Quantum electrodynamics (sort of) (ch. 14)

QM of particles:

Observables: $\vec{x}, \vec{p}$ $\quad [x_j, p_k] = i\hbar\,\delta_{jk}$

$\vec{S}$ $\quad [S_j, S_k] = i\hbar \sum_\ell \epsilon_{jk\ell} S_\ell$

Parameters: $m, e, s, \hbar$

QM of waves/fields: ?? ← quantum field theory

- Classical E&M: fields are $\vec{E}(\vec{x},t)$, $\vec{B}(\vec{x},t)$
  parameters are: $c, \vec{x}, t, \hbar$
- When quantize, fields become operators $\hat{\vec{E}}(\vec{x},t)$ ...
  i.e. an ∞ set of operators labelled by $(\vec{x},t)$.

What we will do:

① Treat E&M field classically & charged particles quantum-mechanically.
This is not self-consistent, but gives the leading result in perturbation series of the full QFT in the parameter $\alpha = \frac{e^2}{\hbar c}$.

② Show how to quantize E&M fields, but will not couple to charged particles...

① Semi-classical electrodynamics

Review classical E&M

(cgs units / gaussian ?)

electric M.E.

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho$$

$$\vec{\nabla}\times\vec{B} - \frac{1}{c}\dot{\vec{E}} = 4\pi\frac{1}{c}\vec{j}$$

magn M.E.

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} + \frac{1}{c}\dot{\vec{B}} = 0$$

} Max's eqns.

$$\vec{F} = q\,\vec{E}(\vec{x}) + q\,\frac{1}{c}\,\dot{\vec{x}}\times\vec{B}(\vec{x})$$

} Lorentz force law

- Solve magnetic M.E. w/ gauge potentials:

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

$$\vec{E} = -\vec{\nabla}\varphi - \frac{1}{c}\dot{\vec{A}}$$

- Solutions for $(\varphi, \vec{A})$ are not unique!

$$\left.\begin{aligned}\vec{A}' &= \vec{A} + \vec{\nabla}\chi \\ \varphi' &= \varphi - \frac{1}{c}\dot{\chi}\end{aligned}\right\}$$

"Gauge invariance"

=

ambiguity in description of EM i.t.o. $(\varphi, \vec{A})$

- Rewrite ME, LFL i.t.o. $(\vec{A}, \varphi)$ and want to write LFL as equation of motion as Euler-Lagrange eqns so can identify the canonical momentum:

– Recall Lagrangian $L = (K.E.) - (P.E.)$ and $\frac{\delta L}{\delta \vec{x}(t)} = 0$ = Eqns of Motion.

To get LFL for particle of charge q mass m need

$$L = \frac{m}{2}\dot{\vec{x}}\cdot\dot{\vec{x}} - q\varphi(\vec{x}(t), t) + \frac{q}{c}\dot{\vec{x}}\cdot\vec{A}(\vec{x}(t), t)$$

NB①: $\varphi(\vec{x}) \doteq \varphi(\vec{x}(t))$, $\vec{A}(\vec{x}) \doteq \vec{A}(\vec{x}(t))$

NB②: There is also a Lagrangian for Maxwell's eqns:

$$L_{EM} \doteq \int d^3x \, \mathcal{L}_{EM}$$

$$\mathcal{L}_{EM} = \frac{1}{8\pi}\left(\vec{E}\cdot\vec{E} - \vec{B}\cdot\vec{B}\right)$$ i.t.o. $(\varphi, \vec{A})$

$$\left.\begin{array}{l}\varphi(\vec{x},t)\\ \vec{A}(\vec{x},t)\end{array}\right\}\text{ analogs of } \vec{x}(t)$$

$$\left.\begin{array}{l}\dot{\varphi}(\vec{x},t)\\ \dot{\vec{A}}(\vec{x},t)\end{array}\right\}\text{ analogs of } \dot{\vec{x}}(t)$$

- Once you have $L(\vec{x},\dot{\vec{x}})$, the canonical momentum conjugate to $\vec{x}$ is

$$\vec{p} \doteq \frac{\partial L}{\partial \dot{\vec{x}}}$$

and the Hamiltonian (energy i.t.o. $\vec{x},\vec{p}$)

$$H \doteq \left\{\vec{p}\cdot\dot{\vec{x}} - L(\vec{x},\dot{\vec{x}})\right\}\Big|_{\dot{\vec{x}}=\dot{\vec{x}}(\vec{p},\vec{x})}$$

Then quantize by saying $\vec{p},\vec{x}$ are hermitian ops satisfying "canonical commutation relations"

$$[\hat{x},\hat{p}] = i\hbar$$

(NB: can do analogous w/ EM fields...)

- Result of this is

$$\vec{p} \doteq m\dot{\vec{x}} + \frac{q}{c}\vec{A}(\vec{x}) \qquad \Rightarrow$$

$$\boxed{H = \frac{1}{2m}\left(\vec{p} - \frac{q}{c}\vec{A}(\vec{x})\right)\cdot\left(\vec{p} - \frac{q}{c}\vec{A}(\vec{x})\right) + q\,\varphi(\vec{x})}$$

where $\vec{p}, \vec{x}$ are operators satisfying

$$[x_j, p_k] = i\hbar\,\delta_{jk} \qquad (\text{i.e. usual})$$

- How about spin? Answ:

$$\boxed{\begin{aligned}
&\hat{H} = \frac{1}{2m}\left(\vec{p} - \frac{q}{c}\vec{A}(\vec{x})\right)\cdot\left(\vec{p} - \frac{q}{c}\vec{A}(\vec{x})\right) + q\,\varphi(\vec{x}) - \frac{gq}{2mc}\,\vec{S}\cdot\vec{B}(\vec{x}) \\
&[x_j, p_k] = i\hbar\,\delta_{jk} \qquad\qquad [S_j, S_k] = i\hbar\sum_{\ell}\epsilon_{jk\ell}S_\ell \\
&[x_j, x_k] = [p_j, p_k] = 0 = [S_j, x_k] = [S_j, p_k]
\end{aligned}}$$

Quantum particle in classical EM field

- Hilbert space o-n basis:

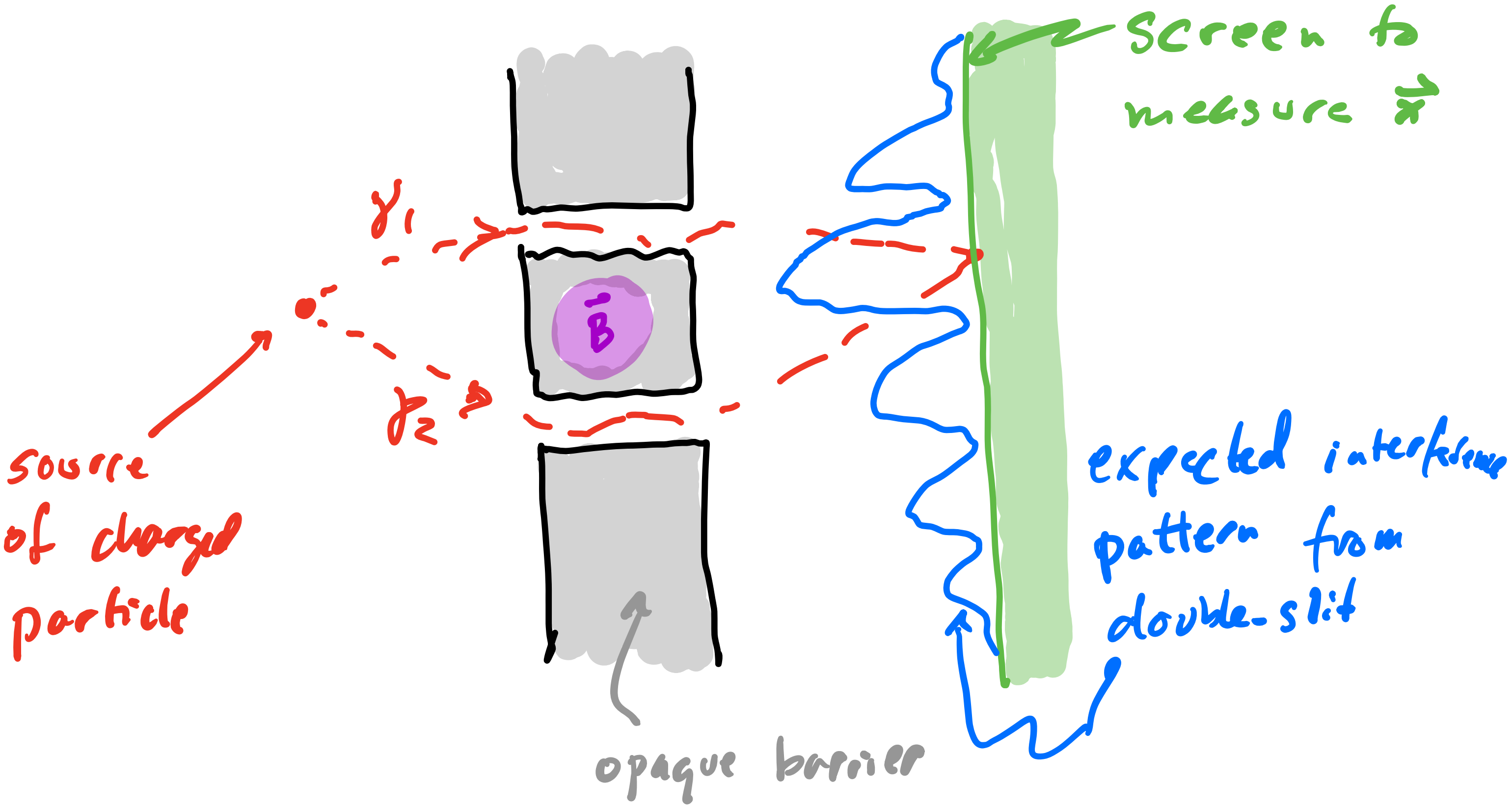
$$\mathcal{H} = \{\,|\vec{x}, m\rangle\,\} \qquad \begin{array}{l}\vec{x} \in \mathbb{R}^3 \\ m \in \{-s, -s+1, \dots, s\}\end{array}$$

with $s =$ fixed $\in \{0, \frac{1}{2}, 1, \frac{3}{2}, \dots\}$

= property of particle = "spin"

Aharanov-Bohm effect

- Start from path integral description of QM & consider following experiment:



- As a (crude) approximation, pretend there are only 2 paths the particle can take to arrive at a point $\vec{x}$:

γ_1 & γ_2.

The Path integral says we are to sum over these paths to compute the amplitude.

In terms of states this just means

$$|\psi\rangle = |\psi_{\gamma_1}\rangle + |\psi_{\gamma_2}\rangle$$

is the state of the particle and the probability of observing it at $\vec{x}$ is

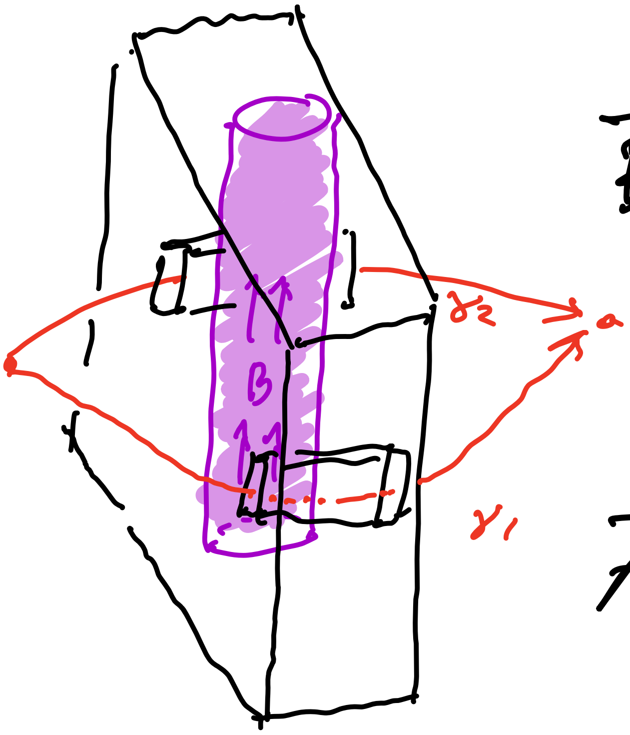
$$\text{Prob}(\vec{x}) = |\langle \vec{x}|\psi\rangle|^2$$

$$= |\langle\vec{x}|\psi_{\gamma_1}\rangle + \langle\vec{x}|\psi_{\gamma_2}\rangle|^2$$

- Since sum inside the $|\cdot|^2$, get interference.

- Now want to use path integral to compute relative phase between $|\psi_{\gamma_1}\rangle$ and $|\psi_{\gamma_2}\rangle$ due to $\vec{B}$-field.

- Key fact:



$\vec{B} = \vec{\nabla} \times \vec{A}$ ⇒ forces to have $\vec{A} \neq 0$ outside of cylinder!

$\vec{A}' = \vec{A} + \vec{\nabla}\chi$ doesn't change this

- Path integral

$$\langle \vec{x} \,|\, \hat{U}(t-t_0) \,|\, \vec{x}_0 \rangle = \int_{\vec{y}(t_0)=\vec{x}_0}^{\vec{y}(t)=\vec{x}} \mathcal{D}\vec{y}(t)\; e^{\frac{i}{\hbar}\int_{t_0}^{t} d\tilde{t}\, L(\vec{y}(\tilde{t}))}$$

$$\psi(\vec{x},t) = \langle \vec{x} | \psi(t) \rangle = \langle \vec{x} | \hat{U}(t-t_0) \,|\, \psi(t_0) \rangle$$

$$= \int d^3x_0 \,\langle \vec{x} | \hat{U}(t-t_0) | \vec{x}_0 \rangle \langle \vec{x}_0 | \psi(t_0) \rangle$$

$$= \int d^3x_0 \int_{\vec{x}_0}^{\vec{x}} \mathcal{D}\vec{y}(t)\, e^{\frac{i}{\hbar}\int_{t_0}^{t} d\tilde{t}\, \underset{\uparrow}{L[\vec{y}]}}\; \psi(\vec{x}_0, t_0)$$

$$L = \frac{m}{2}(\dot{\vec{x}})^2 + \frac{q}{c}\dot{\vec{x}}\cdot\vec{A}(\vec{x})$$

$$\psi_{\vec{A}}(\vec{x},t) = \int_{\vec{x}_0}^{\vec{x}} d\vec{y}\, e^{\frac{iq}{\hbar c}\int_{t_0}^{t}\dot{\vec{y}}\cdot\vec{A}(\vec{y})d\tilde{t}}\ \psi_{\vec{A}=0}(\vec{x},t)$$

$\Rightarrow$ State picks up additional phase

$$e^{\frac{iq}{\hbar c}\int_{t_0}^{t}\vec{A}(\vec{y})\cdot\dot{\vec{y}}\,d\tilde{t}}$$

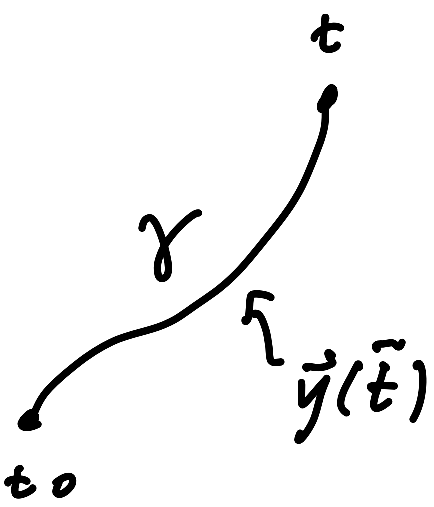
due to presence of $\vec{B}$-field.

- Now examine this phase:

$$\int_{t_0}^{t}\vec{A}(\vec{y}(\tilde{t}))\cdot\dot{\vec{y}}(\tilde{t})\,d\tilde{t}$$

$$= \int_{t_0}^{t_1}\vec{A}(\vec{y})\cdot\frac{d\vec{y}}{d\tilde{t}}\,d\tilde{t}$$

$$= \int_{\gamma}\vec{A}(\vec{y})\cdot d\vec{y}$$



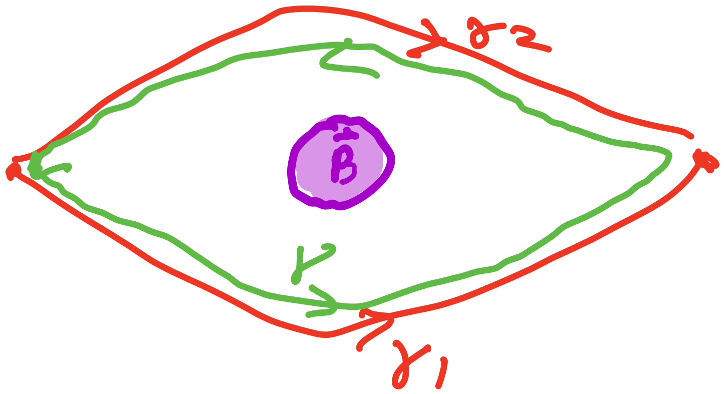
so we get rule

$$\boxed{\psi_{\vec{A}}(\vec{x}) = \sum_{\gamma} e^{\frac{iq}{\hbar c}\int_{\gamma}\vec{A}\cdot d\vec{y}}\, \psi_{\vec{A}=0}(\vec{x})}$$

- Apply this to Aharanov-Bohm experiment

$$\gamma_1 : \quad |\psi_1\rangle_0 \xrightarrow[\vec{B}\text{-field}]{\text{add in}} |\psi_1\rangle_{\vec{A}} = e^{\frac{iq}{\hbar c}\int_{\gamma_1}\vec{A}\cdot d\vec{x}}\, |\psi_1\rangle_0$$

$$\gamma_2 : \quad |\psi_2\rangle_0 \xrightarrow[\vec{B}\text{-field}]{\text{add in}} |\psi_2\rangle_{\vec{A}} = e^{\frac{iq}{\hbar c}\int_{\gamma_2}\vec{A}\cdot d\vec{x}}\, |\psi_2\rangle_0$$



$$\gamma \doteq \gamma_1 - \gamma_2$$

$$|\psi\rangle = |\psi_1\rangle_A + |\psi_2\rangle_A$$

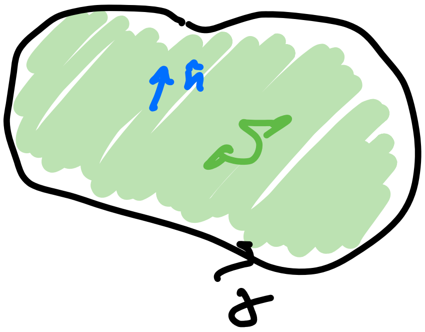
$$= e^{\frac{iq}{\hbar c}\int_{\gamma_2}\vec{A}\cdot d\vec{x}}\left[e^{\frac{iq}{\hbar c}\left[\int_{\gamma_1}\vec{A}\cdot d\vec{x} - \int_{\gamma_2}\vec{A}\cdot d\vec{x}\right]} |\psi_1\rangle_0 + |\psi_2\rangle_0 \right]$$

$$\int_{\gamma_1} \vec{A}\cdot d\vec{x} - \int_{\gamma_2} \vec{A}\cdot d\vec{x} = \oint_{\gamma} \vec{A}\cdot d\vec{x}$$

$$= \int_S (\vec{\nabla}\times\vec{A})\cdot d\vec{S} \quad \text{(Stoke thm)}$$

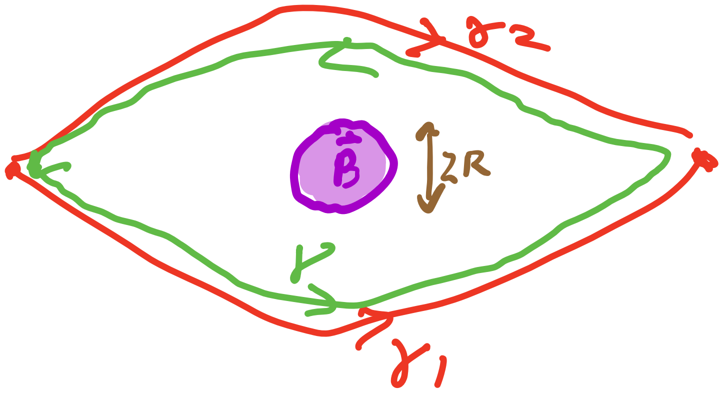
where S is a (any!) surface with boundary

$\partial S = \gamma$:



$$\int_S (\vec{\nabla}\times\vec{A})\cdot d\vec{S} = \int_S \vec{B}\cdot d\vec{S} \doteq \Phi(S)$$

"Magnetic flux through S"



$$\vec{B} = B\hat{z} \Rightarrow \int_S \vec{B}\cdot d\vec{S} = \pi R^2 B .$$

$\therefore$ When turn on $\vec{B}$-field in solenoid the interfering path states gain an additional relative phase

$$e^{\frac{iq}{\hbar c}\pi R^2 B}$$

- This will shift interference pattern, so is observable.

- This is even though charged particles never "see" the $\vec{B}$-field!

Lesson: In QM, phenomena (experiments) depend not only on $\vec{E}$ & $\vec{B}$ fields (classical) but also the field "holonomy"

$$\exp\left\{i\oint_\gamma(\vec{A}\cdot d\vec{x}+\varphi dt)\right\} = \text{"Wilson line"}$$

Atoms in EM fields

- Say we have an atom and we shine light on it.

- The atom is described by, say, the usual H-atom Hamiltonian

$$H_{atom} = \frac{P^2}{2\mu} - e\,\varphi(\vec{x}) = \frac{P^2}{2\mu} - \frac{e^2}{r}$$

governing the state of the electron.

- If the light is bright enough, it is well-described by a classical EM wave.

- Rewrite E&M in terms of its plane wave solutions. I.e. plug into Max's eqns

$$\vec{E} = -\vec{\nabla}\varphi - \frac{1}{c}\dot{\vec{A}} \quad , \quad \vec{B} = \vec{\nabla} \times \vec{A} .$$

Simplify by (partially) fixing the gauge ambiguity in $(\varphi, \vec{A})$ by choosing "Lorentz gauge"

$$0 = \vec{\nabla}\cdot\vec{A} + \frac{1}{c}\dot{\varphi} \tag{L}$$

- Then Max's eqns then become

$$\left.\begin{aligned} &(A) && \frac{1}{c^2}\ddot{\vec{A}} - \nabla^2\vec{A} = 0 \\ &(\varphi) && \frac{1}{c^2}\ddot{\varphi} - \nabla^2\varphi = 0 \end{aligned}\right\}\qquad \text{no sources!}$$

which are just separate wave eqns of each component of $(\varphi, \vec{A})$.

- But notice that (L) does not completely fix the gauge ambiguity

$$\vec{A}' = \vec{A} + \vec{\nabla}\chi \qquad \varphi' = \varphi - \frac{1}{c}\dot{\chi} \tag{g}$$

because

$$\vec{\nabla}\cdot\vec{A}' + \frac{1}{c}\dot{\varphi}' = \vec{\nabla}\cdot\vec{A} + \frac{1}{c}\dot{\varphi} + \left(\nabla^2\chi - \frac{1}{c^2}\ddot{\chi}\right)$$

So Lorentz gauge (L) still allows shift (g) by any χ satisfying

$$\nabla^2\chi - \frac{1}{c}\ddot{\chi} = 0 \, . \tag{\chi}$$

- Say $\varphi(\vec{x},t)$ is solution of $\textcircled{\varphi}$ and at an initial time $t=0$ $\frac{d\varphi}{dt}(\vec{x},0)=0$.

Then

$$\chi(\vec{x},t) \doteq c\cdot\int_0^t\left[\varphi(\vec{x},\tilde{t})-\varphi(\vec{x},0)\right]d\tilde{t}$$

solves $\textcircled{\chi}$.

Proof: $\nabla^2\chi = c\int_0^t\left[\nabla^2\varphi(\tilde{t})-\nabla^2\varphi(0)\right]d\tilde{t} = \frac{1}{c}\int_0^t\left[\frac{d^2\varphi}{d\tilde{t}^2}(\tilde{t})-\cancelto{0}{\frac{d^2\varphi}{d\tilde{t}^2}(0)}\right]d\tilde{t}$

$$= \frac{1}{c}\left[\frac{d\varphi}{dt}(\vec{x},t)-\cancelto{0}{\frac{d\varphi}{dt}(\vec{x},0)}\right] = \frac{1}{c}\frac{d\varphi}{dt}(\vec{x},t)$$

$$\frac{1}{c^2}\frac{d^2}{dt^2}\chi = \frac{1}{c}\frac{d}{dt}\left[\varphi(\vec{x},t)-\varphi(\vec{x},0)\right] = \frac{1}{c}\frac{d\varphi(\vec{x},t)}{dt}.$$

Then can make gauge transformation $\textcircled{g}$ with this χ & still preserve $\textcircled{A}$ & $\textcircled{\varphi}$. But the new φ is

$$\varphi' = \varphi(\vec{x},t) - \frac{1}{c}\dot{\chi}(\vec{x},t)$$

$$= \varphi(\vec{x},t) - \left[\varphi(\vec{x},t)-\varphi(\vec{x},0)\right]$$

$$= \varphi(\vec{x},0).$$

- Net result: we can choose a gauge in which

$$\boxed{\varphi(\vec{x},t) = \varphi(\vec{x},0)}$$ is time-independent

and $\vec{A}$ satisfies

$$\boxed{\nabla^2\vec{A} - \frac{1}{c^2}\ddot{\vec{A}} = 0 \qquad \& \qquad \vec{\nabla}\cdot\vec{A} = 0}$$

(since $\dot{\varphi} = 0$, Lorentz gauge becomes)

This is known as "Coulomb gauge" since now we can take

$$\boxed{\varphi(\vec{x},0) = \frac{e}{r}}$$

the static Coulomb potential that the electron feels, and all the EM radiation is in the $\vec{A}(\vec{x},t)$ field.

- The general solution of the Coulomb-gauge wave equation for $\vec{A}$ is

$$\boxed{\vec{A}(\vec{x},t) = \frac{1}{\sqrt{V}}\sum_{\vec{k}}\sum_{\lambda=1}^{2}\left(a_{\vec{k},\lambda}\,\hat{\varepsilon}(\vec{k},\lambda)\,e^{i(\vec{k}\vec{x}-\omega t)} + \text{complex conj.}\right)}$$

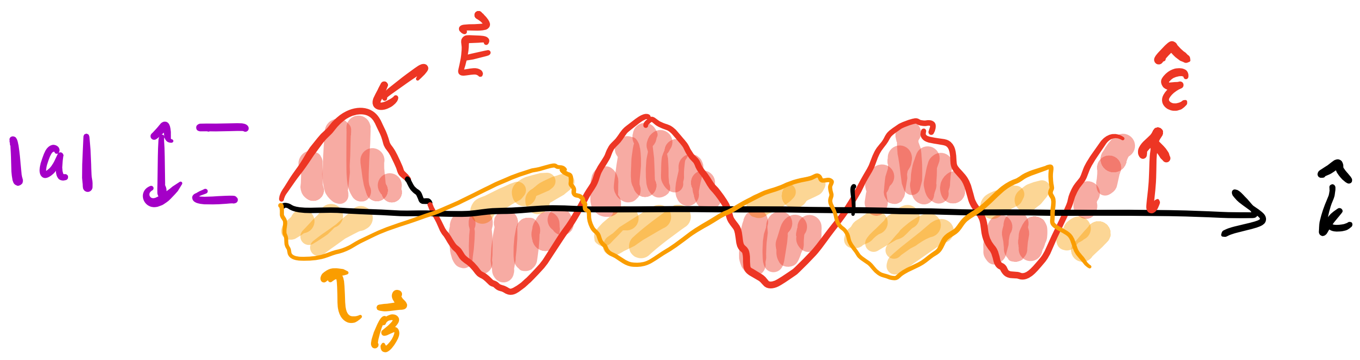
where

- $\omega = ck \quad , \quad k \doteq |\vec{k}|$
- $\hat{\varepsilon}(\vec{k},\lambda)$ are pair of orthonormal polarization vectors, perpendicular to $\vec{k}$:

$$\hat{\varepsilon}(\vec{k},\lambda)\cdot\hat{\varepsilon}(\vec{k},\lambda') = \delta_{\lambda\lambda'} \qquad \lambda,\lambda' \in \{1,2\}$$
$$\vec{k}\cdot\hat{\varepsilon}(\vec{k},\lambda) = 0$$

- $a_{\vec{k},\lambda}$ are complex amplitudes of each plane wave component
- $\frac{1}{\sqrt{V}}$ volume factor is for later convenience.

Picture: $\vec{E} = \underbrace{\frac{e\hat{r}}{r^2}}_{\text{Coul. field}} \underbrace{- \frac{1}{c}\dot{\vec{A}} \, , \quad \vec{B} = \vec{\nabla}\times\vec{A}}_{\text{plane waves}}$



- Then Hamiltonian for the H-atom electron in the presence of lightwave

$$\vec{A}(\vec{x},t) \sim a\,\hat{\varepsilon}\,e^{i(\vec{k}\cdot\vec{x}-\omega t)} + c.c.$$

$$H = \frac{1}{2m}\left(\vec{p}+\frac{e}{c}\vec{A}(\vec{x},t)\right)^2 \underbrace{-e\,\varphi(\vec{x})}_{-\frac{e^2}{r}} + \frac{e}{mc}\vec{S}\cdot\vec{B}(\vec{x},t)$$

$$= \left(\frac{p^2}{2m}-\frac{e^2}{r}\right) + \frac{e}{2mc}\left(\vec{p}\cdot\vec{A}(\vec{x},t)+\vec{A}(\vec{x},t)\cdot\vec{p}+2\vec{S}\cdot\vec{B}(\vec{x},t) + \frac{e}{c}A^2(\vec{x},t)\right)$$

$$= H_0 + a\,H_1(t)$$

For weak EM field, treat "a" as small parameter & try perturbation theory, i.e. expand in power series in a.

But can't use time-independent Schrö eqn $(H|\psi\rangle = E|\psi\rangle)$ b/c H_1 is time-dependent. So have to develop pert. theory for full time-dependant Schrö eqn:

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = H(t)\,|\psi(t)\rangle.$$

Time-dep't perturbation theory

- $H(t) = H_0 + \lambda H_1(t)$ with $\lambda \ll 1$

  $H_0 | E_n^{(0)} \rangle = E_n^{(0)} | E_n^{(0)} \rangle$ given sol'n of time-indep't part.

- Given initial condition at $t=0$:

$$|\psi(0)\rangle = \sum_n |E_n^{(0)}\rangle \underbrace{\langle E_n^{(0)} | \psi(0)\rangle}_{\equiv c_n(0)} = \sum_n c_n(0) |E_n^{(0)}\rangle,$$

then general solution can be written

$$|\psi(t)\rangle = \sum_n c_n(t) e^{-iE_n^{(0)} t/\hbar} |E_n^{(0)}\rangle. \qquad \circledast$$

- If $\lambda = 0$, then $c_n(t) = c_n(0) \Rightarrow \dot{c}_n(t) = 0$,

  so if $\lambda \ll 1$ then expect $|\dot{c}_n(t)|$ small.

Plug $\circledast$ into Schrö eqn to get

$$\boxed{\begin{aligned} & i\hbar \sum_n \left[\dot{c}_n(t) - \frac{iE_n^{(0)}}{\hbar} c_n(t) \right] e^{-iE_n^{(0)} t/\hbar} | E_n^{(0)} \rangle \\ & = \sum_n c_n(t) e^{-iE_n^{(0)} t/\hbar} \left(H_0 + \lambda H_1(t) \right) | E_n^{(0)} \rangle \end{aligned}} \qquad \circledast$$

Can simplify by computing $\langle E_f^{(0)} | \circledast$, use $H_0|E_n^{(0)}\rangle = E_n^{(0)}|E_n^{(0)}\rangle$ & orthonormality

... $\Rightarrow$

$$\boxed{\dot{c}_f(t) = -\frac{i}{\hbar}\sum_n c_n(t)\cdot e^{i(E_f^{(0)}-E_n^{(0)})\frac{t}{\hbar}}\cdot\langle E_f^{(0)}|\lambda\hat{H}_1(t)|E_n^{(0)}\rangle}$$

Now look for a solution for $c_n(t)$ by expanding it in a power series in λ:

$$c_n(t) = c_n^{(0)}(t) + \lambda c_n^{(1)}(t) + \lambda^2 \dots$$

$O(\lambda^0)$: $\dot{c}_f^{(0)}(t) = 0 \Rightarrow c_f^{(0)}(t) = c_f(0)$. ✓

$O(\lambda^1)$:

constants !!

$$\dot{c}_f^{(1)}(t) = -\frac{i}{\hbar}\sum_n c_n(0)e^{i(E_f^{(0)}-E_n^{(0)})t/\hbar}\cdot\langle E_f^{(0)}|\hat{H}_1(t)|E_n^{(0)}\rangle$$

$O(\lambda^2)$: ...

Linear ODE in t for $c_f^{(1)}(t) \Rightarrow$

$$\boxed{c_f^{(1)}(t) = -\frac{i}{\hbar}\sum_n c_n(0)\int_0^t dt'\, e^{i(E_f^{(0)}-E_n^{(0)})t'/\hbar}\cdot\langle E_f^{(0)}|\hat{H}_1(t')|E_n^{(0)}\rangle}$$

- Typically, @ $t=0$ the initial state is an H-atom eigenstate, i.e.

$$|\Psi(0)\rangle = |E_i^{(0)}\rangle \Rightarrow c_n^{(0)} = \delta_{n,i}$$

Then $c_f(t) = c_f^{(0)} + \lambda c_f^{(1)}(t) + \dots = \delta_{f,i} + \lambda c_f^{(1)}(t) + \dots$

$\Rightarrow$

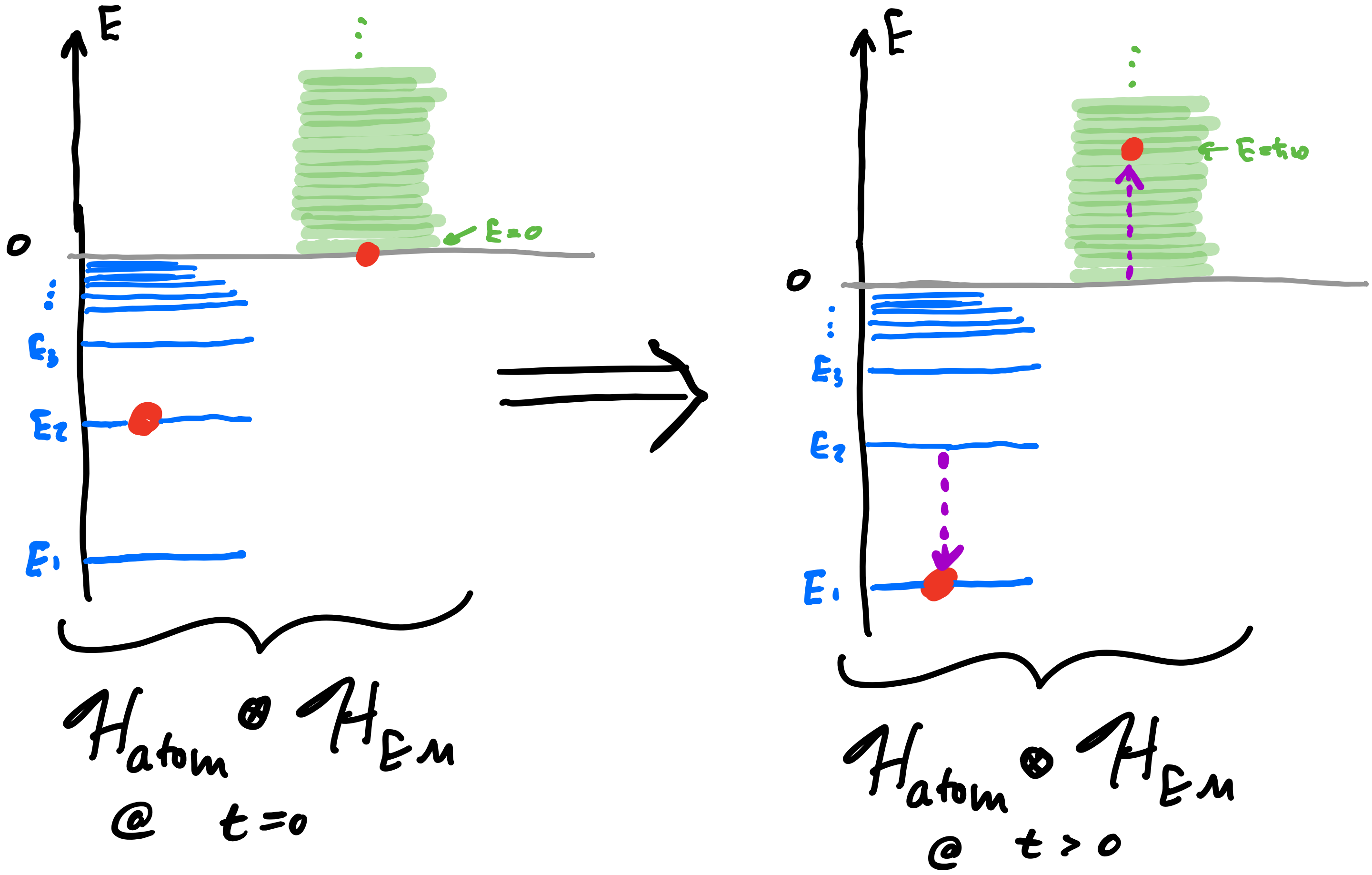
$$\boxed{c_f(t) = \delta_{f,i} - \frac{i}{\hbar}\int_0^t dt' \, e^{i(E_f^{(0)}-E_i^{(0)})t'/\hbar} \cdot \langle E_f^{(0)}|\lambda \hat{H}_1(t')|E_i^{(0)}\rangle}$$

and probability of observing the electron in state $|E_f^{(0)}\rangle$ at time t is

$$\boxed{\text{Prob}(i \to f, t) = |c_f(t)|^2.}$$

Spontaneous emission of light

- Atom & no photon $|E_2\rangle$ Atom & 1 photon $|E_1\rangle$



- Question: What is probability/sec for this to happen?

- So this is not a case of a classical E&M field: to treat this, we have to quantize the E&M field.

- We will motivate in the next lecture the following answer:

In Coulomb gauge $\varphi = 0 = \vec{\nabla}\cdot\vec{A}$, classical plane wave solutions were:

$$\vec{A}(\vec{x},t)=\frac{1}{\sqrt{V}}\sum_{k,\lambda}\left(c_{\vec{k},\lambda}\,\hat{\varepsilon}(\vec{k},\lambda)\,e^{i(\vec{k}\cdot\vec{x}-\omega t)}+c^{*}_{\vec{k},\lambda}\,\hat{\varepsilon}^{*}(\vec{k},\lambda)\,e^{-i(\vec{k}\cdot\vec{x}-\omega t)}\right)$$

Upon quantization $\vec{A}(\vec{x},t)$ becomes an operator $\vec{A}(\vec{x})$

$$\hat{\vec{A}}(\vec{x})=\frac{c\hbar\sqrt{2\pi}}{\sqrt{V\hbar\omega}}\sum_{k,\lambda}\left(\hat{a}_{\vec{k},\lambda}\,\hat{\varepsilon}(\vec{k},\lambda)\,e^{i\vec{k}\cdot\vec{x}}+\hat{a}^{\dagger}_{\vec{k},\lambda}\,\hat{\varepsilon}^{*}(\vec{k},\lambda)\,e^{-i\vec{k}\cdot\vec{x}}\right)$$

where $\{\hat{a}^{\dagger}_{\vec{k},\lambda}\,,\ \hat{a}_{\vec{k},\lambda}\}$ are creation/annihilation operators for 1 photon of energy $\hbar\omega=\hbar ck$ and polarization $\hat{\varepsilon}(\vec{k},\lambda)$.

This gives a rule of thumb: $\omega=ck$

Creating 1 photon with $(\omega,\vec{k},\hat{\varepsilon})$ corresponds to complex classical field

$$\vec{A}(\vec{x},t)\Big|_{\substack{\text{create}\\ \text{photon}}}=\frac{c\hbar\sqrt{2\pi}}{\sqrt{V\hbar\omega}}\,\hat{\varepsilon}^{*}(\vec{k},\lambda)\,e^{-i(\vec{k}\cdot\vec{x}-\omega t)}$$

Annihilating 1 photon with $(\omega,\vec{k},\hat{\varepsilon})$ corresponds to complex classical field

$$\vec{A}(\vec{x},t)\Big|_{\substack{\text{annih.}\\ \text{photon}}}=\frac{c\hbar\sqrt{2\pi}}{\sqrt{V\hbar\omega}}\,\hat{\varepsilon}(\vec{k},\lambda)\,e^{+i(\vec{k}\cdot\vec{x}-\omega t)}$$

(Note: V = volume of box with periodic boundary conditions.)

- We now use this & time-dependent perturbation theory to study spontaneous emission of one photon.

$$H = \frac{1}{2\mu}\left(\vec{p} + \frac{e}{c}\vec{A}(\vec{x},t)\right)^2 - \frac{e^2}{r}$$

$$= \underbrace{\frac{p^2}{2\mu} - \frac{e^2}{r}} + \underbrace{\frac{e}{2\mu c}\left(\vec{p}\cdot\vec{A}(\vec{x},t) + \vec{A}(\vec{x},t)\cdot\vec{p} + \frac{e}{c}A(\vec{x},t)^2\right)}$$

$$\doteq \quad H_0 \quad + \quad H_1(t)$$

- Since $[x_j, p_k] = i\hbar\,\delta_{jk}$, $\Rightarrow$

$$p_j A_k = [p_j, A_k] + A_k p_j$$

$$= \sum_{\ell=1}^{3} \frac{\partial A_k}{\partial x_\ell}[p_j, x_\ell] + A_k p_j$$

$$= -i\hbar\,\frac{\partial A_k}{\partial x_j} + A_k p_j$$

$$\therefore \quad \vec{p}\cdot\vec{A} = -i\hbar\,\underbrace{\cancel{\vec{\nabla}\cdot\vec{A}}}_{\substack{0 \\ \text{in Coulomb gauge}}} + \vec{A}\cdot\vec{p}$$

$$\therefore \quad H_1(t) = \frac{e}{\mu c}\vec{A}(\vec{x},t)\cdot\vec{p} + \frac{e^2}{2\mu c^2}A^2(\vec{x},t) .$$

- Recall from last lecture that 1st-order time-dependent pert. theory says that the amplitude $c_f(t)$ for ending up in the atom state $|a_f\rangle$ at time t if you start in atom state $|a_i\rangle$ at time 0 is

$$c_f(t) = \cancelto{0}{\delta_{f,i}} - \frac{i}{\hbar}\int_0^t dt' \, e^{i(E_f - E_i)t'/\hbar} \langle a_f | H_1(t') | a_i \rangle . \quad \circledast$$

- In our case:

$$|i\rangle = \overbrace{|n_i, l_i, m_i\rangle}^{\text{H-atom state}} \otimes \overbrace{|0\rangle}^{\text{EM field state}} \equiv \overset{\text{short-hand}}{\overbrace{|a_i\rangle}} \otimes |0\rangle$$

$$|f\rangle = |n_f, l_f, m_f\rangle \otimes |1_{\vec{k},\lambda}\rangle \equiv |a_f\rangle \otimes |1_{\vec{k},\lambda}\rangle$$

$|0\rangle \equiv$ "no photon" = no EM field

$|1_{\vec{k},\lambda}\rangle \equiv$ "one photon w/ wave vector $\vec{k}$ and polarization λ"

$\therefore \quad |1_{\vec{k},\lambda}\rangle = a^\dagger_{\vec{k},\lambda} |0\rangle \quad \leftarrow$ creation op. creates 1 photon

& $\quad a_{\vec{k},\lambda} |0\rangle = 0 \quad \leftarrow$ annih. op. kills vacuum

& $\quad a_{\vec{k},\lambda} |1_{\vec{k},\lambda}\rangle = |0\rangle \quad \leftarrow$ annih. op. annihilates 1 photon

Since "$\vec{A} \sim \sum_{\vec{k},\lambda} \left(a_{\vec{k},\lambda} + a^{+}_{\vec{k},\lambda}\right)$"

$$H_1 = \frac{e}{\mu c}\vec{A}\cdot\vec{p} + \frac{e^2}{2\mu c^2}A^2$$

$$\sim \text{"}\sum_{k',\lambda'}\left(a_{k',\lambda'} + a^{+}_{k',\lambda'}\right) + \sum_{k',k''}\left(a+a^{+}\right)^2\text{"}$$

$$\Rightarrow \langle f|H_1|i\rangle \sim \langle 1_{\vec{k},\lambda}|\left[\sum_{k'}\left(\underset{\substack{\downarrow\\0}}{a_{k'}} + a^{+}_{k'}\right) + \sum_{k'k''}\underset{\substack{\downarrow\\0}}{\left(a+a^{+}\right)^2}\right]|0\rangle$$

$$\sim \langle 1_{\vec{k},\lambda}|\; a^{+}_{k,\lambda}\; |0\rangle$$

$\Rightarrow$ Only part of H_1 which contributes to the matrix element is

$$\langle f|H_1|i\rangle = \langle f|\frac{e}{\mu c}\left[\vec{A}\right]_{\left(\substack{\text{creates a photon}\\ \text{with } \vec{k},\lambda}\right)}\cdot\vec{p}\,|i\rangle$$

- Now we plug into our time-dep't pert. thry formula $\circledast$ using our rule of thumb:

$$c_f(t) = -\frac{i}{\hbar}\int_0^t dt'\, e^{i(E_f-E_i)t'/\hbar}\langle a_f|\frac{e}{\mu}\sqrt{\frac{2\pi\hbar}{V\omega}}\,e^{-i(\vec{k}\cdot\vec{x}-\omega t')}\,\hat{\varepsilon}^{*}_{\lambda}\cdot\vec{p}\,|a_i\rangle$$

$$\boxed{\begin{aligned} c_f(t) &= -\frac{i}{\hbar}\left[\int_0^t dt'\, e^{i(E_f+\hbar\omega-E_i)t'/\hbar}\right]\cdot \mathcal{M} \\ \mathcal{M} &\doteq \frac{e}{\mu}\sqrt{\frac{2\pi\hbar}{\omega V}}\;\hat{\varepsilon}^{*}_{\lambda}\cdot\langle a_f|e^{-i\vec{k}\cdot\vec{x}}\,\vec{p}\,|a_i\rangle \end{aligned}}$$

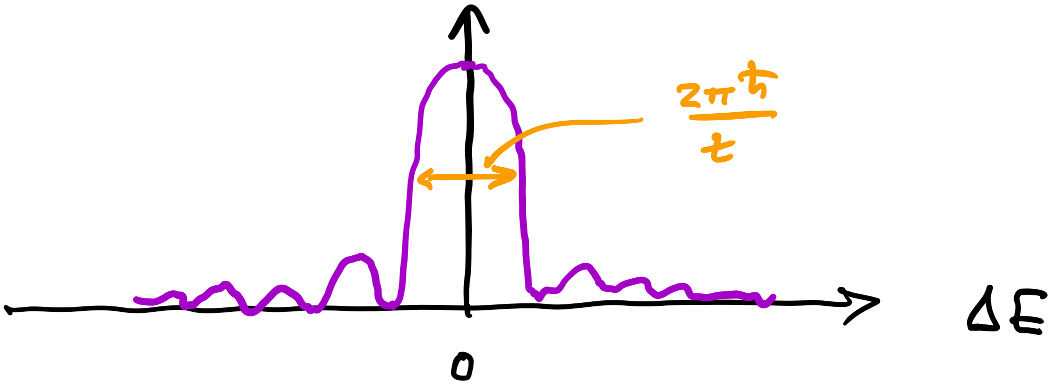
- Green = photon properties
  Blue = electron (atom) properties/operators

- $\mathcal{M}$ = "decay amplitude"
  = time-indep't H-atom matrix element
  which you can compute ... (see text).

- $\Delta E \doteq E_f + \hbar\omega - E_i$
  = total energy final state − tot. energy initial state

$$\int_0^t dt'\, e^{i\Delta E\, t'/\hbar} = \frac{e^{i(\Delta E\cdot t/2\hbar)}}{(\Delta E/2\hbar)}\cdot\sin\left(\frac{\Delta E}{2\hbar}\cdot t\right)$$

Plot $|\cdot|^2 \doteq \dfrac{\sin^2\left(\frac{\Delta E}{2\hbar}t\right)}{(\Delta E/2\hbar)^2}$:



So approximate δ-function $\delta(\Delta E)$, i.e. enforces energy conservation in limit as $t \to \infty$. (i.e. $t \gg 1/\omega$ in practice)

In fact, $\lim_{t\to\infty}\left(\frac{\sin^2(at)}{a^2}\right) = \pi t\,\delta(a)$, so

- Transition probability is

$$P_{i\to f}(t) = |C_f(t)|^2 = \frac{|\mathcal{M}|^2}{\hbar^2}\frac{\sin^2\left(\frac{\Delta E\cdot t}{2\hbar}\right)}{\left(\frac{\Delta E}{2\hbar}\right)^2}$$

$$\approx \frac{2\pi}{\hbar}\cdot t\cdot|\mathcal{M}|^2\,\delta(\Delta E)$$

Define "decay rate" = prob. / time:

$$\boxed{\Gamma_{i\to f} \doteq \lim_{T\to\infty}\frac{P_{i\to f}(T)}{T} = \frac{2\pi}{\hbar}|\mathcal{M}|^2\,\delta(\Delta E)}$$

- In practice, not interested in decay rate to emit photon with an exact value of $\vec{k}$ and polarization λ.

Generally, only detect photon final state with some finite energy (dE) & angular ($d\Omega$) resolutions:

$$\left\{\begin{array}{l} \hbar c\, dk = \hbar\, d\omega = dE < \dfrac{2\pi\hbar}{T} \leftarrow \text{time resolution} \\ \hat{k} = \text{direction of } \vec{k} \in d\Omega \leftarrow \text{angular resolution of detector} \\ \text{and sum over } \lambda \leftarrow \text{don't detect photon polarization} \end{array}\right.$$

So, to compute $\Gamma_{i\to f}$ we should sum over all $\vec{k}, \lambda$ with

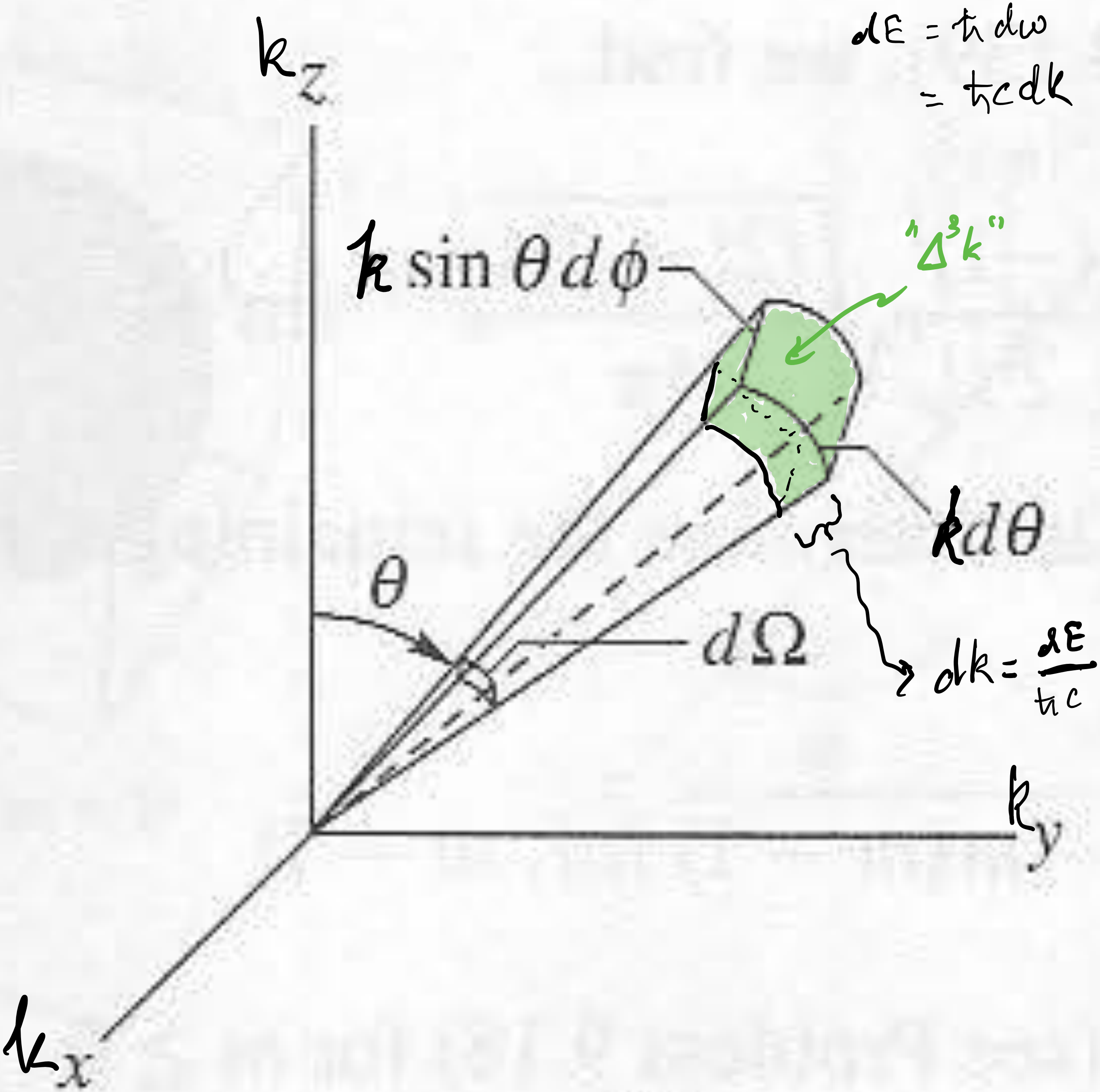
- $k \in [k, k + \frac{1}{\hbar c} dE]$
- $\hat{k} \in d\Omega$
- all $\lambda \in 1, 2$

$\Big\}$ "$\Delta^3 k$" ↑ region in $\vec{k}$-space

$$d\Gamma \doteq \sum_{\lambda} \frac{2\pi}{\hbar} \int_{\Delta^3 k} d^3k \cdot n \cdot |\mathcal{M}|^2 \delta(\Delta E)$$

↑ number of photon states per d^3k-volume = "density of states in $\vec{k}$-space"

↓



$$dE = \hbar\, d\omega = \hbar c\, dk$$

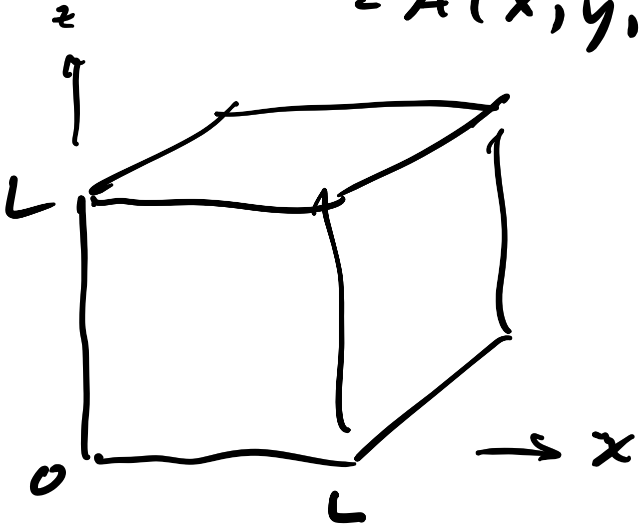
$$dk = \frac{dE}{\hbar c}$$

- Naively, $n = \infty$: there are infinitely many photon states per d^3k-volume since there exist photons with any value of $\vec{k}$.

- But this infinity is from working in infinite volume in space. In real life we only measure things in finite volumes V.

E.g. say we put our experiment in a cubic box of side L, so $V = L^3$. The box give boundary conditions on EM waves. Say we use periodic boundary conditions

$$\begin{aligned}\vec{A}(x,y,z) &= \vec{A}(x+L,y,z) \\ &= \vec{A}(x,y+L,z) \\ &= \vec{A}(x,y,z+L)\end{aligned}$$



(What BC's we use doesn't matter in the end)

Since

$$\vec{A}(\vec{x}) \sim \sum_{\vec{k}} c_k e^{i\vec{k}\cdot\vec{x}}$$

$$\text{periodicity} \Rightarrow e^{i\vec{k}\cdot[\vec{x}+(L\,0\,0)]} = e^{i\vec{k}\cdot\vec{x}}$$

$$\Rightarrow e^{ik_x L} = 1$$

$$\Rightarrow k_x = \frac{2\pi n_x}{L} \quad n_x \in \mathbb{Z}.$$

Similarly for $k_y, k_z \Rightarrow$

$$\vec{k} = \frac{2\pi}{L}\vec{n} \quad \vec{n} \in \mathbb{Z}^3 \quad \left(\begin{matrix}\text{periodic}\\ \text{BC's}\end{matrix}\right)$$

- Therefore, there is 1 allowed $\vec{k}$ in a volume $\left(\frac{2\pi}{L}\right)^3$ of $\vec{k}$-space, i.e.

$$n = \frac{1}{\left(\frac{2\pi}{L}\right)^3} = \frac{L^3}{(2\pi)^3} = \frac{V}{(2\pi)^3} \quad .$$

- So the differential decay rate is

$$d\Gamma = \sum_{\lambda} \frac{2\pi}{\hbar} \int_{\Delta^3 k} d^3k \frac{V}{(2\pi)^3} |\mathcal{M}(\vec{k},\lambda)|^2 \delta(E_f + \hbar ck - E_i)$$

$$= \sum_{\lambda} \frac{V}{(2\pi)^2 \hbar} \int_{k}^{k+\frac{1}{\hbar c}dE} k^2 dk \underbrace{\int d\Omega}_{\Delta\Omega \to \text{take small} = d\Omega} |\mathcal{M}(\vec{k},\lambda)|^2 \delta(E_f + \hbar ck - E_i)$$

$$\cong \frac{V}{(2\pi)^2\hbar} \sum_{\lambda} d\Omega \int_{\varepsilon}^{\varepsilon + dE} \frac{\varepsilon^2 d\varepsilon}{(\hbar c)^3} \delta(\varepsilon - E_i + E_f) \, |\mathcal{M}|^2$$

$$\varepsilon \doteq \hbar c k \doteq \hbar \omega$$

$$= \frac{V \omega^2}{(2\pi\hbar)^2 c^3} d\Omega \sum_{\lambda} |\mathcal{M}(\vec{k},\lambda)|^2 \Bigg|_{\substack{\hbar\omega = \hbar c k \\ = E_i - E_f}}$$

∴ Differential decay rate:

$$\boxed{\frac{d\Gamma}{d\Omega} = \frac{\omega^2 V}{(2\pi\hbar)^2 c^3} \sum_{\lambda} |\mathcal{M}(\vec{k},\lambda)|^2}$$

& Total decay rate:

$$\boxed{\Gamma \doteq \int d\Omega \frac{d\Gamma}{d\Omega} = \frac{\omega^2 V}{(2\pi\hbar)^2 c^3} \sum_{\lambda} \int d\Omega \, |\mathcal{M}(\vec{k},\lambda))|^2}$$

"Fermi golden rule"

- Implicit: ω, $\mathcal{M}$ evaluated at energy-conserving values $k\hbar c = \hbar\omega = E_i - E_f$
- $\int d\Omega$ is over directions of $\vec{k}$
- Factor of V cancels: $|\mathcal{M}|^2 \sim \frac{1}{V} \ldots$!

Quantizing the electromagnetic field

"Canonical quantization"

PARTICLES	FIELDS
$L(q(t),\dot{q}(t))$ $\simeq$	$\int d^3x\,\mathcal{L}(q(\vec{x},t),\dot{q}(\vec{x},t))$
$\downarrow$ Legendre	$\downarrow$ Legendre
$H(q(t),p(t))$	$\mathcal{H}(q(\vec{x},t),p(\vec{x},t))$
$\downarrow$	$\downarrow$
$\hat{H}(\hat{q},\hat{p})+\mathcal{O}(\hbar)$	$\hat{\mathcal{H}}(\hat{q}(\vec{x}),\hat{p}(\vec{x}))+\mathcal{O}(\hbar)$
w/ $\begin{cases}[\hat{q},\hat{p}]=i\hbar\\ [\hat{q},\hat{q}]=[\hat{p},\hat{p}]=0\end{cases}$	$[\hat{q}(\vec{x}),\hat{p}(\vec{y})]=i\hbar\,\delta^3(\vec{x}-\vec{y})$ $[\hat{q}(\vec{x}),\hat{q}(\vec{y})]=[\hat{p}(\vec{x}),\hat{p}(\vec{y})]=0$

- For E&M fields are $\vec{A}(\vec{x},t)$, $\varphi(\vec{x},t)$
- Work in Coulomb gauge: $\vec{\nabla}\cdot\vec{A}=0=\varphi$.
- $\mathcal{L}=\frac{1}{8\pi}(E^2-B^2)=\frac{1}{8\pi}\left(\frac{1}{c^2}\dot{\vec{A}}\cdot\dot{\vec{A}}-(\vec{\nabla}\times\vec{A})\cdot(\vec{\nabla}\times\vec{A})\right)$

- $\mathcal{H} = \frac{1}{8\pi}(E^2 + B^2) = \cdots$ long story ...

- Easier to work directly at the level of solutions to Maxwell's equations:

$$\vec{A}(\vec{x},t) = \frac{1}{\sqrt{V}} \sum_{\vec{k},\lambda} \left(C_{\vec{k},\lambda}\, \hat{\varepsilon}(\vec{k},\lambda)\, e^{i(\vec{k}\cdot\vec{x} - \omega t)} + c.c. \right)$$

with $\omega \doteq kc$, $\hat{\varepsilon}(\vec{k},\lambda) \cdot \hat{\varepsilon}^*(\vec{k},\lambda') = \delta_{\lambda\lambda'}$ $\lambda \in \{1,2\}$

$$\vec{k} \cdot \hat{\varepsilon} = 0$$

and $V = L^3$ the volume of large box with periodic boundary conditions, and.

$$\vec{k} = \frac{2\pi}{L}\vec{n}, \quad \vec{n} \in \mathbb{Z}^3.$$

- So, to quantize want to treat:

$$\vec{A}(\vec{x},0) = \text{operator} \Rightarrow C_{\vec{k},\lambda} = \text{operator}.$$

But what commutation relations should we give $C_{\vec{k},\lambda}$ & $C^\dagger_{\vec{k},\lambda}$?

- Look at Hamiltonian

$$H = \frac{1}{8\pi}\int_V d^3x\,(E^2+B^2) = \cdots$$ (← try it!)

$$= \frac{1}{4\pi}\sum_{\vec{k},\lambda} k^2\left(c^{\dagger}_{\vec{k},\lambda}\, c_{\vec{k},\lambda} + c_{\vec{k},\lambda}\, c^{\dagger}_{\vec{k},\lambda}\right)$$

(Have to use $\int_V d^3x\; e^{i(\vec{k}-\vec{k}')\cdot\vec{x}} = V\,\delta_{\vec{k},\vec{k}'}$.)

Define hermitian operators

$$q_{\vec{k},\lambda} \doteq \frac{1}{c\sqrt{4\pi}}\left(c_{\vec{k},\lambda} + c^{\dagger}_{\vec{k},\lambda}\right)$$

$$p_{\vec{k},\lambda} \doteq \frac{k}{i\sqrt{4\pi}}\left(c_{\vec{k},\lambda} - c^{\dagger}_{\vec{k},\lambda}\right)$$

Then find ...

$$\boxed{H = \sum_{\vec{k},\lambda}\Big(\underbrace{\tfrac{1}{2}\,p^2_{\vec{k},\lambda} + \tfrac{\omega^2}{2}\,q^2_{\vec{k},\lambda}}_{\text{simple harm. oscill. with "}m=1\text{" and frequency }\omega = ck.}\Big).}$$ ①

∴ EM field = ∞ # of decoupled harmonic oscillators!

= "normal modes" of EM field.

- It is now easy to guess how to quantize: just like S.H.O.:

$$\boxed{\begin{aligned} &[q_{\vec{k},\lambda}, p_{\vec{k}',\lambda'}] = i\hbar\, \delta_{\vec{k},\vec{k}'}\, \delta_{\lambda\lambda'} \\ &[q_{\vec{k},\lambda}, q_{\vec{k}',\lambda'}] = [p_{\vec{k},\lambda}, p_{\vec{k}',\lambda}] = 0 \end{aligned}} \qquad \text{②}$$

① & ② = Quantum electromagnetism!

- Alternatively, we can rewrite in terms of SHO creation & annihilation operators

$$\begin{cases} a^\dagger_{\vec{k},\lambda} \doteq \dfrac{1}{c}\sqrt{\dfrac{\omega}{2\pi\hbar}}\; c^\dagger_{\vec{k},\lambda} \\ a_{\vec{k},\lambda} \doteq \dfrac{1}{c}\sqrt{\dfrac{\omega}{2\pi\hbar}}\; c_{\vec{k},\lambda} \end{cases}$$

$$\Rightarrow \boxed{\begin{aligned} &[a_{\vec{k},\lambda}, a^\dagger_{\vec{k}',\lambda'}] = \delta_{\vec{k},\vec{k}'}\, \delta_{\lambda,\lambda'} \\ &[a_{\vec{k},\lambda}, a_{\vec{k}',\lambda'}] = [a^\dagger_{\vec{k},\lambda}, a^\dagger_{\vec{k}',\lambda'}] = 0 \\ \& \quad & H = \sum_{\vec{k},\lambda} \hbar\omega \left(a^\dagger_{\vec{k},\lambda} a_{\vec{k},\lambda} + \frac{1}{2}\right) \end{aligned}}$$

- All that remains is to interpret the physical meaning of $a^\dagger_{\vec{k},\lambda}$ & $a_{\vec{k},\lambda}$ operators.

- From SHO,

$$n_{\vec{k},\lambda} \doteq a^\dagger_{\vec{k},\lambda} a_{\vec{k},\lambda}$$

is the "number operator" for the $\vec{k},\lambda$ mode with eigenvalues

$$n_{\vec{k},\lambda} \in \{0, 1, 2, 3, \cdots\}$$

in an orthonormal eigenbasis

$$\{ |n_{\vec{k},\lambda}\rangle \}.$$

We have this for each mode $\vec{k},\lambda$ so an o-n basis for the whole EM Hilbert space is

$$\{ |n_{\vec{k}_1,\lambda_1}\rangle \otimes |n_{\vec{k}_2,\lambda_2}\rangle \otimes |n_{\vec{k}_3,\lambda_3}\rangle \otimes \cdots$$

$$\doteq \bigotimes_{\vec{k},\lambda} |n_{\vec{k},\lambda}\rangle \quad \}$$

So this is a huge Hilbert space.

- Since $\hat{H} = \sum_{\vec{k},\lambda} \hbar\omega \left(n_{\vec{k},\lambda} + \frac{1}{2}\right)$

$$= \sum_{\vec{k},\lambda} \hbar c k \, n_{\vec{k},\lambda} + \frac{1}{2} \underbrace{\sum_{\vec{k},\lambda} \hbar\omega}_{\text{unobservable } (\infty) \text{ constant} \to \text{drop!}}$$

$$\Rightarrow \quad \hat{H} = \sum_{\vec{k},\lambda} \underbrace{\hbar c k}_{>0} \, \underbrace{n_{\vec{k},\lambda}}_{\in \{0,1,2,\dots\}}$$

∴ Ground state = "vacuum" is

$$|0\rangle \doteq \bigotimes_{\vec{k},\lambda} |0_{\vec{k},\lambda}\rangle \quad \& \quad H|0\rangle = 0.$$

- Now consider "next" eigenstate

$$|1_{\vec{k},\lambda}\rangle = \underbrace{|0..\rangle \otimes |0..\rangle \otimes \cdots \otimes}_{\text{all } n_{\vec{k}',\lambda'}=0} |1_{\vec{k},\lambda}\rangle \otimes \underbrace{\cdots}_{n=0}$$

$$= a^{\dagger}_{\vec{k},\lambda} |0\rangle.$$

Has energy $H|1_{\vec{k},\lambda}\rangle = \hbar c k \, |1_{\vec{k},\lambda}\rangle.$

This is the energy of 1 photon $\hbar\omega = \hbar c k$.
Similarly, the N-photon states would be

$$|N_{(\vec{k}_1,\lambda_1)(\vec{k}_2,\lambda_2)\cdots(\vec{k}_N,\lambda_N)}\rangle \propto \prod_{i=1}^{N} a^{+}_{\vec{k}_i,\lambda_i} |0\rangle .$$

So the Hilbert space includes all possible numbers of photons = "Fock space".

- To confirm our interpretation of these states as photons (particles of light), we should be able to see that they have definite momentum and spin, as well as energy.

$$H = \sum_{\vec{k},\lambda} \hbar\omega\, n_{\vec{k},\lambda} \qquad \omega = ck$$

$$\vec{P} = \frac{1}{4\pi c}\int_V d^3r\, \vec{E}\times\vec{B} = \cdots \qquad (\leftarrow \text{try it!})$$

$$= \sum_{\vec{k},\lambda} \hbar\vec{k}\, n_{\vec{k},\lambda}$$

$$\Rightarrow \quad \vec{P}\,|1_{\vec{k},\lambda}\rangle = \hbar\vec{k}\,|1_{\vec{k},\lambda}\rangle$$

∴ Photon has

$$\boxed{(E_\gamma, \vec{P}_\gamma) = (\hbar\omega, \hbar\vec{k})}$$

$$= (\hbar k c, \hbar\vec{k})$$

S.R. ⇒ $$\underset{\underset{\hbar^2\omega^2}{\|}}{E_\gamma^2} = m_\gamma^2 c^4 + \underset{\underset{\hbar^2 k^2 c^2 = \hbar^2\omega^2}{\|}}{P_\gamma^2}\, c^2$$

$$\Rightarrow \quad m_\gamma = 0 \quad . \checkmark$$

(Photons are massless; they move @ speed of light.)

- Spin

$$\vec{J} = \frac{1}{4\pi c}\int d^3r\, \vec{r}\times(\vec{E}\times\vec{B}) = \cdots \quad \text{messy ?}$$

Take $\vec{k} \propto \hat{z}$; then $\vec{k}\cdot\hat{\varepsilon} = 0$ ⇒

can take basis $\hat{\varepsilon}(\lambda=1) = \hat{x}$, $\hat{\varepsilon}(\lambda=2) = \hat{y}$,

doesn't give J_z eigenstates...

Right basis of polarizations:

$$\hat{\varepsilon}_R \doteq \frac{1}{\sqrt{2}}(\hat{x} + i\hat{y}) \quad \Big\}$$ ‘Right- Δ Left-circular

$$\hat{\varepsilon}_L \doteq \frac{1}{\sqrt{2}}(\hat{x} - i\hat{y}) \Bigg\} \text{polarizations"}$$

Then find $|1_{\vec{k},L}\rangle$ & $|1_{\vec{k},R}\rangle$ are eigenstates of J_z:

$$\boxed{\begin{aligned} J_z\,|1_{\vec{k},R}\rangle &= +\hbar\,|1_{\vec{k},R}\rangle \\ J_z\,|1_{\vec{k},L}\rangle &= -\hbar\,|1_{\vec{k},L}\rangle \end{aligned}}$$

So looks like photon = spin-1, but should have $J_z = 0$ eigenvalue ??

$$s=1 \Rightarrow m \in \{-1, \underset{\text{missing!}}{\underset{\uparrow}{0}}, 1\}$$

- J_x, J_y, J_z do not satisfy ang. mom. (rotation) algebra !?
- In fact, only component of $\vec{J}$ along direction of motion of photon is well-defined

$$h \doteq \frac{1}{k}\vec{k}\cdot\vec{J} \qquad \text{"helicity"}$$

Reason: photons can never be brought to rest, so there is no frame in which they have rotational symmetry, so angular momentum can't even be defined for them!

The only symmetry is rotations around the direction of motion of the photon, which is generated by h.

Relativistic Quantum Mechanics
= Quantum Field Theory

Quantize E & M field ⇒ Hilbert space included all possible numbers of particles (photons).

This is general consequence of special relativity & quantum mechanics: particle number is not conserved, ∴ have to include all possible number of particles in Hilbert space.

- Often say "can create particle-antiparticle pairs from the vacuum". Many particles are their own antiparticle (e.g. photons).

The operators which create/annihilate a particle at a position $\vec{x}$ are called **quantum field operators**

$$\hat{\Phi}(\underset{\uparrow}{\vec{x}})$$

parameters (not operators!)

(E.g. $\hat{\vec{A}}(\vec{x})$ is a quantum field.)

In special relativity, relativistic symmetry (Lorentz covariance) is manifest in **space-time** $x^\mu = (t, \vec{x})$ $\mu = 0, 1, 2, 3$

In our formulation of QM, field operators depend on space & states depend on time:

$$\hat{\Phi}(\vec{x}) \quad , \quad |\Psi(t)\rangle$$

"Schrödinger picture"

This makes Lorentz covariance *non*-manifest.

"Heisenberg Picture"

Re-name $\begin{cases} \hat{\Phi}(\vec{x}) \to \hat{\Phi}_S(\vec{x}) \\ |\Psi(t)\rangle \to |\Psi_S(t)\rangle \end{cases}$ (S ≡ Schrö)

Recall: $|\psi_s(t)\rangle = e^{-i\hat{H}t/\hbar}|\psi_s(0)\rangle$.

Note:
$$\begin{array}{ccc} |\chi_s(t)\rangle & \doteq & \hat{\Phi}_s(\vec{x})\,|\psi_s(t)\rangle \\ \| & & \| \\ e^{-i\hat{H}t/\hbar}|\chi_s(0)\rangle & & \hat{\Phi}_s(\vec{x})\,e^{-i\hat{H}t/\hbar}|\psi_s(0)\rangle \end{array}$$

$$\Rightarrow \quad |\chi_s(0)\rangle = e^{+i\hat{H}t/\hbar}\hat{\Phi}_s(\vec{x})\,e^{-i\hat{H}t/\hbar}|\psi_s(0)\rangle.$$

Define
$$\begin{cases} |\psi_H\rangle \doteq |\psi_s(0)\rangle \\ \hat{\Phi}_H(\vec{x},t) \doteq e^{+i\hat{H}t/\hbar}\hat{\Phi}_s(\vec{x})\,e^{-i\hat{H}t/\hbar} \end{cases}$$

(H ≡ Heisenberg). Then:

- States are time-independent
- Operators have time-dependence

Schrö. eqn: $i\hbar\frac{d}{dt}|\psi_s(t)\rangle = \hat{H}\,|\psi_s(t)\rangle$

Heis. eqn: $i\hbar\frac{d}{dt}\hat{\Phi}_H(\vec{x},t) = -[\hat{H}, \hat{\Phi}_H(\vec{x},t)]$

(Note: $\hat{H}_H = \hat{H}_S = \hat{H}$).

Transl. inv. $\Rightarrow$ $i\hbar\vec{\nabla}_x\hat{\Phi}_H(\vec{x},t) = -[\hat{\vec{P}}, \hat{\Phi}_H(\vec{x},t)]$

$$\Rightarrow \quad \boxed{i\hbar\frac{\partial}{\partial x^\mu}\hat{\Phi}_H(x^\nu) = -[\hat{P}_\mu, \hat{\Phi}_H(x^\nu)]} \quad (*)$$

w/ $x^\mu = (t,\vec{x})$ $\quad \hat{P}_\mu = (\hat{H}, \hat{\vec{P}})$

(*) = Lorentz-covariant "Schrö" eqn.

Relativistic Particles (free – no interactions)

- also described by fields

$\hat{\psi}_{\alpha}(x^{\mu})$ Lorentz group rep'n index (≡ "spin")

create/annihilate particle/antiparticle at x^{μ}

- Scalar (s=0) particles $\hat{\Phi}(x^{\mu})$

Satisfy operator eqn:

$$\partial^2 \doteq \partial^\mu \partial_\mu = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2$$

$$\frac{\hbar^2}{c^2}\partial^2 \hat{\Phi} - m^2 \hat{\Phi} = 0$$

"Klein-Gordon eqn"

$$\leftrightarrow \quad (E^2 - p^2c^2 = m^2c^4)$$

If $\hat{\Phi} = \hat{\Phi}^{\dagger}$ ⟺ scalar particle of mass m which is own antiparticle

If $\hat{\Phi} \neq \hat{\Phi}^{\dagger}$ ⟺ scalar particle & antiparticle mass m

- Dirac (s=½) particle $\hat{\Psi}_{\alpha}(x^{\mu})$ $\alpha \in \{1\,2\,3\,4\}$

Satisfy $i\frac{\hbar}{c}\not{\partial}\hat{\Psi} - m\hat{\Psi} = 0$ "Dirac eqn"

$\not{\partial} \doteq \partial_\mu (\gamma^\mu)^{\alpha}{}_{\beta}$ 4×4 "Dirac matrices"

⇒ spin-½ particle & antiparticle mass m

(e.g. e^-, e^+).

(Majorana ($s=\frac{1}{2}$) particle = own antiparticle.)

- **Massless vector ($h=1$) particle** $\hat{A}_\mu(x^\nu)$

Satisfy Maxwell equns; in Lorentz gauge

$$\hbar^2\partial^2\hat{A}_\mu = 0 \qquad (\partial^\mu\hat{A}_\mu = 0)$$

[Gauge-inv't version: $\sum_\mu \partial^\mu(\partial_\mu\hat{A}_\nu - \partial_\nu\hat{A}_\mu) = 0$.]

- **Massive vector ($s=1$) particle** $\hat{W}_\mu(x)$? ...

Need $\underset{h=1}{(\hat{A}_\mu}, \underset{s=0}{\hat{\Phi})} = \underset{\text{complex}}{(\hat{W}_\mu, \hat{h})} \leftarrow$ real Higgs scalar.

"Higgs scalar"

- **Graviton (massless $h=2$ particle)**

$$\hat{h}_{\mu\nu} = \hat{h}_{\nu\mu}$$

In traceless transverse gauge $(\partial^\mu\hat{h}_{\mu\nu} = \hat{h}_\mu{}^\mu = 0)$

equs of motion: $\partial^2\hat{h}_{\mu\nu} = 0.$

Gauge-ind't version ??? (Einstein equs class.)

Non-relativistic limit

Look at Klein-Gordon field (free relativistic $s=0$ particles)

$$\text{KG:}\quad -\frac{\hbar^2}{c^2}\frac{\partial^2\hat{\phi}}{\partial t^2} + \hbar^2\nabla^2\hat{\phi} - m^2c^2\hat{\phi} = 0$$

For nonrel particles, $E^2 = m^2c^4 + \cancelto{\text{small}}{p^2c^2} \approx m^2c^4$

$$\therefore\quad E \approx E_0 \doteq mc^2.$$

So define

$$\hat{\phi}(\vec{x},t) \doteq \underbrace{e^{-\frac{i}{\hbar}E_0 t}}_{\text{"fast" t-dep}} \, \hat{\phi}_{NR}(\vec{x},t)$$

"slow" t-dep: $\left(|\dot{\phi}_{NR}| \ll \frac{E_0}{\hbar} |\phi_{NR}| \right)$

Plug into KG eqn:

$$-\frac{\hbar^2}{c^2}\ddot{\phi} = \Big(\overbrace{\cancel{\frac{E_0^2}{c^2}\phi_{NR}}}^{\text{big}} \overset{\gg}{+} \overbrace{\frac{2i\hbar}{c^2}E_0\dot{\phi}_{NR}}^{\text{small}} \overset{\gg}{+} \overbrace{\frac{\hbar^2}{c^2}\ddot{\phi}_{NR}}^{\text{v. small}} \Big) e^{-\frac{i}{\hbar}E_0 t}$$

$$+\hbar^2\nabla^2\phi = \qquad + \hbar^2\nabla^2\phi_{NR} \qquad \cdot e^{-\frac{i}{\hbar}E_0 t}$$

$$-m^2c^2\phi = \cancel{-m^2c^2\phi_{NR}} \qquad \cdot e^{-\frac{i}{\hbar}E_0 t}$$

cancel

drop

$$\Rightarrow \quad 0 \approx 2i\hbar m \dot{\hat{\phi}}_{NR} + \hbar^2\nabla^2\hat{\phi}_{NR}$$

$$\Rightarrow \quad i\hbar\dot{\hat{\phi}}_{NR} = -\frac{\hbar^2}{2m}\nabla^2\hat{\phi}_{NR}$$

Interpret $\quad \hat{\phi}_{NR}(\vec{x},t)|0\rangle = \langle \vec{x} | \phi_{NR}(t)\rangle$

vacuum

wave function of single particle

$$\Rightarrow \quad i\hbar\frac{d}{dt}|\phi_{NR}\rangle = \frac{\hat{P}^2}{2m}|\phi_{NR}\rangle$$

$\Rightarrow$ NR Schrö. eqn. for free particle. ✓