-k)_\nu\big](2\pi)^4\delta^4(k+p+q)\,. \tag{7.48}$$

Similarly, one finds for the four-gluon vertex

$$\begin{aligned} &-ig_s(2\pi)^4\delta^4(k+p+q+\ell) \\ &\times\big[\quad f^{abe}f^{cde}\big(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}\big) \\ &\quad + f^{ace}f^{bde}\big(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}\big) \\ &\quad + f^{ade}f^{bce}\big(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma}\big)\big]\,. \end{aligned} \tag{7.49}$$

NB: For a consistent quantization of QCD one should also include in the Lagrangian a gauge-fixing term, as well as "Faddeev-Popov ghosts".

7.4 Asymptotic freedom and confinement

You may have heard that the strong coupling constant $\alpha_s \equiv g_s^2/4\pi$ is "running", i.e. it is a function of the energy scale μ: $\alpha_s = \alpha_s(\mu)$. What is the meaning of this scale dependence?

In QED and QCD, the couplings and masses cannot be predicted by the theory; they must be determined experimentally, e.g. by measuring scattering cross sections. The divergences that appear in calculations of higher-order terms in the perturbation series require the *renormalization* of the theory. One can show that physical observables (such as scattering cross sections) do not depend on the renormalization procedure ("renormalization scheme"); the values for the masses and coupling, however, are scheme dependent. They are "pseudo observables"; their values are not physical. One can use this dependence to improve the convergence of the perturbation series: choose a scheme in which the couplings are small!

If physics at two (or several) different energy scales contributes to a given process, then radiative corrections are typically enhanced by logarithms of the ratios of the energy scales. E.g. $\log(m_b/M_W)$ in B physics. One can sum part of these enhanced corrections to all orders in perturbation theory; this leads to the concept of running couplings and masses ("renormalization-group improved perturbation theory").

Example: Running of the strong coupling constant

$$\alpha_s(\mu) = \frac{\alpha_s(M_Z)}{1 + \alpha_s(M_Z)\frac{\beta_0}{2\pi}\log\frac{\mu}{M_Z}}\,, \tag{7.50}$$

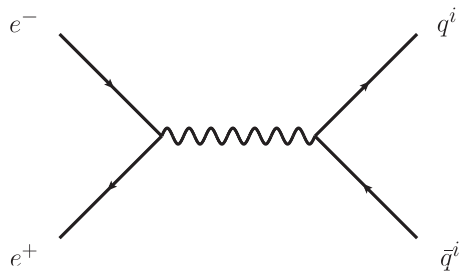
with the one-loop coefficient of the QCD beta function

$$\beta_0 = 11 - \tfrac{2}{3}N_f\,. \tag{7.51}$$

Here, N_f counts the number of fermions. If $\alpha_s(M_Z)$ is extracted from data, one can calculate $\alpha_s(\mu)$ at any other scale (as long as perturbation theory makes sense). Physical observables *do not depend* on μ!

7.5 Production of quark - antiquark pairs

We consider the process $e^+e^- \to q\bar{q}$. This is a simple generalization of $e^+e^- \to \mu^+\mu^-$. The corresponding Feynman diagram is



where $i = 1, 2, 3$ is a color index. We can obtain the cross section from Eq. (6.29) by replacing the muon charge $-e$ by the electric charges of the quarks, $Q_q e$ (with $Q_u = Q_c = Q_t = 2/3$, $Q_d = Q_s = Q_b = -1/3$), and summing over the three color states of the quarks. That gives, in the high-energy limit,

$$\sigma(e^+e^- \to q\bar{q}) \overset{s\to\infty}{\to} 3\Big(\sum_i Q_i^2\Big)\sigma_0\,, \tag{7.52}$$

where

$$\sigma_0 = \sigma(e^+e^- \to \mu^+\mu^-) = \frac{4\pi\alpha^2}{3s}\,. \tag{7.53}$$

The sum in Eq. (7.52) runs over all quark flavors that are kinematically accessible (in dependence on the center-of-mass energy $\sqrt{s}$). The quarks in the final state will hadronize and form two jets of hadrons, with an angular dependence given approximately by Eq. (6.28).

In the high-energy limit, the ratio

$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} \tag{7.54}$$

is proportional to $\left(\sum_i Q_i^2\right)N_c$; that provides an experimental verification of $N_c = 3$. See Fig. 5

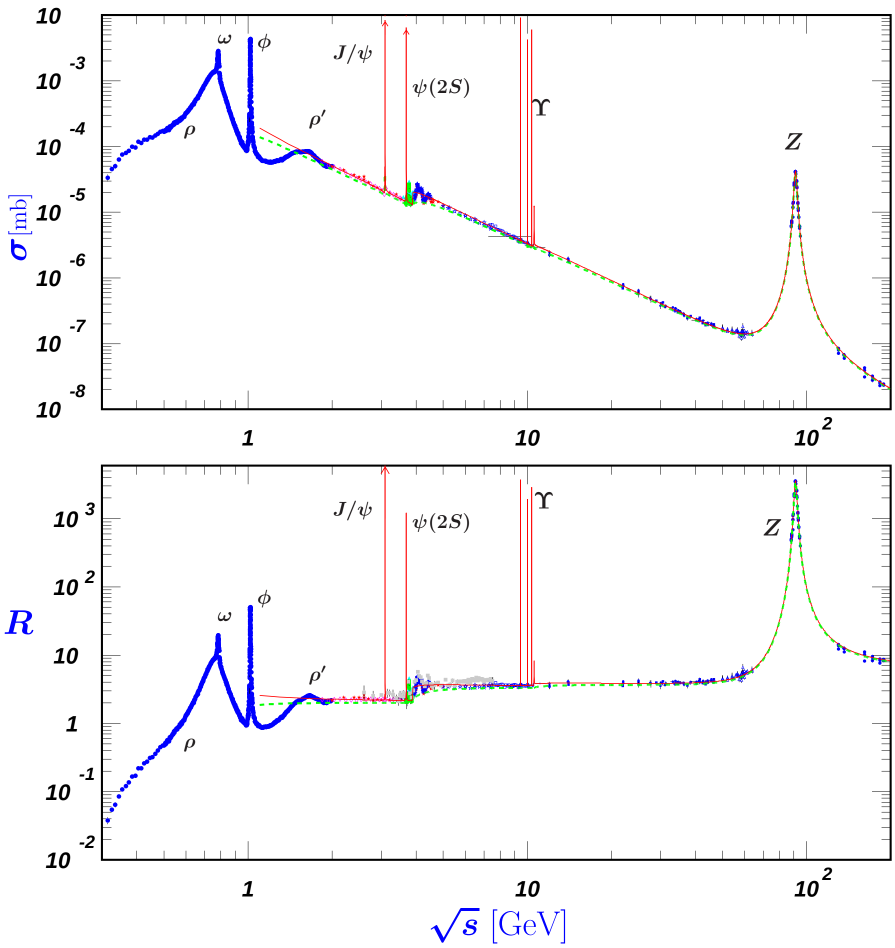
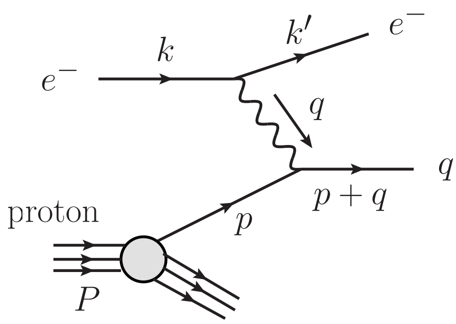


Figure 5: *Cross section $\sigma(e^+e^- \to$ hadrons$)$ and R ratio. From Ref. [6].*

7.6 Deep-inelastic scattering

The next process we would like to consider is electron-proton scattering at large momentum transfer:



The momentum transfer q is space-like,

$$\begin{aligned}(k-k')^2 &= [(k,0,0,k)-(k,\boldsymbol{k})]^2 = (0,-k_x,-k_y,k-k_z)^2 \\ &= -|\boldsymbol{k}|^2 - k^2 + 2k^2\cos\theta = 2k^2(\cos\theta - 1) \leq 0\,,\end{aligned} \tag{7.55}$$

so one frequently defines $Q^2 \equiv -q^2$. We see that Q^2 can be determined by measuring the electron momentum and energy.

We consider the process in the high-energy limit in the c.m.s. The proton carries a light-like momentum along the collision axis. In the "parton picture" we consider the proton to be a loosely bound collection of massless constituents ("partons"); each of them carries a fraction of the proton momentum P, and we write

$$p = \xi P\,, \tag{7.56}$$

with $\xi \in [0,1]$. The electron-proton scattering cross section is then given by the electron-quark cross section, where p is given by Eq. (7.56), multiplied by the probability that the quark carries the momentum fraction ξ, integrated over ξ, and summed over the partons. The probability that the proton contains a parton of type i and carries a fraction of momentum between ξ and $\xi + d\xi$ is given by the non-perturbative, universal parton distributions functions (PDF) $f_f(\xi)d\xi$. The cross section is then given, in terms of the "partonic" cross section, by

$$\sigma(e^-p \to e^-X) = \int_0^1 d\xi \sum_f f_f(\xi)\sigma(e^-q_f \to e^-q_f)\,. \tag{7.57}$$

The cross section for the partonic process can be obtained from $\sigma(e^-\mu^- \to e^-\mu^-)$. Using Eq. (6.79) we find (use $\hat{t} = (k-k')^2 = -2k\cdot k' = -2k^2(1-\cos\theta)$, and so $d\hat{t} = 2k^2 d\cos\theta$)

$$\frac{d\sigma(e^-q_f \to e^-q_f)}{d\hat{t}} = \frac{2\pi\alpha^2 Q_{q_f}^2}{\hat{s}^2}\left[\frac{\hat{s}^2+\hat{u}^2}{\hat{t}^2}\right], \tag{7.58}$$

where $\hat{t}$ etc. are the Mandelstam variables for the partonic process. Using $\hat{t} = (k-k')^2 = -q^2 = Q^2$ and $\hat{s} = (p+k)^2 = 2p\cdot k = 2\xi P\cdot k = \xi(P+k)^2 = \xi s$, as well as $\hat{u} = -\hat{s} - \hat{t}$, we obtain finally

$$\frac{d\sigma(e^-p \to e^-X)}{dQ^2} = \int_0^1 d\xi \sum_f f_f(\xi) Q_{q_f}^2 \frac{2\pi\alpha^2}{Q^4}\left[1 + \left(1 - \frac{Q^2}{\xi s}\right)^2\right]\theta(\xi s - Q^2)\,. \tag{7.59}$$

The theta function implements the kinematic restriction $\hat{s} \geq |\hat{t}|$ (note that, for massless partons and in the C.M.S., Eq. (7.55) reads $\hat{t} = \frac{1}{2}\hat{s}(\cos\theta - 1)$).

Interestingly, it is sufficient to measure the electron momentum in order to determine the ξ dependence. Since the partons have negligible masses, we have

$$0 = (p+q)^2 = 2p\cdot q + q^2 = 2\xi P\cdot q - Q^2\,, \tag{7.60}$$

and so, defining the kinematic quantity

$$x \equiv \frac{Q^2}{2P\cdot q}\,, \tag{7.61}$$

we have simply

$$x = \xi\,. \tag{7.62}$$

Apart from the factor $[1+(1-Q^2/xs)^2]/Q^4$ that characterizes the underlying partonic process, Eq. (7.59) does not depend on Q^2; this is called *Bjorken scaling*.

It is useful to introduce dimensionless variables. In addition to x we define

$$y \equiv \frac{2P\cdot q}{2P\cdot k} = \frac{2P\cdot q}{s}\,. \tag{7.63}$$

In the rest system of the proton this evaluates to

$$y = \frac{q^0}{k^0}\,, \tag{7.64}$$

i.e. the fraction of the electron energy that is transferred to the proton. We can express y in terms of partonic Mandelstam variables,

$$y = \frac{2p\cdot(k-k')}{2p\cdot k} = \frac{\hat{s}+\hat{u}}{\hat{s}}\,, \tag{7.65}$$

such that

$$\frac{\hat{u}}{\hat{s}} = y-1\,. \tag{7.66}$$

Eq. (7.64) implies $0 \le y \le 1$. Now we use

$$xys = Q^2\,, \tag{7.67}$$

and hence

$$d\xi\, dQ^2 = dx\, dQ^2 = dx\frac{dQ^2}{dy}dy = xs\, dxdy\,, \tag{7.68}$$

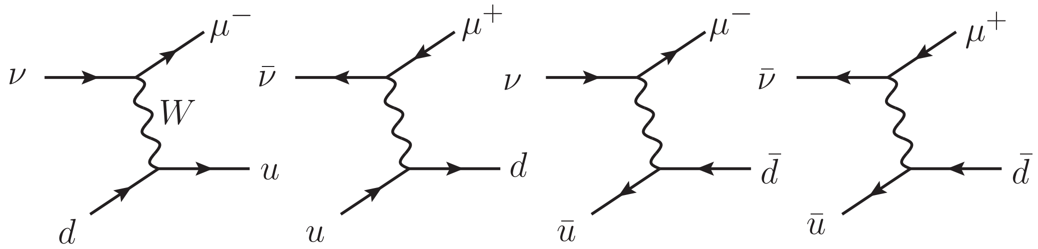
and obtain

$$\frac{d^2\sigma(e^-p\to e^-X)}{dxdy} = \sum_f x f_f(x) Q_{q_f}^2 \frac{2\pi\alpha^2 s}{Q^4}\left[1+\left(1-y\right)^2\right]. \tag{7.69}$$

The factor $\left[1+\left(1-y\right)^2\right]$ is characteristic for scattering on partons of spin 1/2 ("Callan-Gross relation").

7.7 The parton distribution functions

Electron-proton scattering is not sufficient to determine the individual parton distribution functions (see Eq. (7.69)). Deep-inelastic neutrino-proton scattering gives further information. The following processes contribute:



For instance, in the limit that the momentum transfer is much smaller than the W mass, the first diagrams gives

$$\frac{d\sigma(\nu d \to \mu^- u)}{d\hat{t}} = \frac{\pi g^4}{2(4\pi)^2 \hat{s}^2} \frac{\hat{s}^2}{M_W^4} \equiv \frac{G_F^2}{\pi}\,, \tag{7.70}$$

and the second diagram

$$\frac{d\sigma(\bar\nu u \to \mu^+ d)}{d\hat{t}} = \frac{\pi g^4}{2(4\pi)^2 \hat{s}^2} \frac{\hat{u}^2}{M_W^4} = \frac{G_F^2}{\pi}(1-y)^2\,. \tag{7.71}$$

Following the same line of reasoning as before, we find

$$\frac{d^2\sigma(\nu p \to \mu^- X)}{dxdy} = \frac{G_F^2 s}{\pi}\left[x f_d(x) + x f_{\bar u}(x)(1-y)^2\right], \tag{7.72}$$

and

$$\frac{d^2\sigma(\bar\nu p \to \mu^+ X)}{dxdy} = \frac{G_F^2 s}{\pi}\left[x f_u(x)(1-y)^2 + x f_{\bar d}(x)\right]. \tag{7.73}$$

The measurement of a suitable combination of scattering processes allows for the determination of the parton distribution functions. See Fig. 6.

To leading order in α_s the distributions function are independent of Q^2; QCD corrections lead to a small, logarithmic dependence on Q^2, $f_f(x) \to f_f(x, Q^2)$.

The PDFs satisfy several conditions. Since $p \sim [uud]$, we have

$$\int_0^1 dx[f_u(x) - f_{\bar u}(x)] = 2\,, \tag{7.74}$$

and

$$\int_0^1 dx[f_d(x) - f_{\bar d}(x)] = 1\,. \tag{7.75}$$

In the limit of exact isospin symmetry ($u \leftrightarrow d$) we have for the neutron PDFs

$$f_u^n(x) = f_d(x)\,, \qquad f_d^n(x) = f_u(x)\,, \qquad f_{\bar u}^n(x) = f_{\bar d}(x)\,, \qquad \ldots \tag{7.76}$$

(in practice, these relations are valid up to percent corrections). The following relations are exactly valid:

$$f_u^{\bar p}(x) = f_{\bar u}(x)\,, \qquad f_{\bar u}^{\bar p}(x) = f_u(x)\,, \qquad \ldots \tag{7.77}$$

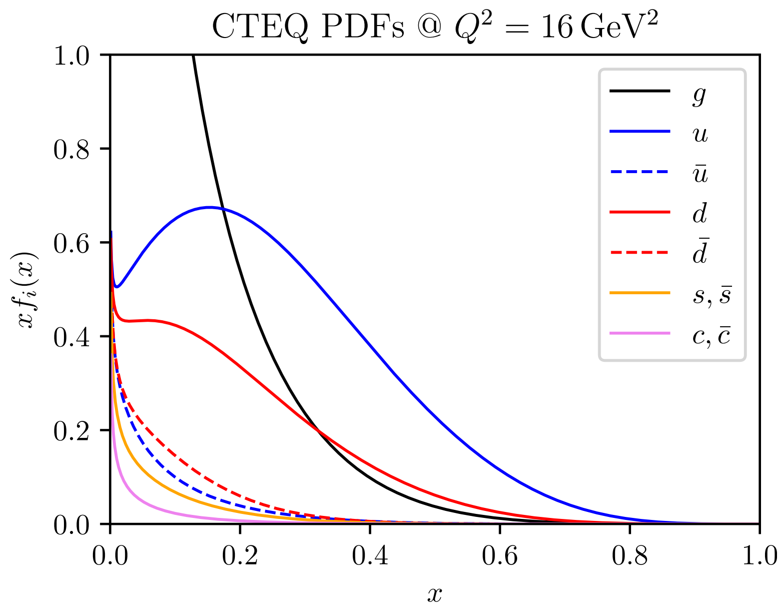


Figure 6: *Plot of the CTEQ parton distribution functions, evaluated for* $Q^2 = 16$ *GeV*^{2}.

The total proton momentum must be carried by all partons, therefore

$$\int_0^1 dx x[f_u(x) + f_d(x) + f_{\bar{u}}(x) + f_{\bar{d}}(x) + f_g(x)] = 1\,. \tag{7.78}$$

Measurements tell us that about half of the proton momentum is carried by neutral partons (gluons).

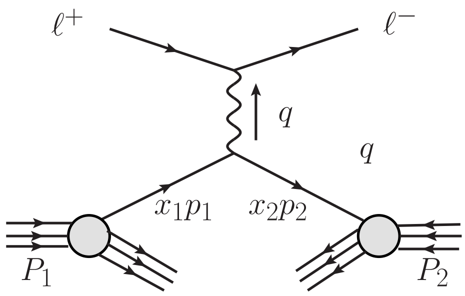
7.8 Hadron collision processes

If in a collision of hadrons a large momentum perpendicular to the collision axis is transferred ("high-p_T"), we can predict the process perturbatively.

7.8.1 Lepton pair production ("Drell-Yan")

We expect a schematic form of the scattering cross section

$$\begin{aligned}&\sigma(p(P_1) + p(P_2) \to \ell^+\ell^- + X)\\ &= \int_0^1 dx_1 \int_0^1 dx_2 \sum_f f_f(x_1) f_{\bar{f}}(x_2) \sigma(q_f(x_1 P_1) + q_{\bar{f}}(x_2 P_2) \to \ell^+\ell^-)\,.\end{aligned} \tag{7.79}$$



The underlying partonic process is $q\bar{q} \to \ell^+\ell^-$. We can read off the partonic cross section from Eqs. (7.52) and (7.53) (averaging rather than summing of initial color states gives the relative factor $1/9$):

$$\sigma(q_f\bar{q}_f \to \ell^+\ell^-) = \frac{1}{3}Q_f^2\frac{4\pi\alpha^2}{3\hat{s}}\,. \tag{7.80}$$

Measurement of the lepton momenta determines the four-momentum of the virtual photon. We can even determine x_1, x_2! Let

$$M^2 = q^2\,. \tag{7.81}$$

We parameterize the component q^0, measured in the proton-proton C.M.S., in terms of the rapidity Y (cf. Eq. (4.39); there, Y was called η):

$$q^0 = M\cosh Y\,. \tag{7.82}$$

The momenta in the proton-proton C.M.S. are ($s = 4E^2$)

$$P_1 = (E,0,0,E)\,, \qquad P_2 = (E,0,0,-E)\,, \tag{7.83}$$

and so

$$q = x_1P_1 + x_2P_2 = \big((x_1+x_2)E,0,0,(x_1-x_2)E\big)\,. \tag{7.84}$$

Using this, we find

$$M^2 = 4x_1x_2E^2 = x_1x_2s\,, \tag{7.85}$$

as well as

$$\cosh Y = \frac{x_1+x_2}{2\sqrt{x_1x_2}} = \frac{1}{2}\left(\sqrt{\frac{x_1}{x_2}} + \sqrt{\frac{x_2}{x_1}}\right), \tag{7.86}$$

and consequently

$$\exp Y = \sqrt{\frac{x_1}{x_2}}\,. \tag{7.87}$$

This allows us to express x_1 and x_2 in terms of M and Y:

$$x_1 = \frac{M}{\sqrt{s}}e^Y\,, \qquad x_2 = \frac{M}{\sqrt{s}}e^{-Y}\,. \tag{7.88}$$

In order to rewrite the integral in Eq. (7.79), we need the Jacobi determinant ($Y = \frac{1}{2}\log(x_1/x_2)$)

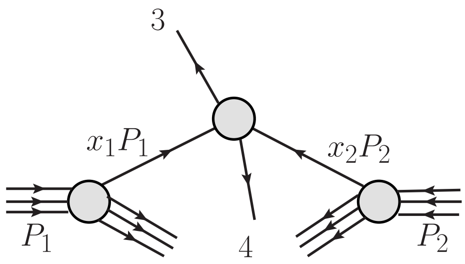
$$\frac{\partial(M^2,Y)}{\partial(x_1,x_2)} = \begin{vmatrix} x_2s & x_1s \\ \frac{1}{2x_1} & -\frac{1}{2x_2}\end{vmatrix} = s = \frac{M^2}{x_1x_2}\,. \tag{7.89}$$

In total, we get ($\hat{s} = q^2 = M^2$)

$$\frac{d^2\sigma(pp \to \ell^+\ell^- + X)}{dM^2 dY} = \sum_f f_f(x_1) f_{\bar{f}}(x_2) Q_f^2 \frac{4\pi\alpha^2}{9M^2} \,. \tag{7.90}$$

7.8.2 General pair production

If we are interested also in the angular distribution of the final state, we can derive a triply differential cross section. Allowing also for hadrons in the final state, the general cross section is given by



$$\frac{d^3\sigma(pp \to 3 + 4 + X)}{dx_1 dx_2 d\hat{t}} = f_1(x_1) f_2(x_2) \frac{d\sigma(1 + 2 \to 3 + 4)}{d\hat{t}} \,. \tag{7.91}$$

As for Drell-Yan, we can express x_1, x_2, $\hat{t}$ in terms of observable parameters. We choose the common transverse momentum $p_\perp \equiv |\boldsymbol{p}_\perp|$ of the final-state partons[5] as well as their longitudinal rapidities y_3, y_4, defined by

$$E_i = p_\perp \cosh y_i \,, \qquad p_{i,\|} = p_\perp \sinh y_i \,. \tag{7.92}$$

The longitudinal rapidities parameterize the boost from the frame where the partons have vanishing longitudinal momentum. They add under Lorentz boosts along the beam axis, while the transverse momentum remains invariant under such a boost.

Let us use an asterisk to denote quantities in the partonic C.M.S. In this frame, the momentum of 3 is given by

$$p_{3,\|*} = \tfrac{1}{2}\sqrt{\hat{s}} \cos\theta_* \,, \qquad p_{3,\perp*} = \tfrac{1}{2}\sqrt{\hat{s}} \sin\theta_* \,, \tag{7.93}$$

in terms of the partonic scattering angle θ_*, while $\boldsymbol{p}_{4*} = -\boldsymbol{p}_{3*}$. Moreover, in the C.M.S. we must have $y_{3*} = -y_{4*} \equiv y_*$. Since rapidities add under successive boosts, we see that $y_3 = Y + y_*$ and $y_4 = Y - y_*$, and so

$$y_* = \tfrac{1}{2}(y_3 - y_4) \,, \qquad Y = \tfrac{1}{2}(y_3 + y_4) \,. \tag{7.94}$$

[5]Note that, to leading order in QCD, the initial partons have parallel momentum, so the transverse momenta of the final-state partons must be equal and opposite, while their longitudinal momenta are not constrained.

To determine the scattering angle, note that the energies of the final-state partons in their C.M.S. is given by $E_* = \frac{1}{2}\sqrt{\hat{s}} = p_\perp \cosh y_*$, or (since $p_{3,\perp *} = p_\perp$ and using Eq. (7.93))

$$\cosh y_* = \frac{1}{\sin\theta_*} . \tag{7.95}$$

Therefore, we can express the partonic Mandelstam variables as

$$\hat{s} = 4p_\perp^2 \cosh^2 y_* = \frac{4p_\perp^2}{\sin^2 y_*} \tag{7.96}$$

and[6]

$$\hat{t} = \tfrac{1}{2}\hat{s}(1-\cos\theta^*) = -2p_\perp^2 \cosh y_* e^{-y_*} . \tag{7.97}$$

Using Eq. (7.88) and evaluating the Lorentz-invariant $M^2 = \hat{s}$ in the partonic C.M.S., this gives

$$x_1 = \frac{2p_\perp}{\sqrt{s}} \cosh y_* e^{Y} , \qquad x_2 = \frac{2p_\perp}{\sqrt{s}} \cosh y_* e^{-Y} . \tag{7.98}$$

Finally, to convert from the set of variables x_1, x_2, $\hat{t}$ to the measurable quantities y_1, y_2, $p_\perp$, we need the Jacobian[7]

$$\begin{aligned}
\frac{\partial(x_1,x_2,\hat{t})}{\partial(y_3,y_4,p_\perp)} &= \begin{vmatrix} \frac{p_\perp}{\sqrt{s}}e^{y_*+Y} & -\frac{p_\perp}{\sqrt{s}}e^{-y_*-Y} & p_\perp^2 e^{-2y_*} \\ \frac{p_\perp}{\sqrt{s}}e^{-y_*+Y} & -\frac{p_\perp}{\sqrt{s}}e^{y_*-Y} & -p_\perp^2 e^{-2y_*} \\ \frac{2}{\sqrt{s}}\cosh y_* e^{Y} & \frac{2}{\sqrt{s}}\cosh y_* e^{-Y} & -4p_\perp \cosh y_* e^{-y_*} \end{vmatrix} \\
&= 2\frac{p_\perp^3}{s}\cosh y_* \begin{vmatrix} e^{y_*} & -e^{-y_*} & e^{-2y_*} \\ e^{-y_*} & -e^{y_*} & -e^{-2y_*} \\ 1 & 1 & -2e^{-y_*} \end{vmatrix} \\
&= 2\frac{p_\perp^3}{s}\cosh y_* |(e^{-y_*}+e^{-3y_*}) - (-e^{-y_*}-e^{-3y_*}) - 2(e^{-3y_*}-e^{y_*})| \\
&= 2\frac{p_\perp^3}{s}\cosh y_*(2e^{y_*}+2e^{-y_*}) = 8\frac{p_\perp^3}{s}\cosh^2 y_* = \frac{2p_\perp \hat{s}}{s} .
\end{aligned} \tag{7.99}$$

Thus, we can write Eq. (7.91) as

$$\frac{d^3\sigma(pp\to 3+4+X)}{dy_3 dy_4 dp_\perp} = f_1(x_1)f_2(x_2)\frac{2p_\perp \hat{s}}{s}\frac{d\sigma(1+2\to 3+4)}{d\hat{t}} . \tag{7.100}$$

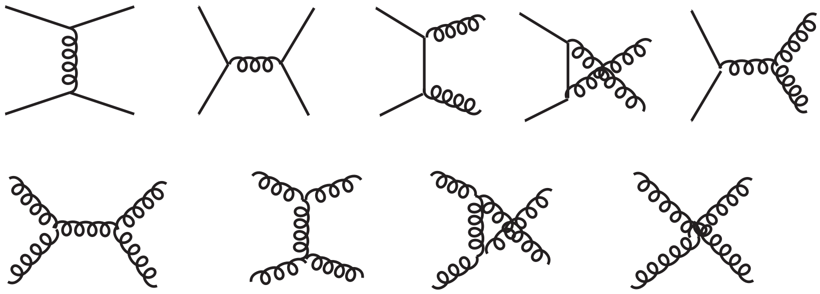
Using $\hat{s} = x_1x_2s$ and $d^2\boldsymbol{p}_\perp = 2\pi p_\perp dp_\perp$ (using rotational symmetry about the beam axis), we find the final form of the cross section for pair production in proton-proton collisions,

$$\frac{d^4\sigma(pp\to 3+4+X)}{dy_3 dy_4 d^2\boldsymbol{p}_\perp} = x_1f_1(x_1)x_2f_2(x_2)\frac{1}{\pi}\frac{d\sigma(1+2\to 3+4)}{d\hat{t}} . \tag{7.101}$$

[6] using $1-\cos = 1-\sqrt{(\cosh^2-1)/\cosh^2} = (\cosh-\sinh)/\cosh$

[7] We have $\partial x_1/\partial y_{3,4} = p_\perp/\sqrt{s}(\pm\sinh y_* + \cosh y_*)e^{Y}$, $\partial x_2/\partial y_{3,4} = p_\perp/\sqrt{s}(\pm\sinh y_* - \cosh y_*)e^{-Y}$, $\partial\hat{t}/\partial y_{3,4} = \mp p_\perp/\sqrt{s}(\sinh y_* - \cosh y_*)e^{-y_*}$,

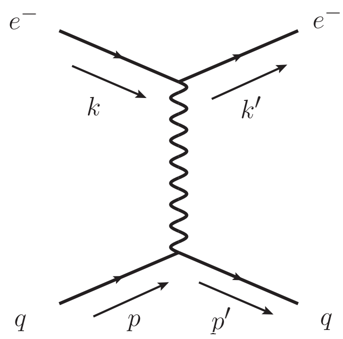
Example: Jet pair production at the LHC



7.9 Elastic electron-proton scattering and form factors

Now we want to calculate the scattering of electrons on protons at very low energies. Is the situation hopeless?

If we naively replace the external quark states by proton states in the process $e^- q \to e^- q$,



we obtain the following transition matrix element:

$$\begin{aligned} & -2i\pi M\delta^4(k+p-k'-p-) \\ = & \frac{(-i)^2}{2!}\int d^4x d^4y \langle 0|a(\boldsymbol{k}', s', e^-)a(\boldsymbol{p}', r', p) \\ & \times e^2 T\{\big(\bar{e}\not{A}e + Q_f\bar{q}_f\not{A}q_f\big)(x)\big(\bar{e}\not{A}e + Q_f\bar{q}_f\not{A}q_f\big)(y)\} \\ & \times a^\dagger(\boldsymbol{k}, s, e^-)a^\dagger(\boldsymbol{p}, r, p)|0\rangle \, . \end{aligned} \tag{7.102}$$

We can calculate the contractions of the electron and photon fields in the usual way; summing over $f = u, d, s$, this gives

$$\begin{aligned} & -2i\pi M\delta^4(k+p-k'-p) \\ = & \frac{ie}{(2\pi)^3}\bar{u}(\boldsymbol{k}', s')\gamma_\mu u(\boldsymbol{k}, s) \\ & \times \int d^4x d^4y \int \frac{d^4q}{(2\pi)^4}\frac{1}{q^2}e^{-iq\cdot(x-y)}e^{-iy\cdot(k-k')}\langle \boldsymbol{p}', r'|J^\mu(x)|\boldsymbol{p}, r\rangle \, , \end{aligned} \tag{7.103}$$

with

$$J^\mu(x) = \sum_f eQ_f \bar{q}_f(x)\gamma^\mu q_f(x)\,. \tag{7.104}$$

The proton matrix element must be evaluated to all orders in α_s! Translational invariance implies

$$\langle \boldsymbol{p}', r'|\mathcal{J}^\mu(x)|\boldsymbol{p}, r\rangle = e^{-i(p-p')\cdot x}\langle \boldsymbol{p}', r'|\mathcal{J}^\mu(0)|\boldsymbol{p}, r\rangle\,. \tag{7.105}$$

Current conservation then requires

$$0 = (p-p')_\mu \langle \boldsymbol{p}', r'|\mathcal{J}^\mu(0)|\boldsymbol{p}, r\rangle\,. \tag{7.106}$$

Setting $\mu = 0$ in Eq. (7.105), integrating over $\boldsymbol{x}$, and recalling Eq. (4.78) gives

$$\langle \boldsymbol{p}', r'|Q|\boldsymbol{p}, r\rangle = (2\pi)^3\delta^3(\boldsymbol{p}'-\boldsymbol{p})\langle \boldsymbol{p}', r'|\mathcal{J}^0(0)|\boldsymbol{p}, r\rangle\,. \tag{7.107}$$

The electric charge of the proton is $+e$, so

$$\langle \boldsymbol{p}, r'|\mathcal{J}^0(0)|\boldsymbol{p}, r\rangle = \frac{e}{(2\pi)^3}\delta_{r'r}\,. \tag{7.108}$$

Lorentz covariance gives further constraints on the form of the matrix element. In general, the matrix element is of the form

$$\langle \boldsymbol{p}', r'|\mathcal{J}^\mu(0)|\boldsymbol{p}, r\rangle = \frac{e}{(2\pi)^3}\bar{u}(\boldsymbol{p}', r')\Gamma^\mu(p', p)u(\boldsymbol{p}, r)\,, \tag{7.109}$$

where $\Gamma^\mu(p', p)$ is a 4×4 matrix, and $u(\boldsymbol{p}, r)$ the proton spinor function. One can decompose the matrix $\Gamma^\mu(p', p)$ in terms of the basis (4.25), with the following contributions:

Scalar: p^μ, p'^μ
Vector: γ^μ, $p^\mu \not{p}$, $p'^\mu \not{p}$, $p'^\mu \not{p}'$, $p^\mu \not{p}'$
Tensor: $[\gamma^\mu, \not{p}]$, $[\gamma^\mu, \not{p}']$, $[\not{p}, \not{p}']p^\mu$, $[\not{p}, \not{p}']p'^\mu$
Axial vector: $\gamma_5\gamma_\rho\epsilon^{\rho\mu\nu\sigma}p_\nu p'_\sigma$

with coefficients that depend on the single Lorentz scalar $p\cdot p'$ (or, alternatively, on $q \equiv (p'-p)^2 = 2m_p^2 - 2p\cdot p'$). Using the Dirac equations

$$\bar{u}(\boldsymbol{p}', r')(\not{p}' - m_p) = 0\,, \qquad (\not{p} - m_p)u(\boldsymbol{p}, r) = 0\,, \tag{7.110}$$

we can eliminate all terms apart from p^μ, p'^μ, and γ^μ (exercise!), and we obtain

$$\begin{aligned}&\bar{u}(\boldsymbol{p}', r')\Gamma^\mu(p', p)u(\boldsymbol{p}, r)\\ =&\bar{u}(\boldsymbol{p}', r')\left[\gamma^\mu F(q^2) + \frac{(p+p')^\mu}{2m_p}G(q^2) + \frac{i(p-p')^\mu}{2m_p}H(q^2)\right]u(\boldsymbol{p}, r)\,.\end{aligned} \tag{7.111}$$

Since $J^\mu(0)$ is Hermitian, we have

$$\langle \boldsymbol{p}', r'|\mathcal{J}^\mu(0)|\boldsymbol{p}, r\rangle^\dagger = \langle \boldsymbol{p}, r|\mathcal{J}^\mu(0)|\boldsymbol{p}', r'\rangle = \bar{u}(\boldsymbol{p}, r)\Gamma^\mu(p, p')u(\boldsymbol{p}', r')\,, \tag{7.112}$$

as well as

$$\langle \boldsymbol{p}', r'|\mathcal{J}^\mu(0)|\boldsymbol{p}, r\rangle^\dagger = \left(\bar{u}(\boldsymbol{p}', r')\Gamma^\mu(p', p)u(\boldsymbol{p}, r)\right)^\dagger = \bar{u}(\boldsymbol{p}, r)\gamma^0\Gamma^{\mu\dagger}(p', p)\gamma^0 u(\boldsymbol{p}', r')\,, \tag{7.113}$$

and it follows that

$$\gamma^0 \Gamma^{\mu\dagger}(p', p)\gamma^0 = \Gamma^\mu(p, p') \,. \tag{7.114}$$

This implies that $F(q^2)$, $G(q^2)$, and $H(q^2)$ are real functions of $q^2 \equiv (p'-p)^2 = 2m_p^2 - 2p\cdot p'$. The current conservation condition (7.106) is automatically satisfied for the first two terms on the right side of Eq. (7.111) since (use the Dirac equation)

$$(p-p')_\mu \gamma^\mu = (\not{p} - m_p) - (\not{p}' - m_p) \,, \tag{7.115}$$

$$(p-p')_\mu (p+p')^\mu = p^2 - p'^2 = 0 \,. \tag{7.116}$$

Since, in general, $(p-p')^2 \neq 0$, we must have $H(q^2) \equiv 0$.

In the limit $\boldsymbol{p} \to \boldsymbol{p}'$ we obtain

$$\langle \boldsymbol{p}, r' | \mathcal{J}^\mu(0) | \boldsymbol{p}, r \rangle = \frac{e}{(2\pi)^3} \bar{u}(\boldsymbol{p}, r') \left[\gamma^\mu F(0) + \frac{p^\mu}{m_p} G(0) \right] u(\boldsymbol{p}, r) \,. \tag{7.117}$$

Now we use $\{\gamma^\mu, \not{p} - m_p\} = 2p^\mu - 2m_p\gamma^\mu$, and hence[8]

$$\bar{u}(\boldsymbol{p}, r')\gamma^\mu u(\boldsymbol{p}, r) = \frac{p^\mu}{m_p} \bar{u}(\boldsymbol{p}, r') u(\boldsymbol{p}, r) \overset{(4.43)}{=} \frac{p^\mu}{p^0} \delta_{r'r} \,. \tag{7.119}$$

This gives

$$\langle \boldsymbol{p}, r' | \mathcal{J}^\mu(0) | \boldsymbol{p}, r \rangle = \frac{e}{(2\pi)^3} \frac{p^\mu}{p^0} \delta_{r'r} \left[F(0) + G(0) \right] \,, \tag{7.120}$$

Comparing with (7.108) gives the normalization condition

$$F(0) + G(0) = 1 \,. \tag{7.121}$$

Frequently, one writes the vertex function in the form

$$\bar{u}(\boldsymbol{p}', r')\Gamma^\mu(p', p) u(\boldsymbol{p}, r) = u(\boldsymbol{p}', r') \left[\gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu} q_\nu}{2m} F_2(q^2) \right] u(\boldsymbol{p}, r) \,, \tag{7.122}$$

where $q \equiv p' - p$ and $\sigma^{\mu\nu} = i[\gamma^\mu, \gamma^\nu]/2$. One can use Gordon's identity (exercise!)

$$\bar{u}(\boldsymbol{p}', r')\gamma^\mu u(\boldsymbol{p}, r) = u(\boldsymbol{p}', r') \left[\frac{(p+p')^\mu}{2m} + \frac{i\sigma^{\mu\nu} q_\nu}{2m} \right] u(\boldsymbol{p}, r) \,, \tag{7.123}$$

to show that

$$F_1(q^2) = F(q^2) + G(q^2) \,, \tag{7.124}$$

$$F_2(q^2) = -G(q^2) \,. \tag{7.125}$$

[8] In the last step, we used

$$\bar{u}(\boldsymbol{p}) u(\boldsymbol{p}) = \frac{m}{p^0} u(\boldsymbol{0})^\dagger D^\dagger(\Lambda)\gamma^0 D(\Lambda) u(\boldsymbol{0}) = \frac{m}{p^0} u(\boldsymbol{0})^\dagger \gamma^0 u(\boldsymbol{0}) = \frac{m}{p^0} u(\boldsymbol{0})^\dagger u(\boldsymbol{0}) = \frac{m}{p^0} \,. \tag{7.118}$$

The normalization condition is now

$$F_1(0) = 1\,. \tag{7.126}$$

The expressions (7.111) or (7.122) can be used in Eq. (7.103) in order to calculate the matrix element. The *form factors* F_1, F_2 (or F, G) be be determined experimentally (exercise!).

To understand the physics better, let's rewrite (7.111) in a third way:

$$\bar{u}(\boldsymbol{p}', r')\Gamma^\mu(p', p)u(\boldsymbol{p}, r) = u(\boldsymbol{p}', r')\left[\frac{(p+p')^\mu}{2m}F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\big(F_1(q^2) + F_2(q^2)\big)\right]u(\boldsymbol{p}, r)\,. \tag{7.127}$$

Consider the nonrelativistic limit $|\boldsymbol{p}|, |\boldsymbol{p}|' \ll m$. The first term in the brackets in Eq. (7.127) is spin independent and leads to Coulomb scattering. The condition (7.126) ensures that the effective proton charge in this limit is $+e$. The second term is spin dependent and corresponds (in the NR limit) to the magnetic moment of the fermion.

Now we consider an interaction with a classical static field $A_{\text{cl}}(\boldsymbol{x}) = (0, \boldsymbol{A}(\boldsymbol{x}))$. The interaction Hamiltonian is

$$H = \int d^4x\, A^\mu_{\text{cl}}(x) J_\mu(x)\,. \tag{7.128}$$

Its matrix element is

$$\begin{aligned}\langle \boldsymbol{p}', r'|H|\boldsymbol{p}, r\rangle &= \int d^4x\, A^\mu_{\text{cl}}(x)e^{-iq\cdot x}\frac{e}{(2\pi)^3}\bar{u}(\boldsymbol{p}', r')\frac{i\sigma_{\mu\nu}q^\nu}{2m}\big(F_1(q^2) + F_2(q^2)\big)u(\boldsymbol{p}, r)\\ &= \int d^4x\, e^{-iq\cdot x}\partial^\nu A^\mu_{\text{cl}}(x)\frac{e}{(2\pi)^3}\bar{u}(\boldsymbol{p}', r')\frac{\sigma_{\mu\nu}}{2m}\big(F_1(q^2) + F_2(q^2)\big)u(\boldsymbol{p}, r)\,,\end{aligned} \tag{7.129}$$

where we integrated by parts and used $q^\nu = i\partial^\nu e^{-iq\cdot x}$. Consider now the contributions with $\mu = 1$, $\nu = 2$ (and $\mu = 2$, $\nu = 1$). We have

$$\sigma_{12} = -\sigma_{21} = \begin{pmatrix}\sigma^3 & 0\\ 0 & \sigma^3\end{pmatrix}\,, \tag{7.130}$$

and so (since the leading term in the interaction is linear in the momentum transfer, to obtain the overall leading term we can set the momenta to zero in the external spinor functions, such that effectively only two components contribute)

$$\sigma_{12}\partial^2 A^1 + \sigma_{21}\partial^1 A^2 = \sigma^3\big(\partial^2 A^1 - \partial^1 A^2\big) = \sigma^3\big(\boldsymbol{\nabla}\times\boldsymbol{A}\big)^3 = \sigma^3 B^3\,, \tag{7.131}$$

with $\boldsymbol{B} = \nabla\times\boldsymbol{A}$. If the magnetic field is nearly homogeneous, we can pull it out of the integral. We then see that the matrix element contains a term proportional to

$$\frac{e}{m_p}\boldsymbol{B}\cdot\boldsymbol{\sigma}(F_1 + F_2)(0) \equiv \boldsymbol{B}\cdot\boldsymbol{\mu}_p\,, \tag{7.132}$$

where

$$\boldsymbol{\mu}_p \equiv g_p\left(\frac{e}{2m_p}\right)\boldsymbol{s}\,, \tag{7.133}$$

with $\boldsymbol{s} = \frac{1}{2}\boldsymbol{\sigma}$ the spin operator for the proton and the *Landé g factor*

$$g_p \equiv 2[F_1(0) + F_2(0)] = 2 + 2F_2(0)\,. \tag{7.134}$$

Measurements give $g_p \approx 2.79$.

For elementary particles like the electron or the muon, $F_1(q^2) = 1$, $F_2(0^2) = 0$ to leading order in QED. Higher-order contribution yield corrections that are (mostly) calculable in perturbation theory. The most famous example is the anomalous magnetic moment of the muon that has recently been measured very precisely at Fermilab.

A model for the form factor

Consider a spherically symmetric, exponentially falling charge distribution

$$\rho(r) = \rho_0 e^{-\mu r}\,, \tag{7.135}$$

with the normalization condition

$$1 = \int d^3x\, \rho(r) = 4\pi\rho_0 \int_0^\infty r^2 dr e^{-\mu r} = \frac{8\pi}{\mu^3}\,, \tag{7.136}$$

so $\rho_0 = \mu^3/8\pi$. Taking the Fourier transform gives the form factor

$$F(\boldsymbol{q}^2) = \frac{1}{\left(1 + \boldsymbol{q}^2/\mu^2\right)^2}\,. \tag{7.137}$$

The mean square radius of the charge distribution is

$$\langle r^2 \rangle \equiv 4\pi\rho_0 \int_0^\infty r^2 dr r^2 e^{-\mu r} = \frac{96\pi\rho_0}{\mu^5} = \frac{12}{\mu^2}\,, \tag{7.138}$$

so we have

$$F(\boldsymbol{q}^2) = \frac{1}{\left(1 + \frac{1}{12}\langle r^2\rangle \boldsymbol{q}^2\right)^2} = 1 - \tfrac{1}{6}\langle r^2\rangle \boldsymbol{q}^2 + \dots\,. \tag{7.139}$$

In relativistic electron-nucleon scattering one typically uses the *Sachs form factors*

$$G_E(q^2) \equiv F_1(q^2) + \frac{q^2}{4m_N^2} F_2(q^2)\,, \tag{7.140}$$

$$G_M(q^2) \equiv F_1(q^2) + F_2(q^2)\,. \tag{7.141}$$

8 Spontaneous symmetry breaking

Spontaneously broken symmetries are symmetries of the underlying theory (the Lagrangian) that are not realized as symmetry transformations on the physical states.

8.1 Degenerate vacua

SSB is always associated with a degeneracy of the vacuum state (e.g. ferromagnetism).

As an example, consider the Lagragian

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2 - V(\phi)\,, \tag{8.1}$$

where V is an even function of ϕ. If $\phi(x) = \bar{\phi}$ is a minimum of V (vacuum state), then so is $\phi(x) = -\bar{\phi}$; we have two degenerate vacua, $|\text{VAC},\pm\rangle$. Is the symmetry really broken? The symmetry of the Lagrangian under $\phi \to -\phi$ implies

$$\langle\text{VAC},+|H|\text{VAC},+\rangle = \langle\text{VAC},-|H|\text{VAC},-\rangle \equiv a\,, \tag{8.2}$$

$$\langle\text{VAC},+|H|\text{VAC},-\rangle = \langle\text{VAC},-|H|\text{VAC},+\rangle \equiv b\,. \tag{8.3}$$

The relevant part of the Hamiltonian can therefore be written

$$H = \begin{pmatrix} a & b \\ b & a \end{pmatrix}\,, \tag{8.4}$$

with eigenstates $|\text{VAC},+\rangle \pm |\text{VAC},-\rangle$ and corresponding energies $a \pm |b|$. These vacuum states are symmetric.

For large volumina, however, the transition matrix elements b are suppressed (tunneling), and the system will be in one of the non-symmetric superpositions $|\text{VAC},+\rangle$ or $|\text{VAC},-\rangle$.

8.2 Spontaneously broken global symmetries

We want to prove *Goldstone's theorem*: For each spontaneously broken continuous symmetry, the spectrum of physical states contains a massless particle with spin zero (a "Nambu-Goldstone boson"; nobel prize 2008).

Consider a theory of N real scalar fields $\phi_n(x)$, with $n = 1,\dots,N$. The Lagrangian is

$$\mathcal{L} = \tfrac{1}{2}\sum_n \partial_\mu\phi_n\partial^\mu\phi_n - V(\phi)\,, \tag{8.5}$$

invariant under some symmetry with infinitesimal transformation

$$\phi_n(x) \to \phi_n(x) + i\epsilon\sum_m t_{nm}\phi_m(x)\,, \tag{8.6}$$

where it_{nm} is a real matrix. The potential must be invariant:

$$\sum_{n,m}\frac{\partial V(\phi)}{\partial\phi_n}t_{nm}\phi_m = 0\,. \tag{8.7}$$

Differentiating with respect to ϕ_ℓ yields

$$\sum_n \frac{\partial V(\phi)}{\partial\phi_n}t_{n\ell} + \sum_{n,m}\frac{\partial^2 V(\phi)}{\partial\phi_\ell\partial\phi_n}t_{nm}\phi_m = 0\,. \tag{8.8}$$

The first term vanishes at the minimum, $\phi(x) = \bar{\phi}$, so

$$\sum_{n,m} \frac{\partial^2 V(\phi)}{\partial\phi_\ell \partial\phi_n}\bigg|_{\phi=\bar{\phi}} t_{nm}\bar{\phi}_m = 0\,. \tag{8.9}$$

The second derivative of the potential gives the masses. If the symmetry is spontaneously broken, then $\sum_m t_{nm}\bar{\phi}_m \neq 0$ is an eigenvector of the mass matrix with eigenvalue zero. There is a massless boson for each "broken" symmetry generator.

8.2.1 Example: the linear sigma model

Consider again N real scalar fields, with Lagrangian

$$\mathcal{L} = \tfrac{1}{2}\sum_n \partial_\mu\phi_n\partial^\mu\phi_n + \frac{\mu^2}{2}\sum_n \phi_n\phi_n - \frac{\lambda}{4}\Big(\sum_n \phi_n\phi_n\Big)^2\,. \tag{8.10}$$

The Lagrangian is invariant under the group $O(N)$ of real rotations of the fields. If $\mu^2 < 0$, the minimum of the potential

$$V(\phi) = -\frac{\mu^2}{2}\sum_n \phi_n\phi_n + \frac{\lambda}{4}\Big(\sum_n \phi_n\phi_n\Big)^2 \tag{8.11}$$

is at $\phi = 0$, which is invariant under $O(N)$ rotations. If $\mu^2 > 0$, however, the minimum is at $\phi(x) = \bar{\phi}$, with

$$\sum_n \bar{\phi}_n^2 = \frac{\mu^2}{\lambda}\,. \tag{8.12}$$

The (tree-level) mass matrix is then given by

$$M_{nm}^2 = \frac{\partial^2 V(\phi)}{\partial\phi_n\partial\phi_m} = -\mu^2\delta_{nm} + \lambda\delta_{nm}\sum_\ell \bar{\phi}_\ell^2 + 2\lambda\bar{\phi}_n\bar{\phi}_m = 2\lambda\bar{\phi}_n\bar{\phi}_m\,. \tag{8.13}$$

It has an eigenvector $\bar{\phi}$ with non-vanishing eigenvalue

$$m^2 = 2\lambda\sum_\ell \bar{\phi}_\ell^2 = 2\mu^2\,, \tag{8.14}$$

and $N-1$ eigenvectors, orthogonal to $\bar{\phi}$, with eigenvalue zero. The "symmetry breaking pattern" is $O(N) \to O(N-1)$ (the latter group leaves the vacuum invariant), so there are

$$\tfrac{1}{2}N(N-1) - \tfrac{1}{2}(N-1)(N-2) = N-1 \tag{8.15}$$

Goldstone bosons. We can choose "coordinates" in field space such that

$$\bar{\phi} = (0, 0, \ldots, 0, v)\,, \tag{8.16}$$

with $v = \mu/\sqrt{\lambda}$, and define shifted fields

$$\phi(x) = (\boldsymbol{\pi}(x), v + \sigma(x))\,. \tag{8.17}$$

The Lagrangian (8.10) is then

$$\mathcal{L} = \tfrac{1}{2}\big(\partial_\mu\boldsymbol{\pi}\big)^2 + \tfrac{1}{2}\big(\partial_\mu\sigma\big)^2 - \tfrac{1}{2}\big(2\mu^2\big)\sigma^2 - \sqrt{\lambda}\mu\sigma^3 - \sqrt{\lambda}\mu\boldsymbol{\pi}^2\sigma - \frac{\lambda}{4}\sigma^4 - \frac{\lambda}{2}\boldsymbol{\pi}^2\sigma^2 - \frac{\lambda}{4}\big(\boldsymbol{\pi}^2\big)^2\,. \tag{8.18}$$

Note that the $N-1$ fields $\boldsymbol{\pi}$ are massless. For $N = 4$, this leads to a model of pions.

8.2.2 Chiral symmetry of QCD

Consider QCD with two quark flavors (up and down) in the massless limit:

$$\mathcal{L} = \bar{u} i \not{D} u + \bar{d} i \not{D} d = \bar{u}_L i \not{D} u_L + \bar{u}_R i \not{D} u_R + (u \to d)\,. \tag{8.19}$$

This Lagrangian is invariant under the global chiral symmetry $SU(2)_L \times SU(2)_R \times U(1) \times U(1)_A$. To see this, we define doublets

$$Q = \begin{pmatrix} u \\ d \end{pmatrix}, \tag{8.20}$$

then we can write

$$\mathcal{L} = \bar{Q}_L i \not{D} Q_L + \bar{Q}_R i \not{D} Q_R\,, \tag{8.21}$$

and we see that the Lagrangian is invariant under the rotations $Q_L \to U_L Q_L$, $Q_R \to U_R Q_R$, with $U_L \in SU(2)_L$, $U_R \in SU(2)_R$. Furthermore, the Lagrangian is invariant under $Q \to e^{i\alpha} Q$. For $e^{i\alpha\gamma_5} in U(1)_A$ we have

$$\bar{q} i \not{D} q = q^\dagger \gamma^0 i \not{D} q \to q^\dagger e^{-i\alpha\gamma_5} \gamma^0 i \not{D} e^{i\alpha\gamma_5} q = q^\dagger \gamma^0 i \not{D} e^{-i\alpha\gamma_5} e^{i\alpha\gamma_5} q = \bar{q} i \not{D} q\,. \tag{8.22}$$

However, the full symmetry is not realized:

1. $U(1)$: baryon number conservation; $J^\mu = \bar{Q}\gamma^\mu Q$
2. $U_L = U_R \in SU(2)_L \times SU(2)_R$: isospin (approximate); $J^{\mu a} = \frac{1}{2}\bar{Q}\gamma^\mu \sigma^a Q$
3. $U(1)_A$ is broken by quantum effects ("anomalies")
4. the non-vectorial part of $SU(2)_L \times SU(2)_R$ is spontaneously broken.

The pions can be interpreted as the Goldstone bosons associated with the spontaneously broken axial symmetry.

8.3 Spontaneously broken local symmetries

The Goldstone bosons are unphysical if SSB occurs in a theory with a local gauge symmetry; instead, the corresponding gauge bosons become massive. This is the *Higgs mechanism.*

8.3.1 Example: Abelian Higgs mechanism

We consider a complex scalar field with self couplings and electromagnetic interactions,

$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + |D_\mu \phi|^2 - V(\phi)\,, \tag{8.23}$$

where, as usual, $D_\mu = \partial_\mu + ieA_\mu$. Assume that the Lagrangian is invariant under the local $U(1)$ transformation

$$\phi(x) \to e^{i\alpha(x)}\phi(x)\,, \qquad A_\mu(x) \to A_\mu(x) - \frac{1}{e}\partial_\mu \alpha(x)\,. \tag{8.24}$$

For instance, if we choose

$$V(\phi) = -\mu^2\phi^*\phi + \frac{\lambda}{2}\big(\phi^*\phi\big)^2\,, \tag{8.25}$$

with $\mu^2 > 0$, we find a non-vanishing vacuum expectation value ("vev") for ϕ (cf. the linear sigma model), and the $U(1)$ symmetry is spontaneously broken. The minimum of the potential is at $|\phi| = \sqrt{\mu^2/\lambda}$, e.g.

$$\langle\phi\rangle_0 \equiv \bar{\phi} = \sqrt{\frac{\mu^2}{\lambda}}\,. \tag{8.26}$$

Again, we expand the Lagrangian about $\bar{\phi}$. For this, we decompose ϕ into its real and imaginary parts,

$$\phi(x) = \bar{\phi} + \frac{1}{\sqrt{2}}\big(\phi_1(x) + i\phi_2(x)\big)\,. \tag{8.27}$$

Then the potential becomes

$$V(\phi) = -\frac{1}{2}\frac{\mu^4}{\lambda} + \frac{1}{2}\cdot 2\mu^2\phi_1^2 + \mathcal{O}(\phi_i^3)\,. \tag{8.28}$$

The field ϕ_1 has mass $2\mu^2$; the field ϕ_2 is massless (Goldstone boson). The kinetic term becomes

$$\begin{aligned}
|D_\mu\phi|^2 &= \big(D_\mu\phi\big)^* D^\mu\phi = \big(\partial_\mu - ieA_\mu\big)\phi^*\big(\partial_\mu + ieA_\mu\big)\phi \\
&= \tfrac{1}{2}\big(\partial_\mu\phi_1\big)^2 + \tfrac{1}{2}\big(\partial_\mu\phi_2\big)^2 + \frac{1}{\sqrt{2}}\partial_\mu(\phi_1 - i\phi_2)ieA^\mu\bar{\phi} - \frac{1}{\sqrt{2}}\partial_\mu(\phi_1 + i\phi_2)ieA^\mu\bar{\phi} \\
&\quad + e^2\bar{\phi}^2 A_\mu A^\mu + \text{(cubic and quartic terms)} \\
&= \tfrac{1}{2}\big(\partial_\mu\phi_1\big)^2 + \tfrac{1}{2}\big(\partial_\mu\phi_2\big)^2 + \sqrt{2}e\bar{\phi}A_\mu\partial^\mu\phi_2 + \tfrac{1}{2}m_A^2 A_\mu A^\mu + \dots\,.
\end{aligned} \tag{8.29}$$

We have a photon-Goldstone coupling $\propto \sqrt{2\mu^2/\lambda}e$ and a photon mass term with $m_A^2 = 2e^2\mu^2/\lambda$.

NB: The Goldstone boson is *unphysical* and can be eliminated by a suitable choice of gauge ("unitarity gauge"): choose $e^{i\alpha(x)}$ such that $\phi(x)$ is real at each point x. The degree of freedom of the original field ϕ_2 appears as the third degree of freedom for the now massive photon (three spin-z components vs. two helicity states!). (In abhorrent language, one often says that the photon became massive by eating the Goldstone boson.)

8.3.2 Example: non-Abelian Higgs mechanism

We now consider a triplet of $SU(2)$ gauge fields, A_μ^a, and a $SU(2)$ doublet of complex scalar fields, Φ. The covariant derivative acting on Φ is

$$D_\mu\Phi = \partial_\mu\Phi - igA_\mu^a\tau^a\Phi\,, \tag{8.30}$$

where $\tau^a = \sigma^a/2$. If Φ obtains a vev, we can perform a $SU(2)$ rotation to bring it into the form

$$\langle \Phi \rangle_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} . \tag{8.31}$$

The kinetic term in the Lagrangian then yields a mass term

$$|D_\mu \Phi|^2 \supset \frac{1}{2} g^2 (0, v) \tau^a \tau^b \begin{pmatrix} 0 \\ v \end{pmatrix} A^a_\mu A^{b\mu} = \frac{g^2 v^2}{8} A^a_\mu A^{a\mu} , \tag{8.32}$$

with equal mass $m_A = gv/2$ for all three gauge bosons. None of the three generators leaves the vacuum invariant.

9 The weak interaction

In order to describe the weak interaction, we extend our previous example by a $U(1)$ symmetry; hence, the gauge symmetry is $SU(2) \times U(1)$. We assign the $U(1)$ charge $+1/2$ to the scalar field; in total,

$$\Phi \to e^{i\alpha^a \tau^a} e^{i\beta/2} \Phi . \tag{9.1}$$

Assume that again Φ obtains a vev that can be rotated into the form

$$\langle \Phi \rangle_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} ; \tag{9.2}$$

this is invariant under a gauge transformation with generator

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \tfrac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \tfrac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{9.3}$$

(this corresponds to $\alpha_1 = \alpha_2 = 0$, $\alpha_3 = \beta$). Thus, we expect one massless and three massive gauge bosons.

9.1 Weak gauge-boson masses

The covariant derivative of the scalar is now

$$D_\mu \Phi = \partial_\mu \Phi - igA^a_\mu \tau^a \Phi - \tfrac{i}{2} g' B_\mu \Phi , \tag{9.4}$$

where A^a_μ and B_μ are the $SU(2)$ and $U(1)$ gauge fields, respectively. The kinetic term contains the mass terms:

$$|D_\mu \Phi|^2 \supset \frac{1}{2} (0, v) \big(gA^a_\mu \tau^a + \tfrac{1}{2} g' B_\mu \big) \big(gA^{b\mu} \tau^b + \tfrac{1}{2} g' B^\mu \big) \begin{pmatrix} 0 \\ v \end{pmatrix} . \tag{9.5}$$

Now we use

$$(0, v) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} = (0, v) \begin{pmatrix} 0 \\ v \end{pmatrix} = v^2 , \tag{9.6}$$

$$(0,v)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0\\ v\end{pmatrix} = (0,v)\begin{pmatrix}v\\ 0\end{pmatrix} = 0\,, \tag{9.7}$$

$$(0,v)\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}v\\ 0\end{pmatrix} = 0\,, \tag{9.8}$$

$$(0,v)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0\\ v\end{pmatrix} = -v^2\,, \tag{9.9}$$

as well as $\{\tau^a, \tau^b\} = \delta^{ab}/2$, and find

$$\begin{aligned} |D_\mu \Phi|^2 \supset \frac{1}{2}\frac{v^2}{4}\Big[g^2\big(A^1_\mu\big)^2 + g^2\big(A^2_\mu\big)^2 + g^2\big(A^3_\mu\big)^2 - 2gg' A^3_\mu B^\mu + g'^2 B_\mu B^\mu\Big] \\ = \frac{1}{2}\frac{v^2}{4}\Big[g^2\big(A^1_\mu\big)^2 + g^2\big(A^2_\mu\big)^2 + \big(gA^3_\mu - g' B_\mu\big)^2\Big]\,. \end{aligned} \tag{9.10}$$

We find three massive gauge boson fields:

$$W^\pm_\mu = \frac{1}{\sqrt{2}}\big(A^1_\mu \mp i A^2_\mu\big)\,, \tag{9.11}$$

with mass $M_w = gv/2$, and

$$Z^0_\mu = \frac{1}{\sqrt{g^2 + g'^2}}\big(gA^3_\mu - g' B_\mu\big)\,, \tag{9.12}$$

with mass $M_Z = \sqrt{g^2 + g'^2}v/2$. The field orthogonal to Z^0_μ,

$$A_\mu = \frac{1}{\sqrt{g^2 + g'^2}}\big(g' A^3_\mu + g B_\mu\big)\,, \tag{9.13}$$

remains massless; we identify it with the photon field.

We now express the covariant derivative

$$D_\mu \Phi = \partial_\mu \Phi - igA^a_\mu \tau^a \Phi - ig' Y B_\mu \Phi\,, \tag{9.14}$$

with the $U(1)$ *hypercharge* Y in terms of the fields in the mass eigenbasis:

$$\begin{aligned} D_\mu \Phi = \partial_\mu \Phi - \frac{ig}{\sqrt{2}}\big(W^+_\mu \tau^+ + W^-_\mu \tau^-\big) \\ - \frac{i}{\sqrt{g^2 + g'^2}} Z_\mu \big(g^2 \tau^3 - g'^2 Y\big) - \frac{igg'}{\sqrt{g^2 + g'^2}} A_\mu \big(\tau^3 + Y\big)\,, \end{aligned} \tag{9.15}$$

where

$$\tau^\pm = \tau^1 \pm i\tau^2\,. \tag{9.16}$$

We now identify the electromagnetic coupling constant as

$$e = \frac{gg'}{\sqrt{g^2 + g'^2}}\,, \tag{9.17}$$

and the electric charge as

$$Q = \tau^3 + Y\,. \tag{9.18}$$

Further, we define the *weak mixing angle* θ_w by

$$\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_w & -\sin\theta_w \\ \sin\theta_w & \cos\theta_w \end{pmatrix} \begin{pmatrix} A^3_\mu \\ B_\mu \end{pmatrix}\,, \tag{9.19}$$

where

$$\cos\theta_w = \frac{g}{\sqrt{g^2+g'^2}}\,, \qquad \sin\theta_w = \frac{g'}{\sqrt{g^2+g'^2}}\,, \tag{9.20}$$

Using

$$g^2\tau^3 - g'^2 Y = (g^2+g'^2)\tau^3 - g'^2 Q\,, \tag{9.21}$$

we have finally

$$D_\mu\Phi = \partial_\mu\Phi - i\frac{g}{\sqrt{2}}\big(W^+_\mu\tau^+ + W^-_\mu\tau^-\big)\Phi - \frac{ig}{\cos\theta_w}Z_\mu\big(\tau^3 - Q\sin^2\theta_w\big)\Phi - ieQA_\mu\Phi\,, \tag{9.22}$$

where

$$g = \frac{e}{\sin\theta_w}\,. \tag{9.23}$$

The W and Z masses are not independent; we have

$$M_W = M_Z\cos\theta_w\,. \tag{9.24}$$

9.2 Weak interactions of fermions

We know from experiment that W boson couple only to left-handed fermion fields. Therefore, we decompose all quark and lepton fields into their LH and RH parts, $\psi = \psi_L + \psi_R$, and require that the ψ_L transform as doublets and the ψ_R as singlets under $SU(2)$. Then we choose the hypercharges such that we obtain the correct electric charge (recall $Q = \tau^3 + Y$):

	$E_L = \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L$	$Q_L = \begin{pmatrix} u \\ d \end{pmatrix}_L$	e_R	u_R	d_R
Y	$-\frac{1}{2}$	$\frac{1}{6}$	-1	$\frac{2}{3}$	$-\frac{1}{3}$

N.B. These gauge quantum numbers imply that a fermion mass term would break the gauge symmetry:

$$\mathcal{L}_{\text{mass}} = -m\bar\psi\psi = -m(\bar\psi_L + \bar\psi_R)(\psi_L + \psi_R)\,. \tag{9.25}$$

Using $\bar\psi_L \equiv (P_L\psi^\dagger)\gamma^0 = \psi^\dagger P_L\gamma^0 = \bar\psi P_R$ we get

$$\mathcal{L}_{\text{mass}} = -m(\bar\psi_L\psi_R + \bar\psi_R\psi_L)\,. \tag{9.26}$$

It follows that all fermions must be massless.

The kinetic term for the (first-generation) fermions is, thus,

$$\mathcal{L}_{\text{kin.}} = \bar E_L i\not{D} E_L + \bar Q_L i\not{D} Q_L + \bar e_R i\not{D} e_R + \bar u_R i\not{D} u_R + \bar d_R i\not{D} d_R\,, \tag{9.27}$$

with the corresponding covariant derivatives D_μ; for instance,

$$\bar{Q}_L i\not{D} Q_L = \bar{Q}_L i\gamma^\mu \big(\partial_\mu - igA^a_\mu \tau^a - \tfrac{i}{6} g' B_\mu\big) Q_L \,, \tag{9.28}$$

etc. (Remember that there are no RH neutrinos in the SM!)

We can express Eq. (9.27) in terms of mass eigenstates and obtain

$$\mathcal{L}_{\text{kin.}} = \bar{e} i\not{\partial} e + \bar{\nu}_e i\not{\partial} \nu_e + \bar{u} i\not{\partial} u + \bar{d} i\not{\partial} d + g\big(W^+_\mu J^{+\mu}_W + W^-_\mu J^{-\mu}_W + Z^+_\mu J^\mu_Z\big) + eA_\mu J^\mu_{\text{EM}} \,, \tag{9.29}$$

with

$$J^{+\mu}_W = \frac{1}{\sqrt{2}}\big(\bar{\nu}_L \gamma^\mu e_L + \bar{u}_L \gamma^\mu d_L\big) \,, \tag{9.30}$$

$$J^{-\mu}_W = \frac{1}{\sqrt{2}}\big(\bar{e}_L \gamma^\mu \nu_L + \bar{d}_L \gamma^\mu u_L\big) \,, \tag{9.31}$$

$$\begin{aligned} J^\mu_Z = \frac{1}{\cos\theta_w}\Big[& \tfrac{1}{2}\bar{\nu}_L \gamma^\mu \nu_L + \bar{e}_L \gamma^\mu (\sin^2\theta_w - \tfrac{1}{2}) e_L + \bar{e}_R \gamma^\mu \sin^2\theta_w e_R \\ & + \bar{u}_L \gamma^\mu (-\tfrac{2}{3}\sin^2\theta_w + \tfrac{1}{2}) u_L + \bar{u}_R \gamma^\mu (-\tfrac{2}{3}\sin^2\theta_w) u_R \\ & + \bar{d}_L \gamma^\mu (\tfrac{1}{3}\sin^2\theta_w - \tfrac{1}{2}) d_L + \bar{d}_R \gamma^\mu (\tfrac{1}{3}\sin^2\theta_w) d_R \Big] \,, \end{aligned} \tag{9.32}$$

$$J^\mu_{\text{EM}} = \bar{e}\gamma^\mu(-1)e + \bar{u}\gamma^\mu(\tfrac{2}{3})u + \bar{d}\gamma^\mu(-\tfrac{1}{3})d \,. \tag{9.33}$$

9.3 Yukawa interaction and Higgs sector

The fermion masses in the SM arise from SSB. Consider, for instance, the electron. The following term is gauge invariant:

$$\mathcal{L} \supset -\lambda_e \bar{E}_L \Phi e_R + \text{h.c.} \,, \tag{9.34}$$

with the Higgs doublet field Φ. Replacing Φ by the vev (9.2), we obtain a mass term for the electron:

$$\mathcal{L} \supset -\frac{\lambda_e v}{\sqrt{2}}\big(\bar{e}_L e_R + \bar{e}_R e_L\big) \,. \tag{9.35}$$

The same procedure works for the down quark. For the up quark, however, we need a "trick". It is straightforward to verify that

$$i\sigma^2 \cdot \boldsymbol{\sigma} \cdot i\sigma^2 = \boldsymbol{\sigma}^* \,, \tag{9.36}$$

where

$$i\sigma^2 \equiv \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \,. \tag{9.37}$$

Now we can construct gauge-invariant mass terms for both the up and the down quarks:

$$\mathcal{L} \supset -\lambda_d \bar{Q}_L \Phi d_R - \lambda_u \bar{Q}_L \epsilon \Phi^* u_R + \text{h.c.} \,. \tag{9.38}$$

The combination $\Phi^c \equiv \epsilon\Phi^*$ is the charge-conjugated Higgs field. The second term is $SU(2)$ invariant, since

$$\begin{aligned}\bar{Q}_L\epsilon\Phi^* \to \bar{Q}_L e^{-i\boldsymbol{\alpha}\cdot\boldsymbol{\tau}}\epsilon(e^{i\boldsymbol{\alpha}\cdot\boldsymbol{\tau}})^*\Phi^* &= \bar{Q}_L e^{-i\boldsymbol{\alpha}\cdot\boldsymbol{\tau}}\epsilon e^{-i\boldsymbol{\alpha}\cdot\boldsymbol{\tau}^*}\Phi^* \\ &= \bar{Q}_L e^{-i\boldsymbol{\alpha}\cdot\boldsymbol{\tau}} e^{+i\boldsymbol{\alpha}\cdot\boldsymbol{\tau}}\epsilon\Phi^* = \bar{Q}_L\epsilon\Phi^*\,,\end{aligned} \tag{9.39}$$

where $e^{i\boldsymbol{\alpha}\cdot\boldsymbol{\tau}} \in SU(2)$ and $\epsilon\cdot\epsilon = -1$. You can check that the hypercharges work out, too.

If we replace Φ by its vev in Eq. (9.38) and use

$$\epsilon\langle\Phi^*\rangle_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}v\\ 0\end{pmatrix}\,, \tag{9.40}$$

we find the mass terms

$$\mathcal{L} \supset -\frac{\lambda_u v}{\sqrt{2}}\bar{u}_L u_R - \frac{\lambda_d v}{\sqrt{2}}\bar{d}_L d_R + \text{h.c.}\,. \tag{9.41}$$

Hence, the fermion masses are

$$m_e = \frac{\lambda_e v}{\sqrt{2}}\,, \qquad m_u = \frac{\lambda_u v}{\sqrt{2}}\,, \qquad m_d = \frac{\lambda_d v}{\sqrt{2}}\,. \tag{9.42}$$

Later, we will generalize the construction to all three fermion generations.

By choosing a suitable gauge (unitarity gauge) we can eliminate the unphysical Goldstone bosons and write the full Higgs field in the broken phase as

$$\Phi(x) = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v + h(x)\end{pmatrix}\,, \tag{9.43}$$

with the physical Higgs field $h(x)$. The Higgs Lagrangian is then given by

$$\mathcal{L}_{\text{Higgs}} = |D_\mu\Phi|^2 + \mu^2\Phi^\dagger\Phi - \lambda(\Phi^\dagger\Phi)^2\,, \tag{9.44}$$

and the minimum of the potential is at

$$v = \sqrt{\frac{\mu^2}{\lambda}}\,. \tag{9.45}$$

Inserting Eq. (9.43) into Eq. (9.44) gives

$$\mathcal{L}_{\text{Higgs}} \supset -\mu^2 h^2 - \lambda v h^3 - \tfrac{1}{4}\lambda h^4 = -\tfrac{1}{2}M_h^2 h^2 - \sqrt{\frac{\lambda}{2}}M_h h^3 - \tfrac{1}{4}\lambda h^4\,, \tag{9.46}$$

with the Higgs mass

$$M_h = \sqrt{2}\mu = \sqrt{2\lambda}v\,. \tag{9.47}$$

The kinetic terms in Eq. (9.44) give the gauge-boson mass terms and their Higgs interactions:

$$\mathcal{L}_{\text{Higgs}} \supset \tfrac{1}{2}(\partial_\mu h)^2 + \left[M_W^2 W_\mu^+ W^{-\mu} + \tfrac{1}{2}M_Z^2 Z^2\right]\left(1 + \frac{h}{v}\right)^2\,. \tag{9.48}$$

9.4 The CKM matrix

The kinetic term for the three fermion generations is

$$\mathcal{L}_{\text{kin.}} = \sum_{k=1}^{3} \left(\bar{L}_L^k i \not{D} L_L^k + \bar{Q}_L^k i \not{D} Q_L^k + \bar{\ell}_R^k i \not{D} \ell_R^k + \bar{u}_R^k i \not{D} u_R^k + \bar{d}_R^k i \not{D} d_R^k \right) . \tag{9.49}$$

where i is a generation index. This Lagrangian has a large $U(3)^5$ *flavor symmetry*: unitary $U(3)$ rotations among the three generations, for L_L, Q_L, ℓ_R, u_R, and d_R. The most general Yukawa interaction Lagrangian for the three generations is

$$\mathcal{L}_{\text{Yuk.}} = - \sum_{i,j=1}^{3} \left[\hat{Y}_{ij}^e \bar{L}_L^i \Phi \ell_R^j + \hat{Y}_{ij}^d \bar{Q}_L^i \Phi d_R^j + \hat{Y}_{ij}^u \bar{Q}_L^i \Phi^c u_R^j \right] + \text{h.c.} , \tag{9.50}$$

with general, complex 3×3 matrices $\hat{Y}^a$. Which of the $3 \cdot 2 \cdot 3^2 = 54$ parameters are physical?

Leptons

The kinetic term for the leptons is invariant under

$$e_R \to R e_R , \qquad \bar{e}_R \to \bar{e}_R R^\dagger , \qquad L_L \to S L_L , \qquad \bar{L}_L \to \bar{L}_L S^\dagger , \tag{9.51}$$

with $R, S \in U(3)$; i.e., $\hat{Y}^e$ is equivalent to $Y^e \equiv S \hat{Y}^e R^\dagger$. By a suitable choice of R and S, Y^e can be made diagonal, real, and non-negative ("Cartan decomposition"). Hence, we have

$$\mathcal{L}_{\text{Yuk.},\ell} = - \sum_{i=1}^{3} y_i^e \bar{L}_L^i \Phi \ell_R^i + \text{h.c.} . \tag{9.52}$$

The Cartan decomposition is not unique; instead of R, S we can also use $R' = DR$, $S' = DS$, with

$$D = \begin{pmatrix} e^{i\phi_1} & 0 & 0 \\ 0 & e^{i\phi_2} & 0 \\ 0 & 0 & e^{i\phi_3} \end{pmatrix} . \tag{9.53}$$

(These phase transformations correspond to the conservation of individual lepton number.) The total number of physical parameters is thus $2 \cdot 3^2 - (2 \cdot 3^2 - 3) = 3$ – the three lepton masses.

Quarks

The kinetic term for the quarks is invariant under

$$\begin{aligned} d_R &\to R_d d_R , & \bar{d}_R &\to \bar{d}_R R_d^\dagger , \\ u_R &\to R_u u_R , & \bar{u}_R &\to \bar{u}_R R_u^\dagger , \\ Q_L &\to S_u Q_L , & \bar{Q}_L &\to \bar{Q}_L S_u^\dagger , \end{aligned} \tag{9.54}$$

with $R_s, R_u, S_u \in U(3)$. Again, we can choose $Y^u \equiv S_u \hat{Y}^u R_u^\dagger$ diagonal, real, and non-negative. Then however,

$$S_u \hat{Y}^d R_d^\dagger = \underbrace{S_u S_d^\dagger}_{V} \underbrace{S_d \hat{Y}^d R_d^\dagger}_{Y^d \text{...diag., real, non-neg.}} \equiv V Y^d \tag{9.55}$$

is neither real nor diagonal. Here,

$$V \equiv S_u S_d^\dagger \tag{9.56}$$

is the Cabibbo-Kobayashi-Maskawa (CKM) matrix. It is unitary by construction. The quark Yukawa interaction is now

$$\mathcal{L}_{\text{Yuk.},\ell} = -\left(\sum_{i,j=1}^{3} y_i^d \bar{Q}_L^i \Phi V_{ij} d_R^j + \sum_{i=1}^{3} y_i^u \bar{Q}_L^i \Phi^c u_R^i \right) + \text{h.c.}\,. \tag{9.57}$$

Instead of R_u, R_d, S_u we could also choose $e^{i\phi} R_u$, $e^{i\phi} R_d$, $e^{i\phi} S_u$, so we have $4 \cdot 3^2 - (3 \cdot 3^2 - 1) = 10$ physical parameters – six quark masses, three mixing angles, and one CP-violating phase. (The symmetry under the phase transformation $e^{i\phi}$ corresponds to baryon number conservation.)

In order to find the quark masses, we define

$$Q_L^i \equiv \begin{pmatrix} u_L^i \\ \sum_j V_{ij} d_L^j \end{pmatrix} . \tag{9.58}$$

This diagonalizes the quark mass terms in Eq. (9.57), if we insert the Higgs vev; we find

$$m_u = \frac{y_1^u v}{\sqrt{2}}\,, \quad m_d = \frac{y_1^d v}{\sqrt{2}}\,, \quad m_c = \frac{y_2^u v}{\sqrt{2}}\,, \quad m_s = \frac{y_2^d v}{\sqrt{2}}\,, \quad m_t = \frac{y_3^u v}{\sqrt{2}}\,, \quad m_b = \frac{y_3^d v}{\sqrt{2}}\,. \tag{9.59}$$

In this basis, the CKM matrix appears in the charged-current interactions; Eqs. and (9.31) become

$$J_W^{+\mu} \to \frac{1}{\sqrt{2}} \sum_{ij} \bar{u}_L^i \gamma^\mu V_{ij} d_L^j + \dots\,, \tag{9.60}$$

$$J_W^{-\mu} = \to \frac{1}{\sqrt{2}} \sum_{ij} \bar{d}_L^i \gamma^\mu V_{ij}^\dagger u_L^j + \dots\,. \tag{9.61}$$

In the *neutral* currents, the CKM matrix cancels; there is no tree-level flavor violation in the SM (“no tree-level FCNCs”).

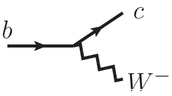
10 Introduction to flavor physics

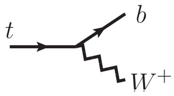
10.1 Phenomenology of the CKM matrix

The usual notation is

$$V = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} . \tag{10.1}$$

Then, for example,



$$\propto V_{cb}\,,$$


$$\propto V_{tb}^*\,. \tag{10.2}$$

The unitarity of V leads to several useful relations. The phenomenologically most interesting is probably the orthogonality condidtion

$$V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0\,. \tag{10.3}$$

This equation defines a triangle in the complex plane – the so-called "unitarity triangle" with the angles

$$\alpha = \arg\left(-\frac{V_{td}V_{tb}^*}{V_{ud}V_{ub}^*}\right), \quad \beta = \arg\left(-\frac{V_{cd}V_{cb}^*}{V_{td}V_{tb}^*}\right), \quad \gamma = \arg\left(-\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*}\right). \tag{10.4}$$

A general, exact parameterization of V is (see the PDG [6])

$$V = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{13}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{-i\delta_{13}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{-i\delta_{13}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{-i\delta_{13}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{-i\delta_{13}} & c_{23}c_{13} \end{pmatrix}, \tag{10.5}$$

with $c_{ij} = \cos\theta_{ij}$, $s_{ij} = \sin\theta_{ij}$ (three angles, one phase). We define

$$\lambda \equiv s_{12}\,, \qquad A \equiv \frac{s_{23}}{\lambda^2}\,, \qquad \rho + i\eta \equiv \frac{s_{13}e^{-i\delta_{13}}}{A\lambda^3}\,. \tag{10.6}$$

Measurements give $\lambda \approx 0.22$, $A \approx 0.8$, $\sqrt{\rho^2+\eta^2} \approx 0.4$, so we can expand in the small parameter λ. This gives the *Wolfenstein parameterization*

$$V = \begin{pmatrix} 1-\frac{1}{2}\lambda^2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\frac{1}{2}\lambda^2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^3 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4)\,. \tag{10.7}$$

(There is an improved parameterization in terms of $\bar\rho$ and $\bar\eta$ that is exactly unitary to all orders in λ; see Ref. [6].) The CKM matrix shows a pronounced hierarchy.

10.2 Neutral meson mixing and CP violation

How does the SM Lagrangian change under the discrete transformations C, $P : x^\mu \to x_\mu$, $T : x^\mu \to -x_\mu$? For $X = C, P, T, CP, CPT$ we have

$$X: \quad \bar{b}\Gamma d \quad \to \quad X\bar{b}\Gamma d X^{-1}\,, \tag{10.8}$$

with

	$\bar b_R d_L(x^\mu)$	$\bar b_R \gamma_\rho d_L(x^\mu)$
C	$\bar d_R b_L(x^\mu)\,\eta_C$	$-\bar d_R \gamma_\rho b_R(x^\mu)\,\eta_C$
P	$\bar b_L d_R(x_\mu)\,\eta_P$	$\bar b_R \gamma^\rho d_R(x_\mu)\,\eta_P$
CP	$\bar d_L b_R(x_\mu)\,\eta_C\eta_P$	$-\bar d_L \gamma^\rho b_L(x_\mu)\,\eta_C\eta_P$
T	$\bar b_R d_L(-x_\mu)\,\eta_T$	$\bar b_L \gamma^\rho d_L(-x_\mu)\,\eta_T$
CPT	$\bar d_L b_R(-x^\mu)\,\eta_C\eta_P\eta_T$	$-\bar d_L \gamma_\rho b_L(-x^\mu)\,\eta_C\eta_P\eta_T$

and the same with $L \leftrightarrow R$. The arbitrary phase factors η_X can be absorbed into the definitions of the quark fields (since the Yukawa part of the Lagrangian is not invariant under this redefinition, this corresponds to a choice of phase convention for the CKM matrix!).

The meson states transform like the corresponding currents, since QCD is invariant under C, P, and T. For instance,

$$CP|\bar B^0(p^\mu)\rangle = -\eta_C\eta_P|B^0(p_\mu)\rangle\,, \qquad CP|B^0(p^\mu)\rangle = -\eta_C^*\eta_P^*|\bar B^0(p_\mu)\rangle\,. \tag{10.9}$$

The vector and scalar fields of the SM transform as follows:

	$A^\mu(x^\rho)$, $G^{a,\mu}(x^\rho)$, $Z^\mu(x^\rho)$	$W^{\pm,\mu}(x^\rho)$	$h(x^\rho)$
C	$-V^\mu(x^\rho)$	$-W^{\mp,\mu}(x^\rho)$	$h(x^\rho)$
P	$V_\mu(x_\rho)$	$W^\pm_\mu(x^\rho)$	$h(x_\rho)$
CP	$-V_\mu(x_\rho)$	$-W^\mp_\mu(x_\rho)$	$h(x_\rho)$
T	$V_\mu(-x_\rho)$	$W^\pm_\mu(-x_\rho)$	$h(-x_\rho)$
CPT	$-V^\mu(-x^\rho)$	$-W^{\mp,\mu}(-x^\rho)$	$h(-x^\rho)$

Therefore, the charged currents transform under CP as (e.g.)

$$V_{ub}\bar u_L\gamma^\mu b_L W^+_\mu + V^*_{ub}\bar b_L\gamma^\mu u_L W^-_\mu \to V_{ub}\bar b_L\gamma^\mu u_L W^-_\mu + V^*_{ub}\bar u_L\gamma^\mu b_L W^+_\mu\,. \tag{10.10}$$

This is the same only for $V_{ub} = V^*_{ub}$. Is this phase actually observable?

Let $|B^0(t)\rangle$ be the state vector of a B meson that has been a B^0 at $t = 0$, i.e. $|B^0(t = 0)\rangle = |B^0\rangle$. Generally, for $t \neq 0$, $|B^0(t)\rangle$ is then a superposition of $|B^0\rangle$ and $|\bar B^0\rangle$. The time evolution is described by

$$i\frac{d}{dt}\begin{pmatrix}|B^0(t)\rangle\\|\bar B^0(t)\rangle\end{pmatrix} = \left(M - \tfrac{i}{2}\Gamma\right)\begin{pmatrix}|B^0(t)\rangle\\|\bar B^0(t)\rangle\end{pmatrix}\,, \tag{10.11}$$

where M and Γ are Hermitian 2×2 matrices. CPT invariance implies $M_{11} = M_{22}$, $\Gamma_{11} = \Gamma_{22}$. The weak interaction induces off-diagonal matrix elements via box diagrams (Fig. 7) The

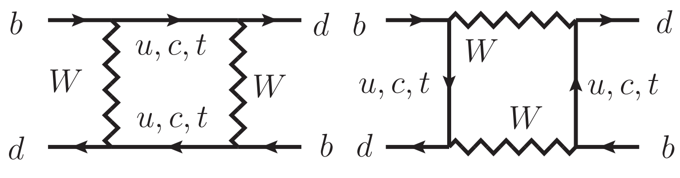


Figure 7: *Leading order box diagrams for B-meson mixing.*

transition amplitude has the form

$$\mathcal{A} = \sum_{i,j=u,c,t} V_{ib} V_{id}^* V_{jb} V_{jd}^* F\Big(\frac{m_i^2}{M_W^2}, \frac{m_j^2}{M_W^2}\Big) \,. \tag{10.12}$$

We define $x_i = m_i^2/M_W^2$, $\lambda_i = V_{ib}V_{id}^*$. CKM unitarity implies $\lambda_u + \lambda_c + \lambda_t = 0$, i.e. the amplitude vanishes for $x_i = 0$ (this is called "Glashow-Iliopoulos-Maiani (GIM) mechanism"). Since $m_t \gg m_c, m_u$ and $\lambda_u \sim \lambda_c \sim \lambda_t$, B-meson mixing is dominated by the λ_t term (careful, this is not necessarily true for other meson systems).

We can diagonalize the Hamiltonian (10.11) to find the weak eigenstates

$$\begin{aligned} |B_L\rangle &= p|B^0\rangle + q|\bar{B}^0\rangle \,, \\ |B_H\rangle &= p|B^0\rangle - q|\bar{B}^0\rangle \,, \end{aligned} \tag{10.13}$$

with $|p|^2 + |q|^2 = 1$. The two states $|B_L\rangle$ and $|B_H\rangle$ are not orthogonal. Their time evolution is given by

$$|B_{H,L}(t)\rangle = \exp\Big[-i\big(M_{H,L} - i\Gamma_{H,L}/2\big)t\Big]|B_{H,L}\rangle \,, \tag{10.14}$$

with $|B_{H,L}(t=0)\rangle = |B_{H,L}\rangle$, and $M_{H,L}$ and $\Gamma_{H,L}$ the masses and life times of the $B_{H,L}$ mesons. Now we define

$$\begin{aligned} m \equiv \frac{M_H + M_L}{2} = M_{11} \,, \qquad \Gamma \equiv \frac{\Gamma_H + \Gamma_L}{2} = \Gamma_{11} \,, \\ \Delta m = M_H - M_L > 0 \,, \qquad \Delta\Gamma = \Gamma_H - \Gamma_L \,. \end{aligned} \tag{10.15}$$

Inverting Eq. (10.13) and inserting Eq. (10.14), we find the time evolution

$$\begin{aligned} |B^0(t)\rangle &= \frac{1}{2p}\Big[e^{-iM_Lt - \Gamma_L t/2}|B_L\rangle + e^{-iM_Ht - \Gamma_H t/2}|B_H\rangle\Big] \,, \\ |\bar{B}^0(t)\rangle &= \frac{1}{2q}\Big[e^{-iM_Lt - \Gamma_L t/2}|B_L\rangle - e^{-iM_Ht - \Gamma_H t/2}|B_H\rangle\Big] \,. \end{aligned} \tag{10.16}$$

Now we can use Eq. (10.13) again to replace $|B_{H,L}\rangle$ on the right side:

$$\begin{aligned} |B^0(t)\rangle &= g_+(t)|B^0\rangle + \frac{q}{p} g_-(t)|\bar{B}^0\rangle \,, \\ |\bar{B}^0(t)\rangle &= \frac{p}{q} g_-(t)|B^0\rangle + g_+(t)|\bar{B}^0\rangle \,, \end{aligned} \tag{10.17}$$

with

$$\begin{aligned} g_+(t) &= e^{-imt} e^{-\Gamma t/2}\Big[\cosh\frac{\Delta\Gamma t}{4}\cos\frac{\Delta m t}{2} - i \sinh\frac{\Delta\Gamma t}{4}\sin\frac{\Delta m t}{2}\Big] \,, \\ g_-(t) &= e^{-imt} e^{-\Gamma t/2}\Big[-\sinh\frac{\Delta\Gamma t}{4}\cos\frac{\Delta m t}{2} + i \cosh\frac{\Delta\Gamma t}{4}\sin\frac{\Delta m t}{2}\Big] \,. \end{aligned} \tag{10.18}$$

Since $\Delta\Gamma \neq 0$, we have $g_\pm(t) \neq 0$ (with the only exception $g_-(t=0) = 0$). A B^0 will never mix back into a pure B^0 state.

M_{12} and Γ_{12} can (in principle) be calculated (e.g. via box diagrams in the SM). The connection to experiment is provided by the relations

$$(\Delta m)^2 - \frac{1}{4}(\Delta\Gamma)^2 = 4|M_{12}|^2 - |\Gamma_{12}|^2\,, \tag{10.19}$$

$$\Delta m \Delta\Gamma = -4\mathrm{Re}\big(M_{12}\Gamma_{12}^*\big)\,, \tag{10.20}$$

$$\frac{q}{p} = -\frac{\Delta m + i\Delta\Gamma/2}{2M_{12} - i\Gamma_{12}} = -\frac{2M_{12}^* - i\Gamma_{12}^*}{\Delta m + i\Delta\Gamma/2}\,. \tag{10.21}$$

Frequently, one defines also

$$\phi \equiv \arg\left(-\frac{M_{12}}{\Gamma_{12}}\right). \tag{10.22}$$

Using Eq. (10.17) we can calculate the time-dependent decay rates $\Gamma(B^0(t)\to f), \Gamma(\bar{B}^0(t)\to f)$ (see Ref. [7]). This allows us to study CP violation. The following question is important: which quantities are physically observable (in the sense that they independent of arbitrary phase conventions)? Let us first define a short notation for the decay amplitudes,

$$A_f \equiv \langle f|B^0\rangle\,, \qquad \bar{A}_f \equiv \langle f|\bar{B}^0\rangle\,, \tag{10.23}$$

and similarly $A_{\bar{f}}$, $\bar{A}_{\bar{f}}$, with the CP-conjugated final state

$$|\bar{f}\rangle = CP|\bar{f}\rangle\,. \tag{10.24}$$

The phases of M_{12}, Γ_{12}, q/p, $\bar{A}_f/A_f$ depend on the phase convention for the CP transformation and / or the CKM matrix. The following quantities are phase-convention independent (and thus observable):

$$\left|\frac{q}{p}\right|, \quad \left|\frac{\bar{A}_{\bar{f}}}{A_f}\right|, \quad \lambda \equiv \frac{q}{p}\frac{\bar{A}_f}{A_f}\,, \quad \phi = \arg\left(-\frac{M_{12}}{\Gamma_{12}}\right), \quad \Delta m\,, \quad \Delta\Gamma\,. \tag{10.25}$$

10.3 Three types of CP violation

10.3.1 CP violation in mixing ($|q/p| \neq 1$)

Eq. (10.21) implies

$$\left|\frac{q}{p}\right|^2 = \left|\frac{2M_{12}^* - i\Gamma_{12}^*}{2M_{12} - i\Gamma_{12}}\right|. \tag{10.26}$$

If $\phi = 0$, then $|q/p| = 1$, while $|q/p| \neq 1$ implies CP violation.

Example: Decay into "wrong-sign" leptons. The final-state lepton of a semileptonic decay of a $B^0 \sim [\bar{b}d]$ always has positive charge, $B^0 \to \ell^+\nu X$. Similarly, $\bar{B}^0 \to \ell^-\bar{\nu}X$. See Fig. 8. Now consider the asymmetries (here, $f = \ell^-\bar{\nu}X$)

$$\begin{aligned} a_{\text{sl}}(t) &= \frac{\Gamma(\bar{B}^0\to\ell^+\nu X) - \Gamma(B^0\to\ell^-\bar{\nu}X)}{\Gamma(\bar{B}^0\to\ell^+\nu X) + \Gamma(B^0\to\ell^-\bar{\nu}X)} \\ &= \frac{|\frac{p}{q}g_-(t)\langle\bar{f}|B\rangle + g_+(t)\langle\bar{f}|\bar{B}\rangle|^2 - |g_+(t)\langle f|B\rangle + \frac{q}{p}g_-(t)\langle f|\bar{B}\rangle|^2}{|\frac{p}{q}g_-(t)\langle\bar{f}|B\rangle + g_+(t)\langle\bar{f}|\bar{B}\rangle|^2 + |g_+(t)\langle f|B\rangle + \frac{q}{p}g_-(t)\langle f|\bar{B}\rangle|^2} \\ &= \frac{|\frac{p}{q}\rangle|^2 - |\frac{q}{p}|^2}{|\frac{p}{q}\rangle|^2 + |\frac{q}{p}|^2} = \frac{1-|q/p|^4}{1+|q/p|^4}\,. \end{aligned} \tag{10.27}$$

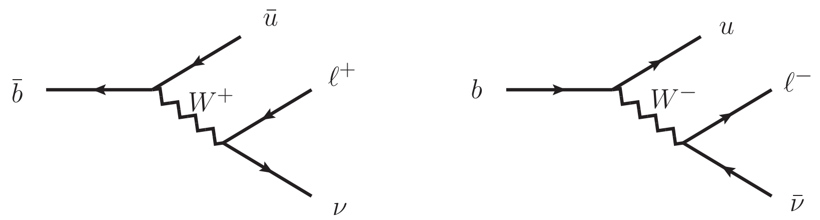


Figure 8: *Decay of a $\bar{b}$ quark vs. decay of a b quark.*

Here, we used $\langle \bar{f}|\bar{B}\rangle = \langle f|B\rangle = 0$ and $|\langle \bar{f}|B\rangle| = |\langle \bar{f}|B\rangle|$.

10.3.2 CP violation in decay ($|\bar{A}_{\bar{f}}/A_f| \neq 1$)

There can be two types of phases in the decay amplitudes $A_f \equiv \langle f|B\rangle$ and $\bar{A}_{\bar{f}} \equiv \langle \bar{f}|\bar{B}\rangle$. *Weak phases* change sign under CP; in the SM, these phases appear only in the CKM matrix. QCD can generate *strong phases* through rescattering in the hadronic final state. Strong phases do not change sign under CP, since QCD is CP invariant. So, in general we can write

$$A_f = \sum_k A_k e^{i(\delta_k + \phi_k)}\,, \qquad \bar{A}_{\bar{f}} = \sum_k A_k e^{i(\delta_k - \phi_k)}\,, \tag{10.28}$$

where the A_k are real, and δ_k and ϕ_k are the strong and weak phases, respectively.

CP is obviously violated if $|\bar{A}_{\bar{f}}/A_f| \neq 1$ (direct CP violation). Direct CP violation can only occur in a process that involves (at least) two amplitudes that differ in both their weak and strong phases. For instance,

$$\left|\frac{A_1 e^{i\delta_1} + A_2 e^{i\delta_1 - i\phi_2}}{A_1 e^{i\delta_1} + A_2 e^{i\delta_1 + i\phi_2}}\right| = \left|\frac{A_1 + A_2 e^{-i\phi_2}}{A_1 + A_2 e^{i\phi_2}}\right| = 1\,. \tag{10.29}$$

Example: CP violation in the decay of charged B mesons.

$$\frac{\Gamma(B^- \to f) - \Gamma(B^+ \to \bar{f})}{\Gamma(B^- \to f) + \Gamma(B^+ \to \bar{f})} = \frac{1 - |\bar{A}_{\bar{f}}/A_f|^2}{1 + |\bar{A}_{\bar{f}}/A_f|^2}\,. \tag{10.30}$$

10.3.3 CP violation in the interference between mixing and decay ($\lambda_f \neq \pm 1$)

Consider the decay into a CP eigenstate f_{CP}. If both B^0 and $\bar{B}^0$ can decay into f_{CP}, there can be CP violation in the interference between the decay without and with mixing, $B^0 \to f_{CP}$ and $B^0 \to \bar{B}^0 \to f_{CP}$.

As an example, consider the general time-dependent asymmetry

$$\begin{aligned} a_f(t) &\equiv \frac{\Gamma(\bar{B}^0 \to f) - \Gamma(B^0 \to \bar{f})}{\Gamma(\bar{B}^0 \to f) + \Gamma(B^0 \to \bar{f})} \\ &= -\frac{(1 - |\lambda_f|^2)\cos(\Delta m t) - 2\mathrm{Im}(\lambda_f)\sin(\Delta m t)}{(1 + |\lambda_f|^2)\cosh(\Delta\Gamma t/2) - 2\mathrm{Re}(\lambda_f)\sinh(\Delta\Gamma t/2)} + \mathcal{O}\left(\left|\frac{\Gamma_{12}}{M_{12}}\right|\sin\phi\right)\,. \end{aligned} \tag{10.31}$$

$a_f(t) \neq 0$ for all three types of CP violation. In some cases we have $|q/p| \approx 1$, $|\bar{A}_f/A_f| \approx 1$. The standard example is $B \to J/\psi K_S$. The dominant decay amplitudes are shown in Fig. 9. Kaon mixing is dominated by the amplitude shown in Fig. 10. B-meson mixing is dominated

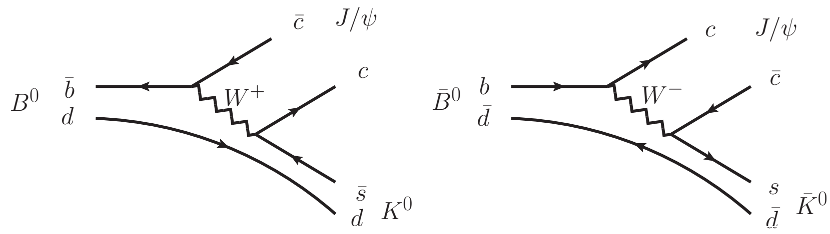


Figure 9: *Dominant decay amplitudes for $B \to J/\psi K_S$.*

by the amplitude shown in Fig. 11. Therefore, we have

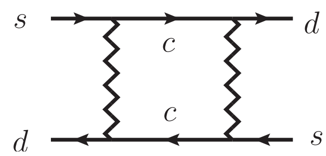


Figure 10: *Dominating box diagrams for K-meson mixing.*

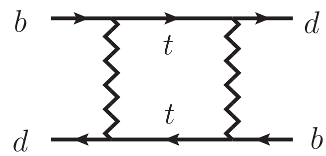


Figure 11: *Dominating box diagrams for B-meson mixing.*

$$\begin{aligned}\lambda_{\psi K_S} &= \frac{q}{p}\frac{\bar{A}_{\psi K_S}}{=} - \left(\frac{V_{tb}^* V_{td}}{V_{tb} V_{td}^*}\right)\left(\frac{V_{cb} V_{cs}^*}{V_{cb}^* V_{cs}}\right)\left(\frac{V_{cs} V_{cd}^*}{V_{cs}^* V_{cd}}\right) \\ &= -\left(\frac{V_{cd} V_{cb}^*}{V_{td} V_{tb}^*}\right)^* \Big/ \left(\frac{V_{cd} V_{cb}^*}{V_{td} V_{tb}^*}\right) = -e^{-2i\beta}\,.\end{aligned} \tag{10.32}$$

For $|q/p| \approx 1$, $|\bar{A}_f/A_f| \approx 1$ (here, only one ampitude contributes!) we have

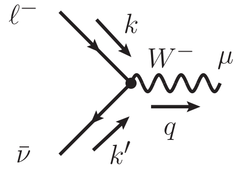
$$a_f(t) = \frac{\mathrm{Im}(\lambda_f)\sin(\Delta m t)}{\cosh(\Delta\Gamma t/2) - \mathrm{Re}(\lambda_f)\sinh(\Delta\Gamma t/2)} \approx \mathrm{Im}(\lambda_f)\sin(\Delta m t)\,, \tag{10.33}$$

where in the last line we used $\Delta\Gamma t \approx 0$. One can extract the angle β by measuring $a_{\psi K_S}(t)$.

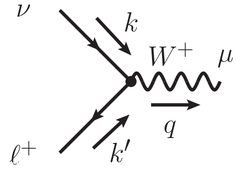
10.4 CP violation in the neutral kaon system*

10.5 Weak effective Hamiltonian

Let's calculate the decay rate for $\mu^- \to e^- \bar{\nu}_e \nu_\mu$. We can read off the necessary Feynman rules from Eq. (9.4) and Eq. (9.31):

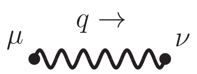


$$: \ \tfrac{ig}{\sqrt{2}}\gamma^\mu P_L (2\pi)^4 \delta^4(k + k' - q)\,, \tag{10.34}$$



$$: \ \tfrac{ig}{\sqrt{2}}\gamma^\mu P_L (2\pi)^4 \delta^4(k + k' - q)\,. \tag{10.35}$$

The W propagator in 't Hooft-Feynman gauge is



$$: \frac{1}{(2\pi)^4}\frac{-i\eta_{\mu\nu}}{q^2 - M_W^2 + i\epsilon}\,. \tag{10.36}$$

Using this, we can calculate the decay amplitude (see Fig. 12). Performing the momentum

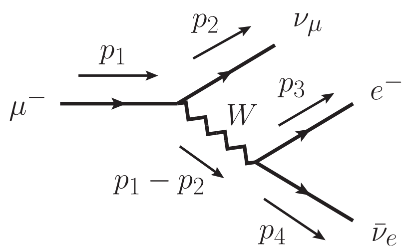


Figure 12: *Leading order contribution to muon decay.*

integration gives

$$-2\pi i\mathcal{M} = \frac{(2\pi)^8}{(2\pi)^{10}}\left(\frac{ig}{\sqrt{2}}\right)^2 (-i)\frac{\bar{u}(\boldsymbol{p}_2, s_2)\gamma^\mu P_L u(\boldsymbol{p}_1, s_1)\bar{u}(\boldsymbol{p}_3, s_3)\gamma_\mu P_L v(\boldsymbol{p}_4, s_4)}{(p_1 - p_2)^2 - M_W^2}\,. \tag{10.37}$$

We have $m_e, m_\mu \ll M_W$, and hence also $(p_1 - p_2)^2 \ll M_W^2$. In excellent approximation, therefore,

$$\begin{aligned} -2\pi i\mathcal{M} &= -\frac{ig^2}{2(2\pi)^2}\frac{\bar{u}(\boldsymbol{p}_2, s_2)\gamma^\mu P_L u(\boldsymbol{p}_1, s_1)\bar{u}(\boldsymbol{p}_3, s_3)\gamma_\mu P_L v(\boldsymbol{p}_4, s_4)}{M_W^2} \\ &\equiv -i\frac{4G_F}{\sqrt{2}(2\pi)^2}\bar{u}(\boldsymbol{p}_2, s_2)\gamma^\mu P_L u(\boldsymbol{p}_1, s_1)\bar{u}(\boldsymbol{p}_3, s_3)\gamma_\mu P_L v(\boldsymbol{p}_4, s_4)\,, \end{aligned} \tag{10.38}$$

with the *Fermi constant*

$$\frac{G_F}{\sqrt{2}} \equiv \frac{g^2}{8M_W^2}\,. \tag{10.39}$$

Using the master formula (3.22) we can calculate the decay rate.

Note that

$$G_F = \frac{\sqrt{2}g^2}{8M_W^2} = \frac{4\sqrt{2}g^2}{8g^2v^2} = \frac{1}{\sqrt{2}v^2}\,, \tag{10.40}$$

i.e. one can measure the Higgs vev via muon decay.

We can obtain the amplitude Eq. (10.38) directly using the "effective" Lagrangian

$$\mathcal{L} = -\frac{4G_F}{\sqrt{2}}(\bar\nu_\mu\gamma^\mu P_L\mu)(\bar e\gamma_\mu P_L\nu_e)\,. \tag{10.41}$$

This Lagrangian is not renormalizable. It represents a low-energy limit of the SM.

11 Chiral Lagrangian*

11.1 The non-linear sigma model and effective field theory

We start by slightly reformulating the linear sigma model. Recall the original Lagrangian Eq. (8.10):

$$\mathcal{L} = \tfrac{1}{2}\sum_n \partial_\mu\phi_n\partial^\mu\phi_n + \frac{\mu^2}{2}\sum_n \phi_n\phi_n - \frac{\lambda}{4}\Big(\sum_n \phi_n\phi_n\Big)^2\,, \tag{11.1}$$

invariant under the group $O(N)$ of real rotations of the fields. We have seen that if $\mu^2 < 0$, the minimum of the potential is at $\phi = 0$, which is invariant under $O(N)$ rotations. If $\mu^2 > 0$, however, the minimum is at $\phi(x) = \bar\phi$, with

$$\sum_n \bar\phi_n^2 = \frac{\mu^2}{\lambda}\,. \tag{11.2}$$

We will now only discuss the case $N = 4$, and call the four field components $\phi = (\boldsymbol{\pi}, \sigma)$. (Note that for $\mu^2 < 0$, nothing distinguishes the four components.) Now we use the following very useful relation for the Pauli matrices σ^a (exercise!)

$$\sigma^a\sigma^b = \delta^{ab} + i\epsilon^{abc}\sigma^c\,, \tag{11.3}$$

and rewrite the four-component field vector ϕ as a matrix

$$\Sigma \equiv \sigma\mathbb{1} + i\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\,, \tag{11.4}$$

such that

$$\sigma^2 + \boldsymbol{\pi}^2 = \tfrac{1}{2}\mathrm{Tr}\big(\Sigma^\dagger\Sigma\big)\,. \tag{11.5}$$

The Lagrangian (8.10) can the be written as

$$\mathcal{L} = \tfrac{1}{4}\mathrm{Tr}\big(\partial_\mu\Sigma^\dagger\partial^\mu\Sigma\big) + \tfrac{\mu^2}{4}\mathrm{Tr}\big(\Sigma^\dagger\Sigma\big) - \tfrac{\lambda}{16}\mathrm{Tr}\big(\Sigma^\dagger\Sigma\big)^2\,. \tag{11.6}$$

In this form, it is obvious that the Lagrangian has a $SU(2)_L \times SU(2)_R$ symmetry,

$$\Sigma \to \Sigma' = U_L \Sigma U_R^\dagger \,, \tag{11.7}$$

with $U_L \in SU(2)_L$ and $U_R \in SU(2)_R$. As before, for $\mu^2 > 0$ we introduce shifted fields, now denoted by

$$\tilde{\sigma} \equiv \sigma - v \,, \tag{11.8}$$

where $v = \sqrt{\mu^2/\lambda}$, and the Lagrangian becomes

$$\mathcal{L} = \tfrac{1}{2}\Big[\partial_\mu\tilde{\sigma}\partial^\mu\tilde{\sigma} - 2\mu^2\tilde{\sigma}^2\Big] + \tfrac{1}{2}\partial_\mu\boldsymbol{\pi}\cdot\partial^\mu\boldsymbol{\pi} - \lambda v\tilde{\sigma}\big(\tilde{\sigma}^2 + \boldsymbol{\pi}^2\big) - \frac{\lambda}{4}\Big[\big(\tilde{\sigma}^2 + \boldsymbol{\pi}^2\big)^2 - v^4\Big] \,. \tag{11.9}$$

The problem with Lagrangians like Eq. (8.10) or (11.9) is that, in order to calculate scattering amplitudes, we need to include Feynman diagrams to all orders (unless λ happens to be small). However, we can rewrite the Lagrangian (8.10) in a form that allows for an expansion in derivatives. The general strategy is as follows:

- Perform a *local* symmetry trsnaformation to eliminate all NGB (Nambu-Goldstone boson) fields.
- NGB reappear as parameters of symmetry transformation.
- Due to *global* invariance, no dependence on constant NGB fields can remain; every terms that involves NGB fields must contain at least one derivative of the NGB fields.
- Derivatives translate into energy-momentum; this allows for an expansion for small energies / momenta.

As an example, we write $\phi(x)$ in Eq. (8.10) as a rotation of $(0, 0, 0, \sigma)$:

$$\phi_n(x) = R_{n4}\sigma(x) \,, \tag{11.10}$$

where $R(x) \in O(4)$, i.e. $R^T(x)R(x) = \mathbb{1}$. It follows that

$$\sigma(x) = \sqrt{\sum_n \phi_n^2(x)} \,, \tag{11.11}$$

and Eq. (8.10) becomes

$$\mathcal{L} = \frac{1}{2}\sum_{n=1}^{4}\big(R_{n4}\partial_\mu\sigma + \sigma\partial_\mu R_{n4}\big)^2 + \frac{1}{2}\mu^2\sigma^2 - \frac{\lambda}{4}\sigma^4 \,. \tag{11.12}$$

The orthogonality of R implies

$$\sum_n R_{n4}^2 = 1 \,, \qquad \sum_n R_{n4}\partial_\mu R_{n4} = \frac{1}{2}\partial_\mu \sum_n R_{n4}^2 = 0 \,, \tag{11.13}$$

so the Lagrangian simplifies to

$$\mathcal{L} = \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}\sigma^2\sum_{n=1}^{4}\partial_\mu R_{n4}\partial^\mu R_{n4} + \frac{1}{2}\mu^2\sigma^2 - \frac{\lambda}{4}\sigma^4\,. \tag{11.14}$$

For $\mu^2 > 0$, σ has a vev $\langle\sigma\rangle_0 = \mu/\sqrt{\lambda}$, as before.

The NGB fields can be chosen as R_{a4}, with $a = 1, 2, 3$, while R_{44} is given by orthogonality. There are several other representations, related to Eq. (11.14) by a non-linear field redefinition. A general theorem of QFT tells us that on-shell S-matrix elements are invariant under non-linear transformations of the fields in the Lagrangian. An important representation, the so-called "exponential representation", is obtained from Eq. (11.4) via the redefinitions

$$\Sigma = \sigma + i\boldsymbol{\sigma}\cdot\boldsymbol{\pi} = (v + S)U\,, \tag{11.15}$$

where

$$U = \exp(i\boldsymbol{\sigma}\cdot\boldsymbol{\pi}'/v)\,. \tag{11.16}$$

An easy calculation yields

$$\mathcal{L} = \frac{1}{2}\Big[\partial_\mu S\partial^\mu S - 2\mu^2 S^2\Big] + \frac{(v+S)^2}{4}\mathrm{Tr}\big(\partial_\mu U\partial^\mu U\big) - \lambda v S^3 - \lambda 4 S^4\,. \tag{11.17}$$

By construction, U transforms like Σ: for $L \in SU(2)_L$, $R \in SU(2)_R$,

$$U \to LUR^\dagger\,. \tag{11.18}$$

The Lagrangian is still invariant under $SU(2)_L \times SU(2)_R$, but the symmetry is realized non-linearly on the NGB fields, while S is invariant. (Note that the vectorial subgroup, $L = R$, is realized linearly: $\delta\boldsymbol{\pi} = \boldsymbol{\theta}\times\boldsymbol{\pi}$.)

Since S is a scalar under $SU(2)_L \times SU(2)_R$, we can "discard" it without impairing the invariance under $SU(2)_L \times SU(2)_R$ of Eq. (11.17). (Formally, we can take the limit $\mu^2 \to \infty$, $\lambda \to \infty$ with $\mu/\sqrt{\lambda}$ fixed.) The Lagrangian is the simply

$$\mathcal{L} = \frac{v^2}{4}\mathrm{Tr}\big(\partial_\mu U\partial^\mu U\big)\,. \tag{11.19}$$

In fact, we did not have to start with the linear sigma model – the idea of *effective field theory* (EFT) is is to write the most general Lagrangian with a given symmetry (breaking) pattern. The "UV theory" is not unique. Comparison with the symmetries of the QCD Lagrangian at low energies shows that the pions of QCD can be interpreted as the NGB of the spontaneously broken chiral $SU(2)_L \times SU(2)_R$ symmetrt of QCD.

We could add to the Lagrangian (11.19) terms of higher order in the pion interactions, such as $\mathrm{Tr}\big(\partial_\mu U\partial^\mu U\big)^2$ etc. As long as we include *all* interactions allowed by symmetry, the theory will be renormalizable (in the generalized sense that all divergences can be absorbed into an (infinite) number of counterterms).

11.2 Spontaneously broken approximate symmetries

Example: QCD with two light quark flavors. Leads to low-mass spin-zero particles, "pseudo Goldstone bosons".

Assume we can split the potential as

$$V(\phi) = V_0(\phi) + V_1(\phi)\,, \tag{11.20}$$

where V_0 is invariant under some symmetry transformation:

$$\sum_{nm} \frac{\partial V_0(\phi)}{\partial \phi_n} \left(t^a\right)_{nm} \phi_m = 0\,. \tag{11.21}$$

V_1 is a small correction due to explicit symmetry breaking, i.e. the minimum of the potential gets shifted from ϕ_0 (the minimum of V_0) to $\bar{\phi} = \phi_0 + \phi_1$, where ϕ_1 is small compared to ϕ_0 (of first order in the small symmetry-breaking terms). The minimization condition is then

$$\left.\frac{\partial V(\phi)}{\partial \phi_n}\right|_{\phi=\phi_0+\phi_1} = 0\,. \tag{11.22}$$

The zeroth-order term vanishes by construction, so the first-order term must also vanish:

$$\sum_m \left.\frac{\partial V_0(\phi)}{\partial \phi_n \partial \phi_m}\right|_{\phi=\phi_0} \phi_{1,m} + \left.\frac{\partial V_1(\phi)}{\partial \phi_n}\right|_{\phi=\phi_0} = 0\,. \tag{11.23}$$

The condition (8.9) reads here

$$\sum_{n,\ell} \left.\frac{\partial^2 V_0(\phi)}{\partial \phi_n \partial \phi_m}\right|_{\phi=\phi_0} \left(t^a\right)_{n\ell} \phi_{0,\ell} = 0\,. \tag{11.24}$$

Multiplying Eq. (11.23) with $(t^a \phi_0)_n$, summing over n, and using Eq. (11.24) gives

$$\sum_n \left(t^a \phi_0\right)_n \left.\frac{\partial V_1(\phi)}{\partial \phi_n}\right|_{\phi=\phi_0} = 0\,. \tag{11.25}$$

This is called the *vacuum alignment condition.* ϕ_0 can always be chosen such that Eq. (11.25) is satisfied (to linear order). The vacuum alignment condition forces the direction of spontaneous symmetry breaking into alignment with the explicit symmetry breaking terms in the Lagrangian.

Example: breaking of $O(N) \to O(N-1)$. In the absence of explicit breaking, we can use the underlying $O(N)$ symmetry to chose which $O(N-1)$ subgroup is left unbroken. If we add a perturbation $\propto \sum_n u_n \phi_n$, then the Lagrangian is invariant under the *specific* $O(N-1)$ that leaves u invariant. Without Eq. (11.25), we would expect only $O(N-2)$ invariance after SSB (both u and ϕ_0). Eq. (11.25), however, implies that $u \propto \phi_0$ (exercise!), so the unbroken symmetry is $O(N-1)$.

11.3 Pions and kaons as Goldstone bosons

Recall that the QCD Lagrangian for two massless quark flavors has a global $SU(2)_L \times SU(2)_R$ symmetry. What is observed in nature? Pions ($\pi^+ \sim [u\bar{d}]$, $\pi^0 \sim [u\bar{u} - d\bar{d}]$), nucleons ($p \sim [uud]$, $n \sim [udd]$) respect an approximate isospin symmetry (interchange of up and down quarks):

$$\frac{\delta m_\pi}{m_\pi} \sim \frac{139.57 - 134.9766}{139.57} \approx 3\,\%\,, \tag{11.26}$$

$$\frac{\delta m_N}{m_N} \approx 0.1\,\%\,. \tag{11.27}$$

However, a "parity doubling" is not observed. We assume that $SU(2)_L \times SU(2)_R$ is spontaneously broken to $SU(2)_V$ isospin, and interpret the pions as Goldstone bosons. Same works for u, d, s and $SU(3)$ (plus kaons, η). How can we test this hypothesis?

The Lagrangian (11.19) has the correct symmetry structure (including SSB), and can easily be generalized to the case $SU(3)_L \times SU(3)_R$. What about explicit symmetry breaking (QED, quark masses)? Let's define $q = (u, d, s)$ and $\mathcal{M} = \text{diag}(m_u, m_d, m_s)$. Then the QCD quark mass Lagrangian is

$$\mathcal{L} = -\bar{q}\mathcal{M}q = -\Big(\bar{q}_R\mathcal{M}q_L + \bar{q}_L\mathcal{M}q_R\Big)\,. \tag{11.28}$$

Now we "invent" a static external 3×3 matrix field χ, and replace Eq. (11.28) by

$$\mathcal{L} = -\bar{q}\mathcal{M}q = -\Big(\bar{q}_R\chi^\dagger q_L + \bar{q}_L\chi q_R\Big)\,, \tag{11.29}$$

which is the same as Eq. (11.28) for $\chi = \chi^\dagger = \mathcal{M}$. However, Eq. (11.29) is formally invariant under $SU(3)_L \times SU(3)_R$ if χ transforms as

$$\chi \to V_L \chi V_R^\dagger \tag{11.30}$$

for $V_R \in SU(3)_R$, $V_L \in SU(3)_L$. Thus we can construct the most general chiral effective Lagrangian with the same explicit symmetry breaking pattern as in QCD, out of the fields U and χ (this is sometimes called the "spurion method", and χ is called a "spurion field"). We will use the parameterization

$$U(x) = \exp\left(\frac{i\sqrt{2}}{f}\Pi\right), \tag{11.31}$$

with

$$\Pi \equiv \sum_a \lambda^a \pi^a = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2\eta_8}{\sqrt{6}}\,. \end{pmatrix} \tag{11.32}$$

Here, λ^a are the Gell-Mann matrices, normalized as $\text{Tr}(\lambda^a\lambda^b) = \delta^{ab}$, and f is a constant with dimension of mass (to be determined later; essentially, f is the pion decay constant).

For instance, the interaction $\mathrm{Tr}(U^\dagger\chi + U\chi^\dagger)$ is invariant under $SU(3)_L \times SU(3)_R$ and parity

$$U(\boldsymbol{x},t) \leftrightarrow U^\dagger(-\boldsymbol{x},t)\,, \qquad \chi \leftrightarrow \chi^\dagger\,. \tag{11.33}$$

Making the replacement $\chi = \chi^\dagger = \mathcal{M}$, we find the most general pion Lagrangian with a single $\mathcal{M}$ insertion

$$\mathcal{L} = \frac{f^2}{4}\mathrm{Tr}\big(\partial_\mu U^\dagger \partial^\mu U\big) + \frac{B_0 f^2}{2}\mathrm{Tr}\big[\mathcal{M}(U+U^\dagger)\big]\,, \tag{11.34}$$

where B_0 is another low-energy constant. Neither f nor B_0 can be determined from chiral symmetry.

By expanding the mass term in Eq. (11.34), we can read off the entries of the pion mass matrix:

$$M^2_{\pi^\pm} = M^2_{\pi^0} = B_0(m_d + m_u)\,, \tag{11.35}$$

$$M^2_{K^\pm} = B_0(m_s + m_u)\,, \tag{11.36}$$

$$M^2_{K^0} = B_0(m_s + m_d)\,, \tag{11.37}$$

$$M^2_{\eta} = B_0\frac{4m_s + m_d + m_u}{3}\,, \tag{11.38}$$

and a π-η mixing term

$$M^2_{\pi\eta} = B_0\frac{m_u - m_d}{\sqrt{3}}\,. \tag{11.39}$$

From $M_{K^+} \approx 493.7\,\mathrm{MeV}$, $M_{K^0} \approx 497.6\,\mathrm{MeV}$, $M_{\pi^+} \approx 139.57\,\mathrm{MeV}$, $M_{\pi^0} \approx 134.98\,\mathrm{MeV}$ we see that $m_s \gg m_d, m_u$.

To discuss the effects of the up- and down-quark masses, we should include the effects of QED (i.e. replace $\partial_\mu \to D_\mu$). The electromagnetic current is

$$J^\mu = ie\bar{q}\gamma^\mu Q q\,, \tag{11.40}$$

where $Q = \mathrm{diag}(2/3, -1/3, -1/3)$. The vector and axial-vector currents are

$$V^{a\mu} = i\bar{q}\gamma^\mu\lambda^a q\,, \tag{11.41}$$

$$A^{a\mu} = i\bar{q}\gamma^\mu\gamma_5\lambda^a q\,, \tag{11.42}$$

with charges

$$T^a \equiv \int d^3x V^{a0}\,, \tag{11.43}$$

$$X^a \equiv \int d^3x A^{a0}\,, \tag{11.44}$$

which act as the QM generators of the symmetry. Their commutators with the electromagnetic current are[9]

$$[T^a, J^\mu] = -ie\bar{q}\gamma^\mu[Q,\lambda^a]q\,, \tag{11.45}$$

[9] It is straightforward to show that $[T^a, q] = -\lambda^a q$ and $[X^a, q] = -\lambda^a\gamma_5 q$.

$$[X^a, J^\mu] = -ie\bar{q}\gamma^\mu\gamma_5[Q, \lambda^a]q\,. \tag{11.46}$$

We see that J^μ commutes with X^3, X^6, X^7, X^8 and T^3, T^6, T^7, T^8, so the electromagnetic part of the effective Lagrangian is invariant under this $SU(2)\times SU(2)\times U(1)\times U(1)$ subgroup. For zero quark masses, QED effects give no masses to the associated neutral π^0, K^0, $\bar{K}^0$, and η bosons. Similarly, for zero quark masses, K^+ and π^+ transform as a doublet under the unbroken $SU(2)$ subgroup generated by

$$T^6 \sim \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\,, \quad T^7 \sim \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}\,, \quad \sqrt{3}T^8 - T^3 \sim \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{11.47}$$

("u-spin"), so the electromagnetic corrections Δ to the K^+ and π^+ masses are equal. Therefore we have

$$\begin{aligned} M^2_{\pi^\pm} &= B_0(m_d + m_u) + \Delta\,, \\ M^2_{\pi^0} &= B_0(m_d + m_u)\,, \\ M^2_{K^\pm} &= B_0(m_s + m_u) + \Delta\,, \\ M^2_{K^0} &= M^2_{\bar{K}^0} = B_0(m_s + m_d)\,, \\ M^2_\eta &= B_0\frac{4m_s + m_d + m_u}{3}\,. \end{aligned} \tag{11.48}$$

As a consequence, there is one linear relation on the five boson masses [Gell-Mann 1961, Okubo 1962]:

$$3M^2_\eta + 2M^2_{\pi^\pm} - M^2_{\pi^0} = 2M^2_{K^\pm} + 2M^2_{K^0}\,. \tag{11.49}$$

This gives $M_\eta = 566\,\text{MeV}$. The difference to the experimental value is due to mixing among the π^0, η, and η'.

Eq. (11.48) yields the quark mass ratios

$$\frac{m_d}{m_s} = \frac{M^2_{K^0} + M^2_{\pi^\pm} - M^2_{K^\pm}}{M^2_{K^0} - M^2_{\pi^\pm} + M^2_{K^\pm}} \tag{11.50}$$

and

$$\frac{m_u}{m_s} = \frac{2M^2_{\pi^0} - M^2_{K^0} - M^2_{\pi^\pm} + M^2_{K^\pm}}{M^2_{K^0} - M^2_{\pi^\pm} + M^2_{K^\pm}}\,. \tag{11.51}$$

Inserting the meson masses we find $m_d/m_s = 0.049$, $m_u/m_s = 0.028$, and thus $m_d/m_u \approx 2$.

11.4 Weak decays in the chiral EFT

We still need to determine the constant f in Eq. (11.31) and Eq. (11.34). It can be extracted from leptonic pion decay, $\pi^- \to \mu\bar{\nu}_\mu$. The effective Lagrangian is

$$\mathcal{L}_{\text{eff}} = -\frac{4G_F}{\sqrt{2}}V_{ud}(\bar{u}\gamma_\rho P_L d)(\bar{\mu}\gamma^\rho P_L \nu_\mu) + \text{h.c.}\,. \tag{11.52}$$

(There are no QCD corrections to this Lagrangian due to current conservation. Similar effective Lagrangians can be written for hadronic decays, in which case QCD effects can be quite large.)

Lorentz invariance (together with conservation of parity) dictates that the pion-to-vacuum matrix element takes the form

$$\langle 0|\bar{u}\gamma^\rho\gamma_5 d|\pi^-(\boldsymbol{p})\rangle \equiv i\sqrt{2}f_\pi p^\rho\,, \tag{11.53}$$

where f_π is the *pion decay constant*. The $\pi^- \to \mu\bar{\nu}_\mu$ matrix element of Eq. (11.52) then becomes

$$\mathcal{M} = \frac{i}{2}\frac{4G_F}{\sqrt{2}}V_{ud}\bar{u}(\boldsymbol{p}_\mu)\gamma^\rho P_L v(\boldsymbol{p}_{\nu_\mu})\langle 0|\bar{u}\gamma_\rho d|\pi^-(\boldsymbol{p})\rangle = 2iG_F V_{ud} f_\pi \bar{u}(\boldsymbol{p}_\mu)\not{p}_\pi P_L v(\boldsymbol{p}_{\nu_\mu})\,, \tag{11.54}$$

leading to the decay rate (exercise!)

$$\Gamma(\pi^- \to \mu\bar{\nu}_\mu) = \frac{G_F^2}{4\pi}f_\pi^2 m_\mu^2 m_\pi |V_{ud}|^2\left[1 - \frac{m_\mu^2}{m_\pi^2}\right]. \tag{11.55}$$

From $\Gamma^{\text{exp.}}(\pi^- \to \mu\bar{\nu}_\mu) = 3.84 \times 10^7/s$ we find

$$f_\pi = 92\,\text{MeV}\,. \tag{11.56}$$

Now let us calculate the same decay using the chiral Lagrangian. For this, we need to implement the weak interaction. The weak Lagrangian contains a term

$$\mathcal{L} \supset V_{ud}\frac{g}{\sqrt{2}}[\bar{u}_L\gamma^\mu d_L]W_\mu^+ + \text{h.c.} = [\bar{q}_L\gamma^\mu \ell_\mu q_L]W_\mu^+ + \text{h.c.}\,, \tag{11.57}$$

where $\ell_\mu \equiv gTW_\mu^+/\sqrt{2}$ with

$$T = \begin{pmatrix} 0 & V_{ud} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{11.58}$$

(At low energies, it is better to think of W_μ^+ as an external source rather than a dynamical field.) The last term in Eq. (11.57) is invariant under the chiral $SU(3)_L \times SU(3)_R$ if ℓ_μ transforms as $\ell_\mu \to V_L \ell_\mu V_L^\dagger$. Using the spurion method, we construct the corresponding terms in the chiral Lagrangian:

$$\mathcal{L} = \frac{f^2}{4}\text{Tr}\big\{(\partial_\mu U - i\ell_\mu U)^\dagger(\partial^\mu U - i\ell^\mu U)\big\} + \ldots = -\frac{g}{4}f\big\{V_{ud}\partial^\mu\pi^- W_\mu^+ + \text{h.c.}\big\} + \ldots\,. \tag{11.59}$$

Now we can compare the pion-vacuum matrix elements of Eq. (11.57) and Eq. (11.59). Using Eq. (11.53), we find

$$\begin{aligned} &V_{ud}\frac{g}{\sqrt{2}}\langle 0|\bar{u}_L\gamma^\rho d_L|\pi^-(\boldsymbol{p})\rangle = -V_{ud}\frac{g}{2\sqrt{2}}\langle 0|\bar{u}\gamma^\rho\gamma_5 d|\pi^-(\boldsymbol{p})\rangle = -V_{ud}\frac{g}{2}if_\pi p^\rho \\ &= -V_{ud}\frac{g}{4}f\langle 0|\partial^\rho\pi^-|\pi^-(\boldsymbol{p})\rangle = -V_{ud}\frac{g}{4}ifp^\rho\,. \end{aligned} \tag{11.60}$$

We see that $f = 2f_\pi$.

12 Tests of the standard model*

12.1 Custodial symmetry

Question: Is the relation $M_W^2 = \cos^2\theta_w M_Z^2$ "exact"? Consider the Higgs Lagrangian

$$\mathcal{L}_{\text{Higgs}} = |D_\mu\phi|^2 - V(\phi)\,, \tag{12.1}$$

where

$$V(\phi) = -\mu^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2\,, \tag{12.2}$$

and

$$D_\mu\phi = \big(\partial_\mu - ig\tau^a W_\mu^a - i\frac{g'}{2}B_\mu\big)\phi\,. \tag{12.3}$$

$\mathcal{L}_{\text{Higgs}}$ is invariant under local $SU(2)\times U(1)$ transformations, and has an additional approximate global symmetry. To see this, we write the components of the Higgs field as

$$\phi = \begin{pmatrix}\phi^+\\ \phi^0\end{pmatrix}\,. \tag{12.4}$$

Then also

$$\phi^c \equiv \epsilon\phi^* = \begin{pmatrix}\phi^{0*}\\ -\phi^-\end{pmatrix}\,, \tag{12.5}$$

with $(\phi^+)^* = \phi^-$, is a $SU(2)$ doublet. Now we define the "bi-doublet"

$$\Phi = \frac{1}{\sqrt{2}}(\phi^c,\phi) = \frac{1}{\sqrt{2}}\begin{pmatrix}\phi^{0*} & \phi^+\\ -\phi^- & \phi^0\end{pmatrix}\,, \tag{12.6}$$

and write the Higgs Lagrangian

$$\mathcal{L}_{\text{Higgs}} = \text{Tr}\{(D_\mu\Phi)^\dagger D^\mu\Phi\} - V(\Phi)\,, \tag{12.7}$$

with

$$V(\Phi) = -\mu^2\text{Tr}\{\Phi^\dagger\Phi\} + \lambda\big(\text{Tr}\{\Phi^\dagger\Phi\}\big)^2 \tag{12.8}$$

and

$$D_\mu\Phi = \partial_\mu\Phi - ig\tau^a W_\mu^a\Phi + i\frac{g'}{2}B_\mu\Phi\sigma_3\,. \tag{12.9}$$

For instance, consider the term

$$\begin{aligned}\text{Tr}\{\Phi^\dagger\Phi\} &= \frac{1}{2}\text{Tr}\Big\{\begin{pmatrix}\phi^0 & -\phi^+\\ \phi^- & \phi^{0*}\end{pmatrix}\begin{pmatrix}\phi^{0*} & \phi^+\\ -\phi^- & \phi^0\end{pmatrix}\Big\}\\ &= \frac{1}{2}\text{Tr}\Big\{\begin{pmatrix}\phi^0\phi^{0*}+\phi^+\phi^- & \dots\\ \dots & \phi^0\phi^{0*}+\phi^+\phi^-\end{pmatrix}\Big\} = \phi^0\phi^{0*}+\phi^+\phi^- = \phi^\dagger\phi\,.\end{aligned} \tag{12.10}$$

The action of $SU(2)\times U(1)$ on the bi-doublet is

$$SU(2)_L: \quad \Phi \to L\Phi\,, \tag{12.11}$$

$$U(1)_Y: \quad \Phi \to \Phi e^{-\frac{i}{2}\sigma_3\theta}\,. \tag{12.12}$$

Now consider the limiting case $g' \to 0$. The covariant derivative is now

$$D_\mu \Phi = \partial_\mu \Phi - ig\tau^a W_\mu^a \Phi\,. \tag{12.13}$$

In this limit, $\mathcal{L}_{\text{Higgs}}$ has a global $SU(2)_R$ symmetry,

$$SU(2)_R: \quad \Phi \to \Phi R^\dagger\,. \tag{12.14}$$

For instance, we have

$$\text{Tr}\{(D_\mu\Phi)^\dagger D^\mu\Phi\} \to \text{Tr}\{R(D_\mu\Phi)^\dagger D^\mu\Phi R^\dagger\} = \text{Tr}\{(D_\mu\Phi)^\dagger D^\mu\Phi\}\,. \tag{12.15}$$

So, for $g' \to 0$, $\mathcal{L}_{\text{Higgs}}$ has the global symmetry $SU(2)_L \times SU(2)_R$,

$$SU(2)_L \times SU(2)_R: \quad \Phi \to L\Phi R^\dagger\,. \tag{12.16}$$

Eq. (12.12) implies that $U(1)_Y \subset SU(2)_L \times SU(2)_R$.

The vacuum state of the bi-doublet is

$$\langle\Phi\rangle = \frac{1}{\sqrt{2}}(\phi^c, \phi) = \frac{1}{\sqrt{2}}\begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix}\,. \tag{12.17}$$

This vacuum breaks both $SU(2)_L$ and $SU(2)_R$,

$$L\langle\Phi\rangle \neq \langle\Phi\rangle\,, \qquad \langle\Phi\rangle R^\dagger \neq \langle\Phi\rangle\,. \tag{12.18}$$

However, $SU(2)_{L+R}$ (i.e. $L = R$) leaves the vacuum invariant:

$$L\langle\Phi\rangle L^\dagger = \langle\Phi\rangle\,. \tag{12.19}$$

The breaking pattern is, thus,

$$SU(2)_L \times SU(2)_R \to SU(2)_{L+R}\,. \tag{12.20}$$

The group $SU(2)_R$ (or, sometimes, $SU(2)_L \times SU(2)_R$) is called *custodial symmetry* (from Latin "custos", protector). The three massive gauge bosons associated with the breaking pattern (12.20) are W^+, W^-, Z^0, with masses

$$M_W^2 = \frac{1}{4}g^2v^2\,, \qquad M_Z^2 = \frac{1}{4}(g^2 + (g')^2)v^2\,, \tag{12.21}$$

and so

$$\frac{M_W^2}{M_Z^2} = \frac{g^2}{g^2 + (g')^2} = \cos^2\theta_w\,, \tag{12.22}$$

or

$$\rho \equiv \frac{M_W^2}{\cos^2\theta_w M_Z^2} = 1\,. \tag{12.23}$$

Figure 13: *Corrections to the ρ parameter due to Higgs exchange*

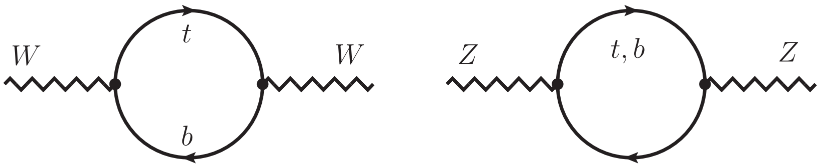


Figure 14: *Corrections to the ρ parameter due to bottom- and top-quark exchange*

In the limit $g' \to 0$, we have $M_W = M_Z$. The custodial symmetry "protects" the ρ parameter from large radiative corrections, in the sense that these corrections must be proportional to powers of g'.

For instance, the one-loop diagrams in Fig. 13 give

$$\hat{\rho} = 1 - \frac{11 G_F M_Z^2 \sin^2\theta_w}{24\sqrt{2}\pi^2} \log\frac{M_h^2}{M_Z^2} \,. \tag{12.24}$$

Recall that $\sin^2\theta_w \to 0$ for $g' \to 0$.

Similarly, the diagrams in Fig. 14 give the contribution

$$\hat{\rho} = 1 + \frac{3G_F}{8\sqrt{2}\pi^2}\left(m_t^2 + m_b^2 - 2\frac{m_t^2 m_b^2}{m_t^2 - m_b^2}\log\frac{m_t^2}{m_b^2}\right) \to 1 \tag{12.25}$$

for $m_t = m_b$. For $y_t = y_b$ we can write the Yukawa Lagrangian as

$$\mathcal{L}_Y \supset -y\bar{Q}_L^3 \phi d_R^3 - y\bar{Q}_L^3 \phi^c u_R^3 + \text{h.c.} = -y\bar{Q}_L^3 \Phi Q_R^3 + \text{h.c.}\,, \tag{12.26}$$

with $y = y_t = y_b$ and $Q_R = (t_R, b_R)^T$; this is invariant under $SU(2)_L \times SU(2)_R$.

12.2 SMEFT

12.3 Electroweak precision tests

13 Neutrino physics*

14 Particle physics and cosmology*

A Spherical harmonics

This section is based on the discussion in Ref. [8]. We set $\hbar = 1$ throughout.

The components of the angular momentum operator $\boldsymbol{L}$ are

$$L_i = -i \sum_{jk} \epsilon_{ijk} x_j \frac{\partial}{\partial k_k} \,. \tag{A.1}$$

Each of the components commutes with $r \equiv \sqrt{\boldsymbol{x}_1^2 + \boldsymbol{x}_2^2 + \boldsymbol{x}_3^2}$, so they can only on the direction of $\boldsymbol{x}$, not its length. In fact, we can show that in polar coordinates

$$x_1 = r \sin\theta \cos\phi \,, \qquad x_2 = r \sin\theta \sin\phi \,, \qquad x_1 = r \cos\theta \,, \tag{A.2}$$

$\boldsymbol{L}$ acts only on the angles. For instance, we have

$$\begin{aligned} \frac{\partial}{\partial \phi} &= \sum_i \frac{\partial x_i}{\partial \phi} \frac{\partial}{\partial x_i} = -r \sin\theta \sin\phi \frac{\partial}{\partial x_1} + r \sin\theta \cos\phi \frac{\partial}{\partial x_2} \\ &= -x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2} = i L_3 \,. \end{aligned} \tag{A.3}$$

To obtain the other two components, we calculate

$$\frac{\partial}{\partial \theta} = \sum_i \frac{\partial x_i}{\partial \theta} \frac{\partial}{\partial x_i} = r \cos\theta \cos\phi \frac{\partial}{\partial x_1} + r \cos\theta \sin\phi \frac{\partial}{\partial x_2} - r \sin\theta \frac{\partial}{\partial x_3} \,, \tag{A.4}$$

and notice that

$$\begin{aligned} \cot\theta \cos\phi \frac{\partial}{\partial \phi} + \sin\phi \frac{\partial}{\partial \theta} &= r \cos\theta \frac{\partial}{\partial x_2} - r \sin\theta \sin\phi \frac{\partial}{\partial x_3} \\ = x_3 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_3} &= -i L_1 \,, \end{aligned} \tag{A.5}$$

and

$$\begin{aligned} \cot\theta \sin\phi \frac{\partial}{\partial \phi} - \cos\phi \frac{\partial}{\partial \theta} &= -r \cos\theta \frac{\partial}{\partial x_1} + r \sin\theta \cos\phi \frac{\partial}{\partial x_3} \\ = -x_3 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_3} &= -i L_2 \,, \end{aligned} \tag{A.6}$$

so we have in total

$$L_1 = i \left(\sin\phi \frac{\partial}{\partial \theta} + \cot\theta \cos\phi \frac{\partial}{\partial \phi} \right) , \tag{A.7}$$

$$L_2 = i \left(-\cos\phi \frac{\partial}{\partial \theta} + \cot\theta \sin\phi \frac{\partial}{\partial \phi} \right) , \tag{A.8}$$

$$L_3 = -i \frac{\partial}{\partial \phi} \,, \tag{A.9}$$

and hence

$$\boldsymbol{L}^2 = -\left[\frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial \phi^2} \right] . \tag{A.10}$$

To find the spectrum of $\boldsymbol{L}^2$, we assume that we can expand the solutions about the origin in terms of homogeneous polynomial in x_1, x_2, x_3 of degree ℓ. In polar coordinates such polynomials have the form

$$\psi(\boldsymbol{x}) = r^\ell Y(\theta, \phi)\,, \tag{A.11}$$

where Y is a homogeneous polynomial of order ℓ in the unit vector

$$\hat{\boldsymbol{x}} \equiv \boldsymbol{x}/r = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)\,. \tag{A.12}$$

As $\boldsymbol{L}^2$ does not act on r, we will drop the factor r^ℓ from now on. From the usual rules of angular momentum in quantum mechanics, we know that the eigenvalues of angular momentum satisfy

$$\boldsymbol{L}^2 Y_\ell^m = \ell(\ell+1) Y_\ell^m \tag{A.13}$$

and

$$L_3 Y_\ell^m = m Y_\ell^m\,, \tag{A.14}$$

with m a positive or negative integer between $-\ell \leq m \leq \ell$, and ℓ a positive integer or half-integer. We can write $Y_\ell^m(\theta,\phi)$ as a homogeneous polynomial in

$$\hat{x}_\pm \equiv \hat{x}_1 \pm i\hat{x}_2 = \sin\theta e^{\pm i\phi}\,, \qquad \hat{x}_3 = \cos\theta\,. \tag{A.15}$$

The functions $Y_\ell^m(\theta,\phi)$ are known as the *spherical harmonics* and can be written in the form

$$Y_\ell^m(\theta,\phi) \propto P_\ell^{|m|}(\theta) e^{im\phi}\,. \tag{A.16}$$

The *associated Legendre polynomials* $P_\ell^{|m|}$ satisfy the differential equation

$$-\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP_\ell^{|m|}}{d\theta}\right) + \frac{m^2}{\sin^2\theta}P_\ell^{|m|} = \ell(\ell+1)P_\ell^{|m|}\,, \tag{A.17}$$

see Eq. (A.10).

We can now simply find the spherical harmonics by writing all homogeneous polynomials in $\hat{\boldsymbol{x}}$ that satisfy the Eqs. (A.13) and (A.14). For instance, Y_0^0 is just a constant. Y_1^m must contain one power of $\hat{x}_\pm$ or $\hat{x}_3$; Eq. (A.14) shows that Y_1^{-1}, Y_1^0, and Y_1^1 must be proportional to x_-, x_3, and x_+, respectively. We find, for $\ell = 0, 1$,

$$Y_0^0 = \sqrt{\frac{1}{4\pi}}\,, \tag{A.18}$$

$$Y_1^{-1} = \sqrt{\frac{3}{8\pi}}\big(\hat{x}_1 - i\hat{x}_2\big) = \sqrt{\frac{3}{8\pi}}\sin\theta e^{-i\phi}\,, \tag{A.19}$$

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\hat{x}_3 = \sqrt{\frac{3}{4\pi}}\cos\theta e^{i\phi}\,, \tag{A.20}$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\big(\hat{x}_1 + i\hat{x}_2\big) = -\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi}\,. \tag{A.21}$$

The prefactors are chosen such that the spherical harmonics are normalized,

$$\int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\phi |Y_\ell^m(\theta,\phi)| = 1\,. \tag{A.22}$$

Being eigenfunctions of the Hermitian operators $\boldsymbol{L}^2$ and L_3 with different eigenvalues, the spherical harmonics are orthogonal. Since they are homogeneous polynomials of order ℓ in $\hat{\boldsymbol{x}}$, they change sign under space-inversion $\hat{\boldsymbol{x}} \to -\hat{\boldsymbol{x}}$ according to

$$Y_\ell^m(\pi-\theta,\pi+\phi) = (-1)^\ell Y_\ell^m(\theta,\phi)\,. \tag{A.23}$$

The spherical harmonics for $m = 0$ are frequently written in terms of the *Legendre polynomials* $P_\ell(\cos\theta)$ as

$$Y_\ell^0(\theta) = \sqrt{\frac{2\ell+1}{4\pi}} P_\ell(\cos\theta)\,. \tag{A.24}$$

(The Legendre polynomials are functions of $\cos\theta$ because they can only be functions of $\hat{x}_3 = \cos\theta$ and $\hat{x}_+\hat{x}_- = 1-\cos^2\theta$. They are normalized as $P_\ell(1) = 1$.)

B Clebsch-Gordan coefficients

The Clebsch-Gordan coefficients appear in the addition of angular momentum in quantum mechanics:

$$|jm\rangle = \sum_{m'm''} C_{j'j''}(jm; m'm'')|j'm'\rangle|j''m''\rangle\,. \tag{B.1}$$

They are non-zero for $|j'-j''| \le j \le j'+j''$ and $m = m'+m''$, and are determined up to a normalization and phase convention. Eq. (B.1) can be inverted to give

$$|j'm'\rangle|j''m''\rangle = \sum_{jm} C_{j'j''}(jm; m'm'')|jm\rangle\,. \tag{B.2}$$

They satisfy the following completeness relations:

$$\sum_{jm} C_{j'j''}(jm; m'm'')C_{j'j''}(jm; \bar{m}'\bar{m}'') = \delta_{m'\bar{m}'}\delta_{m''\bar{m}''}\,, \tag{B.3}$$

$$\sum_{m'm''} C_{j'j''}(jm; m'm'')C_{j'j''}(\bar{j}\bar{m}; m'm'') = \delta_{j\bar{j}}\delta_{m\bar{m}}\,, \tag{B.4}$$

$$\sum_{mm''} C_{j'j''}(jm; m'm'')C_{\bar{j}'j''}(jm; \bar{m}'m'') = \frac{2j+1}{2j'+1}\delta_{j'\bar{j}'}\delta_{m'\bar{m}'}\,. \tag{B.5}$$

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