Electricity and Magnetism

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Abstract

"It doesn't matter what we cover. It matters what you discover."
[Attributed to Viktor Weisskopf, theoretical physicist, 1908 – 2002]

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0 Introductory Remarks

These lecture notes are based on hand-written notes provided by Philip Argyres; the originals can be found on his course homepage.

1 Electrostatics

1.1 The electric field

The force on a charge Q at point $\boldsymbol{r}$ due to other (static) electric charges is

$$\boldsymbol{F} = Q\,\boldsymbol{E}(\boldsymbol{r})\,. \tag{1.1}$$

The charge Q is measured in units of *Coulomb (C)*, the *electric field* $\boldsymbol{E}$ in units of *Newton per Coulomb (N/C)*. (Recall that $N = kg\,m/s^2$). The electric field due to a point charge q, at position $\boldsymbol{r}'$, is

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{q}{4\pi\epsilon_0}\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\,. \tag{1.2}$$

The *permitivity of free space* is $\epsilon_0 = 8.85\times 10^{-12}C^2/Nm^2$ (effectively, this is the definition of the unit "Coulomb").

The electric field due to many charges q_i is the sum of electric fields due to each individual charge:

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\sum_i q_i\frac{\boldsymbol{r}-\boldsymbol{r}_i'}{|\boldsymbol{r}-\boldsymbol{r}_i'|^3}\,. \tag{1.3}$$

For continuous charge distributions, it is frequently useful to define *charge densities*, which we denote as follows:

		density	unit
0-dimensional	point charge	q	C
1-dimensional	line charge	λ	C/m
2-dimensional	surface charge	σ	C/m^2
3-dimensional	volume charge	ρ	C/m^3

The sum over point charges then becomes an integral. For line charge densities:

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int_C d\ell'\,\lambda(\boldsymbol{r}')\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\,, \tag{1.4}$$

where $d\ell' = |d\boldsymbol{\ell}'|$. For surface charge densities:

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0} \int\limits_S da'\, \sigma(\boldsymbol{r}') \frac{\boldsymbol{r} - \boldsymbol{r}'}{|\boldsymbol{r} - \boldsymbol{r}'|^3}\,, \tag{1.5}$$

where $d\ell' = |d\boldsymbol{\ell}'|$. For volume charge densities:

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0} \int\limits_V d^3r'\, \rho(\boldsymbol{r}') \frac{\boldsymbol{r} - \boldsymbol{r}'}{|\boldsymbol{r} - \boldsymbol{r}'|^3}\,. \tag{1.6}$$

1.2 Gauss' law

We calculate the divergence of a static electric field due to a volume charge density:

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \boldsymbol{E} &= \frac{1}{4\pi\epsilon_0} \boldsymbol{\nabla} \cdot \int\limits_V d^3r'\, \rho(\boldsymbol{r}') \frac{\boldsymbol{r} - \boldsymbol{r}'}{|\boldsymbol{r} - \boldsymbol{r}'|^3} \\
&= \frac{1}{4\pi\epsilon_0} \int\limits_V d^3r'\, \rho(\boldsymbol{r}') \boldsymbol{\nabla} \cdot \frac{\boldsymbol{r} - \boldsymbol{r}'}{|\boldsymbol{r} - \boldsymbol{r}'|^3} \\
&= \frac{1}{4\pi\epsilon_0} \int\limits_V d^3r'\, \rho(\boldsymbol{r}') 4\pi\delta(\boldsymbol{r}) = \frac{\rho(\boldsymbol{r})}{\epsilon_0}\,.
\end{aligned} \tag{1.7}$$

This is the (differential form of) Gauss' law. Integrating this equation, we obtain

$$\int\limits_V d^3r\, \boldsymbol{\nabla} \cdot \boldsymbol{E} = \int\limits_V d^3r\, \frac{\rho(\boldsymbol{r})}{\epsilon_0} = \frac{Q_{\text{encl.}}}{\epsilon_0}\,, \tag{1.8}$$

where $Q_{\text{encl.}}$ is the total (net) charge enclosed within the integration volume. Using the divergence theorem, this is equivalent to

$$\oint\limits_{S=\partial V} d\boldsymbol{a} \cdot \boldsymbol{E} = \frac{Q_{\text{encl.}}}{\epsilon_0}\,, \tag{1.9}$$

This is the integral form of Gauss' law.

Next, we calculate the curl of the static electric field.

$$\begin{aligned}
\boldsymbol{\nabla} \times \boldsymbol{E}(\boldsymbol{r}) &= \frac{1}{4\pi\epsilon_0} \boldsymbol{\nabla} \cdot \int\limits_V d^3r'\, \rho(\boldsymbol{r}') \frac{\boldsymbol{r} - \boldsymbol{r}'}{|\boldsymbol{r} - \boldsymbol{r}'|^3} \\
&= \frac{1}{4\pi\epsilon_0} \int\limits_V d^3r'\, \rho(\boldsymbol{r}') \boldsymbol{\nabla} \times \frac{\boldsymbol{r} - \boldsymbol{r}'}{|\boldsymbol{r} - \boldsymbol{r}'|^3} = 0\,.
\end{aligned} \tag{1.10}$$

It follows that

$$\oint\limits_C d\boldsymbol{\ell} \cdot \boldsymbol{E} = 0 \tag{1.11}$$

for any closed contour C. A consequence is that there exists a scalar function $V(\boldsymbol{r})$ (the *electric potential*) such that

$$\boldsymbol{E} = -\boldsymbol{\nabla} \cdot V(\boldsymbol{r})\,. \tag{1.12}$$

In summary, COULOMB's law

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0} \int\limits_V d^3r'\,\rho(\boldsymbol{r}') \frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\,. \tag{1.13}$$

is equivalent[1]

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \frac{\rho(\boldsymbol{r})}{\epsilon_0}\,, \tag{1.14}$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} = 0\,. \tag{1.15}$$

These are the equations for electrostatics.

If a problem has enough symmetry, we can use the integral form of GAUSS' law to quickly derive the electric field. For instance, assume a spherically symmetric volume charge density, $\rho(\boldsymbol{r}) = \rho(r)$, i.e. the charge density only depends on the distance from the origin. We choose a "Gaussian surface" S_r, a sphere of radius r centered on the origin. The sperical symmetry implies that $\boldsymbol{E}(\boldsymbol{r}) = E(r)\hat{\boldsymbol{r}}$. Why? The answer cannot[2] on how we rotate the coordinate system around the origin, so must be independent of the angles θ, φ and θ', φ'. We have

$$\oint\limits_{S_r} d\boldsymbol{a} \cdot \boldsymbol{E}(\boldsymbol{r}) = \int\limits_0^\pi \sin\theta d\theta \int\limits_0^{2\pi} d\varphi\, r^2 \hat{\boldsymbol{r}} \cdot E(r)\hat{\boldsymbol{r}} = 4\pi r^2 E(r)\,, \tag{1.16}$$

and

$$Q_{\text{encl.}} = \int\limits_{V_r} d^3r' \rho(r') = \int\limits_{r'<r} (r')^2 dr' \int\limits_0^\pi \sin\theta d\theta \int\limits_0^{2\pi} d\varphi\, \rho(r') = 4\pi \int\limits_{r'<r} dr'\,(r')^2 \rho(r')\,. \tag{1.17}$$

Using GAUSS' law, we find (for spherically symmetric charge distributions)

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{\hat{\boldsymbol{r}}}{\epsilon_0 r^2} \int\limits_0^r dr'\,(r')^2 \rho(r')\,. \tag{1.18}$$

Next, we assume a cylindrical symmetry (i.e. the charge density depends only on the distance to the z axis, and is translationally invariant in the z direction). Then we choose the

[1] So far, we have only shown that the electrostatic equations follow from COULOMB's law, and not vice versa.

[2] This is actually a subtle point: just because an equation has a symmetry, it does not necessarily follow that the solution has that symmetry.

Gaussian surface $S_{s,z}$ to be a cylinder of height z and radius s centered on the z axis. The cylindrical symmetry implies that $\boldsymbol{E}(\boldsymbol{r}) = E(s)\hat{\boldsymbol{s}}$. We have

$$\begin{aligned}
\oint\limits_{S_{s,z}} d\boldsymbol{a}' \cdot \boldsymbol{E}(\boldsymbol{r}') &= \int\limits_0^s s'ds' \int\limits_0^{2\pi} d\varphi\, \hat{\boldsymbol{z}} \cdot E(s')\hat{\boldsymbol{s}}' \\
&\quad + \int\limits_0^z dz' \int\limits_0^{2\pi} d\varphi\, s\, \hat{\boldsymbol{s}} \cdot E(s)\hat{\boldsymbol{s}} \\
&\quad + \int\limits_0^s s'ds' \int\limits_0^{2\pi} d\varphi\, (-\hat{\boldsymbol{z}}) \cdot E(s')\hat{\boldsymbol{s}}' \\
&= 0 + 2\pi\, z\, s\, E(s)\,,
\end{aligned} \tag{1.19}$$

and

$$Q_{\text{encl.}} = \int\limits_{V_{s,z}} d^3r'\rho(r') = \int\limits_0^z dz' \int\limits_0^s s'ds' \int\limits_0^{2\pi} d\varphi\, \rho(s') = 2\pi z \int\limits_0^s s'ds'\rho(s')\,. \tag{1.20}$$

Using Gauss' law, we find (for cylindrically symmetric charge distributions)

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{\hat{\boldsymbol{s}}}{\epsilon_0 s} \int\limits_0^s ds' s'\rho(r')\,. \tag{1.21}$$

As our last example, we consider planar symmetry (i.e. the charge density is invariant under translations only in the x- or y-directions, $\rho(\boldsymbol{r}) = \rho(z)$). We choose the Gaussian surface $S_{z,A}$ to be a cylinder of arbitrary cross sectional shape A and height z along the z axis. The planar symmetry implies that $\boldsymbol{E}(\boldsymbol{r}) = E(z)\hat{\boldsymbol{z}}$. We have

$$\begin{aligned}
\oint\limits_{S_{z,A}} d\boldsymbol{a}' \cdot \boldsymbol{E}(\boldsymbol{r}') &= \int\limits_A dxdy\, \hat{\boldsymbol{z}} \cdot E(z)\hat{\boldsymbol{z}} \\
&\quad + \int\limits_0^z dz' \int d\ell'\, \hat{\boldsymbol{n}}' \cdot E(z')\hat{\boldsymbol{z}} \\
&\quad + \int\limits_A dxdy\, (-\hat{\boldsymbol{z}}) \cdot E(z)\hat{\boldsymbol{z}}\,.
\end{aligned} \tag{1.22}$$

Here, $\hat{\boldsymbol{n}}'$ is perpendicular to the side of the cylinder, so lies in the x-y plane: $\hat{\boldsymbol{n}}' = a\hat{\boldsymbol{x}} + b\hat{\boldsymbol{y}}$. It follows that $\hat{\boldsymbol{n}}' \times \hat{\boldsymbol{z}} = 0$. Also, note that

$$\int\limits_A dxdy = A\,. \tag{1.23}$$

Therefore,

$$\oint_{S_{z,A}} d\boldsymbol{a}' \cdot \boldsymbol{E}(\boldsymbol{r}') = A\,E(z) - A\,E(0)\,. \tag{1.24}$$

Moreover,

$$Q_{\text{encl.}} = \int_{V_{z,A}} d^3r'\rho(r') = \int_A dxdy \int_0^z dz'\rho(z') = A\int_0^z dz'\rho(z')\,. \tag{1.25}$$

Using Gauss' law, we find (for planar symmetric charge distributions)

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{\hat{\boldsymbol{z}}}{\epsilon_0}\int_0^z dz'\rho(z') + \boldsymbol{E}(0)\,. \tag{1.26}$$

1.3 Discontinuity of the electric field at a surface charge

This is a useful result that follows most easily from the integrated forms of the electrostatic laws.

Consider a surface S with a surface charge density $\sigma(\boldsymbol{r})$. We will show that at a point $\boldsymbol{r} \in S$, the difference between the electric fields just above S, $\boldsymbol{E}_+(\boldsymbol{r})$, and just below S, $\boldsymbol{E}_-(\boldsymbol{r})$, is

$$\boldsymbol{E}_+(\boldsymbol{r}) - \boldsymbol{E}_-(\boldsymbol{r}) = \frac{\sigma(\boldsymbol{r})}{\epsilon_0}\hat{\boldsymbol{n}}\,, \tag{1.27}$$

where $\hat{\boldsymbol{n}}$ is the unit normal vector to S pointing to the "above" ("+") side. Even though there is no symmetry in this problem, we can still use Gauss' law in integrated form by choosing the Gaussian surface to be an arbitrary pill box with surface S' and height ε centered on a point $\boldsymbol{r} \in S$. Gauss' law implies that

$$\oint_{S'} d\boldsymbol{a}' \cdot \boldsymbol{E}(\boldsymbol{r}') = \frac{1}{\epsilon_0}Q_{\text{encl.}}\,. \tag{1.28}$$

We have

$$\oint_{S'} d\boldsymbol{a}' \cdot \boldsymbol{E}(\boldsymbol{r}') = A\hat{\boldsymbol{n}} \cdot \boldsymbol{E}_+(\boldsymbol{r}) - A\hat{\boldsymbol{n}} \cdot \boldsymbol{E}_-(\boldsymbol{r}) + \mathcal{O}(\varepsilon)\,, \tag{1.29}$$

$$\frac{1}{\epsilon_0}Q_{\text{encl.}} = \frac{1}{\epsilon_0}A\sigma(\boldsymbol{r}) + \mathcal{O}(\varepsilon)\,. \tag{1.30}$$

Taking the limit $\varepsilon \to 0$, we find

$$\hat{\boldsymbol{n}} \cdot (\boldsymbol{E}_+(\boldsymbol{r}) - \boldsymbol{E}_-(\boldsymbol{r})) = \frac{1}{\epsilon_0}\sigma(\boldsymbol{r})\,. \tag{1.31}$$

The integrated form of the curl equation is

$$\oint_C d\boldsymbol{\ell}' \cdot \boldsymbol{E}(\boldsymbol{r}') = 0\,. \tag{1.32}$$

Choose C to be a small rectangular loop along $\hat{\boldsymbol{n}}$ and a tangential direction $\hat{\boldsymbol{t}}$ to S. Then

$$\oint_C d\boldsymbol{\ell}' \cdot \boldsymbol{E}(\boldsymbol{r}') = L\hat{\boldsymbol{t}} \cdot \boldsymbol{E}_+(\boldsymbol{r}) - L\hat{\boldsymbol{t}} \cdot \boldsymbol{E}_-(\boldsymbol{r}) + \mathcal{O}(\varepsilon)\,. \tag{1.33}$$

where L is the length of the loop (along the surface), and ε its height. Taking the limit $\varepsilon \to 0$, we find

$$\hat{\boldsymbol{t}} \cdot (\boldsymbol{E}_+(\boldsymbol{r}) - \boldsymbol{E}_-(\boldsymbol{r})) = 0\,, \tag{1.34}$$

for any $\hat{\boldsymbol{t}}$ tangent to S.

This proves Eq. (1.27) since contracting Eq. (1.27) with $\hat{\boldsymbol{n}}$ gives Eq. (1.31), and contracting Eq. (1.27) with $\hat{\boldsymbol{t}}$ gives Eq. (1.34) (using $\hat{\boldsymbol{t}} \cdot \hat{\boldsymbol{n}} = 0$), and since $\{\hat{\boldsymbol{n}}, \hat{\boldsymbol{t}}, \hat{\boldsymbol{t}}'\}$ form a basis at $\boldsymbol{r}$ for any two linearly independent tangent vectors $\hat{\boldsymbol{t}}, \hat{\boldsymbol{t}}'$.

1.4 The electric potential

We have seen that the electrostatic equations imply the existence of an electrostatic potential. Integrating Eq. (1.12) we obtain

$$V(\boldsymbol{r}) - V(\boldsymbol{r}_0) = -\int_{\boldsymbol{r}_0}^{\boldsymbol{r}} \boldsymbol{E} \cdot d\boldsymbol{\ell}\,, \tag{1.35}$$

independent of the path from $\boldsymbol{r}_0$ to $\boldsymbol{r}$. We can add any constant to V: if $V'(\boldsymbol{r}) = V(\boldsymbol{r}) + C$, then $\boldsymbol{E} = -\boldsymbol{\nabla} V' = -\boldsymbol{\nabla} V$. This constant is arbitrary and unobservable. E.g., we can choose it so that $V(\boldsymbol{r}_0) = 0$ at any point $\boldsymbol{r}_0$. If the charge density does not extend to infinity, then a common convention is to choose

$$V(\infty) = 0\,. \tag{1.36}$$

The potential has the unit V (Volts). We have N/C $= [\boldsymbol{E}] = [\boldsymbol{\nabla} V] =$ V/m, so V = Nm/C = J/C (Joule per Coulomb). We can rewrite Eq. (1.14) in terms of the potential:

$$\frac{\rho(\boldsymbol{r})}{\epsilon_0} = \boldsymbol{\nabla} \cdot \boldsymbol{E} = -\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} V) = -\nabla^2 V\,, \tag{1.37}$$

so

$$\nabla^2 V = -\frac{\rho(\boldsymbol{r})}{\epsilon_0}\,. \tag{1.38}$$

This is the Poisson equation (called the Laplace equation if the right side is zero). ∇^2 is called the *Laplacian*. This is a key equation of physics.

We can also rewrite Coulomb's equation in terms of the potential:

$$\begin{aligned} V(\boldsymbol{r}) &= -\int_\infty^{\boldsymbol{r}} d\boldsymbol{\ell}'' \cdot \boldsymbol{E}(\boldsymbol{r}'') = -\frac{1}{4\pi\epsilon_0}\int_\infty^{\boldsymbol{r}} d\boldsymbol{\ell}'' \cdot \left(\int d^3r'\, \rho(\boldsymbol{r}') \frac{\boldsymbol{r} - \boldsymbol{r}'}{|\boldsymbol{r} - \boldsymbol{r}'|^3}\right) \\ &= -\frac{1}{4\pi\epsilon_0}\int d^3r'\, \rho(\boldsymbol{r}') \int_\infty^{\boldsymbol{r}} \frac{d\boldsymbol{\ell}'' \cdot (\boldsymbol{r} - \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3}\,. \end{aligned} \tag{1.39}$$

To do the integral over the $d\boldsymbol{\ell}''$, we can choose any path and any coordinate system:

$$\int_{\infty}^{\boldsymbol{r}} \frac{d\boldsymbol{\ell}'' \cdot (\boldsymbol{r} - \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3} = \int_{r=\infty}^{r=|\boldsymbol{r}-\boldsymbol{r}'|} . \tag{1.40}$$

So

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} . \tag{1.41}$$

Note: since we chose $V(\infty) = 0$ in deriving it, this formula does not apply if $\rho(\boldsymbol{r})$ extends to infinity (it diverges).

1.5 Work and energy in electrostatics

The work required to move a charge Q in a fixed $\boldsymbol{E}$ field is

$$W = \int_{\boldsymbol{r}_0}^{\boldsymbol{r}} d\boldsymbol{\ell} \cdot \boldsymbol{F} = \int_{\boldsymbol{r}_0}^{\boldsymbol{r}} d\boldsymbol{\ell} \cdot (-Q\boldsymbol{E}) = Q\big(V(\boldsymbol{r}) - V(\boldsymbol{r}_0)\big) , \tag{1.42}$$

independent of the path. This is the energy change in moving Q from $\boldsymbol{r}_0$ to $\boldsymbol{r}$. If $\boldsymbol{r}_0 = \infty$ and we set $V(\infty) = 0$, then we have

$$W = QV(\boldsymbol{r}) . \tag{1.43}$$

This is the potential energy of the charge Q in the given electric field $\boldsymbol{E}$.

What is the energy of a point charge distribution? Say we have charges q_i at positions $\boldsymbol{r}_i$, $i = 1, \ldots, n$. Can can imagine assmbling them by bringing them together from infinity one at a time. The work to bring in q_1 is $W_1 = 0$, since $\boldsymbol{E} = 0$. For the second charge,

$$W_2 = q_2 V_1(\boldsymbol{r}_2) = \frac{q_2}{4\pi\epsilon_0} \frac{q_1}{|\boldsymbol{r}_2 - \boldsymbol{r}_1|} . \tag{1.44}$$

For the third charge,

$$W_3 = q_2\big(V_1(\boldsymbol{r}_3) + V_2(\boldsymbol{r}_3)\big) = \frac{q_3}{4\pi\epsilon_0} \left(\frac{q_1}{|\boldsymbol{r}_3 - \boldsymbol{r}_1|} + \frac{q_2}{|\boldsymbol{r}_3 - \boldsymbol{r}_2|} \right) . \tag{1.45}$$

For n charges, we find

$$\begin{aligned} W = \sum_{i=1}^{n} W_i &= \frac{1}{4\pi\epsilon_0} \left\{ 0 + \left(\frac{q_2 q_1}{|\boldsymbol{r}_2 - \boldsymbol{r}_1|} \right) + \left(\frac{q_3 q_1}{|\boldsymbol{r}_3 - \boldsymbol{r}_1|} + \frac{q_3 q_2}{|\boldsymbol{r}_3 - \boldsymbol{r}_2|} \right) + \ldots \right\} \\ &= \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \sum_{j=i+1}^{n} \frac{q_i q_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|} = \frac{1}{2} \sum_{i=1}^{n} q_i \left(\sum_{j=1, j\neq i}^{n} \frac{1}{4\pi\epsilon_0} \frac{q_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|} \right) . \end{aligned} \tag{1.46}$$

The term in brackets in the last expression is the potential at $\boldsymbol{r}_i$ due to all *other* charges $j \neq i$, so

$$W = \frac{1}{2} \sum_{i=1}^{n} q_i V(\boldsymbol{r}_i) . \tag{1.47}$$

If we included the potential due to q_i at $\boldsymbol{r}_i$, we would get

$$\frac{1}{4\pi\epsilon_0}\frac{q_i}{|\boldsymbol{r}_i - \boldsymbol{r}_i|} = \frac{1}{4\pi\epsilon_0}\frac{q_i}{0} = \infty\,. \tag{1.48}$$

This is the infinite energy required to assemble a single point charge. We assume that point charges are given to us "pre-assembled" by nature, so we do not need to do this work.

The energy of a continuous charge distribution can be obtained by taking the continuous limit of Eq. (1.47):

$$W = \frac{1}{2}\int d^3r\,\rho(\boldsymbol{r})V(\boldsymbol{r})\,. \tag{1.49}$$

Here, $V(\boldsymbol{r})$ is the potential due to all charges $\rho(\boldsymbol{r}')$ for $\boldsymbol{r}' \neq \boldsymbol{r}$, but

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int\limits_{\boldsymbol{r}'\neq\boldsymbol{r}} d^3r\,\frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} = \frac{1}{4\pi\epsilon_0}\int d^3r\,\frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\,, \tag{1.50}$$

the last integration including the point $\boldsymbol{r}$. They are the same since

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{1}{4\pi\epsilon_0}\int\limits_{B_\varepsilon(\boldsymbol{r})} d^3r\,\frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} &= \frac{\rho(\boldsymbol{r})}{4\pi\epsilon_0}\lim_{\varepsilon\to 0}\int_0^\varepsilon r^2dr\int_0^\pi \sin\theta d\theta\int_0^{2\pi} d\phi\,\frac{1}{r} \\ &= \frac{\rho(\boldsymbol{r})}{\epsilon_0}\lim_{\varepsilon\to 0}\int_0^\varepsilon r dr = \frac{\rho(\boldsymbol{r})}{\epsilon_0}\lim_{\varepsilon\to 0}\frac{\varepsilon^2}{2} = 0\,,\end{aligned} \tag{1.51}$$

where $B_\varepsilon(\boldsymbol{r})$ is the ball centered on $\boldsymbol{r}$ with radius ε. So for continuous charge distributions we make no mistake by including the point $\boldsymbol{r}$ in the integration.

We can rewrite Eq. (1.49) using Eq. (1.14),

$$\begin{aligned}W &= \frac{\epsilon_0}{2}\int d^3r\,(\boldsymbol{\nabla}\cdot\boldsymbol{E})V = \frac{\epsilon_0}{2}\int d^3r\,\left[\boldsymbol{\nabla}\cdot(\boldsymbol{E}V) - \boldsymbol{E}\cdot\boldsymbol{\nabla}V\right] \\ &= \frac{\epsilon_0}{2}\oint\limits_S d\boldsymbol{a}\cdot(\boldsymbol{E}V) + \frac{\epsilon_0}{2}\int d^3r\,E^2\,.\end{aligned} \tag{1.52}$$

Here, S is an arbitrarily large sphere. Assuming $V(\infty) = 0$ (ρ has compact support), the first integral in the last line vanishes and we have simply

$$W = \frac{\epsilon_0}{2}\int d^3r\,E^2\,. \tag{1.53}$$

The electric field carries energy density $\frac{\epsilon_0}{2}E^2$. This interpretation becomes more convincing in electrodynamics when radiation is considered.

Note that $\boldsymbol{E} = \boldsymbol{E}_1 + \boldsymbol{E}_2$ does not imply $W = W_1 + W_2$:

$$\begin{aligned}W &= \frac{\epsilon_0}{2}\int d^3r\,E^2 = \frac{\epsilon_0}{2}\int d^3r\,(\boldsymbol{E}_1+\boldsymbol{E}_2)^2 \\ &= \frac{\epsilon_0}{2}\int d^3r\,(E_1^2 + 2\boldsymbol{E}_1\cdot\boldsymbol{E}_2 + E_2^2) = W_1 + W_2 + \epsilon_0\int d^3r\,\boldsymbol{E}_1\cdot\boldsymbol{E}_2\,.\end{aligned} \tag{1.54}$$

1.6 Conductors

All[3] *normal matter*[4] in nature is made up of positively and negatively charged particles.

Insulators are materials (collections of normal matter held together somehow) inside of which charges do not move in repsonse to an arbitrarily small applied electric field.

Conductors are materials inside of which charges are free to move. The mobile charges are called the *conduction charges*. The main example are solid metals, in which the conductions charges are electrons, though there are other examples as well. “Free to move” means that if you apply an arbitrarily small electric field, charges will move in response in the interior of the material. The charges are confined to the material. This assumes nothing about how quickly the charges respond (measured by the *resistivity* of the material). In practice, if the applied electric field is large enough, a couple of things can happen: (i) the conductor can run out of mobile charges, in which case it becomes an insulator, and/or (ii) the conduction charges can be pulled out of the material entirely. (Likewise, if the applied electric field in an insulator is large enough, some charges in its interior can be pulled free to become conduction charges.)

In *static* (time-independent) situations, the charges must move inside the conductor in such a way as to make the total electric field zero in the interior of the conductor. If this wasn't the case, then there would be a non-zero electric field inside the conductor, and charges would move, so it wouldn't be static. It follows that $\rho = 0$ inside (static) conductors, because of Eq. (1.14). The only net charge can be on the surface of the conductor. It follows that the potential is constant inside the conductor, because

$$V(\boldsymbol{b}) - V(\boldsymbol{a}) = -\int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{E} \cdot d\boldsymbol{\ell} = 0 \tag{1.55}$$

as long as there is a path connecting $\boldsymbol{a}$ and $\boldsymbol{b}$ inside the conductor. If the conductor is in a region (volume) R with surface $S = \partial R$ with unit normal $\hat{\boldsymbol{n}}$, then just outside S the electric field is perpendicular to S, because if the electric field has any component tangent to S, then it will cause conduction charges to move along S inside the conductor to cancel it.

We always assume the net charge of the conductor is zero. Note that since charge move to the surface of the conductor, there will be a separation of positive and negative charges. If the applied electric field is non-uniform, this will mean that there is a *net force* on the conductor. We assume that something (e.g. some insulators) is holding the conductor in place.

Cavities are empty volumes in the interior of the conductor – i.e., completely surrounded by conductor. If there are no charges inside the cavity, then $\boldsymbol{\nabla} \cdot \boldsymbol{E} = 0$ there, so the electric field lines (flux along tiny cylinders) must reach across the cavity. But then $\oint_C \boldsymbol{E} \cdot d\boldsymbol{\ell} > 0$, which is a contradiction. We conclude that the electric field must vanish inside an empty cavity and there are no surface charges on the cavity surface, and the potential is constant inside the cavity.

[3]Except: *neutronium*, found in neutron stars, and *relativistic* matter made up of photons (light, or electromagnetic radiation) and neutrinos

[4]The universe consists of about 5% normal matter, 27% dark matter, and 68% dark energy.

If there *are* charges inside the cavity, then $\nabla \cdot \boldsymbol{E} \neq 0$ inside and the above argument fails: we can get $\oint_C \boldsymbol{E} \cdot d\boldsymbol{\ell} = 0$. We can choose any closed surface Σ inside the conductor and surrounding the cavity. Then

$$\oint_{\Sigma} \boldsymbol{E} \cdot d\boldsymbol{a} = 0 = \frac{1}{\epsilon_0} Q_{\text{encl.}} \,. \tag{1.56}$$

But the enclosed charge is the sum of the surface charge and the charge inside the cavity, so *the total charge induced on the surface of the cavity equals minus the total charge inside the cavity.*

We can use the boundary condition (1.27) to relate the surface charge and the potential. At the surface of the conductor, take $\hat{\boldsymbol{n}}$ to point out of the conductor. Then $\boldsymbol{E}_- = 0$, since the electric field vanishes inside the conductor. Thus, at the surface of a conductor the field outside is

$$\boldsymbol{E} = \frac{\sigma}{\epsilon_0} \hat{\boldsymbol{n}} \,. \tag{1.57}$$

Using $\boldsymbol{E} = -\nabla V$ this implies $\partial V / \partial n = \hat{\boldsymbol{n}} \cdot \nabla V = -\sigma / \epsilon_0$, or

$$\sigma = -\epsilon_0 \frac{\partial V}{\partial n} \,. \tag{1.58}$$

These will be useful formulas for determining the surface charge σ induced on conductors.

Recalling that $\boldsymbol{F} = q\boldsymbol{E}$, we get that the *pressure* (the force per area) on a surface charge is $\boldsymbol{f} = \sigma \boldsymbol{E}$. But since there is a discontinuity in the electric field at a surface charge one has to be a bit careful. The right answer is

$$\boldsymbol{f} = \frac{\sigma}{2} \left(\boldsymbol{E}_+ + \boldsymbol{E}_- \right) , \tag{1.59}$$

i.e., the *average* of the fields above and below the surface. (See Griffiths for the argument.)

For conductor, since $\boldsymbol{E}_- = 0$, we get that the pressure on a conductor is

$$\boldsymbol{f} = \frac{\sigma}{2} \boldsymbol{E} = \frac{\sigma^2}{2\epsilon_0} \hat{\boldsymbol{n}} = \frac{\epsilon_0}{2} E^2 \hat{\boldsymbol{n}} \,. \tag{1.60}$$

This is an outward-pointing electrostatic pressure on the surface of the conductor. The *net force* on a conductor in an electric field (assuming this pressure does not rip the conductor apart) is

$$\boldsymbol{F} = \oint_S d\boldsymbol{a}\, \boldsymbol{f} = \frac{1}{2\epsilon_0} \oint_S d\boldsymbol{a}\, \sigma^2 = \frac{\epsilon_0}{2} \oint_S d\boldsymbol{a}\, E^2 \,. \tag{1.61}$$

To find this, we need to solve for σ or $\boldsymbol{E}$ everywhere at the surface S of the conductor.

Finally, we discuss capacitors. Consider two conductors, given equal and opposite excess charges $\pm Q$. Then each capacitor will be at some constant potential, call them $V_\pm$. The *capacitance* is defined as

$$C \equiv \frac{Q}{V} \,, \tag{1.62}$$

where $V = V_+ - V_-$ (only potential differences have any meaning, anyway), and with units F = C/V ("Farad" = "Coulomb per Volt"). Capacitance is an interesting quantity because it is independent of Q – if you double Q, you double V – so it depends only on the *geometry* of the two conductors.

The electrostatic energy stored in a capacitor holding charge Q can be computed by building up Q gradually by moving a small charge dq from one conductor to the other. The work done in this step is $dW = V\,dq = (q/C)\,dq$. So the total work (energy) is

$$W = \int_0^Q dq\,\frac{q}{C} = \frac{1}{2}\frac{Q^2}{C}\,, \tag{1.63}$$

using that C is constant (independent of q).

1.7 Laplace's equation

The potential satisfies Poisson's equation (1.38). If we know $\rho(\boldsymbol{r})$, a solution is

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int d^3r'\,\frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\,, \tag{1.64}$$

with $V(\infty) = 0$. We want to show that this is the only solution (with $V(\infty) = 0$). The key is to first consider the equation without charges (Laplace's equation):

$$\nabla^2 V(\boldsymbol{r}) = 0\,. \tag{1.65}$$

Then the above solution is simply $V = 0$. We want to show that this is the only solution with $V(\infty) = 0$. We will prove first that $V(\boldsymbol{r})$ is the average of its values over any sphere $S_R(\boldsymbol{r})$ of radius R centered on $\boldsymbol{r}$:

$$V(\boldsymbol{r}) = \frac{1}{4\pi R^2}\oint_{S_R(\boldsymbol{r})} V\,da\,. \tag{1.66}$$

To show this, we choose $\boldsymbol{r}$ to be the origin of a spherical coordinate system (r', θ', ϕ'). Then, by Eq. (1.65),

$$\begin{aligned}
0 &= \int_{B_R} d^3r'\nabla^2 V(\boldsymbol{r}') = \int_{B_R} d\boldsymbol{a}'\cdot\boldsymbol{\nabla}'V(\boldsymbol{r}') = \int_{B_R} da'\hat{\boldsymbol{r}}'\cdot\boldsymbol{\nabla}'V(\boldsymbol{r}') \\
&= \int\int R^2\sin\theta' d\theta' d\phi' \frac{\partial V}{\partial r} V(\boldsymbol{r}')\bigg|_{r'=R} = R^2\frac{\partial}{\partial R}\left(\int\int \sin\theta' d\theta' d\phi' V(R\hat{\boldsymbol{r}}')\right) \\
&= 4\pi R^2\frac{\partial}{\partial R}\left[\frac{R^2\int\int\sin\theta' d\theta' d\phi' V(R\hat{\boldsymbol{r}}')}{4\pi R^2}\right] = 4\pi R^2\frac{\partial}{\partial R}\left[\frac{\oint_{S_R} da' V}{4\pi R^2}\right]
\end{aligned} \tag{1.67}$$

where B_R is a ball with radius R at the origin, $S_R = \partial B_R$, and we used the divergence theorem in the second step. It follows that the expression in square brackets is independent of R. In the limit $R \to 0$, by Taylor expansion, $V(\boldsymbol{r}') = V(0) + \mathcal{O}(R)$, so

$$\lim_{R\to 0} \frac{\oint_{S_R} da' V}{4\pi R^2} = \lim_{R\to 0} \frac{4\pi R^2[V(0) + \mathcal{O}(R)]}{4\pi R^2} = V(0)\,, \tag{1.68}$$

and so

$$\frac{\oint_{S_R} da' V}{4\pi R^2} = V(0)\,. \tag{1.69}$$

This means that $V(\boldsymbol{r})$ can have no local maxima or minima. Therefore, the extreme values of $V(\boldsymbol{r})$ must occur on the boundaries.

Next, we prove the *first uniqueness theorem*: The solution to Laplace's equation in some volume $\mathcal{R}$ is uniquely determind if V is specified on the boundary surface $\mathcal{S} = \partial\mathcal{R}$. Here is the proof: assume that there are two solutions V_1 and V_2 with the same boundary values. Then $\nabla^2 V_1 = \nabla^2 V_2 = 0$, so $\nabla^2(V_1 - V_2) = 0$, and $V_1 - V_2$ satisfies Laplace's equation. So $V_1 - V_2$ cannot have local maxima or minima except on $\mathcal{S}$. But $V_1 - V_2$ vanishes on the surface, so we must have $V_1 = V_2$.

In particular, if $\mathcal{R}$ is all space, then $\mathcal{S}$ the "sphere at infinity", and if $V(\infty) = 0$, then V must vanish everywhere.

Now consider putting in charges $\rho(\boldsymbol{r})$, so the potential satisfies Eq. (1.38). We fix $\rho(\boldsymbol{r})$ and the boundary values $V|_{\mathcal{S}}$. Then V is unique. Here is the proof: assume $V_1 \neq V_2$ are two solutions. Then $V_1 - V_2$ satisfies Laplace's equation, so we can use the first uniqueness theorem to find $V_1 = V_2$.

Now we prove the *second uniqueness theorem*: in a region $\mathcal{R}$ surrounded by conductors with specified *total* charges, Q_i, on each conductor, and with a specified additional fixed charge density $\rho(\boldsymbol{r})$, the electric field is uniquely determined. Here is the proof: suppose there are two solution $\boldsymbol{E}_1$, $\boldsymbol{E}_2$ in $\mathcal{R}$:

$$\boldsymbol{\nabla} \cdot \boldsymbol{E}_1 = \frac{\rho}{\epsilon_0}\,, \qquad \boldsymbol{\nabla} \cdot \boldsymbol{E}_2 = \frac{\rho}{\epsilon_0}\,. \tag{1.70}$$

If $\mathcal{S}_i$ are surfaces enclosing only the charges Q_i, and $\mathcal{S}_0$ a surface enclosing all charges, the Gauss' theorem tells us that

$$\oint_{\mathcal{S}_i} \boldsymbol{E}_1 \cdot d\boldsymbol{a} = \frac{Q_i}{\epsilon_0}\,, \qquad \oint_{\mathcal{S}_i} \boldsymbol{E}_2 \cdot d\boldsymbol{a} = \frac{Q_i}{\epsilon_0}\,, \tag{1.71}$$

$$\oint_{\mathcal{S}_0} \boldsymbol{E}_1 \cdot d\boldsymbol{a} = \frac{Q_{\text{total}}}{\epsilon_0}\,, \qquad \oint_{\mathcal{S}_0} \boldsymbol{E}_2 \cdot d\boldsymbol{a} = \frac{Q_{\text{total}}}{\epsilon_0}\,. \tag{1.72}$$

Consider $\boldsymbol{E} = \boldsymbol{E}_1 - \boldsymbol{E}_2$. Then $\boldsymbol{\nabla} \cdot \boldsymbol{E} = 0$ and $\oint_{\mathcal{S}_i} \boldsymbol{E} \cdot d\boldsymbol{a} = 0$ for all i. On $\mathcal{S}_i$, V_1 and V_2 are constants, so

$$V|_{\mathcal{S}_i} = (V_2 - V_1)|_{\mathcal{S}_i} = V^{(i)} \tag{1.73}$$

is constant. It follows that

$$
\begin{aligned}
0 &= \sum_i V^{(i)} \oint_{\mathcal{S}_i} \boldsymbol{E} \cdot d\boldsymbol{a} = \oint_{\mathcal{S}_0} V\boldsymbol{E} \cdot d\boldsymbol{a} = \int_{\mathcal{R}} d^3r\, \boldsymbol{\nabla} \cdot (V\boldsymbol{E}) \\
&= \int_{\mathcal{R}} d^3r \left(\boldsymbol{\nabla} V \cdot \boldsymbol{E} + V \boldsymbol{\nabla} \cdot \boldsymbol{E}\right) = -\int_{\mathcal{R}} d^3r\, E^2 \,.
\end{aligned} \tag{1.74}
$$

But $E^2 \geq 0$ everywhere, so the last integral being zero implies that $\boldsymbol{E} = 0$ in $\mathcal{R}$.

1.8 Method of images

The above uniqueness theorems tell us that if we can just find one solution, then we are done. They do not tell us *how* to find a solution. If we are given a fixed charge distribution, $\rho(\boldsymbol{r})$, then the solution (with $V(\infty) = 0$) is given by Eq. (1.64). But in a problem with conductors, we only know the *total charges* Q_i on the conductors, and not the static surface charge distributions $\sigma(\boldsymbol{r})$ on each conductor. The second uniqueness theorem then tells us there is a unique answer. How to find it?

In a very few special cases there is a trick, the "method of images", that allows us to get the solution. Two cases where it works are for an infinite conducting plane and for a conducting sphere. We will only discuss the plane here, and will leave the sphere for the problem set.

So consider a conductor filling $z \leq 0$, with a charge q a distance d above. What is the induced charge $\sigma(\boldsymbol{r})$ on the surface ($z = 0$) and $V(\boldsymbol{r})$ above the surface? We know that the potential is constant on the conductor and, since the overall additive constant in V is undetermined, we can choose $V_{z=0} = 0$. Then, also, $V \to 0$ at infinity. So we want to solve for V in the region $\mathcal{R} = \{z > 0\}$ with $V = 0$ on $\partial\mathcal{R}$, and with a point charge at $\boldsymbol{r} = d\hat{\boldsymbol{z}}$. The uniqueness theorem implies the solution is unique, so if we can find *any* V satisfying these boundary conditions, then we are done.

Trick: put an "imaginary charge" $-q$ at $\boldsymbol{r} = -d\hat{\boldsymbol{z}}$. The image charge is not real: we have removed the conductor and put this fictitious charge in its place. Since these fictitious changes are not in $\mathcal{R}$, we have not messed up the problem there.

From the reflection symmetry $z \to -z$ it should be clear that the potential of the image charge, $-q$, will be equal and opposite to that of q on $z = 0$:

$$
\begin{aligned}
V(\boldsymbol{r}) &= \frac{1}{4\pi\epsilon_0}\left(\frac{q}{|\boldsymbol{r} - d\hat{\boldsymbol{z}}|} + \frac{-q}{|\boldsymbol{r} + d\hat{\boldsymbol{z}}|}\right) \\
&= \frac{q}{4\pi\epsilon_0}\left(\frac{1}{|x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + (z-d)\hat{\boldsymbol{z}}|} - \frac{1}{|x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + (z+d)\hat{\boldsymbol{z}}|}\right),
\end{aligned} \tag{1.75}
$$

and so

$$
V(x, y, z = 0) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{x^2 + y^2 + d^2}} - \frac{1}{\sqrt{x^2 + y^2 + d^2}}\right) = 0\,. \tag{1.76}
$$

So we have found our solution! The induced surface charge is

$$
\sigma = -\epsilon_0 \frac{\partial V}{\partial n} = -\epsilon_0 \frac{\partial V}{\partial z}\bigg|_{z=0} = \ldots = \frac{-qd}{2\pi(x^2 + y^2 + d^2)^{3/2}}\,. \tag{1.77}
$$

The total induced charge is

$$Q = \int da\, \sigma = \frac{-qd}{2\pi} \int \frac{dx\, dy}{2\pi(x^2 + y^2 + d^2)^{3/2}} = \ldots = -q\,. \tag{1.78}$$

The electric field is

$$\boldsymbol{E} = \begin{cases} \frac{q}{4\pi\epsilon_0}\left(\frac{\boldsymbol{r}-d\hat{\boldsymbol{z}}}{|\boldsymbol{r}-d\hat{\boldsymbol{z}}|^3} - \frac{\boldsymbol{r}+d\hat{\boldsymbol{z}}}{|\boldsymbol{r}+d\hat{\boldsymbol{z}}|^3}\right) & z > 0\,, \\ 0 & z < 0\,. \end{cases} \tag{1.79}$$

Note that the electric field for $z < 0$ is *not* the same as for the "image problem"! The force on q is

$$\boldsymbol{f} = q\boldsymbol{E}(d\hat{\boldsymbol{z}}) = q\frac{-q}{4\pi\epsilon_0}\frac{\hat{\boldsymbol{z}}}{(2d)^2} = \frac{-q^2\hat{z}}{16\pi\epsilon_0 d^2}\,. \tag{1.80}$$

The energy stored is

$$W = \frac{\epsilon_0}{2}\int d^3r\, E^2 = \frac{\epsilon_0}{2}\int_{z>0} d^3r\, E^2\,. \tag{1.81}$$

But, by symmetry, q and $-q$ gives $|\boldsymbol{E}(x,y,-z)| = |\boldsymbol{E}(x,y,z)|$, so

$$W = \frac{\epsilon_0}{2}\left(\frac{1}{2}\int_{z>0} d^3r\, E^2_{\text{imag.}}\right) = \frac{1}{2}W_{\text{image problem}} = \frac{1}{2}\left(\frac{-1}{4\pi\epsilon_0}\frac{q^2}{2d}\right) = \frac{-q^2}{16\pi\epsilon_0 d}\,. \tag{1.82}$$

1.9 Multipole expansion

2 Electric fields in matter

3 Magnetostatics

The *magnetic field* $\boldsymbol{B}(\boldsymbol{r})$ is defined by noticing that a charge Q at $\boldsymbol{r}$ moving with velocity $\boldsymbol{v}$ experiences a force

$$\boldsymbol{F} = Q\big(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}\big)\,. \tag{3.1}$$

This is the Lorentz force law; it *defines* both $\boldsymbol{E}$ and $\boldsymbol{B}$.

It implies that magentic forces do no work on charged particles because the force is perpendicular to the velocity of the particle:

$$dW_{\text{mag.}} = \boldsymbol{F}_{\text{mag.}}\cdot d\boldsymbol{\ell} = Q(\boldsymbol{v}\times\boldsymbol{B})\cdot(\boldsymbol{v}dt) = 0\,. \tag{3.2}$$

$\boldsymbol{F}_{\text{mag.}}$ still accelerates particles, but only by changing direction, not speeding up or slowing down. So motion in magnetic fields tends to be in circles.

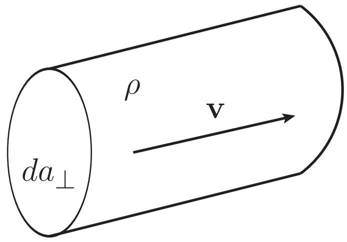

Figure 1: *Flow of charge through a surface.*

3.1 Currents and current densities

If some charges with density $\rho(\boldsymbol{r}, t)$ are moving with velocities $\boldsymbol{v}(\boldsymbol{r}, t)$, then the *charge current density* is

$$\boldsymbol{J}(\boldsymbol{r}, t) = \rho(\boldsymbol{r}, t)\boldsymbol{v}(\boldsymbol{r}, t)\,. \tag{3.3}$$

It has dimension (charge per volume) times (length per time), so charge per area per time, with units (C/s/m^2). The units of the current (charge per time) C/s = A is called "Ampère".

In materials, we can have different types of charges moving at different velocities simultaneously. E.g. electrons with $\rho_e(\boldsymbol{r}, t)$, $\boldsymbol{v}_e(\boldsymbol{r}, t)$ and ions (nuclei) with $\rho_i(\boldsymbol{r}, t)$, $\boldsymbol{v}_i(\boldsymbol{r}, t)$. Then the total charge and current densities are $\rho = \rho_e + \rho_i$ and $\boldsymbol{J} = \rho_e\boldsymbol{v}_e + \rho_i\boldsymbol{v}_i$, respectively. So, in general $\boldsymbol{J} \neq \rho\boldsymbol{v}$! E.g. in a metal $\rho_e = -\rho_i$, so $\rho = 0$, but $\boldsymbol{v}_e \neq 0$ (electrons move) and $\boldsymbol{v}_i = 0$ (nuclei don't move), so $\boldsymbol{J} = \rho_e\boldsymbol{v}_e$.

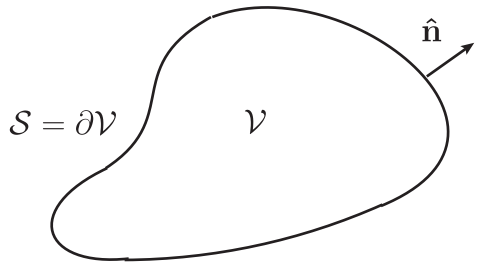

Figure 2: *A volume and its boundary.*

Charge is conserved: any net charge that flows in to / out of a volume $\mathcal{V}$ will increase / decrease the charge inside. The net charge flowing out of $\mathcal{V}$ per unit time is ($\mathcal{S} = \partial\mathcal{V}$) is

$$\oint_{\mathcal{S}} \boldsymbol{J}\cdot\hat{\boldsymbol{n}}\,da = \oint_{\mathcal{S}} \boldsymbol{J}\cdot d\boldsymbol{a}\,. \tag{3.4}$$

The decrease in net charge inside $\mathcal{V}$ per unit time is

$$\frac{d}{dt}\int_{\mathcal{V}} \rho\, d^3r = -\int_{\mathcal{V}} \frac{\partial\rho}{\partial t}\, d^3r\,. \tag{3.5}$$

Charge conservation then means

$$-\int_{\mathcal{V}} \frac{\partial\rho}{\partial t}\, d^3r = \oint_{\mathcal{S}} \boldsymbol{J}\cdot d\boldsymbol{a} = \int_{\mathcal{V}} d^3r\,(\boldsymbol{\nabla}\cdot\boldsymbol{J})\,. \tag{3.6}$$

Since this is valid for all volumes $\mathcal{V}$, we can also write the conservation equation in differential form:

$$-\frac{\partial \rho}{\partial t} = \boldsymbol{\nabla} \cdot \boldsymbol{J}\,. \tag{3.7}$$

For a *continous, steady stream* of moving charged particles we have $\boldsymbol{v}(\boldsymbol{r},t) = \boldsymbol{v}(\boldsymbol{r})$ and $\rho(\boldsymbol{r},t) = \rho(\boldsymbol{r})$. This leads to "steady" or "stationary" currents $\boldsymbol{J}(\boldsymbol{r},t) = \boldsymbol{J}(\boldsymbol{r})$ and static charge densities. So for steady currents and static charge densities we have $\partial_t \boldsymbol{J} = \partial_t \rho = 0$, and charge conservation becomes

$$\boldsymbol{\nabla} \cdot \boldsymbol{J} = 0\,. \tag{3.8}$$

So we cannot choose steady current densities arbitrarily.

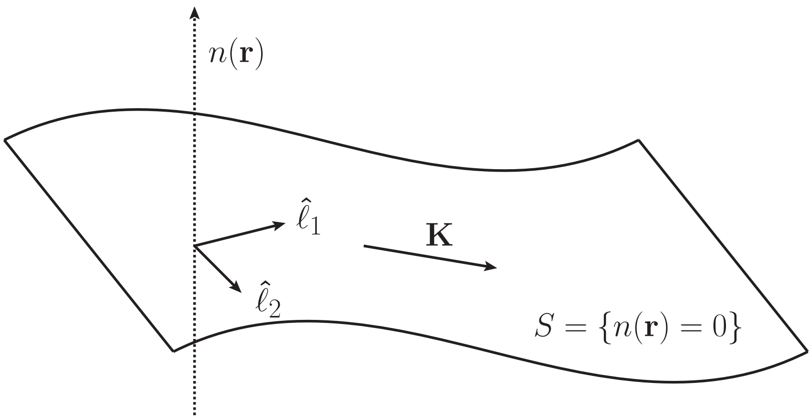

Figure 3: *Surface current density.*

If moving charges are confined to a surface S, then $\boldsymbol{J}(\boldsymbol{r},t) = \boldsymbol{K}(\boldsymbol{r},t)\delta(n)$, where the delta function restricts the current to the surface S, i.e., $S = \{n(\boldsymbol{r}) = 0\}$. $\boldsymbol{K}(\boldsymbol{r},t)$ is a *surface current density,* with dimensions (charge per length per time) and units A/m.

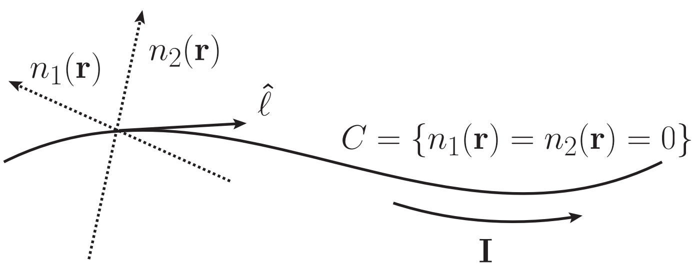

Figure 4: *Line current density.*

If moving charges are confined to a curve C, then $\boldsymbol{J}(\boldsymbol{r},t) = \boldsymbol{I}(\boldsymbol{r},t)\delta(n_1)\delta(n_2)$, where the delta functions restrict the current to the curve C, i.e., $C = \{n_1(\boldsymbol{r}) = n_2(\boldsymbol{r}) = 0\}$. $\boldsymbol{I}(\boldsymbol{r},t)$ is a *line current density* or simply current, with dimensions (charge per time) and units A.

For these to make sense (charge conservation) the charges must flow along (tangential to) the surface S ($\boldsymbol{K} \cdot \hat{\boldsymbol{n}} = 0 \Rightarrow \boldsymbol{K} = k_1 \boldsymbol{\ell}_1 + k_2 \boldsymbol{\ell}_2$) or the curve C ($\boldsymbol{I} \cdot \hat{\boldsymbol{n}}_1 = \boldsymbol{I} \cdot \hat{\boldsymbol{n}}_2 = 0 \Rightarrow \boldsymbol{I} = I\boldsymbol{\ell}$).

Steady-current conservation, $\boldsymbol{\nabla} \cdot \boldsymbol{J} = 0$, implies

$$0 = \boldsymbol{\nabla}_{\parallel} \cdot \boldsymbol{K} = \partial_{\ell_1} K_1 + \partial_{\ell_2} K_2 \,, \tag{3.9}$$

$$0 = \boldsymbol{\nabla}_{\parallel} \cdot \boldsymbol{I} = \partial_{\ell} I \,. \tag{3.10}$$

In particular, it follows that I is constant. Steady line currents, namely constant currents down wires, are the main experimental realization.

For a moving point charge q at $\boldsymbol{r} = \boldsymbol{r}_0(t)$, the current density is $\boldsymbol{J}(\boldsymbol{r}, t) = q\boldsymbol{v}(\boldsymbol{r}, t)\delta(\boldsymbol{r} - \boldsymbol{r}_0(t))$. Even if the velocity of the particle is constant, $\boldsymbol{v}(\boldsymbol{r}, t) = \boldsymbol{v}(\boldsymbol{r})$, we cannot have $\partial_t \boldsymbol{J}(\boldsymbol{r}, t) = 0$! So there is no such thing as a steady current of a single particle.

We can summarize this as follows. For charges:

$$\sum_i (\cdot) q_i \sim \int_C (\cdot) \lambda \, d\ell \sim \int_S (\cdot) \sigma \, da \sim \int_V (\cdot) \rho \, d^3 r \,, \tag{3.11}$$

and for currents:

$$\sum_i (\cdot) q_i \boldsymbol{v}_i \sim \int_C (\cdot) \boldsymbol{I} \, d\ell \sim \int_S (\cdot) \boldsymbol{K} \, da \sim \int_V (\cdot) \boldsymbol{J} \, d^3 r \,. \tag{3.12}$$

Note that

$$\int_C (\cdot) \boldsymbol{I} \, d\ell = \int_C (\cdot) I \, d\boldsymbol{\ell} \,, \tag{3.13}$$

but

$$\int_S (\cdot) \boldsymbol{K} \, da \neq \int_S (\cdot) K \, d\boldsymbol{a} \,. \tag{3.14}$$

3.2 Magnetic force on a current-carrying wire

Assume a wire is carrying a steady current $\boldsymbol{I} = \lambda \boldsymbol{v} = I \hat{\boldsymbol{\ell}}$. The magnetic force on a small section of the wire with charge $dq = \lambda d\ell$ is

$$\boldsymbol{F}_{\text{mag}} = \int_C (\boldsymbol{v} \times \boldsymbol{B}) dq = \int_C (\lambda \boldsymbol{v} \times \boldsymbol{B}) d\ell = \int_C (\boldsymbol{I} \times \boldsymbol{B}) d\ell = I \int_C (d\boldsymbol{\ell} \times \boldsymbol{B}) \,. \tag{3.15}$$

It is found experimentally that a steadily moving q at position $\boldsymbol{r}'$ with velocity $\boldsymbol{v}$ creates a magnetic field at the point $\boldsymbol{r}$. This is the Biot-Savart law:

$$\boldsymbol{B}(\boldsymbol{r}, t) = \frac{\mu_0}{4\pi} \frac{q\boldsymbol{v} \times (\boldsymbol{r} - \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3} \,. \tag{3.16}$$

This law is only approximately true: $\boldsymbol{r}' = \boldsymbol{r}_0 + \boldsymbol{v}t$, with constant velocity $\boldsymbol{v}$ that is small compared to the speed of light. It becomes exact if $\boldsymbol{r}'(t) \to \boldsymbol{r}'(t_r)$ where t_r is the "retarded time" given implictly by $t_r = t - |\boldsymbol{r} - \boldsymbol{r}'(t_r)|/c$. In the limit $c \to \infty$, $t_r = t$. One can compare this to COULOMB's law (1.13) which also only applies if the particle's velocity is small compared to the speed of light.

Here, the *permeability of free space* is defined as $\mu_0 \equiv 4\pi \times 10^{-7}\,\mathrm{NA}^{-2}$ exactly. (It defines the relation between A and C). The unit of the magnetic field is *Tesla* T = N/(Am) or *Gauss* (1 Gauss = 10^{-4} T). Also: $\mu_0\epsilon_0 = c^{-2}$ (exactly) and $c \equiv 299,792,458$ m/s (exactly; this defines the meter).

Note that

$$\frac{q\boldsymbol{v}\times(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3} = \int d^3r'\,\frac{q\boldsymbol{v}(\boldsymbol{r}')\delta(\boldsymbol{r}'-\boldsymbol{r}_0(t))\times(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3} = \int d^3r'\,\frac{\boldsymbol{J}(\boldsymbol{r}')\times(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\,, \tag{3.17}$$

so the BIOT-SAVART law for a general current density is

$$\boldsymbol{B}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi}\int d^3r'\,\frac{\boldsymbol{J}(\boldsymbol{r}')\times(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\,. \tag{3.18}$$

This is exact for steady currents (no time dependence). For a steady line current, $\boldsymbol{I} = I d\boldsymbol{\ell}$, we have

$$\boldsymbol{B}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi} I \int \frac{d\boldsymbol{\ell}'\times(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\,. \tag{3.19}$$

There is something funny going on here: moving charges create a $\boldsymbol{B}$ field, but motion is relative: one can go to an inertial frame where the charge is stationary. Does the $\boldsymbol{B}$ field go away? Yes! Electric and magnetic fields are frame dependent: different inertial observers will see different values of $\boldsymbol{E}$ and $\boldsymbol{B}$! (It turns out (special relativity) that $\boldsymbol{E}\cdot\boldsymbol{B}$ and $E^2 - c^2B^2$ are frame-independent.)

3.3 Differential equations satisfied by the magnetic field

Starting from the BIOT-SAVART law for currents, Eq. (3.18), we find

$$\boldsymbol{\nabla}\cdot\boldsymbol{B} = \frac{\mu_0}{4\pi}\int d^3r\,\boldsymbol{\nabla}_{\boldsymbol{r}}\cdot\frac{\boldsymbol{J}(\boldsymbol{r}')\times(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3} = 0\,. \tag{3.20}$$

Now we use this to calculate

$$\begin{aligned}\boldsymbol{\nabla}\times\boldsymbol{B} &= \frac{\mu_0}{4\pi}\int d^3r\,\boldsymbol{\nabla}_{\boldsymbol{r}}\times\frac{\boldsymbol{J}(\boldsymbol{r}')\times(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\\ &= \frac{\mu_0}{4\pi}\int d^3r\,\bigg\{\boldsymbol{J}(\boldsymbol{r}')\bigg(\boldsymbol{\nabla}_{\boldsymbol{r}}\cdot\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\bigg) - \frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\big(\boldsymbol{\nabla}_{\boldsymbol{r}}\cdot\boldsymbol{J}(\boldsymbol{r}')\big)\\ &\qquad + \bigg(\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\cdot\boldsymbol{\nabla}_{\boldsymbol{r}}\bigg)\boldsymbol{J}(\boldsymbol{r}') - \big(\boldsymbol{J}(\boldsymbol{r}')\cdot\boldsymbol{\nabla}_{\boldsymbol{r}}\big)\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\bigg\}\,.\end{aligned} \tag{3.21}$$

In the first term of the last expression, we have

$$\boldsymbol{\nabla}_{\boldsymbol{r}}\cdot\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3} = 4\pi\delta(\boldsymbol{r}-\boldsymbol{r}')\,. \tag{3.22}$$

The second and third terms vanish, and also the last term is zero, since

$$\begin{aligned} -\big(\boldsymbol{J}(\boldsymbol{r}')\cdot\boldsymbol{\nabla}_{\boldsymbol{r}}\big)\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3} &= +\big(\boldsymbol{J}(\boldsymbol{r}')\cdot\boldsymbol{\nabla}_{\boldsymbol{r}'}\big)\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3} \\ = +\big(\boldsymbol{J}(\boldsymbol{r}')\cdot\boldsymbol{\nabla}_{\boldsymbol{r}'}\big)\frac{x-x'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\hat{\boldsymbol{x}}+\dots \\ = +\boldsymbol{\nabla}_{\boldsymbol{r}'}\cdot\left[\frac{x-x'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\boldsymbol{J}(\boldsymbol{r}')\right]\hat{\boldsymbol{x}} - \frac{x-x'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\hat{\boldsymbol{x}}\boldsymbol{\nabla}_{\boldsymbol{r}'}\cdot\boldsymbol{J}(\boldsymbol{r}')+\dots\,. \end{aligned} \tag{3.23}$$

The second term vanishes due to current conservation, and integrating over all space we have

$$\int d^3r'\big(-\boldsymbol{J}(\boldsymbol{r}')\cdot\boldsymbol{\nabla}_{\boldsymbol{r}}\big)\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3} = \int\limits_{S_\infty}\left[d\boldsymbol{a}\cdot\frac{x-x'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\boldsymbol{J}(\boldsymbol{r}')\right]\hat{\boldsymbol{x}}+\dots=0\,, \tag{3.24}$$

where we used the divergence theorem and integrated over a sphere with very large radius where the integrand vanishes (similar for the terms proportional to $\hat{\boldsymbol{y}}$ and $\hat{\boldsymbol{z}}$). In total, we find Ampère's law:

$$\boldsymbol{\nabla}\times\boldsymbol{B} = \mu_0\boldsymbol{J}\,. \tag{3.25}$$

Integrating over an arbitrary open surface S and using Stokes' theorem, we find

$$\oint\limits_C \boldsymbol{B}\cdot d\boldsymbol{\ell} = \mu_0\int\limits_S \boldsymbol{J}\cdot d\boldsymbol{a} \equiv \mu_0 I_{\text{encl.}}\,, \tag{3.26}$$

for any C and any S such that $C=\partial S$, and $I_{\text{encl.}}$ is the total current passing through S. Just as with the integral form of Gauss' law (1.9), the integral form of Ampère's law is useful when the problem has a lot of symmetry, such as infinite straight lines, infinite planes, infinite solenoids, circular solenoids ("toroids").

3.4 The vector potential

Reminder: We have seen that since the electric field satisfies $\boldsymbol{\nabla}\times\boldsymbol{E}=0$, we could find an electric potential V such that

$$\boldsymbol{E} = -\boldsymbol{\nabla}V\,. \tag{3.27}$$

The potential is not unique: any $V' = V + c$, where c is a constant scalar field, gives the same electric field $\boldsymbol{E}$. If we use Eq. (3.27) in Gauss' law, $\boldsymbol{\nabla}\cdot\boldsymbol{E} = \rho/\epsilon_0$, we obtain the Poisson equation, $\nabla^2\rho = -\rho/\epsilon_0$.

A similar procedure works for magnetic fields as follows. Since the divergence of the magnetic field vanishes, $\boldsymbol{\nabla}\cdot\boldsymbol{B}=0$, we can find a *magnetic vector potential* $\boldsymbol{A}$ such that

$$\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}\,. \tag{3.28}$$

The vector potential is also not unique. Adding to $\boldsymbol{A}$ the gradient of any scalar field,

$$\boldsymbol{A}' = \boldsymbol{A} + \boldsymbol{\nabla}\lambda(\boldsymbol{r}) \tag{3.29}$$

gives the same magnetic field: $\boldsymbol{\nabla} \times \boldsymbol{A}' = \boldsymbol{\nabla} \times \boldsymbol{A} + \boldsymbol{\nabla} \times \boldsymbol{\nabla}\lambda(\boldsymbol{r}) = \boldsymbol{B}$.

Ampère's law, expressed in terms of the vector potential, becomes $\mu_0 \boldsymbol{J} = \boldsymbol{\nabla} \times \boldsymbol{B} = \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A}$. This can be simplified using the "gauge freedom" (3.29) to choose the vector potential such that $\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$ (this is called *Coulomb gauge*).

(That this is always possible can be seen as follows. If $\boldsymbol{\nabla} \cdot \boldsymbol{A} \neq 0$, we define $\boldsymbol{A}' = \boldsymbol{A} + \boldsymbol{\nabla}\lambda(\boldsymbol{r})$, with λ chosen such that $\nabla^2\lambda = -\boldsymbol{\nabla} \cdot \boldsymbol{A}$. This is just Poisson's equation, with source term $\boldsymbol{\nabla} \cdot \boldsymbol{A}$. Then we have $\boldsymbol{\nabla} \cdot \boldsymbol{A}' = \boldsymbol{\nabla} \cdot \boldsymbol{A} + \nabla^2\lambda = \boldsymbol{\nabla} \cdot \boldsymbol{A} - \boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$.)

In Coulomb gauge, Ampère's law becomes simply

$$\nabla^2 \boldsymbol{A} = -\mu_0 \boldsymbol{J} \,. \tag{3.30}$$

This is, component by component, just a version of Poisson's equation. Taking over the general form of the solution, we find the vector potential in terms of the current densities,[5]

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int d^3r' \frac{\boldsymbol{J}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \,. \tag{3.31}$$

This is not as useful as the electric potential (one reason being that it is still a vector), but the vector potential is useful when formulating electrodynamics in a Lorentz-invariant way, and it is essential in the quantization of electrodynamics.

One of the main applications is the multipole expansion.

3.4.1 Multipole expansion of the vector potential

We can derive the multipole expansion by using the identity

$$\frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} = \frac{1}{\sqrt{r^2 + (r')^2 - 2rr'\cos\theta'}} = \frac{1}{r} \sum_{\ell=0}^{\infty} \left(\frac{r'}{r}\right)^{\ell} P_\ell(\cos\theta') \tag{3.32}$$

in Eq. (3.31). Here, P_ℓ are the Legendre polynomials, and $\cos\theta' \equiv \hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{r}}'$. We obtain

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} \boldsymbol{M}_\ell(\hat{\boldsymbol{r}}) \,, \tag{3.33}$$

where

$$\boldsymbol{M}_\ell(\hat{\boldsymbol{r}}) \equiv \frac{\mu_0}{4\pi} \int d^3r' (r')^\ell \boldsymbol{J}(\boldsymbol{r}') P_\ell(\cos\theta') \tag{3.34}$$

is the ℓth magnetic multipole.

There are no magnetic charges (as far as we know), so the magnetic monopole always vanishes:

$$\boldsymbol{M}_0(\hat{\boldsymbol{r}}) = \frac{\mu_0}{4\pi} \int d^3r' \boldsymbol{J}(\boldsymbol{r}') = 0 \,. \tag{3.35}$$

(This can be shown as follows. For steady currents we have $\boldsymbol{\nabla} \cdot \boldsymbol{J} = 0$. Let's consider the volume integral over the divergence of the current density times an *arbitrary* function

[5] When using this expression, we will always tacitly assume that the currently densities vanish sufficiently rapidly at infinity, such that the integral exists.

f. Since the current is localized, using the divergence theorem and choosing the boundary sufficiently far from the current density, we see that the integral must be zero:

$$0 = \int d^3r'\ \boldsymbol{\nabla}' \cdot (f\boldsymbol{J}) = \int d^3r'\ (f\boldsymbol{\nabla}' \cdot \boldsymbol{J} + \boldsymbol{J} \cdot \boldsymbol{\nabla}' f)\,. \tag{3.36}$$

It follows, since $\boldsymbol{\nabla}' \cdot \boldsymbol{J} = 0$, that

$$0 = \int d^3r'\ \boldsymbol{J} \cdot \boldsymbol{\nabla}' f\,, \tag{3.37}$$

for any function $f(\boldsymbol{r}')$. Now we choose $f = x_i'$, $i = 1, 2, 3$, so that $\boldsymbol{\nabla}' f = \hat{\boldsymbol{r}}_i$, the unit vector in i-direction. We find that

$$0 = \int d^3r'\ J_i\,, \tag{3.38}$$

and so $\boldsymbol{M}_0(\hat{\boldsymbol{r}}) = 0$.)

For the special case of a current loop, we can write the volume integral as a line integral over the current loop, and the absence of a magnetic monopole implies

$$0 = \int d^3r'\, \boldsymbol{J}(\boldsymbol{r}') = \oint_C \boldsymbol{I} \cdot d\boldsymbol{\ell}' \tag{3.39}$$

and so, since this is valied for any current,

$$\oint_C d\boldsymbol{\ell}' = 0\,. \tag{3.40}$$

We will now consider the next term in the expansion, the *magnetic dipole*:

$$\begin{aligned}\boldsymbol{M}_1(\boldsymbol{r}) &= \int d^3r'\, r' P_1(\cos\theta')\boldsymbol{J}(\boldsymbol{r}') = \int d^3r'\, r'\cos\theta' \boldsymbol{J}(\boldsymbol{r}') = \int d^3r'\, r'(\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{r}}')\boldsymbol{J}(\boldsymbol{r}') \\ &= \int d^3r'\, (\hat{\boldsymbol{r}} \cdot \boldsymbol{r}')\boldsymbol{J}(\boldsymbol{r}')\,.\end{aligned} \tag{3.41}$$

Let's consider the ith Cartesian component of this equation:

$$(M_1)_i = \hat{\boldsymbol{r}} \cdot \int d^3r'\, \boldsymbol{r}' J_i(\boldsymbol{r}') = \frac{1}{r}\sum_{j=1}^{3} x_j \int d^3r'\, x_j' J_i(\boldsymbol{r}')\,. \tag{3.42}$$

Now we use Eq. (3.37) with $f = x_i' x_j'$ to find

$$\begin{aligned}0 &= \int d^3r'\, \boldsymbol{J}(\boldsymbol{r}') \cdot \boldsymbol{\nabla}(x_i' x_j') = \int d^3r'\, \boldsymbol{J}(\boldsymbol{r}') \cdot (x_i' \hat{\boldsymbol{r}}_j' + x_j' \hat{\boldsymbol{r}}_i') \\ &= \int d^3r' \left[x_i' J_j(\boldsymbol{r}') + x_j' J_i(\boldsymbol{r}')\right].\end{aligned} \tag{3.43}$$

We can use this to write

$$\int d^3r'\, x'_j J_i(\boldsymbol{r}') = \frac{1}{2}\int d^3r' \left[x'_j J_i(\boldsymbol{r}') - x'_i J_j(\boldsymbol{r}')\right]. \tag{3.44}$$

Inserting this into the expression for $(M_1)_i$, Eq. (3.42), gives

$$\begin{aligned}(M_1)_i &= -\frac{1}{2r}\sum_{j=1}^{3} x_j \int d^3r' \left[x'_i J_j(\boldsymbol{r}') - x'_j J_i(\boldsymbol{r}')\right] = -\frac{1}{2r}\sum_{j=1}^{3} x_j \epsilon_{ijk} \int d^3r' \left(\boldsymbol{r}' \times \boldsymbol{J}(\boldsymbol{r}')\right)_k \\ &= -\frac{1}{2r}\left[\boldsymbol{r} \times \int d^3r' \left(\boldsymbol{r}' \times \boldsymbol{J}(\boldsymbol{r}')\right)\right]_i ,\end{aligned} \tag{3.45}$$

and, hence,

$$\boldsymbol{M}_1 = -\frac{1}{2}\hat{\boldsymbol{r}} \times \int d^3r' \left(\boldsymbol{r}' \times \boldsymbol{J}(\boldsymbol{r}')\right). \tag{3.46}$$

It is conventional to define the *magnetic moment density*

$$\boldsymbol{\mathcal{M}}(\boldsymbol{r}') \equiv \frac{1}{2}\boldsymbol{r}' \times \boldsymbol{J}(\boldsymbol{r}'), \tag{3.47}$$

and the *magnetic moment*

$$\boldsymbol{m} \equiv \frac{1}{2}\int d^3r' \left(\boldsymbol{r}' \times \boldsymbol{J}(\boldsymbol{r}')\right), \tag{3.48}$$

such that $\boldsymbol{M}_1 = -\hat{\boldsymbol{r}} \times \boldsymbol{m}$. (This is the analog of the electric dipole moment $\boldsymbol{p}$.)

For a current loop C we have

$$\boldsymbol{M}_1 = -\frac{1}{2}\hat{\boldsymbol{r}} \times \left(I \oint_C \boldsymbol{r}' \times d\boldsymbol{\ell}'\right) = \frac{I}{2}\hat{\boldsymbol{r}} \times \left(\oint_C d\boldsymbol{\ell}' \times \boldsymbol{r}'\right) \tag{3.49}$$

and

$$\boldsymbol{m} = \frac{I}{2}\oint_C \boldsymbol{r}' \times d\boldsymbol{\ell}' \equiv I \int_S d\boldsymbol{a}, \tag{3.50}$$

where S is any surface with $\partial S = C$.

A "pure" dipole can be obtained in the limits $\int_S d\boldsymbol{a} \to 0$ and $I \to \infty$.

If the magnetic dipole is aligned with the z direction, $\boldsymbol{m} = m\hat{\boldsymbol{z}} = m(\cos\theta\hat{\boldsymbol{r}} - \sin\theta\hat{\boldsymbol{\theta}})$, then

$$\boldsymbol{A}_{\text{dipole}}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\hat{\boldsymbol{\varphi}}, \tag{3.51}$$

and using Eq. (B.6),

$$\boldsymbol{B}_{\text{dipole}}(\boldsymbol{r}) = \boldsymbol{\nabla} \times \boldsymbol{A}_{\text{dipole}}(\boldsymbol{r}) = \frac{\mu_0 m}{4\pi r^3}\left(2\cos\theta\,\hat{\boldsymbol{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right) = \frac{\mu_0}{4\pi r^3}\left[3(\boldsymbol{m}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}} - \boldsymbol{m}\right]. \tag{3.52}$$

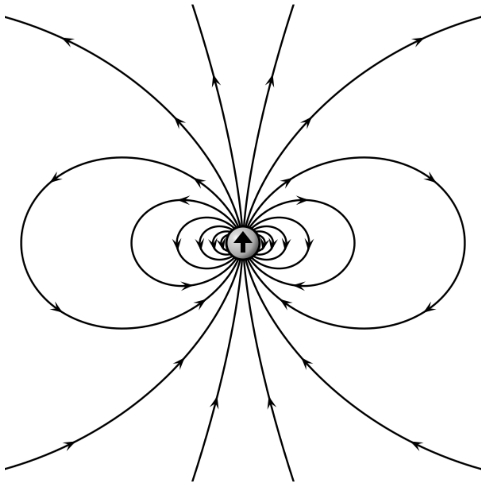

Figure 5: *The field of a magnetic dipole. [Image credit: Geek3 - Own work, CC BY-SA 3.0]*

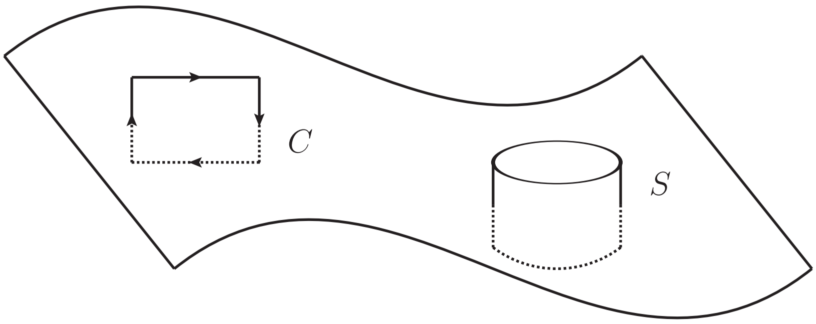

Figure 6: *Boundary conditions on the magnetic field and vector potential.*

3.5 Boundary conditions on B and A

We consider a surface current density $\boldsymbol{k}$. We must have $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$, and hence

$$\int_S \boldsymbol{B} \cdot d\boldsymbol{a} = 0\,. \tag{3.53}$$

Taking the limit of a very small surface, we see that the orthogonal components of the magnetic field above and below the surface must cancel each other:

$$\boldsymbol{B}^{\perp}_{\text{above}} = \boldsymbol{B}^{\perp}_{\text{below}}\,. \tag{3.54}$$

Similarly, we must have $\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$, and hence

$$\oint_C \boldsymbol{B} \cdot d\boldsymbol{\ell} = I_{\text{encl.}}\,, \tag{3.55}$$

where $I_{\text{encl.}}$ denotes the current flowing through the loop. Taking the limit of a very small loop, we see that the difference of the parallel components of the magnetic field above and below the surface must add up to $\mu_0 \boldsymbol{K}$:

$$\boldsymbol{B}^{\parallel}_{\text{above}} - \boldsymbol{B}^{\parallel}_{\text{below}} = \mu_0 \boldsymbol{K} \,. \tag{3.56}$$

These two conditions can be combined into the single equality

$$\boldsymbol{B}_{\text{above}} - \boldsymbol{B}_{\text{below}} = \mu_0 \big(\boldsymbol{K} \times \hat{\boldsymbol{n}}\big) \,, \tag{3.57}$$

where $\hat{\boldsymbol{n}}$ is a normal vector orthogonal to the surface. For the vector potential, this implies

$$\frac{\partial}{\partial n}\boldsymbol{A}_{\text{above}} - \frac{\partial}{\partial n}\boldsymbol{A}_{\text{below}} = -\mu_0 \boldsymbol{K} \,. \tag{3.58}$$

(We can compare this to the case of the electric field. For a discontinuity at a surface charge, we had

$$\boldsymbol{E}_{\text{above}} - \boldsymbol{E}_{\text{below}} = \frac{\sigma}{\epsilon_0}\hat{\boldsymbol{n}} \,, \tag{3.59}$$

$$\frac{\partial V}{\partial n}\boldsymbol{E}_{\text{above}} - \frac{\partial V}{\partial n}\boldsymbol{E}_{\text{below}} = -\frac{\sigma}{\epsilon_0} \,.) \tag{3.60}$$

4 Magnetic fields in matter

Magnetism of matter is ultimately a quantum-mechanical phenomenon. We can gain some understanding from a classical description. In this picture, atoms or molecules of matter can be approximated as "point dipoles" $\boldsymbol{m}$. Without applied magnetic field, some material's atoms have no dipole moment (this happens mostly in atoms with even numbers of electrons), but most have "permanent magentic dipole moments". We can think of these dipole moments as due to electrons orbiting the nuclei, forming somthing like small current loops, or due to the electron spins (intrinsic angular momentum of electrons). Both really need quantum mechanics and relativity to understand.

Diamagnetism Diamagnetic material has $\boldsymbol{m} = 0$ when $\boldsymbol{B} = 0$. An applied magnetic field induces $\boldsymbol{m} \propto -\boldsymbol{B}$ (pointing opposite to B).

Paramagnetism Paramagnetic material has atoms with permanent dipoles $\boldsymbol{m} \neq 0$ when $\boldsymbol{B} = 0$. They are aligned (or randomly distributed) such that the net (average) dipole moment vanishes for external field $\boldsymbol{B} = 0$. The force and torque on a dipole iin an applied magnetic field $\boldsymbol{B}$ are given by $\boldsymbol{F} = (\boldsymbol{m} \cdot \boldsymbol{\nabla})\boldsymbol{B}$ and $\boldsymbol{N} = \boldsymbol{m} \times \boldsymbol{B}$, respectively. $\boldsymbol{N}$ tends to rotate $\boldsymbol{m}$ to be parallel to $\boldsymbol{B}$, and the average magnetic dipole moment is now nonzero, $\langle \boldsymbol{m} \rangle \neq 0$. So paramagnetic material tends to develop a net dipole moment $\boldsymbol{m} \propto +\boldsymbol{B}$ (aligning with applied $\boldsymbol{B}$). Paramagnetism tends to be much stronger than diamagnetism.

Ferromagnetism Ferromagnetic material has permanent dipoles that do not cancel or average; $\langle \boldsymbol{m} \rangle \neq 0$ even if $\boldsymbol{B} = 0$. Many ferromagnetic solids look paramagnetic at large scales because ferromagnetic *domains* are randomly misaligned, so average $\langle \boldsymbol{m} \rangle = 0$. Applying $\boldsymbol{B}$ leads to paramagnetic response as $\boldsymbol{m}$s tend to align. Sufficiently strong external magnetic fields can move or merge domains so they are mostly aligned. Then, when the external field turns off, the material remains feromagnetic ("hysteresis" = history dependence). The reason that permanent magnets (ferromagnetism) occurs is that it is energetically favorable for neighbouring dipoles to align: $E(\uparrow\uparrow) < E(\uparrow\downarrow)$. The reason is essentially quantum mechanical. There are situations in which the reverse is true and dipoles tend to "anti-align": $\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow$. Then $\langle \boldsymbol{m} \rangle = 0$ if $\boldsymbol{B} = 0$ and the material looks paramagnetic. This is called *anti-ferromagnetism.*

At high temperatures, thermal motion tends to randomize dipoles overcoming the ferromagnetic alignment.

4.1 Linear media

Here, we consider diamagnetic and paramagnetic materials with small applied external magnetic fields. Then we expect a linear response: $\langle \boldsymbol{m} \rangle \propto \boldsymbol{B}$, just as with dielectrics.

We define the *magnetization*

$$\boldsymbol{M} = \langle \boldsymbol{m} \rangle / \text{unit volume} \,. \tag{4.1}$$

For a dipole,

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \frac{\boldsymbol{m} \times \boldsymbol{r}}{|\boldsymbol{r}|^3} \,. \tag{4.2}$$

Averaging over many dipoles, we find

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int d^3r' \frac{\boldsymbol{M}(\boldsymbol{r}') \times (\boldsymbol{r} - \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3} \,. \tag{4.3}$$

Using the identity

$$\boldsymbol{\nabla}' \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} = \frac{\boldsymbol{r} - \boldsymbol{r}'}{|\boldsymbol{r} - \boldsymbol{r}'|^3} \tag{4.4}$$

we can write Eq. (4.3) as

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int d^3r' \left[\boldsymbol{M}(\boldsymbol{r}') \times \boldsymbol{\nabla}' \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} \right] . \tag{4.5}$$

Integrating by parts then gives

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \left\{ \int d^3r' \, \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} \big(\boldsymbol{\nabla}' \times \boldsymbol{M}(\boldsymbol{r}') \big) - \int d^3r' \, \boldsymbol{\nabla}' \times \frac{\boldsymbol{M}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \right\} . \tag{4.6}$$

The second term can be written as a surface integral,[6] and we find

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int d^3r' \, \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} \big[\boldsymbol{\nabla}' \times \boldsymbol{M}(\boldsymbol{r}')\big] + \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{M}(\boldsymbol{r}') \times d\boldsymbol{a}'}{|\boldsymbol{r}-\boldsymbol{r}'|} \,. \tag{4.9}$$

This expression motivates the definition of the *bound current density*

$$\boldsymbol{J}_b \equiv \boldsymbol{\nabla} \times \boldsymbol{M} \,, \tag{4.10}$$

(note that $\boldsymbol{\nabla} \cdot \boldsymbol{J}_b = 0$), and the *bound surface current*

$$\boldsymbol{K}_b = \boldsymbol{M} \times \hat{\boldsymbol{n}} \,. \tag{4.11}$$

With these definitions, the vector potential becomes

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int d^3r' \, \frac{\boldsymbol{J}_b(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} + \frac{\mu_0}{4\pi} \int da \, \frac{\boldsymbol{K}_b(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} \,. \tag{4.12}$$

The potential of the material is the same as that produced by the current density inside the volume, and the surface current density at the surface of the material, where the magnetization goes to zero discontinuously.

If we subject some magnetic material to an external magnetic field produced by some "free current" $\boldsymbol{J}_f$ (which may flow inside the material), the total current is the sum of the free current and the induced current $\boldsymbol{J}_b$,

$$\boldsymbol{J} = \boldsymbol{J}_f + \boldsymbol{J}_b \,. \tag{4.13}$$

Ampère's law tells us that

$$\frac{1}{\mu_0}\big(\boldsymbol{\nabla} \times \boldsymbol{B}\big) = \boldsymbol{J} = \boldsymbol{J}_f + \boldsymbol{J}_b = \boldsymbol{J}_f + \boldsymbol{\nabla} \times \boldsymbol{M} \,. \tag{4.14}$$

We can then define a field

$$\boldsymbol{H} \equiv \frac{1}{\mu_0}\boldsymbol{B} - \boldsymbol{M} \,, \tag{4.15}$$

such that Ampère's law becomes

$$\boldsymbol{\nabla} \times \boldsymbol{H} = \boldsymbol{J}_f \,. \tag{4.16}$$

[6] The divergence theorem, applied to $\boldsymbol{A} \times \boldsymbol{c}$ with $\boldsymbol{c}$ an arbitrary constant, gives

$$\int_V \boldsymbol{\nabla} \cdot (\boldsymbol{A} \times \boldsymbol{c}) = \int_V \boldsymbol{c} \cdot (\boldsymbol{\nabla} \times \boldsymbol{A}) = \oint_S (\boldsymbol{A} \times \boldsymbol{c}) \cdot d\boldsymbol{a} = -\oint_S \boldsymbol{c} \cdot (\boldsymbol{A} \times d\boldsymbol{a}) \,, \tag{4.7}$$

and so

$$\int_V \boldsymbol{\nabla} \times \boldsymbol{A} = -\oint_S \boldsymbol{A} \times d\boldsymbol{a} \,. \tag{4.8}$$

Note that, while $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$, the divergence of $\boldsymbol{H}$ does not, in general, vanish: instead, we have $\boldsymbol{\nabla} \cdot \boldsymbol{H} = -\boldsymbol{\nabla} \cdot \boldsymbol{M}$. This implies the boundary conditions

$$H^{\perp}_{\text{above}} - H^{\perp}_{\text{below}} = -\left(M^{\perp}_{\text{above}} - M^{\perp}_{\text{below}}\right) . \tag{4.17}$$

Eq. (4.16) implies

$$\boldsymbol{H}^{\parallel}_{\text{above}} - \boldsymbol{H}^{\parallel}_{\text{below}} = \boldsymbol{K}_b \times \hat{\boldsymbol{n}} . \tag{4.18}$$

For linear (isotropic, homogeneous) media, we have

$$\boldsymbol{M} = \chi_m \boldsymbol{H} , \tag{4.19}$$

where the constant χ_m is called the *magnetic susceptibility* of the material. The material is diamagnetic for $\chi_m < 0$, and paramagnetic for $\chi_m > 0$. We can also write

$$\boldsymbol{B} = \mu \boldsymbol{H} , \tag{4.20}$$

with the *permeability* $\mu \equiv \mu_0(1 + \chi_m)$.

Let's compare electrostatics in an insulator and magnetostatics in a magnetic material:

insulators	magnetic materials
$\boldsymbol{\nabla} \cdot \boldsymbol{D} = \rho_f$	$\boldsymbol{\nabla} \cdot \boldsymbol{B} = \rho_f$
$\boldsymbol{\nabla} \times \boldsymbol{E} = 0$	$\boldsymbol{\nabla} \times \boldsymbol{H} = \boldsymbol{J}_f$
$\boldsymbol{D} \equiv \epsilon_0 \boldsymbol{E} + \boldsymbol{P}$	$\boldsymbol{H} \equiv \frac{1}{\mu_0}\boldsymbol{B} - \boldsymbol{M}$
$\Delta D^{\perp} = \sigma_f$ (@ boundary)	$\Delta B^{\perp} = 0$ (@ boundary)
$\Delta E^{\parallel} = 0$ (@ boundary)	$\Delta H^{\parallel} = \boldsymbol{K}_f \times \hat{\boldsymbol{n}}$ (@ boundary)
$\boldsymbol{P} = \epsilon_0 \chi_e \boldsymbol{E}$	$\boldsymbol{M} = \chi_m \boldsymbol{H}$
$\boldsymbol{D} = \epsilon \boldsymbol{E}$	$\boldsymbol{B} = \mu \boldsymbol{H}$
$\epsilon = \epsilon_0(1 + \chi_e)$	$\mu = \mu_0(1 + \chi_m)$

The last three lines apply for linear materials.

4.2 Conductors

Recall that for electrostatics, in a conductor: (a) $\boldsymbol{E} = \rho = 0$ in interior (V =constant); (b) $E^{\parallel}_{\text{out}} = 0$ at the boundary, and (c) $E^{\perp}_{\text{out}} = \sigma$ at the boundary; and therefore, $\boldsymbol{E}_{\text{out}} = \frac{\sigma}{\epsilon_0}\hat{\boldsymbol{n}}$ at the boundary. The potential V is continuous across the boundary, but $\partial V/\partial n = \sigma/\epsilon_0$. Here, σ is the induced surface charge on the boundary.

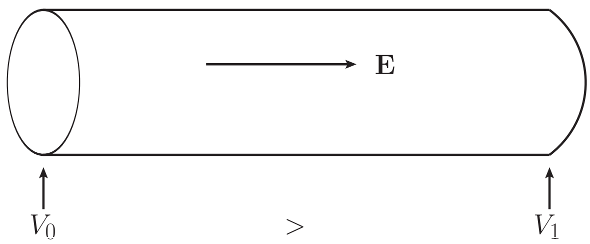

Figure 7: *Electric field inside a conductor in the presence of a potential difference.*

But we can also have steady currents in conductors, giving sources for magnetostatic fields. Currents are a response of conductions charges (electrons) to applied $\boldsymbol{E}$ field inside the conductor. If one can add / subtract charges at the ends of the wire, so as to keep the potential difference $V_0 - V_1$ constant (e.g., use a battery), then we keep a steady $\boldsymbol{E} \neq 0$ inside the metal! This $\boldsymbol{E}$ accelerates conduction electrons, forming currents.

In conductors there are frictional forces on the conduction electrons (e.g. scattering off impurities…) which slow them down (dissipating their energy as heat). A given $\boldsymbol{E}$ then accelerates electrons to a steady velocity v at which the frictional force counterbalances $e\boldsymbol{E}$. Then we get a steady current $\boldsymbol{J} = \rho_e \boldsymbol{v}$.

What is the relation $\boldsymbol{J} = \boldsymbol{J}(\boldsymbol{E})$? In linear (sufficiently small applied $\boldsymbol{E}$), homogeneous, isotropic conductor we have Ohm's law, $\boldsymbol{J} = \sigma \boldsymbol{E}$. Here, the constant σ is the *conductivity* (not the surface charge!). Its inverse, $1/\sigma$, is the *resistivity*. (If the conductor moves with a steady velocity in a magnetic field $\boldsymbol{B}$, then $\boldsymbol{J} = \sigma(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$.)

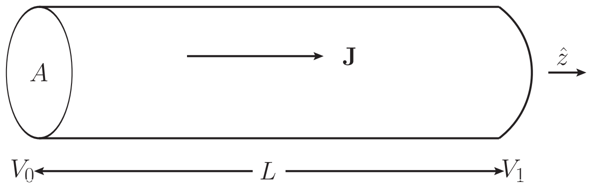

Figure 8: *Ohm's law applied to a wire.*

Applying this to a wire, we obtain the more familiar version of Ohm's law. Aligning the wire (with cross sectional area A) in the z direction, we have

$$AJ\hat{\boldsymbol{z}} = A\boldsymbol{J} = A\sigma \boldsymbol{E} = A\sigma \frac{V_2 - V_1}{L} \hat{\boldsymbol{z}}\,, \tag{4.21}$$

so

$$\underbrace{AJ}_{\text{current } I} = \underbrace{\frac{A\sigma}{L}}_{1/R} \underbrace{(V_2 - V_1)}_{V \text{ voltage difference}}\,, \tag{4.22}$$

or $V = RI$.

Conductivities can vary between 0 (insulator) and ∞ (superconductor). Conductors can be diamagnetic (e.g. Ag), paramagnetic (e.g. Al), or ferromagnetic (e.g. Fe).

4.3 Superconductors

Can think of superconductors as the $\sigma \to \infty$ limit of conductors. There is no friction: an applied $\boldsymbol{E} \neq 0$ in conductor accelerates charges without limit, $\boldsymbol{E} \propto \partial \boldsymbol{J}/\partial t$, so there is no Ohm's law. What about the magnetic response of superconductors? They still satisfy $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0, \boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$ for steady currents (they can have a steady current if $\boldsymbol{E} = 0$ inside the superconductor).

4.3.1 London equations

Inside the superconductor, we have the London equations:

$$\mu_0 \frac{\partial \boldsymbol{J}}{\partial t} = \frac{1}{\lambda^2}\boldsymbol{E}\,, \tag{4.23}$$

$$\mu_0 \boldsymbol{\nabla} \times \boldsymbol{J} = -\frac{1}{\lambda^2}\boldsymbol{B}\,. \tag{4.24}$$

The *London penetration depth* λ has dimensions of length. For typical superconductors, $50\,\text{nm} \lesssim \lambda \lesssim 500\,\text{nm}$. (Recall that the typical size of an atom is of the order of $0.1\,\text{nm}$.)

These two equations can only be derived using quantum mechanics. However, Eq. (4.23) is intuitive using the Lorentz force law:

$$\boldsymbol{F} = m_e \dot{\boldsymbol{v}} = -e\boldsymbol{E}\,, \tag{4.25}$$

so

$$\dot{\boldsymbol{J}} = -n_e e \dot{\boldsymbol{v}} = \frac{n_e e^2}{m_e}\boldsymbol{E}\,, \tag{4.26}$$

where n_e is the electron number density. Now we can calculate $\boldsymbol{\nabla}\times$ Eq. (4.23) and use Faraday's law, $\boldsymbol{\nabla} \times \boldsymbol{E} = -\partial \boldsymbol{B}/\partial t$, to find

$$0 = \mu_0 \boldsymbol{\nabla} \times \frac{\partial \boldsymbol{J}}{\partial t} - \frac{1}{\lambda^2}\boldsymbol{\nabla} \times \boldsymbol{E} = \mu_0 \boldsymbol{\nabla} \times \frac{\partial \boldsymbol{J}}{\partial t} + \frac{1}{\lambda^2}\frac{\partial \boldsymbol{B}}{\partial t}\,, \tag{4.27}$$

or

$$0 = \frac{\partial}{\partial t}\left(\mu_0 \boldsymbol{\nabla} \times \boldsymbol{J} + \frac{1}{\lambda^2}\boldsymbol{B}\right). \tag{4.28}$$

Eq. (4.24) is consistent with this result.

4.3.2 London penetration and Meissner effect

Now let's apply Ampère's law, $\mu_0 \boldsymbol{J} = \boldsymbol{\nabla} \times \boldsymbol{B}$, to the second London equation (4.24):

$$0 = \boldsymbol{\nabla} \times \big(\boldsymbol{\nabla} \times \boldsymbol{B}\big) + \frac{1}{\lambda^2}\boldsymbol{B} = \boldsymbol{\nabla}\big(\boldsymbol{\nabla} \cdot \boldsymbol{B}\big) - \nabla^2 \boldsymbol{B} + \frac{1}{\lambda^2}\boldsymbol{B}\,. \tag{4.29}$$

Using $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$, we obtain

$$\nabla^2 \boldsymbol{B} = \frac{1}{\lambda^2}\boldsymbol{B}\,. \tag{4.30}$$

This equation leads to the Meissner-Ochsenfeld effect, the expulsion of magnetic fields from the superconductor. To illustrate this in a simple setting, we consider a large, planar superconductor, and apply an external magnetic field. Because of the setup, we expect the magnetic field to depend only on the direction orthognal to the surface of the superconductor, which we take to be the z direction. For definiteness, we choose the direction of the applied, constant external magnetic field in the x direction, $\boldsymbol{B} = B_0 \hat{\boldsymbol{x}}$ (for $z \geqq 0$). Then Eq. (4.30) becomes

$$\nabla^2 \boldsymbol{B}(z) = \frac{\partial^2}{\partial z^2} B(z)\hat{\boldsymbol{x}} = \frac{1}{\lambda^2} B(z)\hat{\boldsymbol{x}}\,, \tag{4.31}$$

or

$$B''(z) = \frac{1}{\lambda^2} B(z) \,. \tag{4.32}$$

The general solution to this equation is

$$B(z) = \alpha e^{z/\lambda} + \beta e^{-z/\lambda} \,. \tag{4.33}$$

We determine the coefficients α and β by the boundary conditions. For $z \to \infty$, the second term would blow up, so $\beta = 0$. For $z = 0$, we must have $B(z) = B_0$, so $\alpha = B_0$. This gives the solution

$$B(z) = B_0 e^{z/\lambda} \,. \tag{4.34}$$

The solution implies that $\boldsymbol{B} \to 0$ exponentially fast inside the superconductor: this is the *Meissner-Ochsenfeld effect*. The current is restricted to within the London penetration depth of the surface of the superconductor.

Superconductors expel magnetic fields, this leads to "magnetic levitation". Another (probably more relevant) techonological application are superconducting electromagnets. Inside the superconductor, we can have steady currents without applied electric fields, without energy loss due to heating.

5 Maxwell's equations

5.1 Overview

We start with a review of the laws of electro- and magnetostatics.

Newton's law:

$$\boldsymbol{F} = m \frac{d^2\boldsymbol{r}}{dt^2} \,. \tag{5.1}$$

Lorentz force:

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \,. \tag{5.2}$$

Equivalently, the force density f on the charge density ρ and the current density $\boldsymbol{J}$, due to $\boldsymbol{E}$ and $\boldsymbol{B}$, is given by $\boldsymbol{f} = \rho(\boldsymbol{E} + \boldsymbol{J} \times \boldsymbol{B})$.

Charge conservation:

$$\boldsymbol{\nabla} \cdot \boldsymbol{J} = -\frac{d\rho}{dt} \,. \tag{5.3}$$

In electro-/magnetostatics, $d\rho/dt = d\boldsymbol{J}/dt = 0$, so $\boldsymbol{\nabla} \cdot \boldsymbol{J} = 0$.

The field equations determine $\boldsymbol{E}$ and $\boldsymbol{B}$ due to ρ and $\boldsymbol{J}$. For the electric field, we have

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \frac{1}{\epsilon_0}\rho \quad \text{Gauss} \,, \tag{5.4}$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} = 0 \,. \tag{5.5}$$

These two laws together imply Coulomb's law:

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} (\boldsymbol{r} - \boldsymbol{r}') \,. \tag{5.6}$$

For the magnetic field, we have

$$\nabla \cdot \boldsymbol{B} = 0\,, \tag{5.7}$$

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} \quad \text{Ampère}\,. \tag{5.8}$$

These two laws together imply the Biot-Savart law:

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int d^3r' \frac{\boldsymbol{J}(\boldsymbol{r}') \times (\boldsymbol{r} - \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|}\,. \tag{5.9}$$

These laws apply only in static situations! In non-static situations ("electrodynamics"), the following laws will need to be corrected:

$$\nabla \times \boldsymbol{E} = 0 \quad \Longrightarrow \quad \nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = 0 \qquad \text{Faraday}\,, \tag{5.10}$$

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} \quad \Longrightarrow \quad \nabla \times \boldsymbol{B} - \mu_0 \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} = \mu_0 \boldsymbol{J} \qquad \text{Maxwell}\,. \tag{5.11}$$

Also the right side of Eq. (5.1) needs to be corrected (Einstein).

5.2 Faraday's law on electromagnetic induction

In a conducting wire, we have Ohm's law:

$$\mathcal{E} = I\,R\,, \tag{5.12}$$

where I is the current, R is the resistance, and $\mathcal{E}$ is the *electromotive force* ("EMF" – work per charge):

$$\mathcal{E} \equiv -\int_a^b d\boldsymbol{\ell} \cdot \boldsymbol{E}\,. \tag{5.13}$$

On the other hand, if you move a wire through a magnetic field, the conduction electrons will feel a force by the Lorentz force law, $\boldsymbol{F} = -e(\boldsymbol{v} \times \boldsymbol{B})$, performing the work per charge

$$\mathcal{E}_{\text{mag}} \equiv \oint_C d\boldsymbol{\ell} \cdot (\boldsymbol{v} \times \boldsymbol{B}) = -\oint_C \boldsymbol{B} \cdot (\boldsymbol{v} \times d\boldsymbol{\ell})\,. \tag{5.14}$$

The change in surface spanning C in time dt is

$$\int_{dS} da = S(t + dt) - S(t)\,, \tag{5.15}$$

so

$$\mathcal{E}_{\text{mag}} dt = -\int_{dS} \boldsymbol{B} \cdot d\boldsymbol{a} \equiv -d\Phi\,, \tag{5.16}$$

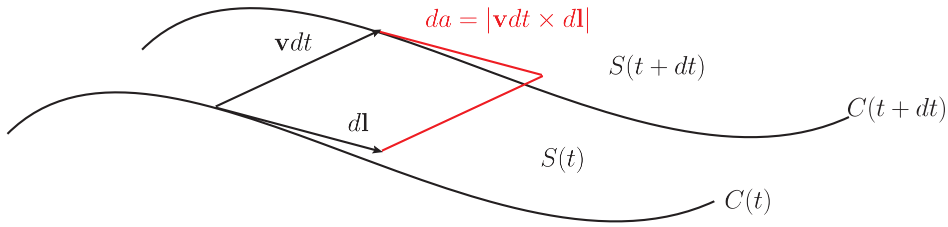

Figure 9: *The change in surface spanning C in time dt*

or

$$\mathcal{E}_{\rm mag} dt = -\frac{d\Phi}{dt}\,, \tag{5.17}$$

where

$$\Phi \equiv \int_S \boldsymbol{B} \cdot d\boldsymbol{a} \tag{5.18}$$

is the *magnetic flux* through surface S. But the EMF developed in this way, $\mathcal{E}_{\rm mag}$, must equal the electric EMF, $\mathcal{E}$, so

$$\mathcal{E} = \oint_C d\boldsymbol{\ell} \cdot \boldsymbol{E} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\int_S \boldsymbol{B} \cdot d\boldsymbol{a}\,. \tag{5.19}$$

By the equivalence of inertial frames, we expect it should not depend on whether wire is moving or $\boldsymbol{B}$ is changing:

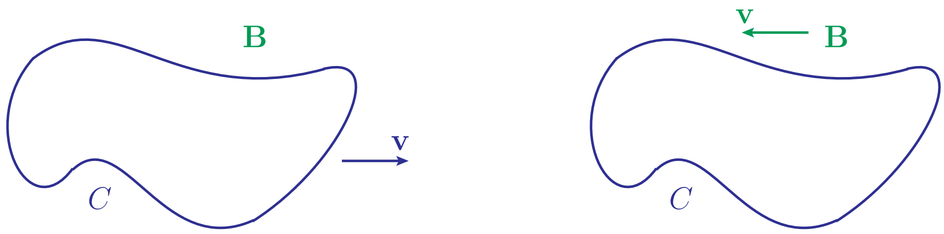

Figure 10: *Moving loop in magnetic field.*

So if we choose C and S fixed, then we expect

$$\oint_C d\boldsymbol{\ell} \cdot \boldsymbol{E} = -\int_S \frac{\partial \boldsymbol{B}}{\partial t} \cdot d\boldsymbol{a}\,, \tag{5.20}$$

or, by Stokes' theorem,

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}\,. \tag{5.21}$$

This is Faraday's law: a changing $\boldsymbol{B}$ field *induces* an $\boldsymbol{E}$ field, and so induces a current I to flow in a wire by $\mathcal{E} = I\,R$. By Ampère's law, this induced I will source a magnetic field $\boldsymbol{B}'$

and thus a contribution to the magnetic flux Φ'. The induced current I will flow in a direction such that the induced flux Φ' *opposes* the change in flux, $\partial\Phi/\partial t$, that induced it (Lenz's rule).

Note the formal similarity:

$$\oint_C d\boldsymbol{\ell}\cdot\boldsymbol{E} = -\frac{\partial\Phi}{\partial t} = -\int_S \frac{\partial\boldsymbol{B}}{\partial t}\cdot d\boldsymbol{a} \quad \text{Faraday}\,, \tag{5.22}$$

$$\oint_C d\boldsymbol{\ell}\cdot\boldsymbol{B} = \mu_0 I_{\text{encl.}} = \int_S \mu_0\boldsymbol{J}\cdot d\boldsymbol{a} \quad \text{Ampère}\,. \tag{5.23}$$

So we can use the same kind of symmetry arguments to determine the induced $\boldsymbol{E}$ for a given $-\partial\boldsymbol{B}/\partial t$. But the induced $\boldsymbol{E}$ causes a current, and the current sources a $\boldsymbol{B}$ field, and the changing $\boldsymbol{B}$ induces $\boldsymbol{E}$, which causes a current… So in general we have to solve self-consistently or in a "quasi-static" approximation which ignores the change in current due to the induced $\boldsymbol{E}$.

A closed current $I(t)$ sources $\boldsymbol{B}\propto I(t)$ and hence $\Phi(t)\propto I(t)$, where Φ is the flux through the circuit. We write $\Phi(t) = L\,I(t)$, where L is a constant, the *inductance*, which is a property of the geometry of the circuit. Then

$$\mathcal{E} = -\frac{d\Phi}{dt} = -L\frac{dI}{dt}\,. \tag{5.24}$$

So, for circuits, we have *resistors*, $\mathcal{E} = IR$; *inductors*, $\mathcal{E} = -L\,dI/dt$; and *capacitors*, $\mathcal{E}C = q$, so $d\mathcal{E}/dtC = I$. Charge conservation implies $I = I_1 + I_2$. These give a set of coupled differential equations which determine $\mathcal{E}(t)$, $I(t)$ in circuits. See Griffiths for a detailed discussion with many examples.

5.3 Energy in electric and magnetic fields

We calculate the work, W, required to assemble a charge configuration ρ, and current $\boldsymbol{J}$. We build it up by moving charges $q = \rho\,dV$ from infinity:

$$\begin{aligned}\frac{dW}{dt} &= -\frac{d}{dt}(\mathcal{E}q) = -q\frac{d\mathcal{E}}{dt} - \mathcal{E}\frac{dq}{dt} = q\frac{dV}{dt} - \mathcal{E}I = q\frac{d}{dt}\left(\frac{q}{C}\right) - IL\frac{dI}{dt}\\ &= \frac{d}{dt}\left(\frac{q^2}{2C} + \frac{LI^2}{2}\right),\end{aligned} \tag{5.25}$$

and so

$$W = \frac{q^2}{2C} + \frac{LI^2}{2} = \frac{1}{2}qV + \frac{1}{2}I\Phi\,. \tag{5.26}$$

We can rewrite this for continuous charge and current distributions, $q\to\rho\,dV$ and $Id\boldsymbol{\ell}\to\boldsymbol{J}dV$, so

$$\frac{1}{2}qV \to \frac{1}{2}\int d^3r\,\rho V\,, \tag{5.27}$$

$$\frac{I}{2}\Phi = \frac{I}{2}\int_S \boldsymbol{B}\cdot d\boldsymbol{a} = \frac{I}{2}\int_S (\boldsymbol{\nabla}\times\boldsymbol{A})\cdot d\boldsymbol{a} = \frac{I}{2}\int_C \boldsymbol{A}\cdot d\boldsymbol{\ell} \to \frac{1}{2}\int d^3r\, \boldsymbol{J}\cdot\boldsymbol{A}\,, \tag{5.28}$$

and so the work is

$$W = \frac{1}{2}\int d^3r \Big(\rho V + \boldsymbol{J}\cdot\boldsymbol{A}\Big)\,. \tag{5.29}$$

We can rewrite this in terms of the electric and magnetic fields:

$$\int d^3r\,\rho V = \epsilon_0 \int d^3r\,(\boldsymbol{\nabla}\cdot\boldsymbol{E})V = -\epsilon_0\int d^3r\,\boldsymbol{E}\cdot\boldsymbol{\nabla}V = \epsilon_0\int d^3r\,\boldsymbol{E}\cdot\boldsymbol{E}\,, \tag{5.30}$$

$$\int d^3r\,\boldsymbol{J}\cdot\boldsymbol{A} = \frac{1}{\mu_0}\int d^3r\,(\boldsymbol{\nabla}\times\boldsymbol{B})\cdot\boldsymbol{A} = \frac{1}{\mu_0}\int d^3r\,\boldsymbol{B}\cdot(\boldsymbol{\nabla}\times\boldsymbol{A}) = \frac{1}{\mu_0}\int d^3r\,\boldsymbol{B}\cdot\boldsymbol{B}\,. \tag{5.31}$$

(Here, we used Eq. (A.15) in the second equality.) It follows that

$$W = \frac{1}{2}\int d^3r\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)\,. \tag{5.32}$$

5.4 Maxwell's equations

Together with Faraday's law, the field equations become

$$\frac{\rho}{\epsilon_0} = \boldsymbol{\nabla}\cdot\boldsymbol{E}\,, \qquad 0 = \boldsymbol{\nabla}\cdot\boldsymbol{B}\,, \tag{5.33}$$

$$0 = \boldsymbol{\nabla}\times\boldsymbol{E} + \frac{\partial\boldsymbol{B}}{\partial t}\,, \qquad \mu_0\boldsymbol{J} = \boldsymbol{\nabla}\times\boldsymbol{B}\,. \tag{5.34}$$

Are these compatible with charge conservation, $\boldsymbol{\nabla}\cdot\boldsymbol{J} = -\partial\rho/\partial t$? Taking the divergence of Ampère's law gives

$$\boldsymbol{\nabla}\cdot\boldsymbol{J} = \frac{1}{\mu_0}\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\boldsymbol{B}) = 0\,. \tag{5.35}$$

We want on the right side

$$-\frac{\partial\rho}{\partial t} = -\frac{\partial}{\partial t}(\epsilon_0\boldsymbol{\nabla}\cdot\boldsymbol{E}) = \boldsymbol{\nabla}\cdot(-\epsilon_0\frac{\partial\boldsymbol{E}}{\partial t})\,. \tag{5.36}$$

We replace the right side of Ampère's law: $\boldsymbol{\nabla}\times\boldsymbol{B} \to \boldsymbol{\nabla}\times\boldsymbol{B} - \mu_0\epsilon_0\partial\boldsymbol{E}/\partial t$, so that

$$\boldsymbol{\nabla}\cdot\boldsymbol{J} = \frac{1}{\mu_0}\boldsymbol{\nabla}\cdot\left(\boldsymbol{\nabla}\times\boldsymbol{B} - \mu_0\epsilon_0\frac{\partial\boldsymbol{E}}{\partial t}\right) = 0 - \epsilon_0\boldsymbol{\nabla}\cdot\frac{\partial\boldsymbol{E}}{\partial t} = -\frac{\partial\rho}{\partial t}\,. \tag{5.37}$$

In this way, we get a consistent set of field equations:

$$\boldsymbol{\nabla}\cdot\boldsymbol{J} = -\frac{\partial\rho}{\partial t}\,, \tag{5.38}$$

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})\,, \tag{5.39}$$

$$\frac{\rho}{\epsilon_0} = \nabla \cdot \boldsymbol{E}\,, \qquad 0 = \nabla \cdot \boldsymbol{B}\,, \tag{5.40}$$

$$0 = \nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t}\,, \qquad \mu_0 \boldsymbol{J} = \nabla \times \boldsymbol{B} - \mu_0 \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}\,. \tag{5.41}$$

MAXWELL's new term implies that a changing electric field induces a magnetic field.

We can make the physical content more obvious by defining units intelligently. So far, we have introduced separate and arbitrary units for charge, electric, and magnetic fields. Say we redefine them as

$$\begin{cases} q \mapsto \alpha q \Rightarrow \rho \mapsto \alpha\rho\,, \boldsymbol{J} \mapsto \alpha \boldsymbol{J} \\ \boldsymbol{E} \mapsto \beta \boldsymbol{E} \\ \boldsymbol{B} \mapsto \gamma \boldsymbol{B} \end{cases} \tag{5.42}$$

for some constant α, β, γ. Then our equations become

$$\nabla \cdot \boldsymbol{J} = -\frac{\partial \rho}{\partial t}\,, \qquad \boldsymbol{F} = q(\alpha\beta \boldsymbol{E} + \alpha\gamma \boldsymbol{v} \times \boldsymbol{B})\,, \tag{5.43}$$

$$\frac{\alpha\rho}{\beta\epsilon_0} = \nabla \cdot \boldsymbol{E}\,, \qquad 0 = \nabla \cdot \boldsymbol{B}\,, \tag{5.44}$$

$$0 = \nabla \times \boldsymbol{E} + \frac{\gamma}{\beta}\frac{\partial \boldsymbol{B}}{\partial t}\,, \qquad \alpha\mu_0 \boldsymbol{J} = \gamma \nabla \times \boldsymbol{B} - \beta\mu_0\epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}\,. \tag{5.45}$$

We choose $\alpha\beta = 1$ to keep $\boldsymbol{F} = q\boldsymbol{E}$. Then we choose $\alpha/(\beta\epsilon_0) = 4\pi$ to make GAUSS' law simple. It follows that $\alpha = \sqrt{4\pi\epsilon_0}$ and $\beta = 1/\sqrt{4\pi\epsilon_0}$. Finally, we choose $\gamma = \sqrt{\mu_0/(4\pi)}$. Then the equations become

$$\nabla \cdot \boldsymbol{J} = -\frac{\partial \rho}{\partial t}\,, \qquad \boldsymbol{F} = q\left(\boldsymbol{E} + \frac{1}{c}\boldsymbol{v} \times \boldsymbol{B}\right)\,, \tag{5.46}$$

$$4\pi\rho = \nabla \cdot \boldsymbol{E}\,, \qquad 0 = \nabla \cdot \boldsymbol{B}\,, \tag{5.47}$$

$$0 = \nabla \times \boldsymbol{E} + \frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t}\,, \qquad \frac{4\pi}{c}\boldsymbol{J} = \nabla \times \boldsymbol{B} - \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t}\,, \tag{5.48}$$

where

$$c \equiv \frac{1}{\sqrt{\mu_0\epsilon_0}} \tag{5.49}$$

with units of velocity. This makes it clear that the only fundamental constant in EM is the *speed of light*,

$$c \approx 3 \times 10^8 \text{m/s}\,. \tag{5.50}$$

In "cgs" units, $[E] = [B] =$ force/charge. The quasi-static limit corresponds to $v \ll c$.

Finally, we discuss one more "versions" of MAXWELL's equations. In linear, homogeneous, isotropic and *dispersionless* matter (dielectrics, and para- or diamagnetics), we have

$$\rho_f = \nabla \cdot \boldsymbol{D}\,, \qquad 0 = \nabla \cdot \boldsymbol{B}\,, \tag{5.51}$$

$$0 = \nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t}\,, \qquad \boldsymbol{J}_f = \nabla \times \boldsymbol{H} - \frac{\partial \boldsymbol{D}}{\partial t}\,, \tag{5.52}$$

where $\boldsymbol{D} = \epsilon \boldsymbol{E}$, $\boldsymbol{H} = \boldsymbol{B}/\mu$, and $\epsilon = \epsilon_0(1 + \chi_e)$, $\mu = \mu_0(1 + \chi_m)$ are constants. More general matter has $\boldsymbol{D} = \epsilon \boldsymbol{E} + \boldsymbol{P}$, $\boldsymbol{H} = \boldsymbol{B}/\mu - \boldsymbol{M}$, and

$$P_i = \sum_{i=1}^{3} \epsilon_0 [\chi_e(\omega, \boldsymbol{r})]_i^j E_j + \mathcal{O}(E^2) , \tag{5.53}$$

$$M_i = \sum_{i=1}^{3} [\chi_m(\omega, \boldsymbol{r})]_i^j H_j + \mathcal{O}(H^2) . \tag{5.54}$$

Here, the ω dependence parameterizes time-dependent effects (dispersion), the $\boldsymbol{r}$ dependence represents inhomogeneities, the tensor indices indicate anisotropies, and the the higher-order terms parameterize non-linearities.

MAXWELL's equations, as differential equations, must be supplemented by boundary conditions at the edges of the system in question in order to have a unique solution. In the limit where continuous charge and current distributions become discontinuous (e.g. at sharp surfaces, etc.), boundary (or "matching") conditions can be deduced from MAXWELL's equations. Easiest from the integral form of the equations:

$$\oint_S \boldsymbol{E} \cdot d\boldsymbol{a} = \frac{1}{\epsilon_0} Q_{\text{encl.}} , \qquad \oint_S \boldsymbol{B} \cdot d\boldsymbol{a} = 0 , \tag{5.55}$$

$$\oint_C \boldsymbol{E} \cdot d\boldsymbol{\ell} = -\frac{\partial}{\partial t} \oint_S \boldsymbol{B} \cdot d\boldsymbol{a} , \qquad \oint_C \boldsymbol{B} \cdot d\boldsymbol{\ell} = \mu_0 I_{\text{encl.}} + \mu_0 \epsilon_0 \frac{\partial}{\partial t} \oint_S \boldsymbol{E} \cdot d\boldsymbol{a} . \tag{5.56}$$

It follows that

$$\epsilon_0 E_{\text{in}}^{\perp} - \epsilon_0 E_{\text{out}}^{\perp} = \sigma , \qquad B_{\text{in}}^{\perp} - B_{\text{out}}^{\perp} = 0 , \tag{5.57}$$

$$\boldsymbol{E}_{\text{in}}^{\parallel} - \boldsymbol{E}_{\text{out}}^{\parallel} = \boldsymbol{0} , \qquad \frac{1}{\mu_0} \boldsymbol{B}_{\text{in}}^{\parallel} - \frac{1}{\mu_0} \boldsymbol{B}_{\text{out}}^{\parallel} = \boldsymbol{K} \times \hat{\boldsymbol{n}} , \tag{5.58}$$

just as in electro- and magnetostatics.

6 Conservation laws

Charge conservation means that no (net) charge is destroyed or created – it just moves around:

$$\frac{\partial \rho}{\partial t} = -\boldsymbol{\nabla} \cdot \boldsymbol{J} . \tag{6.1}$$

The first term represents the change in charge density per time, the second term the rate of flow (current) out of the volume. Integrating over all space, we find the total charge

$$\int d^3 r \, \rho \equiv Q , \tag{6.2}$$

and

$$-\int d^3r\,\nabla\cdot\boldsymbol{J} = \oint_\infty \boldsymbol{J}\cdot d\boldsymbol{a} = 0\,. \tag{6.3}$$

It follows that

$$\frac{dQ}{dt} = 0\,, \tag{6.4}$$

the total charge is conserved.

We will find analogous descriptions of *energy*, *momentum*, and *angular momentum* carried by the elecromagnetic fields: they are not created or destroyed, they just move around.

6.1 Energy

We have seen (Eq. (5.32)) that the total energy carried by the electromagnetic field is

$$U = \frac{1}{2}\int d^3r\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)\,. \tag{6.5}$$

We can write this in terms of the *energy density*

$$\rho_U = \frac{\epsilon_0}{2}E^2 + \frac{1}{2\mu_0}B^2\,, \tag{6.6}$$

so that

$$U = \int d^3r\,\rho_U\,. \tag{6.7}$$

Maxwell's equations in vacuum impliy the local conservation of energy:

$$-\frac{\partial\rho_U}{\partial t} = \nabla\cdot\boldsymbol{S}\,, \tag{6.8}$$

where

$$\boldsymbol{S} \equiv \frac{1}{\mu_0}\boldsymbol{E}\times\boldsymbol{B} \tag{6.9}$$

is the *Poynting vector* (the energy current density).

Here is the proof:

$$\begin{aligned}\nabla\cdot\boldsymbol{S} &= \frac{1}{\mu_0}\nabla\cdot(\boldsymbol{E}\times\boldsymbol{B}) = \frac{1}{\mu_0}\boldsymbol{B}\cdot(\nabla\times\boldsymbol{E}) - \frac{1}{\mu_0}\boldsymbol{E}\cdot(\nabla\times\boldsymbol{B})\\ &= \frac{1}{\mu_0}\boldsymbol{B}\cdot\left(\frac{\partial\boldsymbol{B}}{\partial t}\right) - \frac{1}{\mu_0}\boldsymbol{E}\cdot\left(\frac{\partial\boldsymbol{E}}{\partial t}\right) = -\frac{\partial\rho_U}{\partial t}\,.\end{aligned} \tag{6.10}$$

If we include sources (charges) in Maxwell's equations, the same argument gives instead

$$-\frac{\partial\rho_U}{\partial t} = \nabla\cdot\boldsymbol{S} + \boldsymbol{E}\cdot\boldsymbol{J}\,, \tag{6.11}$$

where the last term represents the work per volume per time done on the charges. In other words, the energy of the electromagnetic field is conserved: it only changes by moving around ($\nabla\cdot\boldsymbol{S}$) or adding kinetic energy to charges ($\boldsymbol{E}\cdot\boldsymbol{J}$).

6.2 Momentum

The same applies to the mmentum carried by the electromagnetic field. Let $\boldsymbol{p}(\boldsymbol{r}, t)$ be the *momentum density* of the electromagnetic field. Each component p_i, $i = 1, 2, 3$, should be conserved, so we expect conservation equation of the form

$$-\frac{\partial p_i}{\partial t} = \boldsymbol{\nabla} \cdot \boldsymbol{T}_i\,, \tag{6.12}$$

for some *momentum current density vectors* $\boldsymbol{T}_i$. If we write these vectors in components (with unit basis vectors $\hat{\boldsymbol{e}}_i$),

$$\boldsymbol{T}_i = \sum_{j=1}^{3} T_{ij} \hat{\boldsymbol{e}}_j\,, \tag{6.13}$$

then

$$-\frac{\partial p_i}{\partial t} = \sum_{j=1}^{3} \frac{\partial T_{ij}}{\partial x_j}\,. \tag{6.14}$$

$T_{ij}(\boldsymbol{r}, t)$ is called the *electromagnetic stress tensor.* T_{ij} represents the jth component of the momentum per unit time per unit area moving in ith direction. If $i = j$ then dp_i/dt per area is parallel to $\hat{\boldsymbol{e}}_i$. But $dp_i/dt = F_i$ is the force, so T_{ii} is the force per area or the *pressure* in ith direction.

If there is charged matter present, the momentum carried by the electromagnetic field can change by accelerating the particles:

$$-\frac{\partial p_i}{\partial t} = \frac{\partial p_i^{\text{(particles)}}}{\partial t} + \sum_{j=1}^{3} \frac{\partial T_{ji}}{\partial x_j}\,. \tag{6.15}$$

But

$$\frac{\partial p_i^{\text{(particles)}}}{\partial t} = f_i\,, \tag{6.16}$$

the electromagnetic force density on the particles,

$$\boldsymbol{f} = \rho \boldsymbol{E} + \boldsymbol{J} \times \boldsymbol{B}\,. \tag{6.17}$$

So, if we can find p_i and T_{ji} as functions of the electric and magnetic fields, such that

$$-\frac{\partial p_i}{\partial t} = f_i + \sum_{j=1}^{3} \frac{\partial T_{ji}}{\partial x_j}\,, \tag{6.18}$$

by virtue of Maxwell's equations, then p_i is the momentum density of the electromagnetic field, and T_{ji} is the electromagnetic stress tensor. We find that

$$\boldsymbol{p} = \mu_0 \epsilon_0 \boldsymbol{S} = \epsilon_0 \boldsymbol{E} \times \boldsymbol{B}\,, \tag{6.19}$$

$$T_{ji} = T_{ij} = \epsilon_0 \big(E_i E_j - \tfrac{1}{2} \delta_{ij} E^2 \big) + \frac{1}{\mu_0} \big(B_i B_j - \tfrac{1}{2} \delta_{ij} B^2 \big)\,. \tag{6.20}$$

Here is the proof:

$$
\begin{aligned}
\boldsymbol{f} = \rho \boldsymbol{E} + \boldsymbol{J} \times \boldsymbol{B} &= (\epsilon_0 \boldsymbol{\nabla} \cdot \boldsymbol{E})\boldsymbol{E} + \left(\frac{1}{\mu_0}\boldsymbol{\nabla} \times \boldsymbol{B} - \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}\right) \times \boldsymbol{B} \\
&= (\epsilon_0 \boldsymbol{\nabla} \cdot \boldsymbol{E})\boldsymbol{E} + \frac{1}{\mu_0}(\boldsymbol{\nabla} \cdot \boldsymbol{B})\boldsymbol{B} - \frac{1}{\mu_0}\boldsymbol{B} \times (\boldsymbol{\nabla} \times \boldsymbol{B}) - \epsilon_0 \frac{\partial}{\partial t}(\boldsymbol{E} \times \boldsymbol{B}) + \epsilon_0 \boldsymbol{E} \times \frac{\partial \boldsymbol{B}}{\partial t} \\
&= (\epsilon_0 \boldsymbol{\nabla} \cdot \boldsymbol{E})\boldsymbol{E} + \frac{1}{\mu_0}(\boldsymbol{\nabla} \cdot \boldsymbol{B})\boldsymbol{B} - \frac{1}{\mu_0}\boldsymbol{B} \times (\boldsymbol{\nabla} \times \boldsymbol{B}) - \frac{\partial \boldsymbol{p}}{\partial t} + \epsilon_0 \boldsymbol{E} \times (\boldsymbol{\nabla} \times \boldsymbol{E})\,.
\end{aligned}
\tag{6.21}
$$

Now we use the identity (A.23) to find

$$
\begin{aligned}
\boldsymbol{f} &= -\frac{\partial \boldsymbol{p}}{\partial t} + \epsilon_0\left\{(\boldsymbol{\nabla} \cdot \boldsymbol{E})\boldsymbol{E} + (\boldsymbol{E} \cdot \boldsymbol{\nabla})\boldsymbol{E} - \tfrac{1}{2}\boldsymbol{\nabla}(\boldsymbol{E}^2)\right\} \\
&\qquad + \frac{1}{\mu_0}\left\{(\boldsymbol{\nabla} \cdot \boldsymbol{B})\boldsymbol{B} + (\boldsymbol{B} \cdot \boldsymbol{\nabla})\boldsymbol{B} - \tfrac{1}{2}\boldsymbol{\nabla}(\boldsymbol{B}^2)\right\} \\
&= -\frac{\partial \boldsymbol{p}}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{T}\,.
\end{aligned}
\tag{6.22}
$$

6.3 Angular momentum

We define the *angular momentum density* of the electromagnetic field as

$$
\boldsymbol{\ell} \equiv \boldsymbol{r} \times \boldsymbol{p}(\boldsymbol{r}) = \epsilon_0 \boldsymbol{r} \times (\boldsymbol{E} \times \boldsymbol{B})\,. \tag{6.23}
$$

So even static electric and magnetic fields can carry angular momentum!

Angular momentum is also locally conserved: there is an angular momentum current density tensor $\boldsymbol{\mathcal{L}}_i \equiv \boldsymbol{r} \times \boldsymbol{T}_i$, and

$$
-\frac{\partial \ell_i}{\partial t} = \boldsymbol{\nabla} \times \boldsymbol{\mathcal{L}}_i\,. \tag{6.24}
$$

Or, in component notation,

$$
\ell_i = \sum_{j,k=1}^{3} \epsilon_{ijk} x_j p_k\,, \qquad \mathcal{L}_{ij} = \sum_{k,l=1}^{3} \epsilon_{ikl} x_k T_{lj}\,. \tag{6.25}
$$

In conclusion, electromagnetic fields carry energy, momentum, and angular momentum, just as does matter.

7 Electromagnetic waves

7.1 General comments

Waves are, vaguely, any disturbance of a continuous system which can carry energy from one place to another. An example of a continuous system are electric and magnetic fields, $\boldsymbol{E}(\boldsymbol{r}, t)$ and $\boldsymbol{B}(\boldsymbol{r}, t)$. More generally, continuous systems are described by *fields* $f(\boldsymbol{r}, t)$, satisfying some *equations of motion.*

We generally expect equations of motion to be *local*: the values of the field are determined by the values of the field infinitesimally close by. It follows that the equations of motion can be written as *differential equations* (like MAXWELL's equations).

If we assume that the size of the disturbance is small enough, then we can approximate the equations of motion as *linear* differential equations (like MAXWELL's equations),

$$\mathcal{D}(\boldsymbol{r}, t, \boldsymbol{\nabla}, \partial/\partial t) f(\boldsymbol{r}, t) = 0\,, \tag{7.1}$$

where $\mathcal{D}$ is some differential operator.

If we assume that the system is *homogeneous* (translation invariant in space and time), then the equations of motion cannot depend explicitly on $\boldsymbol{r}, t$, and we have

$$\mathcal{D}(\boldsymbol{\nabla}, \partial/\partial t) f(\boldsymbol{r}, t) = 0\,, \tag{7.2}$$

like MAXWELL's equations in vacuum.

We assume the equations of motion are at most *second order in derivatives*: in t because of Newtonian mechanics (accelerations are functions of positions and velocities), and in $\boldsymbol{r}$ to have invariance under change of inertial frame $\boldsymbol{r} \to \boldsymbol{r} + \boldsymbol{v}t$, i.e. "relativity":

$$\left[A_{ij}\frac{\partial^2}{\partial x_i \partial x_j} + B_i\frac{\partial^2}{\partial x_i \partial t} + C\frac{\partial^2}{\partial t^2} + D_i\frac{\partial}{\partial x_i} + E\frac{\partial}{\partial t} + F\right] f(\boldsymbol{r}, t) = 0\,. \tag{7.3}$$

(This is unlike MAXWELL's equations which are *first order*. But we will see that by solving for some fields, we can recast a system of first-order linear differential equations as second-order differential equations.)

If we assume *isotropy* (rotational invariance) and *time reversal invariance*, then $A_{ij} = A\delta_{ij}$, and $B_i = D_i = E = 0$, so

$$\left[A\boldsymbol{\nabla}^2 + C\frac{\partial^2}{\partial t^2} + F\right] f(\boldsymbol{r}, t) = 0\,. \tag{7.4}$$

Energy conservation follows from the above assumptions: the equations of motion imply there exists an energy density ρ_U and an energy density current $\boldsymbol{S}$ such that

$$-\frac{\partial \rho_U}{\partial t} = \boldsymbol{\nabla} \cdot \boldsymbol{S}\,. \tag{7.5}$$

They are (check this!)

$$\rho_U = -\frac{C}{2A}\left(\frac{\partial f}{\partial t}\right)^2 + \tfrac{1}{2}(\boldsymbol{\nabla} f) \cdot (\boldsymbol{\nabla} f) - \frac{F}{2A} f^2\,, \tag{7.6}$$

$$\boldsymbol{S} = \frac{\partial f}{\partial t}\boldsymbol{\nabla} f\,. \tag{7.7}$$

(Momentum conservation also follows, so there is a conserved momentum density $\boldsymbol{p}$ and stress tensor T_{ij}, just as with MAXWELL's equations.)

We assume *stability* of the system: small disturbances of the system do not grow with time. This means that all the terms in the energy density, ρ_U, are positive. Then energy conservation means that they cannot grow without bound. Thus rename $A/C \equiv -v^2$, $A/F \equiv -\ell^2$:

$$\left[\boldsymbol{\nabla}^2 - \frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \frac{1}{\ell^2}\right] f(\boldsymbol{r}, t) = 0\,. \tag{7.8}$$

This is the *dispersive wave equation*, with v the *wave velocity* and ℓ the *dispersion length*.

Thus, small disturbances of practically any stable continuous system will have waves which will satisfy (to some approximation) the above wave equation (7.8).

The combination

$$\boldsymbol{\nabla}^2 - \frac{1}{v^2}\frac{\partial^2}{\partial t^2} \equiv \Box \tag{7.9}$$

is called the *d'Alembertian*.

Electromagnetic waves in vacuum are *dispersionless*, $\ell = \infty$, and travel with speed of light, $v = c$. But electromagnetic waves in matter have $\ell < \infty$, $v \neq c$.

7.2 Dispersionless waves in one dimension

We have $1/\ell = 0$ and $\boldsymbol{\nabla} \to d/dz$, so the wave equation becomes

$$0 = \frac{d^2 f}{dz^2} - \frac{1}{v^2}\frac{d^2 f}{dt^2} = \left(\frac{d}{dz} - \frac{1}{v}\frac{d}{dt}\right)\left(\frac{d}{dz} + \frac{1}{v}\frac{d}{dt}\right) f\,. \tag{7.10}$$

We change variables to $u = z - vt$, $w = z + vt$, so

$$z = \tfrac{1}{2}(w + u)\,, \qquad t = \tfrac{1}{2v}(w - u)\,, \tag{7.11}$$

and

$$\frac{\partial}{\partial u} = \frac{\partial z}{\partial u}\frac{\partial}{\partial z} + \frac{\partial t}{\partial u}\frac{\partial}{\partial t} = \frac{1}{2}\left(\frac{\partial}{\partial z} - \frac{1}{v}\frac{\partial}{\partial t}\right), \tag{7.12}$$

$$\frac{\partial}{\partial w} = \frac{\partial z}{\partial w}\frac{\partial}{\partial z} + \frac{\partial t}{\partial w}\frac{\partial}{\partial t} = -\frac{1}{2}\left(\frac{\partial}{\partial z} + \frac{1}{v}\frac{\partial}{\partial t}\right), \tag{7.13}$$

so the wave equation becomes simply

$$\frac{\partial^2 f}{\partial u \partial w} = 0\,. \tag{7.14}$$

The general solution is $f(u, w) = g(u) + h(w)$, so

$$f(z, t) = g(z - vt) + h(z + vt)\,, \tag{7.15}$$

for arbitrary g, h. This is a superposition of a right-moving and a left-moving wave.

We can *Fourier transform* an arbitrary function:

$$\varphi(z) = \int_{-\infty}^{\infty} dk\, \tilde{A}(k) e^{ikz} \quad \Leftrightarrow \quad \tilde{A}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dz\, \varphi(z) e^{-ikz} \,. \tag{7.16}$$

In other words, we can decompose it into a sum of sinusoidal functions $e^{ikz} = \cos(kz) + i\sin(kz)$. Here, k is the *wave number*,

$$k \equiv \frac{2\pi}{\lambda}\,, \tag{7.17}$$

where $\lambda \geq 0$ is the *wave length*.

Fourier transforms relate complex functions. If φ is real, then $\tilde{A}$ is complex, but satisfies

$$\varphi(z) \in \mathbb{R} \quad \Leftrightarrow \quad \tilde{A}(k)^* = \tilde{A}(-k)\,. \tag{7.18}$$

Let's apply this to $g(z - vt)$ and $h(z + vt)$:

$$g(z - vt) = \int_{-\infty}^{\infty} dk\, \tilde{B}(k) e^{ik(z-vt)}\,, \tag{7.19}$$

$$h(z + vt) = \int_{-\infty}^{\infty} dk\, \tilde{C}(k) e^{ik(z+vt)}\,, \tag{7.20}$$

with

$$\tilde{B}(k)^* = \tilde{B}(-k)\,, \qquad \tilde{C}(k)^* = \tilde{C}(-k)\,. \tag{7.21}$$

Then the general solution is

$$f(z,t) = \int_{-\infty}^{\infty} dk\, \left[\tilde{B}(k) e^{ik(z-vt)} + \tilde{C}(k) e^{ik(z+vt)}\right]. \tag{7.22}$$

Defining the *(angular) frequency*

$$\omega \equiv v|k| \geq 0\,, \tag{7.23}$$

we obtain

$$
\begin{aligned}
f(z,t) &= \int_0^\infty dk\,\left[\tilde{B}(k)e^{i(kz-\omega t)} + \tilde{C}(k)e^{i(kz+\omega t)}\right] \\
&\quad + \int_{-\infty}^0 dk\,\left[\tilde{B}(k)e^{i(kz+\omega t)} + \tilde{C}(k)e^{i(kz-\omega t)}\right] \\
&= \int_0^\infty dk\,\left[\tilde{B}(k)e^{i(kz-\omega t)} + \tilde{C}(-k)^* e^{i(kz+\omega t)}\right] \\
&\quad + \int_{-\infty}^0 dk\,\left[\tilde{B}(-k)^* e^{i(kz+\omega t)} + \tilde{C}(k)e^{i(kz-\omega t)}\right] \\
&= \int_0^\infty dk\,\tilde{B}(k)e^{i(kz-\omega t)} + \int_{-\infty}^0 dk\,\tilde{C}(k)e^{i(kz-\omega t)}\big] \\
&\quad + \int_0^\infty dk\,\tilde{B}(k)^* e^{-i(kz-\omega t)} + \int_{-\infty}^0 dk\,\tilde{C}(k)^* e^{-i(kz-\omega t)}\big]\,.
\end{aligned}
\tag{7.24}
$$

Now define

$$
\tfrac{1}{2}\tilde{A}(k) \equiv \begin{cases} \tilde{B}(k) & \text{for } k > 0\,, \\ \tilde{C}(k) & \text{for } k < 0\,. \end{cases} \tag{7.25}
$$

Then we have

$$
f(z,t) = \frac{1}{2}\int_{-\infty}^{\infty} dk\,\tilde{A}(k)e^{i(kz-\omega t)} + \text{c.c.}\,. \tag{7.26}
$$

Here, "c.c." means the complex conjugate of the preceding expression. This packages left-moving ($k < 0$) and right-moving ($k > 0$) waves together.

Griffiths defines the *complex waveform*

$$
\tilde{f}(z,t) \equiv \int_{-\infty}^{\infty} dk\,\tilde{A}(k)e^{i(kz-\omega t)}\,, \tag{7.27}
$$

where $\tilde{A}(k)$ is a complex amplitude, so that the real waveform is

$$
f(z,t) = \text{Re}[\tilde{f}(z,t)]\,. \tag{7.28}
$$

This language and notation is universally used in physics and we will use it to describe electromagnetic waves below. The point is that since our wave equations are linear, we are free to work with complex solutions and just take the real part at the end. Explicitly, we have

$$
\begin{aligned}
f(z,t) = \mathrm{Re}[\tilde{f}(z,t)] &= \int\limits_{-\infty}^{\infty} dk\, \mathrm{Re}[\tilde{A}(k) e^{i(kz-\omega t)}] \\
&= \int\limits_{-\infty}^{\infty} dk \left\{ \mathrm{Re}[\tilde{A}(k)]\mathrm{Re}[e^{i(kz-\omega t)}] - \mathrm{Im}[\tilde{A}(k)]\mathrm{Im}[e^{i(kz-\omega t)}] \right\} \\
&\equiv \int\limits_{-\infty}^{\infty} dk \left\{ R(k)\cos(kz-\omega t) - I(k)\sin(kz-\omega t) \right\} .
\end{aligned}
\tag{7.29}
$$

If we define

$$
A(k) \equiv \sqrt{R(k)^2 + I(k)^2}\,, \qquad \delta(k) \equiv -\arctan\left(\frac{I(k)}{R(k)}\right), \tag{7.30}
$$

then

$$
R(k) = A(k)\cos(\delta(k))\,, \qquad I(k) = -A(k)\sin(\delta(k))\,. \tag{7.31}
$$

This is equivalent to

$$
\tilde{A}(k) = A(k) e^{-i\delta(k)} \tag{7.32}
$$

and

$$
\tilde{f}(z,t) = \int\limits_{-\infty}^{\infty} dk\, A(k)\cos(kz - \omega t - \delta(k))\,, \tag{7.33}
$$

with *amplitude* $A(k)$ and *phase shift* $\delta(k)$.

7.3 Electromagnetic waves in vacuum

MAXWELL's equations in vacuum are (we use the notation $\epsilon_0\mu_0 = 1/c^2$)

$$
\boldsymbol{\nabla}\cdot\boldsymbol{E} = 0\,, \qquad \boldsymbol{\nabla}\cdot\boldsymbol{B} = 0\,, \tag{7.34}
$$

$$
\boldsymbol{\nabla}\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t}\,, \qquad \boldsymbol{\nabla}\times\boldsymbol{B} = -\frac{1}{c^2}\frac{\partial\boldsymbol{E}}{\partial t}\,. \tag{7.35}
$$

Taking the curl of the equation (7.35), and then using Eq. (A.22) and Eqs. (7.34), we find

$$
\boldsymbol{\nabla}^2\boldsymbol{E} - \frac{1}{c^2}\frac{\partial^2\boldsymbol{E}}{\partial t^2} = \boldsymbol{\nabla}^2\boldsymbol{B} - \frac{1}{c^2}\frac{\partial^2\boldsymbol{B}}{\partial t^2} = 0\,. \tag{7.36}
$$

All components of $\boldsymbol{E}$ and $\boldsymbol{B}$ satisfy the wave equation with velocity c.

7.3.1 Plane-wave decomposition

We can decompose the solution into *plane waves* by Fourier transforming as in the one-dimensional case above. The only changes are the *wave vector* $k \to \boldsymbol{k}$, the definition of the wave number via $k \equiv |\boldsymbol{k}| = 2\pi/\lambda$, and the *direction of wave propagation*, $\hat{\boldsymbol{k}} \equiv \boldsymbol{k}/k$. The angular frequency is again defined as $\omega = ck$. The exponential factor becomes $e^{i(kz-\omega t)} \to e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}$, the integration measure $dk \to d^3k$, and the amplitude function becomes $\tilde{A}(k) \to \tilde{E}_0(\boldsymbol{k})\hat{\boldsymbol{n}}(\boldsymbol{k})$, where E_0 is a complex amplitude and $\hat{\boldsymbol{n}}$ is the *polarization vector* (the direction of the $\boldsymbol{E}$ field). It follows that the complex waveforms for the electric and magnetic fields are

$$\tilde{\boldsymbol{E}}(\boldsymbol{r},t) = \int d^3k\, \tilde{E}_0(\boldsymbol{k})\hat{\boldsymbol{n}}(\boldsymbol{k})e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}\,, \tag{7.37}$$

$$\tilde{\boldsymbol{B}}(\boldsymbol{r},t) = \int d^3k\, \tilde{B}_0(\boldsymbol{k})\hat{\boldsymbol{m}}(\boldsymbol{k})e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}\,. \tag{7.38}$$

These satisfy the wave equation, but we still need to check MAXWELL's equation. For instance,

$$\begin{aligned}\boldsymbol{\nabla}\cdot\tilde{\boldsymbol{E}}(\boldsymbol{r},t) &= \int d^3k\, \tilde{E}_0(\boldsymbol{k})(\hat{\boldsymbol{n}}(\boldsymbol{k})\cdot\boldsymbol{\nabla})e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} \\ &= i\int d^3k\, \tilde{E}_0(\boldsymbol{k})(\hat{\boldsymbol{n}}(\boldsymbol{k})\cdot\boldsymbol{k})e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} = 0\,,\end{aligned} \tag{7.39}$$

so $\hat{\boldsymbol{n}}(\boldsymbol{k})\cdot\boldsymbol{k} = 0$ – the wave must be *transverse*. Similarly,

$$\begin{aligned}\boldsymbol{\nabla}\times\tilde{\boldsymbol{E}}(\boldsymbol{r},t) &= -\int d^3k\, \tilde{E}_0(\boldsymbol{k})(\hat{\boldsymbol{n}}\times\boldsymbol{\nabla})e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} \\ &= -i\int d^3k\, \tilde{E}_0(\boldsymbol{k})(\hat{\boldsymbol{n}}\times\boldsymbol{k})e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} = 0\,,\end{aligned} \tag{7.40}$$

and

$$\begin{aligned}\frac{\partial\tilde{\boldsymbol{B}}(\boldsymbol{r},t)}{\partial t} &= -\int d^3k\, \tilde{B}_0(\boldsymbol{k})\hat{\boldsymbol{m}}\frac{\partial}{\partial t}e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} \\ &= -i\int d^3k\, \tilde{B}_0(\boldsymbol{k})\hat{\boldsymbol{m}}\omega e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} = 0\,,\end{aligned} \tag{7.41}$$

so we must have

$$\omega\hat{\boldsymbol{m}}(\boldsymbol{k})\tilde{B}_0(\boldsymbol{k}) = \boldsymbol{k}\times\hat{\boldsymbol{n}}(\boldsymbol{k})E_0(\boldsymbol{k})\,. \tag{7.42}$$

That means we must have $\hat{\boldsymbol{m}}(\boldsymbol{k}) = \boldsymbol{k}\times\hat{\boldsymbol{n}}(\boldsymbol{k})$ (the magnetic field is orthogonal to the electric field) and $\tilde{B}_0(\boldsymbol{k}) = \tilde{E}_0(\boldsymbol{k})/c$. One can check in a similar fashion that the other two MAXWELL equations are also satisfied. The net result is

$$\tilde{\boldsymbol{E}}(\boldsymbol{r},t) = \int d^3k\, \tilde{E}_0(\boldsymbol{k})\hat{\boldsymbol{n}}(\boldsymbol{k})e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}\,, \tag{7.43}$$

$$\tilde{\boldsymbol{B}}(\boldsymbol{r},t) = \frac{1}{c}\int d^3k\, \tilde{E}_0(\boldsymbol{k})\boldsymbol{k}\times\hat{\boldsymbol{n}}(\boldsymbol{k})e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}\,, \tag{7.44}$$

with $\omega = ck$ and $\boldsymbol{k} \times \hat{\boldsymbol{n}} = 0$. Electromagnetic waves can be written as a superposition of plane waves.

To get real fields, we just take the real parts. Note that $\tilde{E}_0$ and $\hat{\boldsymbol{n}}$ can be complex, while $\boldsymbol{k}$, $\boldsymbol{r}$, ω, t are real. So

$$\boldsymbol{E}(\boldsymbol{r}, t) = \int d^3k\, \tilde{E}_{\boldsymbol{k}}(\boldsymbol{r}, t)\,, \qquad \boldsymbol{B}(\boldsymbol{r}, t) = \frac{1}{c} \int d^3k\, \boldsymbol{k} \times \tilde{E}_{\boldsymbol{k}}(\boldsymbol{r}, t)\,, \tag{7.45}$$

where the electric field plane wave is

$$\boldsymbol{E}(\boldsymbol{r}, t) = E_0(\boldsymbol{k}) \hat{\boldsymbol{n}}(\boldsymbol{k}) \cos(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t - \delta(\boldsymbol{k}))\,, \tag{7.46}$$

with $E_0 \cos\delta - iE_0 \sin\delta = \tilde{E}_0$. (Here we have chosen $\hat{\boldsymbol{n}}(\boldsymbol{k})$ to be real.)

7.3.2 Non-plane wave decomposition

Plane waves form a *basis* of solutions to the wave equation, but they are not the only such basis. Just as in separation of variables in electrostatics we found that sinusoidal solutions $e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$ were convenient for rectilinear boundary conditions, but that spherical harmonics $r^\ell P_\ell(\cos\theta)$, $r^{-\ell-1} P_\ell(\cos\theta)$ were convenient for spherical boundary conditions, the same is true for waves. For example, (ignoring polarization) outgoing spherical waves:

$$\tilde{E}(r, \theta, \phi, t) \sim \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \tilde{E}_{\ell,m} h_\ell(kr) Y_{\ell,m}(\theta, \phi)\,, \tag{7.47}$$

where the h_ℓ are HANKEL *functions* and the $Y_{\ell,m}$ are *spherical harmonics.*

7.3.3 Beams of light

Light is an electromagnetic wave. Typically, light is a superposition of plane waves of many different $\boldsymbol{k}$ and $\hat{\boldsymbol{n}}(\boldsymbol{k})$ and amplitudes and phases. A *collimated* beam is one in which all $\boldsymbol{k}$ point in the same direction. A *polarized* beam is one where all $\hat{\boldsymbol{n}}$ point in the same direction. A *monochromatic* beam is one in which all $k = |\boldsymbol{k}| = \omega/c$ are the same. A *coherent* beam is one in which all phases $\delta(\boldsymbol{k})$ are the same. A collimated, polarized, monochromatic, and coherent beam is given by a single FOURIER mode:

$$\tilde{\boldsymbol{E}}(\boldsymbol{r}, t) = \tilde{E}_0 \hat{\boldsymbol{n}} e^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)}\,. \tag{7.48}$$

7.3.4 Energy and momentum of electromagnetic waves

The energy density is

$$u = \frac{\epsilon_0}{2} E^2 + \frac{1}{2\mu_0} B^2 = \frac{\epsilon_0}{2} \left(E^2 + c^2 B^2\right)\,. \tag{7.49}$$

We have to be careful using complex waveforms because $(\mathrm{Re}\tilde{E})^2 \neq \mathrm{Re}(\tilde{E}^2)$. For a single plane-wave mode

$$u = \frac{\epsilon_0}{2} \left(E_0^2 \cos^2(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t - \delta) + c^2 \frac{1}{c^2} E_0^2 \cos^2(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t - \delta)\right) = \epsilon_0 E_0^2 \cos^2(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t - \delta)\,. \tag{7.50}$$

Similarly, we find the energy density current (Poynting vector)

$$\boldsymbol{S} = c\,u\hat{\boldsymbol{k}}\,. \tag{7.51}$$

This is an energy density times a velocity, so electromagnetic waves can carry energy at the speed of light. The momentum density is

$$\boldsymbol{p} = \frac{1}{c^2}\boldsymbol{S} = \frac{u}{c}\hat{\boldsymbol{k}}\,. \tag{7.52}$$

The momentum of electromagnetic waves has magnitude energy divided by the speed of light c.

If calculate the average over time, defined by

$$\langle X\rangle \equiv \lim_{T\to\infty}\frac{1}{T}\int_0^T dt\,X(t)\,, \tag{7.53}$$

then, since

$$\langle\cos^2(\omega t+\delta)\rangle = \lim_{T\to\infty}\frac{1}{T}\int_0^T dt\ \cos^2(\omega t+\delta) = \frac{1}{2}\,, \tag{7.54}$$

we have

$$\langle u\rangle = \frac{1}{2}\epsilon_0 E_0^2\,, \tag{7.55}$$

$$\langle\boldsymbol{S}\rangle = \frac{c}{2}\epsilon_0 E_0^2\hat{\boldsymbol{k}}\,, \tag{7.56}$$

$$I \equiv \langle S\rangle = \frac{c}{2}\epsilon_0 E_0^2\,, \tag{7.57}$$

$$\langle\boldsymbol{p}\rangle = \frac{1}{2c}\epsilon_0 E_0^2\hat{\boldsymbol{k}}\,, \tag{7.58}$$

$$P \equiv \langle\boldsymbol{p}\rangle c = \frac{1}{2}\epsilon_0 E_0^2 = \frac{I}{c}\,. \tag{7.59}$$

Here, I is the *intensity* (power per area), and P is the *radiation pressure.*

One can do the same for angular momentum. It turns out that a collimated monochromatic wave carries *angular momentum density*

$$\boldsymbol{\mathcal{J}} = \pm\frac{\epsilon_0}{2\omega}|\tilde{E}_\pm|^2\hat{\boldsymbol{k}}\,, \tag{7.60}$$

where $\tilde{E}_\pm$ are the complex amplitudes of waves with *circular polarization*

$$\tilde{\boldsymbol{E}}_\pm(\boldsymbol{r},t) = \tilde{E}_\pm\hat{\boldsymbol{n}}_\pm e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}\,, \tag{7.61}$$

where

$$\hat{\boldsymbol{n}}_\pm \equiv \frac{1}{\sqrt{2}}\big(\hat{\boldsymbol{n}}_1 \pm i\hat{\boldsymbol{n}}_2\big)\,, \tag{7.62}$$

and $\hat{\boldsymbol{n}}_1$, $\hat{\boldsymbol{n}}_2$, $\hat{\boldsymbol{k}}$ form a real orthonormal basis of unit vectors. In terms of real polarizations, circular polarization is a superposition of a plane wave polarized in the $\hat{\boldsymbol{n}}_1$ direction and another of equal amplitude polarized in the $\hat{\boldsymbol{n}}_2$ direction, but with a $\pi/2$ *phase shift*; for instance,

$$\boldsymbol{E}_+(\boldsymbol{r},t) = E_+\big[\hat{\boldsymbol{x}}\cos(kz-\omega t) + \hat{\boldsymbol{y}}\sin(kz-\omega t)\big]\,. \tag{7.63}$$

Energy and momeetum in a superposition of plane waves: since u, $\boldsymbol{p}$, …are quadratic in the fields, one cannot simply add the u of each plane wave. If $\boldsymbol{E} = \boldsymbol{E}_1 + \boldsymbol{E}_2$, then

$$u = \epsilon_0 E^2 = \epsilon_0\big(E_1^2 + E_2^2 + 2\boldsymbol{E}_1\cdot\boldsymbol{E}_2\big) \neq u_1 + u_2\,. \tag{7.64}$$

The cross terms between plane-wave components are called *interference terms.* If one time averages, different frequencies do not interfere:

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} dt\, e^{-i\omega_1 t}e^{+i\omega_2 t} = \propto \delta(\omega_1-\omega_2)\,. \tag{7.65}$$

But different $\hat{\boldsymbol{k}}$, $\hat{\boldsymbol{n}}(\hat{\boldsymbol{k}})$, $E_0(\hat{\boldsymbol{k}})$ will, in general, interfere.

7.4 Electromagnetic waves in matter

7.4.1 Electromagnetic waves in linear media

In a linear, homogeneous, isotropic, dielectric, dia-/paramagnetic material with no free charges or currents, Maxwell's equations are the same as in vacuum but with $\epsilon_0 \to \epsilon$ and $\mu_0 \to \mu$. So electromagnetic waves travel just as in vacuum, but with speed $v = 1/\sqrt{\epsilon\mu}$. Thus, such materials are transparent. We define the *index of refraction* of the material to be

$$n \equiv \frac{v}{c} = \sqrt{\frac{\epsilon_0\mu_0}{\epsilon\mu}}\,. \tag{7.66}$$

For most materials $n < 1$ or $v < c$. What happens at the boundary between two such materials? Expect: reflection and transmission. This is governed by the boundary conditions

$$\epsilon_1 E_1^\perp = \epsilon_2 E_2^\perp\,, \qquad E_1^\parallel E_2^\parallel\,, \tag{7.67}$$

$$B_1^\perp = B_2^\perp\,, \qquad \frac{1}{\mu_1}B_1^\parallel\frac{1}{\mu_2}B_2^\parallel\,. \tag{7.68}$$

Consider an incident plane wave in $+z$ direction normal (perpendicular) to interface at $z=0$ and polarized in x direction (see Fig. 11):

$$\tilde{\boldsymbol{E}}_I(z,t) = \tilde{E}_{0,I}e^{i(kz-\omega t)}\hat{\boldsymbol{x}}\,, \tag{7.69}$$

$$\tilde{\boldsymbol{B}}_I(z,t) = \frac{1}{v_1}\tilde{E}_{0,I}e^{i(kz-\omega t)}\hat{\boldsymbol{y}}\,. \tag{7.70}$$

$E_{I\perp} = B_{I\perp} = 0$, so boundary conditions imply that all $\perp$ components vanish, so waves can only be in $\pm z$ direction. By assumption there is no incoming wave from the right moving in the $-z$ direction. So there will only be a reflected and a transmitted wave:

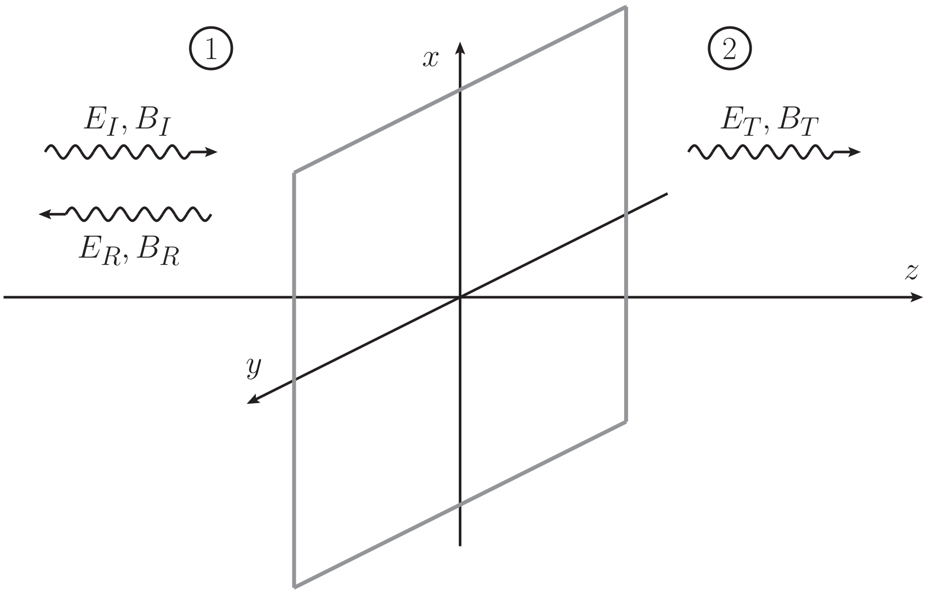

Figure 11: *Transmission and reflection of a plane wave (normal incidence).*

8 Potentials and fields

9 Radiation

10 Special Relativity

A Survey of Mathematical Topics

A.1 Differential Calculus

Given a function $f : \mathbb{R}^d \to \mathbb{R}$, $f(x_1, \ldots, x_d) \in \mathbb{R}$, we define the *exterior derivative*

$$df \equiv \frac{\partial f}{\partial x_1} dx_1 + \ldots + \frac{\partial f}{\partial x_d} dx_d \,. \tag{A.1}$$

Here, $\partial f / \partial x_i$ is the derivative w.r.t. x_i, keeping all other x_j fixed. The exterior derivative is linear,

$$d(af + bg) = a\, df + b\, dg \,, \tag{A.2}$$

where $a, b \in \mathbb{R}$ are constants and f, g are functions, and satisfies the *Leibniz rule*

$$d(fg) = f\, dg + g\, df \,. \tag{A.3}$$

We can write the exterior derivative in term of the *gradient* $\boldsymbol{\nabla} f$ as $df = (\boldsymbol{\nabla} f) \cdot d\boldsymbol{\ell}$, where (in Cartesian coordinates)

$$\boldsymbol{\nabla} f \equiv \frac{\partial f}{\partial x_1} \hat{\boldsymbol{x}}_1 + \ldots + \frac{\partial f}{\partial x_d} \hat{\boldsymbol{x}}_d \,, \tag{A.4}$$

and

$$d\boldsymbol{\ell} \equiv dx_1\,\hat{\boldsymbol{x}}_1 + \ldots + dx_d\,\hat{\boldsymbol{x}}_d\,. \tag{A.5}$$

Note that these are vector-valued functions. The geometrical interpretation is that $\boldsymbol{\nabla} f$ points into the direction of maximum increase of f, and $|\boldsymbol{\nabla} f|$ is the slope of f in that direction. We define the *gradient differential operator* (referred to as "nabla")

$$\boldsymbol{\nabla} \equiv \sum_{i=1}^{d} \hat{\boldsymbol{x}}_i \frac{\partial}{\partial x_1}\,. \tag{A.6}$$

It takes scalar functions to vector-valued functions.

If f is a scalat function and $\boldsymbol{v}$ a vector-valued function (a *vector field*), we can form, in addition to the gradient, the *divergence* of a vector field $\boldsymbol{\nabla}\cdot\boldsymbol{v}$ (a scalar function), and the *curl* of a vector field $\boldsymbol{\nabla}\times\boldsymbol{v}$ (another vector field). In Cartesian components, the divergence is given by

$$\boldsymbol{\nabla}\cdot\boldsymbol{v} = \sum_{i=1}^{d} \frac{\partial v_i}{\partial x_1}\,. \tag{A.7}$$

The divergence is proportional to the rate $\boldsymbol{v}$ "spreads out" in space. The curl is given, in Cartesian components, by

$$\boldsymbol{\nabla}\times\boldsymbol{v} = \det\begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{pmatrix} \tag{A.8}$$

$|\boldsymbol{\nabla}\times\boldsymbol{v}|$ is proportional to the "vorticity" of the vector field. It is defined only in three dimensions.

$\boldsymbol{\nabla}$ is a linear differential operator, so

$$\boldsymbol{\nabla}(af + bg) = a\boldsymbol{\nabla} f + b\boldsymbol{\nabla} g\,, \tag{A.9}$$
$$\boldsymbol{\nabla}\cdot(a\boldsymbol{v} + b\boldsymbol{w}) = a\boldsymbol{\nabla}\cdot\boldsymbol{v} + b\boldsymbol{\nabla}\cdot\boldsymbol{w}\,, \tag{A.10}$$
$$\boldsymbol{\nabla}\times(a\boldsymbol{v} + b\boldsymbol{w}) = a\boldsymbol{\nabla}\times\boldsymbol{v} + b\boldsymbol{\nabla}\times\boldsymbol{w}\,, \tag{A.11}$$

for $a, b \in \mathbb{R}$ constants. It satisfies the following useful *product rules*,

$$\boldsymbol{\nabla}(fg) = f\boldsymbol{\nabla} g + g\boldsymbol{\nabla} f\,, \tag{A.12}$$
$$\boldsymbol{\nabla}\cdot(f\boldsymbol{v}) = f\boldsymbol{\nabla}\cdot\boldsymbol{v} + \boldsymbol{v}\cdot\boldsymbol{\nabla} f\,, \tag{A.13}$$
$$\boldsymbol{\nabla}(\boldsymbol{v}\cdot\boldsymbol{w}) = \boldsymbol{v}\times(\boldsymbol{\nabla}\times\boldsymbol{w}) + \boldsymbol{w}\times(\boldsymbol{\nabla}\times\boldsymbol{v}) + (\boldsymbol{\nabla}\cdot\boldsymbol{v})\boldsymbol{w} + (\boldsymbol{\nabla}\cdot\boldsymbol{w})\boldsymbol{v}\,, \tag{A.14}$$
$$\boldsymbol{\nabla}\cdot(\boldsymbol{v}\times\boldsymbol{w}) = \boldsymbol{w}\cdot(\boldsymbol{\nabla}\times\boldsymbol{v}) - \boldsymbol{v}\cdot(\boldsymbol{\nabla}\times\boldsymbol{w})\,, \tag{A.15}$$
$$\boldsymbol{\nabla}\times(f\boldsymbol{v}) = f\boldsymbol{\nabla}\times\boldsymbol{v} - \boldsymbol{v}\times\boldsymbol{\nabla} f\,, \tag{A.16}$$
$$\boldsymbol{\nabla}\times(\boldsymbol{v}\times\boldsymbol{w}) = (\boldsymbol{w}\cdot\boldsymbol{\nabla})\boldsymbol{v} - (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{w} - \boldsymbol{w}(\boldsymbol{\nabla}\cdot\boldsymbol{v}) + \boldsymbol{v}(\boldsymbol{\nabla}\cdot\boldsymbol{w})\,. \tag{A.17}$$

Applying the nabla operator twice results in

$$\boldsymbol{\nabla}\cdot\boldsymbol{\nabla} f \equiv \nabla^2 f = \sum_{i=1}^{d} \frac{\partial^2 f}{\partial x_i^2}\,, \tag{A.18}$$

$$\nabla^2 \boldsymbol{v} = \sum_{i=1}^{d} \nabla^2 v_i \hat{\boldsymbol{x}}_i \,, \tag{A.19}$$

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} f = 0 \,, \tag{A.20}$$

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times f) = 0 \,, \tag{A.21}$$

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{v}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{v}) - \nabla^2 \boldsymbol{v} \,. \tag{A.22}$$

Another useful equation that follows simply from Eq. (A.14) is

$$\boldsymbol{v} \times \big(\boldsymbol{\nabla} \times \boldsymbol{v}\big) = \tfrac{1}{2}\boldsymbol{\nabla}(\boldsymbol{v}^2) - (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} \,. \tag{A.23}$$

A.2 Integral Calculus

The *fundamental theorem of calculus* is

$$\int\limits_{\text{region}} \text{derivative of } f = \int\limits_{\text{boundary of region}} f \qquad . \tag{A.24}$$

In three dimensions, we get many different versions. Scalar integrals:

$$\int\limits_{C} d\boldsymbol{\ell} \cdot \boldsymbol{\nabla} f = \int\limits_{\partial C} f \,, \tag{A.25}$$

$$\int\limits_{S} d\boldsymbol{a} \cdot (\boldsymbol{\nabla} \times \boldsymbol{v}) = \int\limits_{\partial S} d\boldsymbol{\ell} \cdot \boldsymbol{v} \,, \tag{A.26}$$

$$\int\limits_{V} dV (\boldsymbol{\nabla} \cdot \boldsymbol{v}) = \int\limits_{\partial V} d\boldsymbol{a} \cdot \boldsymbol{v} \,. \tag{A.27}$$

Vector-valued integrals:

$$\int\limits_{S} d\boldsymbol{a} \times \boldsymbol{\nabla} f = \int\limits_{\partial S} d\boldsymbol{\ell}\, f \,, \tag{A.28}$$

$$\int\limits_{V} dV\, \boldsymbol{\nabla} f = \int\limits_{\partial V} d\boldsymbol{a}\, f \,, \tag{A.29}$$

$$\int\limits_{V} dV (\boldsymbol{\nabla} \times \boldsymbol{v}) = \int\limits_{\partial V} d\boldsymbol{a} \times \boldsymbol{v} \,. \tag{A.30}$$

To compute these, we need to know the following definitions: C, S, V are oriented curves, surfaces, volumes, and ∂C, ∂S, ∂V their boundaries; $\int_{\partial C}$ is a "zero-dimensional" integral (the usual familiar integral), $\int_C d\boldsymbol{\ell}$ a one-dimensional *line integral*, $\int_S d\boldsymbol{a}$ a two-dimensional *surface integral*, $\int_V dV$ a three-dimensional *volume integral*.

Integrating the LEIBNIZ rule and using the fundamental theorem yields *integration-by-parts* identities:

$$\int_R g\,df + \int_R f\,dg = \int_R d(fg) = \int_{\partial R} fg\,. \tag{A.31}$$

For a single variable, this gives the familiar result

$$\int_a^b dx\left(g\frac{df}{dx} + f\frac{dg}{dx}\right) = fg\Big|_a^b\,, \tag{A.32}$$

or,

$$\int_a^b dx\,f\frac{dg}{dx} = -\int_a^b dx\,g\frac{df}{dx} + fg\Big|_a^b\,. \tag{A.33}$$

In three dimensions, we just apply the same logic to any three-dimensional LEIBNIZ rule. For instance, Eq. (A.16) implies

$$\int_S d\boldsymbol{a}\cdot\boldsymbol{\nabla}\times(f\boldsymbol{v}) = \int_{\partial S} d\boldsymbol{\ell}\cdot\boldsymbol{v}\,f = \int_S d\boldsymbol{a}\cdot(\boldsymbol{\nabla}f\times\boldsymbol{v}) + \int_S d\boldsymbol{a}\cdot(\boldsymbol{\nabla}\times\boldsymbol{v})f\,. \tag{A.34}$$

A.2.1 Line integrals

We describe an *oriented curve* from point $\boldsymbol{a}$ to point $\boldsymbol{b}$ as the set of points $C = \{\boldsymbol{r}(s), 0 \le s \le 1\}$, with $\boldsymbol{r}(0) = \boldsymbol{a}$ and $\boldsymbol{r}(1) = \boldsymbol{b}$. Then

$$d\boldsymbol{\ell}(s) \equiv \frac{d\boldsymbol{r}(s)}{ds}ds \tag{A.35}$$

is the infinitesimal tangent vector at s, and we define the line integral as the sum of infinitesimal tangent vectors,

$$\int_C d\boldsymbol{\ell} = \int_0^1 ds\,\frac{d\boldsymbol{r}(s)}{ds}\,. \tag{A.36}$$

The *boundary of a curve* are just the two endpoints, $\partial C = \{\boldsymbol{a}, \boldsymbol{b}\}$. The integral over the boundary is just the "usual" integral,

$$\int_{\partial C} f = f(\boldsymbol{b}) - f(\boldsymbol{a}) = f\Big|_{\boldsymbol{a}}^{\boldsymbol{b}}\,. \tag{A.37}$$

Note that is C is a *closed curve*, then $\boldsymbol{b} = \boldsymbol{a}$, so $\partial C = \emptyset$ (the empty set).

In Cartesian coordinates, we can represent the curve as

$$C = \{\boldsymbol{r}(s) = x(s)\hat{\boldsymbol{x}} + y(s)\hat{\boldsymbol{y}} + z(s)\hat{\boldsymbol{z}}\,, 0 \le s \le 1\}\,. \tag{A.38}$$

Then the line integral becomes

$$\int_C d\boldsymbol{\ell} = \int_0^1 ds\,\frac{d\boldsymbol{r}(s)}{ds} = \int_0^1 ds\left(\frac{\partial x}{\partial s}\hat{\boldsymbol{x}} + \frac{\partial y}{\partial s}\hat{\boldsymbol{y}} + \frac{\partial z}{\partial s}\hat{\boldsymbol{z}}\right) \equiv \int_C \left(dx\,\hat{\boldsymbol{x}} + dy\,\hat{\boldsymbol{y}} + dz\,\hat{\boldsymbol{z}}\right). \tag{A.39}$$

A.2.2 Surface integrals

We can describe an *oriented surface* as the set of points $S = \{\boldsymbol{r}(s,t), 0 \le s \le 1, 0 \le t \le 1\}$, with

$$\boldsymbol{r}(s,t) = x(s,t)\hat{\boldsymbol{x}} + y(s,t)\hat{\boldsymbol{y}} + z(s,t)\hat{\boldsymbol{z}}\,. \tag{A.40}$$

The *infinitesimal normal surface area* is given by

$$d\boldsymbol{a} \equiv ds\,dt\,\frac{d\boldsymbol{r}}{ds} \times \frac{d\boldsymbol{r}}{dt}\,, \tag{A.41}$$

and the surface integral is

$$\int\limits_S d\boldsymbol{a} = \int\limits_0^1 ds \int\limits_0^1 dt\,\frac{d\boldsymbol{r}}{ds} \times \frac{d\boldsymbol{r}}{dt}\,. \tag{A.42}$$

The *boundary of a surface* is a closed curve. We often write the integral as

$$\int\limits_{\partial S} d\boldsymbol{\ell} \equiv \oint\limits_{\partial S} d\boldsymbol{\ell}\,. \tag{A.43}$$

If the surface is closed, then its boundary is the empty set.

In Cartesian coordinates, the surface integral becomes

$$\int\limits_S d\boldsymbol{a} = \int\limits_0^1 ds \int\limits_0^1 dt\,\frac{d\boldsymbol{r}}{ds} \times \frac{d\boldsymbol{r}}{dt} = \int\limits_0^1 ds \int\limits_0^1 dt\,\det\begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t} \end{pmatrix}\,. \tag{A.44}$$

Surfaces can be decsribed implicitly, as the solution of an equation: $S = \{\boldsymbol{r}|u(\boldsymbol{r}) = 0\}$, where u is a scalar function. If we can find *any* two functions $s(\boldsymbol{r})$, $t(\boldsymbol{r})$, such that

$$0 < \boldsymbol{\nabla} s \cdot \big(\boldsymbol{\nabla} t \times \boldsymbol{\nabla} u\big) = \det\begin{pmatrix} \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} & \frac{\partial s}{\partial z} \\ \frac{\partial t}{\partial x} & \frac{\partial t}{\partial y} & \frac{\partial t}{\partial z} \\ \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \end{pmatrix} \equiv J\,, \tag{A.45}$$

then (s,t) parameterizes S, $d\boldsymbol{a} \propto \boldsymbol{\nabla} u$, and

$$\int\limits_S d\boldsymbol{a} = \int ds\,dt\,J^{-1}\,\boldsymbol{\nabla} u\big|_{u=0}\,. \tag{A.46}$$

A.2.3 Volume integrals

We can describe an *oriented volume* as the set of points $V = \{\boldsymbol{r}(s,t,u), 0 \le s,t,u \le 1\}$. E.g., we can just take $(s,t,u) = (x,y,z)$ as the Cartesian coordinates. The *infinitesimal volume element* is given by

$$dV \equiv ds\,dt\,du\,\frac{d\boldsymbol{r}}{ds} \cdot \left(\frac{d\boldsymbol{r}}{dt} \times \frac{d\boldsymbol{r}}{du}\right), \tag{A.47}$$

and the volume integral is

$$\int_V dV = \int_0^1 ds \int_0^1 dt \int_0^1 du \, \frac{d\boldsymbol{r}}{ds} \cdot \left(\frac{d\boldsymbol{r}}{dt} \times \frac{d\boldsymbol{r}}{du} \right). \tag{A.48}$$

The *boundary of a volume* is a closed surface. We often write the integral as

$$\int_{\partial V} d\boldsymbol{a} \equiv \oint_{\partial V} d\boldsymbol{a}\,. \tag{A.49}$$

If the surface is closed, then its boundary is the empty set. The orientation of the surface is inherited from the orientation of the volume: $d\boldsymbol{a}$ points *out of* V if $dV > 0$, and $d\boldsymbol{a}$ points *into* V if $dV < 0$.

In Cartesian coordinates, the volume integral becomes

$$\int_V dV = \int_0^1 ds \int_0^1 dt \int_0^1 du \, \frac{d\boldsymbol{r}}{ds} \cdot \left(\frac{d\boldsymbol{r}}{dt} \times \frac{d\boldsymbol{r}}{du} \right) = \int \int \int ds\, dt\, du \,\det \begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \end{pmatrix}. \tag{A.50}$$

A.3 Separation of variables

A systematic method for finding series solutions of Laplace's equation when the boundary conditions are on surfaces described by constant values of coordinates in some coordinate system.

Cartesian: boundaries at x = constant or y = constant or z = constant. So, good in "rectangular" domains. Note that x_i, y_i, z_i could be at infinity.

Spherical: boundaries at r = constant or ϕ = constant or θ = constant.

Cylindrical: boundaries at r = constant or ϕ = constant or z = constant.

The idea is to look for solutions of $\nabla^2 V = 0$ such that: $V(x, y, z) = X(x) \cdot Y(y) \cdot Z(z)$ (Cartesian), $V(r, \phi, \theta) = R(r) \cdot \Phi(\phi) \cdot \Theta(\theta)$ (spherical), or $V(r, \phi, z) = R(r) \cdot \Phi(\phi) \cdot Z(z)$ (cylindrical). Then we will find that $\nabla^2 V = 0$ separates into three *ordinary* differential equations for the factor functions (X, Y, Z) etc., each depending on a real constant (k_x^2, k_y^2, k_z^2). These ordinary differential equations can be solved once and for all and we get a family of *special functions*, e.g.:

Cartesian: $X_k \sim e^{\pm ikx} \sim \sin(kx), \cos(kx)$ or $X_k \sim e^{\pm kx} \sim \sinh(kx), \cosh(kx)$, and the same for Y_k, Z_k.

Spherical: $R_k \sim r^{\ell_\pm}$, with $\ell_\pm = -(1 \mp \sqrt{1 + 4k^2})/2$, $\Theta(\theta) \sim P_\ell^m(\cos\theta)$ (associated Legendre polynomials), $\Phi(\phi) \sim e^{\pm ik\phi}$ or $\Phi(\phi) \sim e^{\pm k\phi}$.

Cylindrical: $R_k \sim J_k, Y_k$ (Bessel functions), $\Phi(\phi) \sim e^{\pm ik\phi}$ or $\Phi(\phi) \sim e^{\pm k\phi}$, $Z(z) \sim e^{\pm ikz}$ or $Z(z) \sim e^{\pm kz}$.

"Special functions" have the very nice properties of *orthogonality* and *completeness*. If $N_k(x)$ is a set of special functions, then, heuristically,

$$\int dx \, N_k(x) N_{k'}(x) = \delta_{kk'} \tag{ON}$$

(orthogonality) and

$$\sum_k N_k(x) N_k(x') = \delta(x - x') \tag{CO}$$

(completeness). Eq. (CO) implies that any function $f(x)$ can be written as a linear combination of the $N_k(x)$:

$$\begin{aligned} f(x) &= \int dx'\, f(x')\delta(x - x') = \int dx'\, f(x') \sum_k N_k(x) N_k(x') \\ &= \sum_k N_k(x) \Big(\int dx'\, f(x') N_k(x') \Big) \equiv \sum_k c_k N_k(x)\,. \end{aligned} \tag{A.51}$$

Eq. (ON) implies that the coefficients c_k can be determined uniquely:

$$\begin{aligned} \int dx\, f(x) N_k(x) &= \int dx \Big(\sum_{k'} c_{k'} N_{k'}(x) \Big) N_k(x) \\ &= \sum_{k'} c_{k'} \int dx\, N_{k'}(x) N_k(x) = \sum_{k'} c_{k'} \delta_{kk'} = c_k\,. \end{aligned} \tag{A.52}$$

(This is just like expanding any vector in a basis, $\boldsymbol{v} = \sum_k v_k \boldsymbol{e}_k$, with basis vectors $\boldsymbol{e}_k$ and coefficients c_k. In fact, the sets of special functions form an *infinite-dimensional* vector space.)

A familiar example is $N_k(x) \to N_m(\phi) = e^{in\phi}$, with $n \in \mathbb{Z}$. In this case, the orthogonality relation is

$$\frac{1}{2\pi} \int_0^{2\pi} d\phi\, e^{in\phi} e^{-im\phi} = \delta_{nm}\,, \tag{A.53}$$

and the completeness relation is

$$\sum_n e^{in\phi} e^{-in\phi'} = 2\pi\delta(\phi - \phi')\,. \tag{A.54}$$

It follows that

$$f(\phi) = \sum_n c_n e^{in\phi}\,, \tag{A.55}$$

with

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} d\phi\, f(\phi) e^{-in\phi}\,. \tag{A.56}$$

This is the *Fourier series* for periodic functions.

Similarly, if $N_k(x) = e^{ikx}$, with $k \in \mathbb{R}$, then the orthogonality relation is

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} dx\, e^{ikx} e^{-ik'x} = \delta(k - k')\,, \tag{A.57}$$

and the completeness relation is

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ikx}e^{-ikx'} = \delta(x-x')\,. \tag{A.58}$$

It follows that

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, \tilde{f}(\phi)e^{ikx}\,, \tag{A.59}$$

with

$$\tilde{f}(x) = \int_{-\infty}^{\infty} dx\, f(x)e^{-ikx}\,. \tag{A.60}$$

This is the familiar *Fourier transform.*

These examples make it clear that the precise set of special functions used depends on the boundary conditions. For instance, if if the boundaries are at $x = x_0$ and $x = x_1$, and both x_0 and x_1 are finite, then we get a Fourier series, but if $x_0 = \pm\infty$ and/or $x_1 = \pm\infty$, then we get a Fourier transform.

A.3.1 An example: Cartesian coordinates

Here, we have $\nabla^2 V = 0$ such that: $V(x,y,z) = X(x)\cdot Y(y)\cdot Z(z)$. Explicitly,

$$\begin{aligned} 0 &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)X(x)Y(y)Z(z) \\ &= \left(\partial_x^2 X\right)YZ + \left(\partial_x^2 Y\right)XZ + \left(\partial_x^2 Z\right)XY\,, \end{aligned} \tag{A.61}$$

so

$$0 = \frac{\partial_x^2 X}{X} + \frac{\partial_y^2 Y}{Y} + \frac{\partial_z^2 Z}{Z}\,. \tag{A.62}$$

As the first term is only a function of x, the second only of y, and the third only of z, we must have

$$\frac{\partial_x^2 X}{X} = -k_x^2\,, \quad \frac{\partial_y^2 Y}{Y} = -k_y^2\,, \quad \frac{\partial_z^2 Z}{Z} = -k_z^2\,, \tag{A.63}$$

with $k_x^2 + k_y^2 + k_z^2 = 0$. (This implies that not all of the k_i^2 can be positive!)

The solutions to Eq. (A.63) are

$$X(x) = \tilde{A}\cos(k_x x) + \tilde{B}\sin(k_x x) = Ae^{ikx} + Be^{-ikx}\,, \tag{A.64}$$

if $k_x^2 > 0$, and

$$X(x) = \tilde{A}\cosh(\kappa_x x) + \tilde{B}\sinh(\kappa_x x) = Ae^{\kappa x} + Be^{-\kappa x}\,, \tag{A.65}$$

if $\kappa_x^2 = -k_x^2 < 0$ (so $\kappa_x^2 > 0$). Similarly for Y, Z. The values that the k_x^2, k_y^2, k_z^2 can take are determined by the *boundary conditions.*

A.3.2 An example: spherical coordinates

As an example, we solve LAPLACE's equation in the region $r_1 < r < r_2$, with boundary conditions

$$V(r = r_1, \theta, \phi) = 0\,, \tag{A.66}$$

$$V(r = r_2, \theta, \phi) = g_2(\theta, \phi)\,. \tag{A.67}$$

We have (see Eq. (B.7))

$$0 = \nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}\,. \tag{A.68}$$

Inserting $V(r, \phi, \theta) = R(r) \cdot \Phi(\phi) \cdot \Theta(\theta)$ and multiplying by $r^2/V(r, \phi, \theta)$, we have

$$0 = \frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{1}{\Theta\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) + \frac{1}{\Phi\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}\,. \tag{A.69}$$

The first term is a function of r only, while the second and third terms depend only on θ and ϕ, so we must have

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) = \lambda R\,, \tag{R}$$

and

$$\frac{1}{\Theta\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) + \frac{1}{\Phi\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} = -\lambda\,, \tag{A.70}$$

with $\lambda \in \mathbb{R}$. Multiplying the last equation by $\sin^2\theta$, we find

$$\frac{\sin\theta}{\Theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) + \lambda\sin^2\theta + \frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2} = 0\,. \tag{A.71}$$

Since now the last term depends on θ only, it must be constant,

$$\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2} \equiv -m^2\,. \tag{A.72}$$

It follows then that

$$\frac{\partial^2\Phi}{\partial\phi^2} = -m^2\Phi\,. \tag{Φ}$$

and

$$\frac{\sin\theta}{\Theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) + \lambda\Theta\sin^2\theta = m^2\Theta\,. \tag{Θ}$$

Let's solve Eq. (Φ) first. The solutions are

$$\Phi_m = e^{im\phi}\,, \quad m \in \mathbb{R}\,, \tag{A.73}$$

for $m^2 > 0$, or

$$\Phi_\mu = e^{i\mu\phi}\,, \quad \mu \in \mathbb{R}\,, \tag{A.74}$$

for $m^2 = -mu^2 < 0$. What are the boundary conditions on $\Phi_{m,\mu}$? Periodicity in ϕ requires that

$$\Phi_{m,\mu}(\phi + 2\pi) = \Phi_{m,\mu}(\phi) \,. \tag{A.75}$$

There is no solution (apart from $\mu = 0$) for Φ_μ. For Φ_m we get $e^{2\pi i m} = 1$, so $m \in \mathbb{Z}$.

Now let's look at Eq. (Θ). What are the boundary conditions? We want $\Theta(\theta)$ to be regular at $\theta = 0, \pi$. It follows that $\lambda = \ell(\ell+1)$ with $\ell \in \mathbb{Z}$ and $\ell > |m|$. The solutions of Eq. (Θ) are then the *associated Legendre polynomials* $P_\ell^m(\cos\theta, \sin\theta)$.

For this course, we will stick to situations where only $m = 0$ contributes. This means only $\Theta_0(\phi) = 1$ is allowed, which means we are restricting to problems where there is no ϕ dependence, i.e., there is rotational symmetry about the z axis.

So, we have to modify our problem to have boundary condition $V(r = r_2, \theta, \phi) = g_2(\theta)$ (i.e., no ϕ dependence). When $m = 0$, the solutions of Eq. (Θ) that are regular at $\theta = 0, \pi$ are the *Legendre polynomials*

$$\Theta_\ell(\theta) = P_\ell(\cos\theta) \,, \tag{A.76}$$

with $0 \leq \ell \in \mathbb{Z}$, and

$$P_\ell(x) = \frac{1}{2^\ell \ell!}\left(\frac{d}{dx}\right)^\ell (x^2 - 1)^\ell \,. \tag{A.77}$$

For instance,

$$P_0(x) = 1 \,, \tag{A.78}$$
$$P_1(x) = x \,, \tag{A.79}$$
$$P_2(x) = \tfrac{1}{2}(3x^2 - 1) \,, \tag{A.80}$$
$$P_3(x) = \tfrac{1}{2}(5x^3 - 3x) \,, \tag{A.81}$$
$$P_4(x) = \tfrac{1}{8}(35x^4 - 30x^2 + 3) \,, \tag{A.82}$$
$$P_5(x) = \tfrac{1}{8}(63x^5 - 70x^3 + 15x) \,. \tag{A.83}$$

There are normalized to $P_\ell(1) = 1$ and satisfy

$$P_\ell(-x) = (-1)^\ell P_\ell(x) \,, \tag{A.84}$$

as well as

$$\int_{-1}^{1} dx\, P_m(x) P_n(x) = \int_0^\pi \sin\theta\, d\theta\, P_m(\cos\theta) P_n(\cos\theta) = \frac{2}{2n+1}\delta_{mn} \tag{A.85}$$

and

$$\sum_{\ell=0}^{\infty} \frac{2\ell+1}{2} P_\ell(x) P_\ell(x') = \delta(x - x') \,. \tag{A.86}$$

Now look at Eq. (R):

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right) = \ell(\ell+1) R \,. \tag{A.87}$$

Notice that this equation is invariant under the rescaling $r \to \alpha r$, so we guess the solutions $R \propto r^a$. Plugging this in we find $a(a+1) = \ell(\ell+1)$, so $a = \ell$ or $a = -\ell - 1$. So the general solution is

$$R_\ell(r) = A_\ell r^\ell + B_\ell r^{-\ell-1} \,. \tag{A.88}$$

Putting this all together, we have

$$V(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + B_\ell r^{-\ell-1}\right) P_\ell(\cos\theta) \,. \tag{A.89}$$

(This solution is only good if (i) there is no ϕ dependence, and (ii) it covers the full range $0 \le \theta \le \pi$.) Now we apply the boundary conditions:

$$0 = V(r_1,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r_1^\ell + B_\ell r_1^{-\ell-1}\right) P_\ell(\cos\theta) \,. \tag{A.90}$$

Since the $P_\ell(\cos\theta)$ form a complete set of functions, we have

$$A_\ell r_1^\ell + B_\ell r_1^{-\ell-1} = 0 \tag{A.91}$$

for all $\ell \ge 0$. Next,

$$g_2(\theta) = V(r_2,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r_2^\ell + B_\ell r_2^{-\ell-1}\right) P_\ell(\cos\theta) \,. \tag{A.92}$$

Now we use the orthogonality of the $P_\ell(\cos\theta)$:

$$\begin{aligned} &\int_0^\pi \sin\theta \, d\theta \, g_2(\theta) P_n(\cos\theta) \\ &= \sum_{\ell=0}^{\infty} \left(A_\ell r_2^\ell + B_\ell r_2^{-\ell-1}\right) \int_0^\pi \sin\theta \, d\theta \, P_\ell(\cos\theta) P_n(\cos\theta) \\ &= \sum_{\ell=0}^{\infty} \left(A_\ell r_2^\ell + B_\ell r_2^{-\ell-1}\right) \frac{2\delta_{\ell n}}{2n+1} = \frac{2}{2n+1}\left(A_n r_2^n + B_n r_2^{-n-1}\right) \end{aligned} \tag{A.93}$$

Eqs. (A.91) and (A.93) determine A_ℓ, B_ℓ for all ℓ.

B Curvilinear coordinate systems

B.1 Spherical coordinates

Relation to Cartesian coordinates:

$$x = r\sin\theta\cos\phi \,, \quad y = r\sin\theta\sin\phi \,, \quad z = r\cos\theta \,. \tag{B.1}$$

Infinitesimal line element:

$$d\boldsymbol{\ell} = dr\,\hat{\boldsymbol{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\,\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}\,. \tag{B.2}$$

Infinitesimal volume element:

$$dV = r^2\sin\theta\,dr\,d\theta\,d\phi\,. \tag{B.3}$$

Gradient: For a scalar field f, we have

$$\boldsymbol{\nabla} f = \frac{\partial f}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}}\,. \tag{B.4}$$

Divergence: For a vector field $\boldsymbol{A}$, we have

$$\boldsymbol{\nabla}\cdot\boldsymbol{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\big(r^2A_r\big) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\big(A_\theta\sin\theta\big) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}\,. \tag{B.5}$$

Curl: For a vector field $\boldsymbol{A}$, we have

$$\begin{aligned}\boldsymbol{\nabla}\times\boldsymbol{A} = {}& \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\big(A_\phi\sin\theta\big) - \frac{\partial A_\theta}{\partial\phi}\right]\hat{\boldsymbol{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}\big(rA_\phi\big)\right]\hat{\boldsymbol{\theta}} \\ &+ \frac{1}{r}\left[\frac{\partial}{\partial r}\big(rA_\theta\big) - \frac{\partial A_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}\,.\end{aligned} \tag{B.6}$$

Laplacian:

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}\,. \tag{B.7}$$

B.2 Cylindrical coordinates

Infinitesimal line element:

$$d\boldsymbol{\ell} = dr\,\hat{\boldsymbol{r}} + r\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\boldsymbol{z}}\,. \tag{B.8}$$

Infinitesimal volume element:

$$dV = r\,dr\,d\phi\,dz\,. \tag{B.9}$$

Gradient: For a scalar field f, we have

$$\boldsymbol{\nabla} f = \frac{\partial f}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial f}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial f}{\partial z}\hat{\boldsymbol{z}}\,. \tag{B.10}$$

Divergence: For a vector field $\boldsymbol{A}$, we have

$$\boldsymbol{\nabla}\cdot\boldsymbol{A} = \frac{1}{r}\frac{\partial}{\partial r}\big(rA_r\big) + \frac{1}{r}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}\,. \tag{B.11}$$

Curl: For a vector field $\boldsymbol{A}$, we have

$$\boldsymbol{\nabla}\times\boldsymbol{A} = \left[\frac{1}{r}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right]\hat{\boldsymbol{r}} + \left[\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}\big(rA_\phi\big) - \frac{\partial A_r}{\partial\phi}\right]\hat{\boldsymbol{z}}\,. \tag{B.12}$$

Laplacian:

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial\phi^2} + \frac{\partial^2 f}{\partial z^2}\,. \tag{B.13}$$