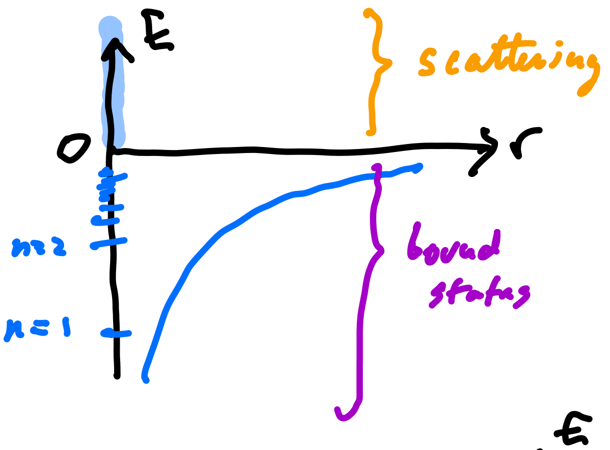
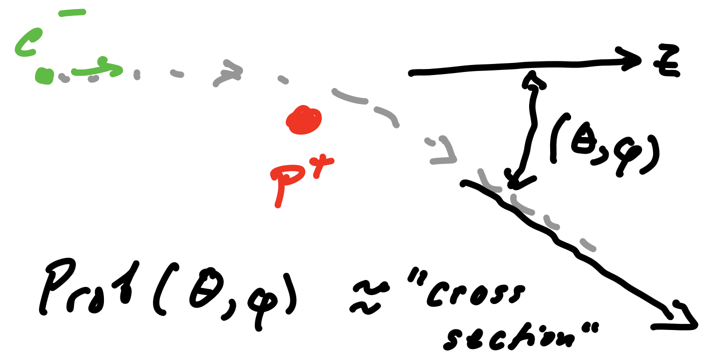
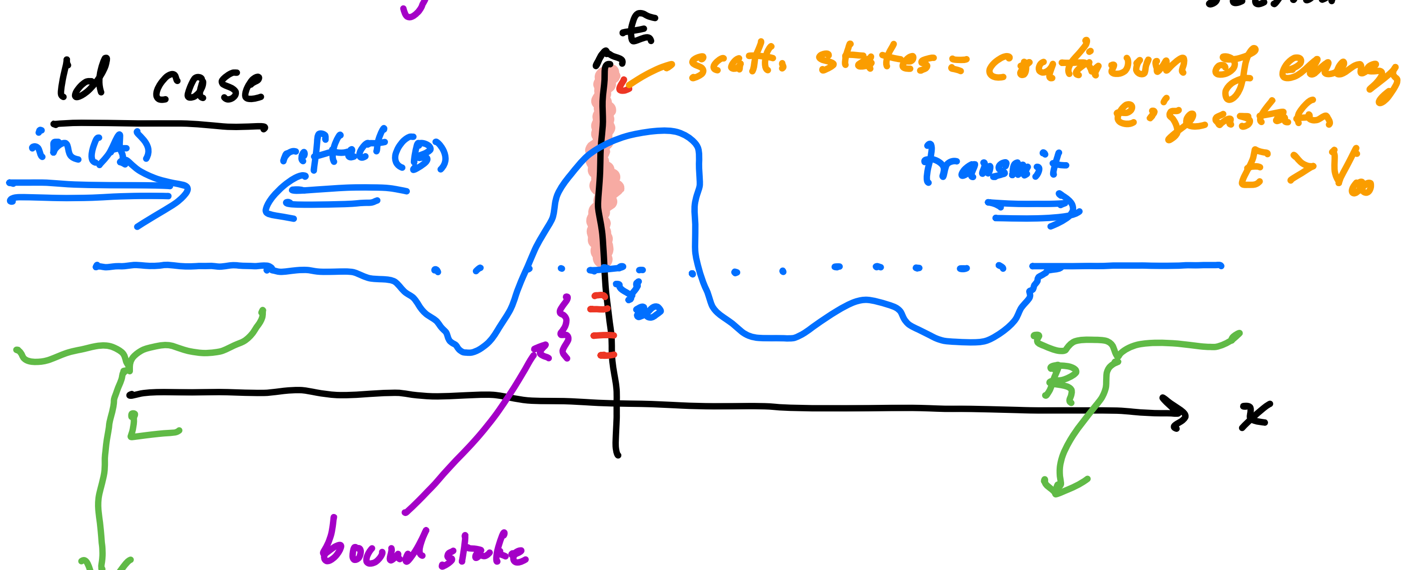
Scattering (Ch. 13)





1d case



$$-\frac{\hbar^2}{2m}\psi_L''(x) + V_\infty \psi_L(x) = E\,\psi_L(x) \qquad -\frac{\hbar^2}{2m}\psi_R''(x) + V_\infty \psi_R(x) = E\,\psi_R(x)$$

$\psi_L(x) \equiv$ asymptotic $\psi(x)$, $x \to -\infty$ $\qquad \psi_R(x) \equiv$ asymp. $\psi(x)$, $x \to +\infty$

$$\psi_L'' = -\frac{2m}{\hbar^2}(E - V_\infty)\,\psi_L$$

$$\psi_L = A e^{ikx} + B e^{-ikx}$$

$A e^{ikx}$: particle moving to right

$B e^{-ikx}$: " " " left

$$k \equiv \sqrt{\frac{2m(E - V_\infty)}{\hbar^2}}$$

Incoming b.c. from the left:

$$\psi_L = A e^{ikx} + B e^{-ikx}$$

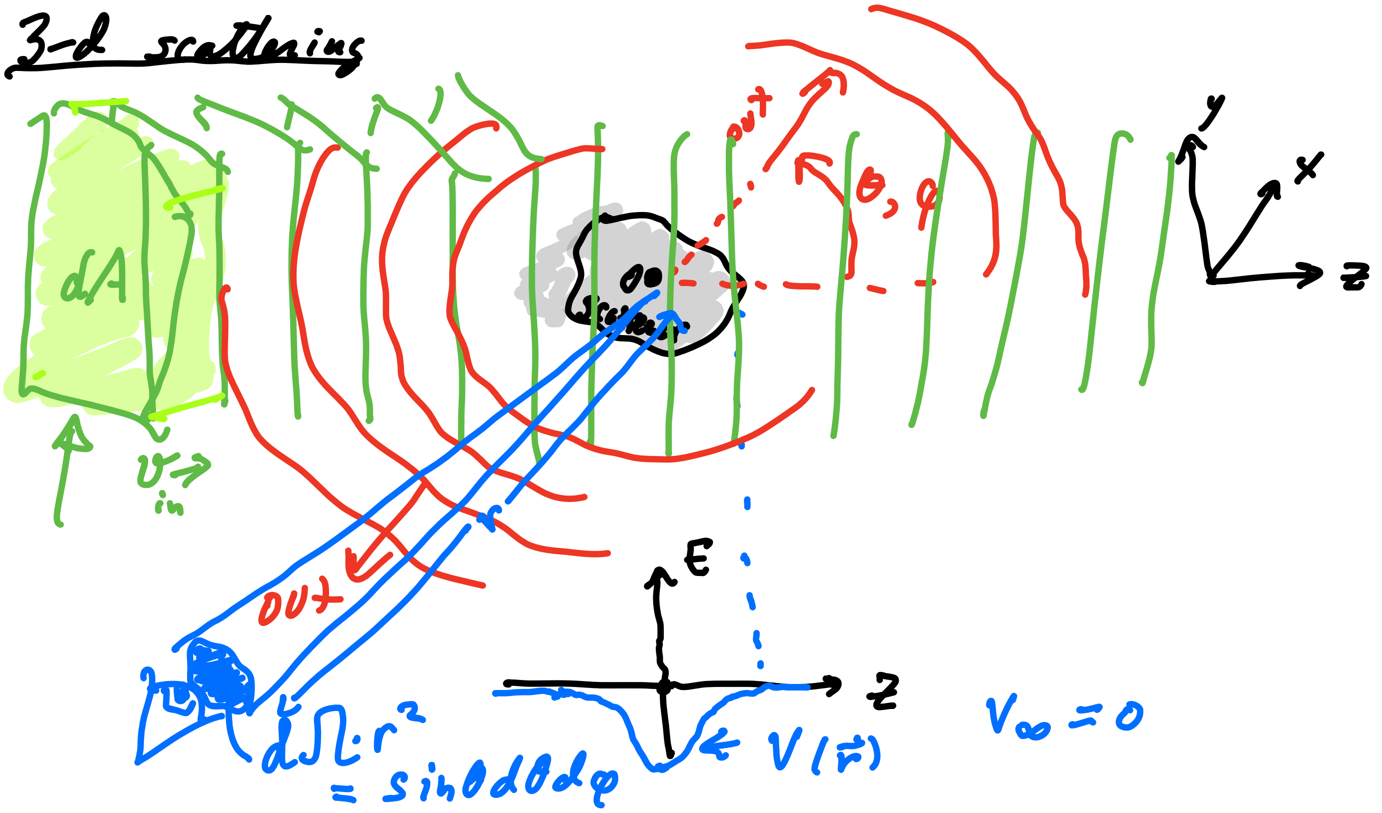
$\llcorner \propto$ amplitude to reflect

$\psi_R = C e^{ikx}$ ← $\propto$ " " transmit.

Prob. reflect: $R = \left|\frac{B}{A}\right|^2$

" transmit: $T = \left|\frac{C}{A}\right|^2$ $(R+T=1)$. "unitarity"

3-d scattering



To find b.c.: solve energy eigenvalue problem 'at' $r \to \infty$, where $V(r) \to 0$.

$$H \to \frac{p^2}{2\mu} = \text{free particle}$$

$$\psi(\vec{r}) = A e^{i\vec{k}\cdot\vec{r}} \qquad E = \frac{\hbar^2 k^2}{2\mu} \qquad \boxed{k = \frac{1}{\hbar}\sqrt{2\mu E}}$$

Given E, there are ∞'ly-many $\vec{k}$'s

General sol'n: $\psi(\vec{r}) = \int \underbrace{d\Omega}_{\vec{k} = (k, \theta, \varphi)} A(\theta\varphi) e^{i\vec{k}\cdot\vec{r}}$

$\vec{k} = (k, \theta, \varphi)$ polar coordinates

$\psi = A e^{ikz} + \ldots \iff$ incoming particle moving in $+z$ direction.

Solve $\frac{p^2}{2\mu}\psi = E\psi$ in spherical coords

$$\psi = R(r)\, Y_{lm}(\theta,\varphi) \qquad u(r) \equiv r \cdot R(r)$$

$$-\frac{\hbar^2}{2\mu}\frac{d^2u}{dr^2} + \cancelto{\text{drop}}{\frac{l(l+1)\hbar^2}{2\mu r^2}}\, u = E u$$

$\underset{\sim\sim}{r \to \infty}$

$$u = \underbrace{Be^{ikr}}_{\text{outgoing}} + \underbrace{\cancelto{0}{Ce^{-ikr}}}_{\text{incoming}} \qquad k = \frac{\sqrt{2\mu E}}{\hbar}$$

scatt. b.c.

$$(r \to \infty) \qquad \psi(\vec r) \approx \ldots + \boxed{f(\theta,\varphi)}\,\frac{e^{ikr}}{r}$$

this is what we want to compute

Summarize asymptotic $(r \to \infty)$ form of ψ $\qquad k = \frac{\sqrt{2\mu E}}{\hbar}$

$$\boxed{\psi(\vec r) \overset{r\to\infty}{\approx} A e^{ikz} + f(\theta,\varphi)\,\frac{e^{ikr}}{r}}$$

$$\boxed{\text{expect} \quad \mathrm{Prob}(\theta,\varphi) \approx \frac{|f(\theta,\varphi)|^2}{|A|^2}} \longrightarrow \text{make precise.}$$

Relation of $f(\theta,\varphi)$ to scattering probabilities

Schrö. Eq. $\quad -\frac{\hbar^2}{2\mu}\nabla^2\psi + V\psi = i\hbar\frac{\partial\psi}{\partial t}$

(probs 13.1 & 13.2 of Townsend)

$$0 = \frac{\partial}{\partial t}(\psi^*\psi) + \vec{\nabla}\cdot\vec{j}$$

local conservation law

$$\frac{\partial}{\partial t}(|\psi|^2) = -\vec{\nabla}\cdot\vec{j}$$

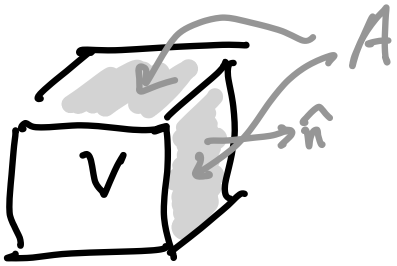
$|\psi|^2 = \psi^*_{(\vec{r})}\psi_{(\vec{r})}$ = probability density of finding a particle at position $\vec{r}$

= prob./vol.

$$\vec{j} = \frac{\hbar}{2\mu i}\left(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right)$$

= probability current density

= $\frac{\text{prob.}}{\text{vol.}} \times \text{velocity} = \frac{\text{prob}}{\text{area}\cdot\text{time}}$



$$\int d^2a\,\hat{n}\cdot\vec{j} = \frac{\text{prob leaving } V}{\text{time}}$$

A = boundary V

$$\int_V d^3\vec{r}\,|\psi|^2 = \frac{\text{prob}}{\text{vol}}\cdot V = \text{prob.}$$

$$\frac{d}{dt}\left(\int_V d^3r\, |\psi|^2\right) = -\int_V d^3r\, \vec{\nabla}\cdot\vec{j}$$

$$= -\int_A d^2a\ \hat{n}\cdot\vec{j}$$

A = boundary V

w/ outward normal

As $r\to\infty$ $\quad \psi \overset{r\to\infty}{\sim} Ae^{ikz} + \underbrace{A\,f(\theta,\varphi)}\ \frac{e^{ikr}}{r}$

definition!

$$|\psi|^2 \overset{r\to\infty}{\sim} |A|^2$$

$$\vec{j} \overset{r\to\infty}{\sim} \frac{\hbar k}{\mu r^2} |f(\theta,\varphi)|^2\, |A|^2\, \hat{r}$$

Differential cross-section

$$d\sigma \equiv \frac{\#\text{ particles going out into solid angle } d\Omega \text{ per second}}{\#\ \text{''}\ \ \text{''}\ \text{ in per unit area per second}}$$

$$\equiv \frac{(\text{prob. part. out per second per area})\cdot(r^2 d\Omega)}{(\text{prob. part. in per vol}).(\text{velocity in})}$$

$$\equiv \lim_{r\to\infty} \frac{|\vec{j}(r)|\ r^2 d\Omega}{|\psi(\vec{r})|^2\cdot\left(\frac{\hbar k}{\mu}\right)} \sim \frac{T^{-1}L^{-2}L^2}{L^{-3}\ L\,T^{-1}} \sim L^2$$

$$= \frac{\frac{\hbar k}{\mu r^2} |f(\theta,\varphi)|^2 |A|^2 r^2 d\Omega}{|A|^2 \cdot \left(\frac{\hbar k}{\mu}\right)}$$

$$\boxed{d\sigma = |f(\theta,\varphi)|^2 d\Omega} \qquad d\Omega = \sin\theta \, d\theta \, d\varphi$$

$$\frac{d\sigma}{d\Omega}(\theta,\varphi) \approx \text{“prob. of scattering to } (\theta,\varphi)\text{”}$$

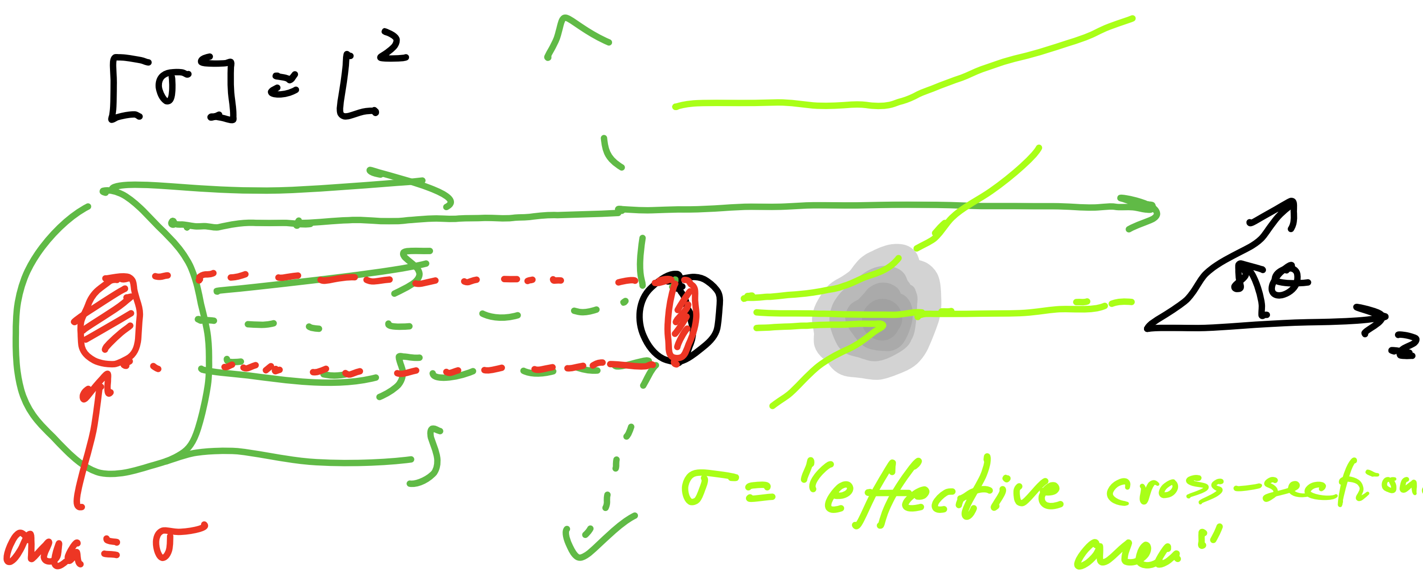
$$f(\theta,\varphi) \equiv \text{“scattering amplitude”.}$$

Total x-section

$$\sigma = \int \left(\frac{d\sigma}{d\Omega}\right) d\Omega = \int d\Omega \, |f(\theta,\varphi)|^2$$

$$= \text{“prob. of } \underline{\text{any}} \text{ scattering”}$$

$$[\sigma] = L^2$$



σ = "effective cross-sectional area"

Note: $\frac{d\sigma}{d\Omega}(\theta,\varphi)$ is ok as long as $\theta \neq 0$

$\theta = 0$ = "forward scattering"
= "transmission"

Computing $f(\theta,\varphi)$

- Goal: solve the time-independent Schröd. eqn:

$\hat{H}|\psi\rangle = E|\psi\rangle$

(assuming central pot'l for simplicity)

$$\Leftrightarrow -\frac{\hbar^2}{2\mu}\nabla^2\psi(\vec{r}) + V(r)\psi(\vec{r}) = E\,\psi(\vec{r})$$

$$\Leftrightarrow \boxed{(\nabla^2 + k^2)\psi(\vec{r}) = \frac{2\mu}{\hbar^2}V(r)\,\psi(\vec{r})} \quad (*)$$

with $\hbar^2 k^2 \doteq 2\mu E$, and with boundary conditions (leading terms):

$$\boxed{\psi(\vec{r}) \xrightarrow[r\to\infty]{} A\left(e^{ikz} + f(\theta,\varphi)\frac{e^{ikr}}{r}\right)} \quad (BC)$$

where $f(\theta,\varphi)$ unknown.

- For $r \to \infty$ $V(r) \to 0$, so the r.h.s. of (*) is small. This suggest that we treat the r.h.s. of (*) as a "perturbation".

Rewrite (*) as

$$\boxed{(\nabla^2 + k^2)\,\psi(\vec{r}) = J(\vec{r})} \qquad (J)$$

where we have simply defined the "source" $J(\vec{r}) \doteq \frac{2\mu}{\hbar^2} V(r)\,\psi(\vec{r})$, and pretend that $J(\vec{r})$ is known.

- Solve (J) by Greens function method:

(1): Solve

$$\boxed{(\nabla^2 + k^2)\,A(\vec{r}) = 0} \qquad (H)$$

for "homogeneous sol'n" $A(\vec{r})$

(2): Solve

$$\boxed{(\nabla_r^2 + k^2)\, G(\vec{r}, \vec{r}\,') = \delta^3(\vec{r} - \vec{r}\,')} \qquad \text{G}$$

for "Greens function" $G(\vec{r}, \vec{r}\,')$.

- In steps (1) & (2) impose boundary conditions BC on $G(\vec{r}, \vec{r}\,')$ and $A(\vec{r})$.

(3): Then general solution to J is

$$\boxed{\psi(\vec{r}) = A(\vec{r}) + \int d^3\vec{r}\,'\, G(\vec{r}, \vec{r}\,')\, J(\vec{r}\,')}$$

Proof:

$$(\nabla_r^2 + k^2)\psi(\vec{r}) = (\nabla^2 + k^2) A(\vec{r})$$

$$+ \int d^3\vec{r}\,' \,(\nabla_r^2 + k^2) G(\vec{r}, \vec{r}\,')\, J(\vec{r}\,')$$

$$= 0 + \int d^3\vec{r}\,'\, \delta^3(\vec{r} - \vec{r}\,')\, J(\vec{r}\,')$$

$$= J(\vec{r}) \quad . \checkmark$$

Step (1): Solve for $A(\vec{r})$.

- (H) is free particle equation so general solution is superposition of momentum eigenstates

$$A(\vec{r}) = A e^{i\vec{k}\cdot\vec{r}} + B e^{-i\vec{k}\cdot\vec{r}}$$

(really: sum over all $\vec{k}$ ...)

- (BC) says as $r \to \infty$ leading term is $A e^{ikz}$, so only this plane wave term can survive, $\therefore$

$$\boxed{A(\vec{r}) = A e^{ikz}} \qquad (1)$$

Step (2): solve for $G(\vec{r},\vec{r}')$.

- $G(\vec{r},\vec{r}') = G(\vec{r}-\vec{r}')$ by translation invariance

So (G) becomes

$$(\nabla_r^2 + k^2)\, G(\vec{r}-\vec{r}') = \delta^3(\vec{r}-\vec{r}')$$

Set $\vec{r}\,'=0$ w.l.o.g. (will put it back at end).
So G becomes

$$(\nabla^2+k^2)\,G(\vec{r}) = \delta^3(\vec{r}). \qquad G'$$

- $G(\vec{r}) = G(r)$ by rotational invariance.

Then $\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)$ + angular parts,
so G' becomes

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dG}{dr}\right)+k^2G = 0 \quad \text{for } r\neq 0$$

- General solution is

$$G = C\,\frac{e^{+ikr}}{r} + D\,\frac{e^{-ikr}}{r}$$

but BC $\Rightarrow$ $D=0$, so

$$G(r) = C\,\frac{e^{ikr}}{r}$$

w/ C = undetermined constant.

- Claim: $\boxed{(\nabla^2+k^2)\left(\frac{e^{ikr}}{r}\right) = -4\pi\,\delta^3(\vec{r})}$

To prove, want to show that

$$I \doteq \int d^3r \, f(\vec{r}) \, (\nabla^2 + k^2)\left(\frac{e^{ikr}}{r}\right) = -4\pi f(0)$$

for any smooth function $f(\vec{r})$.

Since $(\nabla^2 + k^2)\left(\frac{e^{ikr}}{r}\right) = 0$ for $r \neq 0$,

$$I = \lim_{\epsilon \to 0} \int_{r<\epsilon} d^3r \, f(\vec{r}) \, (\nabla^2 + k^2)\left(\frac{e^{ikr}}{r}\right)$$

$$= \lim_{\epsilon \to 0} \int_{r<\epsilon} d^3r \, f(0) \, (\nabla^2 + k^2)\left(\frac{1}{r}\right)$$

$$= f(0) \cdot \overbrace{\lim_{\epsilon \to 0} \int_{r<\epsilon} d^3r \, \nabla^2\left(\frac{1}{r}\right)}^{\text{"} I_1 \text{"}}$$

$$+ k^2 f(0) \cdot \underbrace{\lim_{\epsilon \to 0} \int_{r<\epsilon} d^3r \cdot \frac{1}{r}}_{\text{"} I_2 \text{"}}$$

$$I_2 = \lim_{\epsilon \to 0} \int_{r<\epsilon} d^3r \cdot \frac{1}{r} = 4\pi \lim_{\epsilon \to 0} \int_0^{\epsilon} r^2 dr \, \frac{1}{r}$$

$$= 4\pi \lim_{\epsilon \to 0} \frac{\epsilon^2}{2} = 0.$$

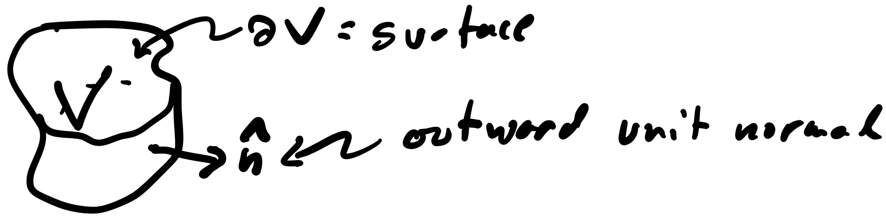
Evaluating I_1 is trickier.

$$I_1 = \lim_{\epsilon\to 0} \int_{r<\epsilon} d^3r \ \vec{\nabla}\cdot\vec{\nabla}\left(\frac{1}{r}\right)$$

$$= \lim_{\epsilon\to 0} \int_{r=\epsilon} r^2 d\Omega \ \hat{r}\cdot\vec{\nabla}\left(\frac{1}{r}\right)$$

by divergence theorem:

$$\left(\int_V d^3r \ \vec{\nabla}\cdot\vec{g} = \int_{\partial V} d^2a \ \hat{n}\cdot\vec{g} \right)$$



Now $\vec{\nabla} g(r) = \hat{r}\frac{dg}{dr}$, so

$$I_1 = \lim_{\epsilon\to 0} \int \epsilon^2 d\Omega \ \hat{r}\cdot\hat{r}\left(-\frac{1}{r^2}\right)\Big|_{r=\epsilon}$$

$$= -\lim_{\epsilon\to 0} \int d\Omega \frac{\epsilon^2}{\epsilon^2} = -4\pi.$$

$$\therefore \quad I = f(0)\, I_1 + k^2 f(0)\, I_2 = -4\pi f(0). \checkmark$$

$$\therefore \quad G(r) = -\frac{1}{4\pi}\frac{e^{ikr}}{r}.$$

Replace $r \to |\vec{r}-\vec{r}'|$, get

$$\boxed{G(\vec{r},\vec{r}\,') = -\frac{1}{4\pi}\,\frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}} \qquad (2)$$

Step (3): Therefore solution to (J) is

$$\boxed{\psi(\vec{r}) = A e^{ikz} - \frac{1}{4\pi}\int d^3\vec{r}\,'\ \frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}\ J(\vec{r})}$$

This is the Greens function solution to (J) with the scattering boundary conditions (BC).

- But, recall $J(\vec{r}) \doteq \frac{2\mu}{\hbar^2} V(\vec{r})\psi(\vec{r})$.

Plugging into above, get "solution" to Schrö eqn:

$$\boxed{\psi(\vec{r}) = \underbrace{A e^{ikz}}_{\psi_{inc}} \underbrace{- \frac{1}{4\pi}\int d^3\vec{r}\,'\ \frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}\cdot\frac{2\mu}{\hbar^2} V(r')\psi(\vec{r}\,')}_{\psi_{scatt}}}$$

This isn't really a solution, since $\psi(\vec{r})$ appears on both sides: it is just a re-writing of Schrö's eqn as an integral eqn! (Incorporates BC's).

Born approximation

- Replace $\psi(\vec{r}\,') \longrightarrow Ae^{ikz'}$ on the r.h.s.

  Idea is that as $r\to\infty$ $\psi(\vec{r}) \approx Ae^{ikz}$ + small

  since $\left|Ae^{ikz}\right| \gg \left|Af(\theta,\varphi)\frac{e^{ikr}}{r}\right|$.

- So

$$\psi_{\text{Born}}(\vec{r}) \overset{r\to\infty}{\approx} Ae^{ikz} + \int d^3\vec{r}\,'\, G(\vec{r},\vec{r}\,')\frac{2\mu}{\hbar^2}V(r')Ae^{ikz'}$$

$$= A\left(e^{ikz} + \frac{2\mu}{\hbar^2}\int d^3\vec{r}\,' G(\vec{r},\vec{r}\,')V(r')e^{ikz'}\right)$$

$$= A\left(e^{ikz} - \frac{\mu}{2\pi\hbar^2}\int d^3r'\, \frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}\underbrace{V(r')}_{\neq 0 \text{ only near } \vec{r}\,'=0}e^{ikz'}\right)$$

Therefore can assume $r \gg r'$ inside integral.

- $|\vec{r}-\vec{r}\,'| = \sqrt{r^2+(r')^2-2\,\vec{r}\cdot\vec{r}\,'}$

$$= r\sqrt{1+\underbrace{\left(\frac{r'}{r}\right)^2 - 2\hat{r}\cdot\frac{\vec{r}'}{r}}_{\text{small}}}$$

$$\approx r\left(1 - \hat{r}\cdot\frac{\vec{r}'}{r} + \mathcal{O}\left(\frac{r'}{r}\right)^2\right)$$

$$\approx r - \hat{r}\cdot\vec{r}' + r\,\mathcal{O}\left(\frac{r'}{r}\right)^2$$

- So

$$\frac{1}{|\vec{r}-\vec{r}'|} \approx \frac{1}{r - \hat{r}\cdot\vec{r}' + \cdots}$$

$$= \frac{1}{r\left(1-\hat{r}\cdot\frac{\vec{r}'}{r}+\cdots\right)}$$

$$= \frac{1}{r}\left(1-\hat{r}\cdot\frac{\vec{r}'}{r}+\cdots\right)^{-1}$$

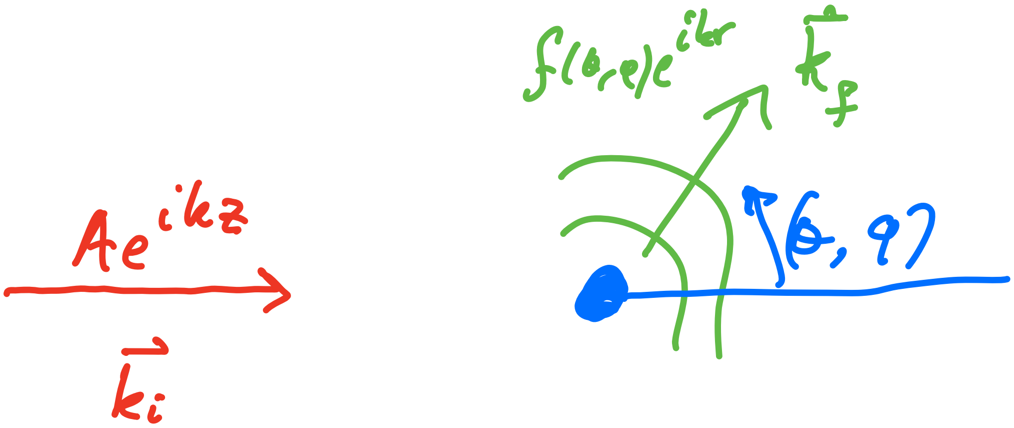
$$= \frac{1}{r}\left(1+\hat{r}\cdot\frac{\vec{r}'}{r}+\cdots\right)$$

- $$e^{ik|\vec{r}-\vec{r}'|} \approx e^{ikr}\,e^{-ik\,\hat{r}\cdot\vec{r}'}$$

$$\doteq e^{ikr}\,e^{-i\vec{k}_f\cdot\vec{r}'}$$

where define $\boxed{\vec{k}_f \doteq k\hat{r}}$

& define $\boxed{\vec{k}_i \doteq k\hat{z}}$



- $e^{ikz'} = e^{ik\hat{z}\cdot\vec{r}\,'} = e^{i\vec{k}_i\cdot\vec{r}\,'}$.

Plug all these into rhs of Ψ_{Born}, get

$$\boxed{\Psi_{Born}(\vec{r}) \overset{r\to\infty}{\approx} A\Big(\underbrace{e^{ikz}}_{inc} \underbrace{- \frac{\mu}{2\pi\hbar^2}\frac{e^{ikr}}{r}\int d^3r'\, V(\vec{r}\,')e^{i\vec{q}\cdot\vec{r}\,'}}_{scatt}\Big)}$$

where defined momentum transfer$/\hbar$

$$\boxed{\vec{q} \equiv \vec{k}_i - \vec{k}_f = \text{momentum transfer}/\hbar}$$

$= \underbrace{k(\hat{z} - \hat{r})}$ (θ,φ)-dependence in here.

- So, scattering amplitude, in Born approx:

$$\boxed{f_{Born}(\theta,\varphi) = -\frac{\mu}{2\pi\hbar^2}\int d^3\vec{r}\,'\, V(\vec{r}\,')\, e^{i\vec{q}\cdot\vec{r}\,'}}$$

3d Fourier transform of scattering potential!

• Validity of Born approximation

$$|\psi_{inc}| \gg |\psi_{scatt}| \quad ?$$

$$1 \overset{?}{\gg} \left|\frac{\psi_{scatt}}{\psi_{inc}}\right| = \left| \frac{-\frac{\mu}{2\pi\hbar^2}\int d^3r' \, \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} V(r')\, \psi_{inc}(r')}{\psi_{inc}(r)}\right|$$

for $r \approx 0$ (near scatterer) $\psi_{inc}(r) = Ae^{ikz} \approx A$

$$\approx \frac{\mu}{2\pi\hbar^2}\left|\int_0^\infty dr' \int_0^{2\pi} d\varphi' \int_0^{\pi} \sin\theta' \, d\theta' \, (r')^2 \, \frac{e^{ikr'}}{r'} V(r') \, e^{ikr'\cos\theta'}\right|$$

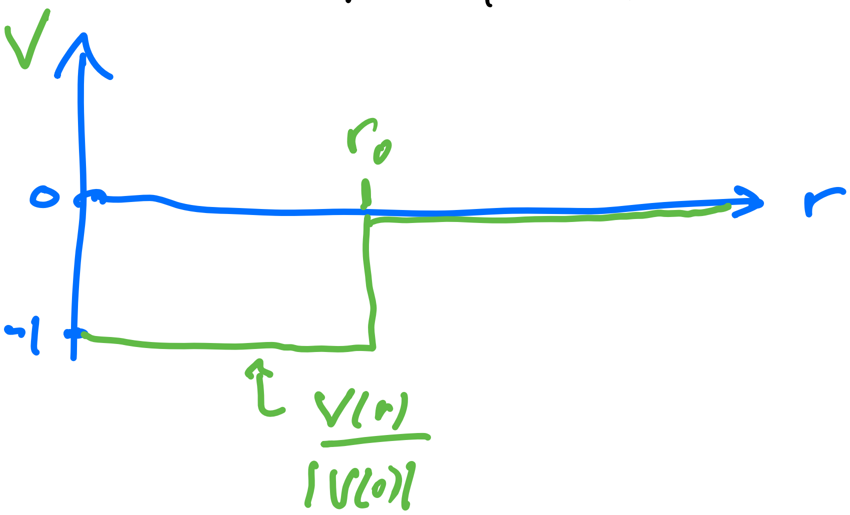
$$= \left|\frac{2\mu}{\hbar^2 k}\int_0^\infty dr' \, e^{ikr'} \, V(r') \, \sin kr'\right|$$

$$\approx \begin{cases} (k\to\infty): \left|\dfrac{2\mu}{\hbar^2 k}\displaystyle\int_0^{1/k} dr' \cdot kr' \cdot V(r')\right| \approx \dfrac{2\mu|V(0)|}{\hbar^2 k^2} = \dfrac{|V(0)|}{E} \\[2ex] (k\to 0): \left|\dfrac{2\mu}{\hbar^2 k}\displaystyle\int_0^{\infty} dr' \cdot kr' \cdot V(r')\right| \approx \dfrac{\mu}{\hbar^2}|V(0)|\, a^2 = \dfrac{|V(0)|}{2E} k^2 a^2 \end{cases}$$

where defined $a^2 \doteq \left|2\int_0^\infty dr \cdot r \, \frac{V(r)}{V(0)}\right|$.

a has interpretation as "range of potential":

For example, for spherical well pot'l



$$a^2 = 2\left|\int_0^\infty dr \cdot r\,\frac{V(r)}{V(0)}\right| = 2\int_0^{r_0} dr \cdot r = r_0^2$$

So $a = r_0$. ✓

- So expect Born approx good if

(1) $E \gg |V(0)|$ as $k \to \infty$

or

(2) $E \gg |V(0)|\, k^2 a^2$ as $k \to 0$

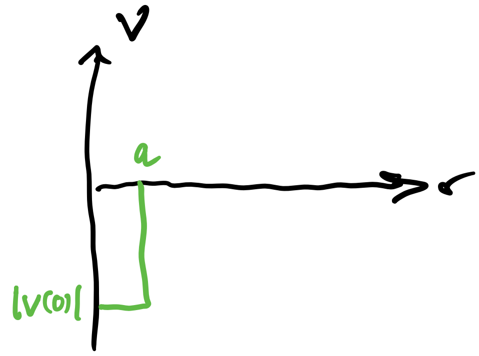
- Limit (1): $E = \frac{\hbar^2 k^2}{2\mu} \Rightarrow$

so good if $k^2 \gg \frac{2\mu |V(0)|}{\hbar^2}$ as $k \to \infty$

i.e. if pot'l $V(0)$ is finite, Born approx. is good in high-energy limit.

• Limit (2): $E = \frac{\hbar^2 k^2}{2\mu} \Rightarrow$

$$\frac{\hbar^2}{2\mu a^2} \gg |V(0)|$$



i.e. if pot'l much "narrower" than it is "deep", then Born approx. is good in low-energy limit.

Example: Yukawa pot'l

• $$V(r) = g \frac{e^{-m_0 r}}{r}$$ "short range potential"

$\frac{1}{m_0}$ = "screening length"

- if $r \ll \frac{1}{m_0} \rightarrow m_0 r \ll 1 \Rightarrow e^{-m_0 r} \approx 1$

  $\rightarrow$ like Coulomb.

- if $r \gg \frac{1}{m_0} \rightarrow m_0 r \gg 1 \Rightarrow e^{-m_0 r} \rightarrow 0$

  $\rightarrow$ no pot'l

- Born approx:

$$f(\theta,\varphi) = -\frac{\mu g}{2\pi\hbar^2}\int d^3r' \, \frac{e^{-m_0 r'}}{r'} \, e^{i\vec{q}\cdot\vec{r}'}$$

with $\quad \vec{q} = \vec{k}_i - \vec{k}_f = k(\hat{z} - \hat{r})$

- Choose z'-axis to be along $\vec{q}$

$$\Rightarrow \quad \vec{q} = q\hat{z}'$$

$$\therefore \quad \vec{q}\cdot\vec{r}' = q\,\hat{z}'\cdot\vec{r}' = q(\cos\theta')\,r'$$

Then

$$f(\theta,\varphi) = \frac{-\mu g}{2\pi\hbar^2}\int_0^\infty (r')^2 dr' \int_0^{2\pi} d\varphi' \int_0^\pi \sin\theta' d\theta' \, \frac{e^{-m_0 r'}}{r'} \, e^{iq(\cos\theta')r'}$$

$$= -\frac{\mu g}{2\pi\hbar^2} 2\pi \int_0^\infty dr' \, r' e^{-m_0 r'} \int_{-1}^{1} d(\cos\theta') \, e^{iqr'\cos\theta'}$$

$$= -\frac{\mu g}{\hbar^2}\int_0^\infty dr' \, r' e^{-m_0 r'} \frac{2}{qr'} \sin qr'$$

$$= \frac{-2\mu g}{\hbar^2(m_0^2+q^2)} .$$

- $q^2 = (\vec{k}_i - \vec{k}_f)^2 = k^2(\hat{z}-\hat{r})^2$

$$= k^2(1+1-2\,\hat{z}\cdot\hat{r})$$

$$= 2k^2(1-\cos\theta) = 4k^2\sin^2\left(\frac{\theta}{2}\right)$$

$$\Rightarrow \quad f(\theta,\varphi) = \frac{-2\mu g}{\hbar^2\left(m_0^2 + 4k^2\sin^2\frac{\theta}{2}\right)} .$$

so

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_{\text{Yuk}} = |f(\theta,\varphi)|^2 = \frac{4\mu^2 g^2}{\hbar^4\left[m_0^2+4k^2\sin^2\frac{\theta}{2}\right]^2}}$$

Limits

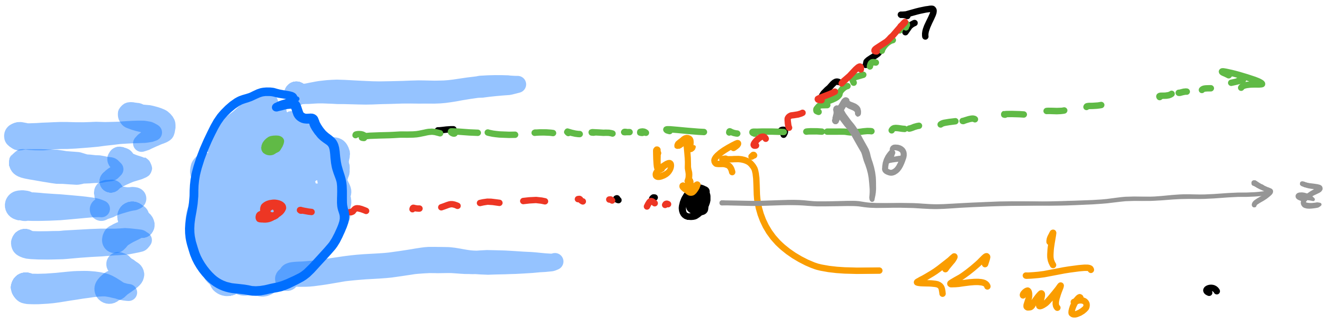
(1) $m_0 \to 0 \Rightarrow$ coulomb potential

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Coul.}} = \frac{\mu^2 g^2}{4\hbar^4 k^4 \sin^4\frac{\theta}{2}} \qquad \hbar^2 k^2 = 2\mu E$$

$$= \frac{g^2}{16E^2\sin^4\frac{\theta}{2}} = \text{“Rutherford scatt. x-section”}$$

(2) high-energy fixed angle scattering

- take E (& k) large, keep θ fixed
- same as Rutherford x-section



$b \ll \frac{1}{m_0}$ at fixed θ in limit $E \to \infty$.

(3) Total x-section

$$\sigma \doteq \int d\Omega \, \frac{d\sigma}{d\Omega}$$

$$= \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta \, d\theta \cdot \frac{4\mu^2 g^2}{\hbar^4 \left[m_0^2 + 4k^2 \sin^2\frac{\theta}{2}\right]^2}$$

$$= \frac{8\pi\mu^2 g^2}{\hbar^4} \int_{-1}^{1} \frac{d(\cos\theta)}{\left[m_0^2 + 2k^2(1-\cos\theta)\right]^2} < \frac{16\pi\mu^2 g^2}{\hbar^4} \, \frac{1}{m_0^4}$$

- But for $m_0 = 0$ (Coulomb scattering)

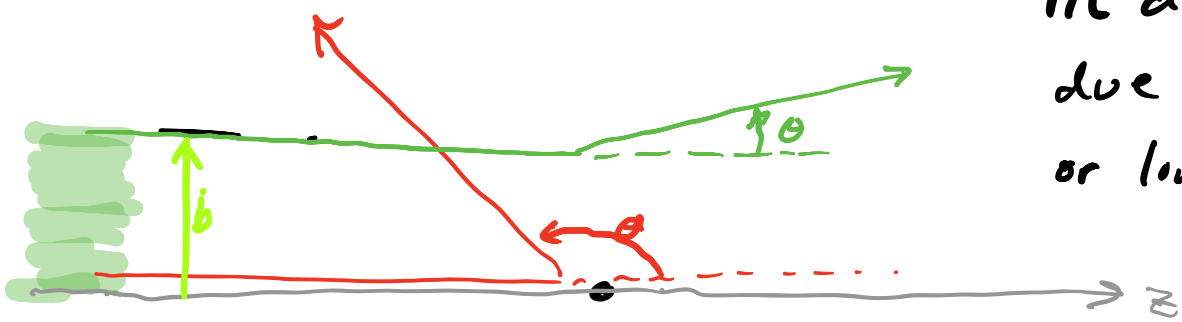
$$\sigma = \frac{2\pi\mu^2 g^2}{k^4 \hbar^4} \int_{-1}^{1} \frac{d(\cos\theta)}{(1-\cos\theta)^2} = \infty \,!$$

- Divergence comes from $\cos\theta \to 1$

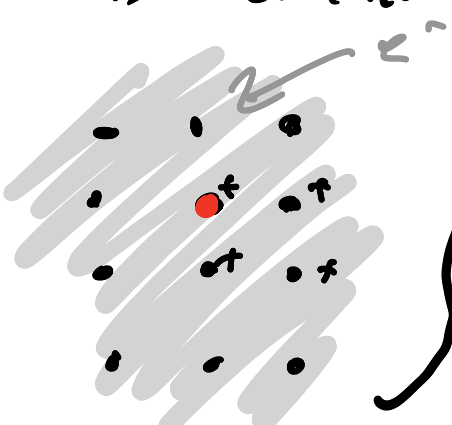
  i.e. $\theta \to 0$ = almost no scattering:

- This comes from the tiny deflection from particles with impact parameter $b \to \infty$

"IR divergence" due to low-energy or long distances



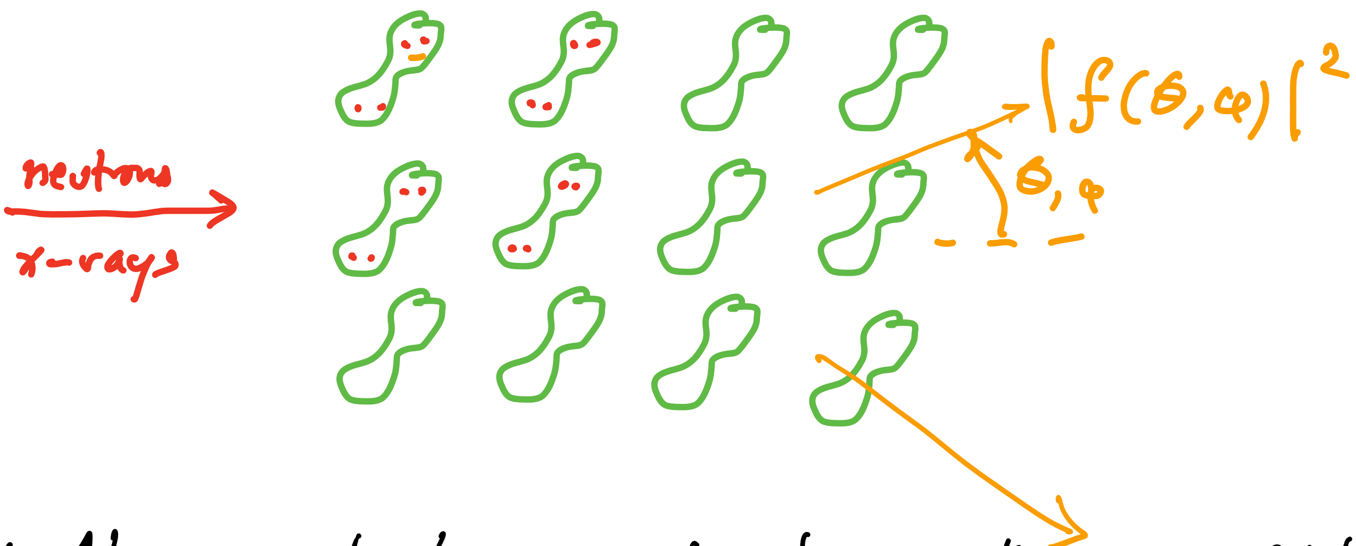
- In real materials, this IR behavior is "screened" or "cut off":



real materials are charge-neutral on atomic scales, so Coulomb potential is really Yukawa with $m_0 \sim \frac{1}{a_0}$.

Partial Wave expansion (§13.4)

- Found Born approx $f(\theta,\varphi) \sim$ F.T. of $V(\vec{r})$
  Good at large E:



- Now want to approximate scattering amplitudes at small $E>0$. The idea is to write $f(\theta,\varphi)$ in terms of "partial waves"

① $f(\theta,\varphi) = \sum_{\ell=0}^{\infty} (\text{partial wave})_\ell$

and then keep only the low-ℓ terms

this lecture, exact

② approx: $\approx \sum_{\ell=0,1} (\text{partial wave})_\ell$

good for $0<E$ small.

next lecture

① Expand $f(\theta,\varphi)$ in spherical harmonics:

$$f(\theta,\varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} c_{\ell m} Y_{\ell m}(\theta,\varphi)$$

$$\int d\Omega\, Y^*_{\ell' m'}(\theta,\varphi) f(\theta,\varphi) = \sum_{\ell,m} c_{\ell m} \underbrace{\int d\Omega\, Y^*_{\ell' m'} Y_{\ell m}}_{\delta_{\ell\ell'}\delta_{mm'}}$$

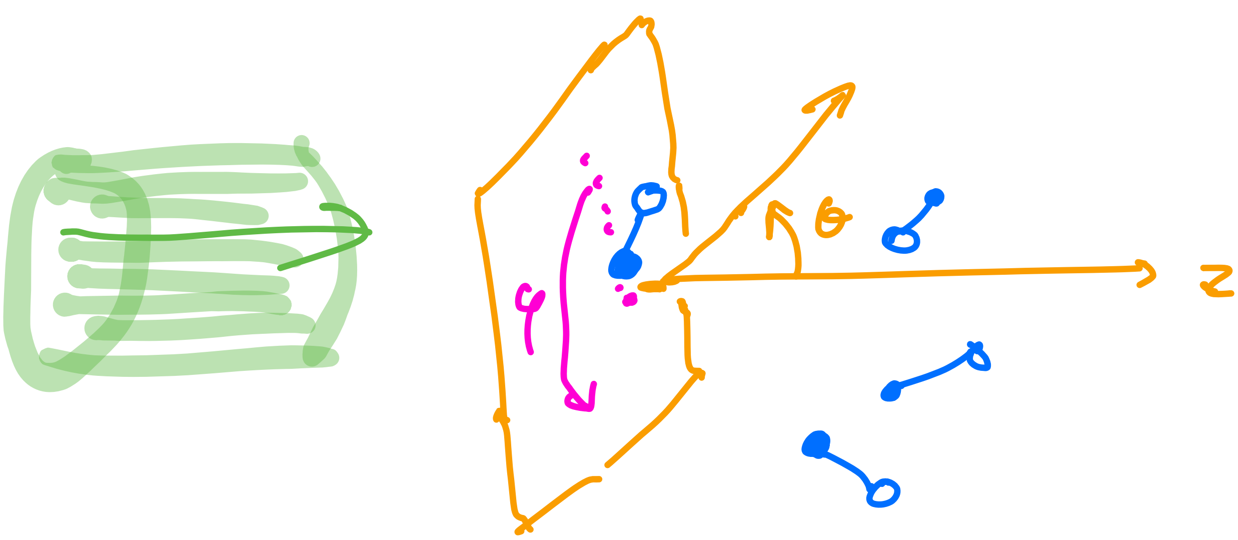
$$= c_{\ell' m'} .$$

Recall $Y_{\ell,m}(\theta,\varphi) = \langle \theta,\varphi | \ell, m \rangle$ &

$$\delta_{\ell\ell'}\delta_{mm'} = \langle \ell' m' | \ell m \rangle = \int d\Omega \langle \ell', m' | \theta,\varphi \rangle \langle \theta,\varphi | \ell, m \rangle$$

$$= \int d\Omega\, Y^*_{\ell' m'}(\theta,\varphi)\, Y_{\ell,m}(\theta,\varphi) .$$

• Simplify: Assume $f(\theta,\varphi) = f(\theta)$



Recall $Y_{l,m}(\theta,\varphi) = Q(\theta)\, e^{im\varphi}$,

so no φ-dependence $\Leftrightarrow$ $m=0$ &

$$f(\theta) = \sum_{l=0}^{\infty} c_{l0}\, Y_{l,0}(\theta)$$

with $Y_{l,0} = \sqrt{\frac{2l+1}{4\pi}}\, P_l(\cos\theta)$

[Legendre polynomials:
$\deg(P_l) = l$ $\quad P_l(-x) = (-)^l P_l(x)$]

Rewrite this as

$$\boxed{f(\theta) = \sum_{l=0}^{\infty} (2l+1)\cdot a_l(k)\cdot P_l(\cos\theta)}$$

partial wave expansion

conventional factor

emphasize E-dependence

Compute $a_l(k)$ from Schrö's eqn?

$\hat{H}|\psi\rangle = E|\psi\rangle$ with boundary conditions

BC $\quad \psi(\vec{r}) \xrightarrow[\sim\sim]{r\to\infty} A\left(e^{ikz} + f(\theta)\,\frac{e^{ikr}}{r}\right)$

As $r\to\infty$, $V(r)\to 0$, so start by solving Schrö's eqn. with $V=0$ (free particle)

$$\psi = \sum_{l,m} R_l(r)\, Y_{lm}(\theta,\varphi) \quad \&$$

$$(*) \quad \frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} - \frac{l(l+1)}{r^2}R + k^2 R = 0$$

$$(\hbar^2 k^2 \doteq 2\mu E)$$

Basis of solutions to (*) are the spherical Bessel and Neumann functions

Spherical Bessel $\quad j_l(\rho) \equiv (-\rho)^l \left(\frac{1}{\rho}\frac{d}{d\rho}\right)^l \left(\frac{\sin\rho}{\rho}\right)$

spherical Neumann $\quad \eta_l(\rho) \equiv -(-\rho)^l \left(\frac{1}{\rho}\frac{d}{d\rho}\right)^l \left(\frac{\cos\rho}{\rho}\right)$

$$\rho \equiv kr.$$

(Section 10.4 of Townsend.)

$\therefore$ General solution for $\psi(r)$ as $r\to\infty$

$$\psi(\vec{r}) \xrightarrow[r\to\infty]{} \sum_{\ell=0}^{\infty}\left[A_\ell j_\ell(kr) + B_\ell \eta_\ell(kr)\right] P_\ell(\cos\theta)$$

$$\xrightarrow[r\to\infty]{\sim} \sum_{\ell=0}^{\infty}\left[A_\ell \frac{\sin(kr-\frac{\pi\ell}{2})}{kr} - B_\ell \frac{\cos(kr-\frac{\pi\ell}{2})}{kr}\right] P_\ell(\cos\theta)$$

where used the leading large-r form of $j_\ell(kr)$ & $\eta_\ell(kr)$.

- Can rewrite (A_ℓ, B_ℓ) as (C_ℓ, δ_ℓ) using the identity

$$\boxed{A\sin\alpha - B\cos\alpha = C\sin(\alpha+\delta)}$$

exercise!

"phase shift"

- So

$$\boxed{\psi(\vec{r}) \xrightarrow[r\to\infty]{\sim} \sum_{\ell=0}^{\infty} \frac{C_\ell}{2ikr}\left[\underbrace{e^{i(kr-\frac{\pi\ell}{2}+\delta_\ell)}}_{\sim \frac{e^{ikr}}{r}} - e^{-i(kr-\frac{\pi\ell}{2}+\delta_\ell)}\right] P_\ell(\cos\theta)}$$ (*)

$\sim \frac{e^{ikr}}{r}$ outgoing spherical wave ✓

$\sim e^{ikz}$ incoming plane wave?

- Want to extract incoming plane wave in this expansion ... see Ch. 10 Townsend ...

$$e^{ikz} = e^{ikr\cos\theta} = \sum_{l=0}^{\infty} i^l (2l+1) \cdot j_l(kr) \cdot P_l(\cos\theta)$$

$$\xrightarrow[r\to\infty]{\sim} \sum_{l=0}^{\infty} \frac{i^l(2l+1)}{2ikr}\left[e^{i(kr-\frac{\pi l}{2})} - e^{-i(kr-\frac{\pi l}{2})}\right] P_l(\cos\theta)$$

- Therefore, from BC have

$$\psi(\vec{r}) \xrightarrow[r\to\infty]{\sim} A\left(e^{ikz} + f(\theta)\frac{e^{ikr}}{r}\right)$$

$$= A\left\{ \sum_{l=0}^{\infty} \frac{i^l(2l+1)}{2ikr}\left[e^{i(kr-\frac{\pi l}{2})} - e^{-i(kr-\frac{\pi l}{2})}\right] P_l(\cos\theta) + \sum_{l=0}^{\infty} \frac{(2l+1)}{r} a_l(k) e^{ikr} P_l(\cos\theta) \right\}$$

partial wave expansion

$$= A \sum_{l=0}^{\infty} \frac{2l+1}{2ikr}\left[\left(i^l + 2ik e^{\frac{i\pi l}{2}} a_l(k)\right) e^{i(kr-\frac{\pi l}{2})} - i^l e^{-i(kr-\frac{\pi l}{2})}\right] P_l(\cos\theta)$$

Compare this to (*), get

$$\begin{cases} C_l e^{i\delta_l} = A(2l+1)\left(i^l + 2ik \overbrace{e^{\frac{i\pi l}{2}}}^{=i^l} a_l(k)\right) \\ C_l e^{-i\delta_l} = A(2l+1)\, i^l \end{cases}$$

Divide them to get

$$e^{2i\delta_\ell} = 1 + 2ik\, a_\ell(k)$$

$$\Rightarrow \boxed{a_\ell(k) = \frac{1}{k} e^{i\delta_\ell(k)} \sin(\delta_\ell(k))}$$

Partial waves c.t.o. phase shifts

$$\Rightarrow |a_\ell|^2 = \frac{\sin^2\delta_\ell}{k^2}$$

- So the partial wave expansion becomes

$$\boxed{f(\theta) = \sum_{\ell=0}^{\infty} \frac{(2\ell+1)}{k} e^{i\delta_\ell(k)} \sin(\delta_\ell(k))\, P_\ell(\cos\theta)}$$

PWE

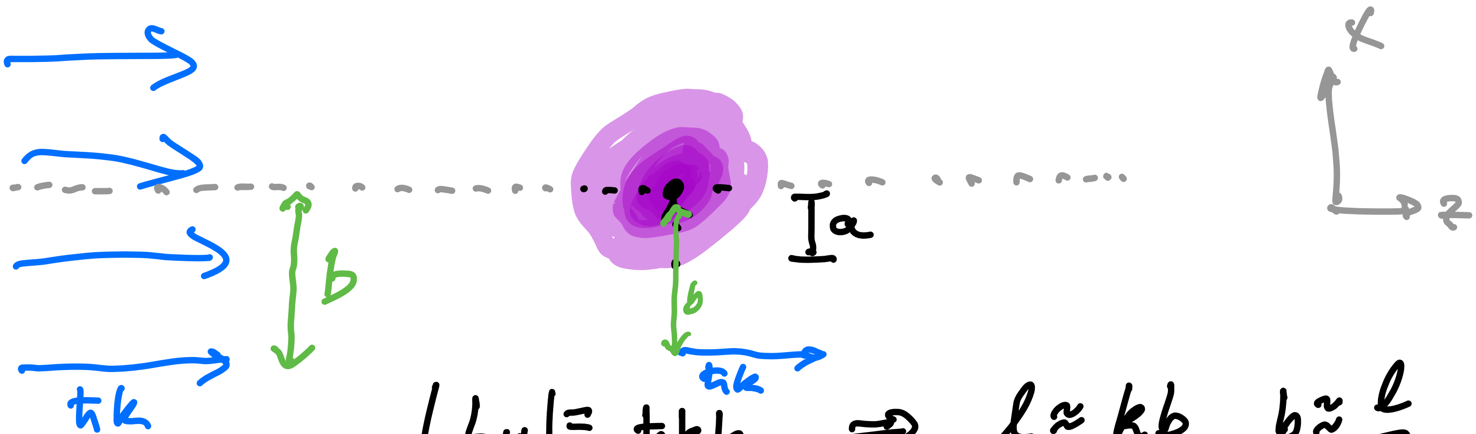
- We have just shifted our problem from computing $a_\ell(k)$ to computing the phase shifts $\delta_\ell(k)$ ...

- But this kind of reasoning, using the asymptotic form of spherical Bessel functions, is how we will solve for $a_\ell(k)$ or $\delta_\ell(k)$ in examples ...

② Partial wave approximation

- If the scatterer is of finite size, a, then $a_\ell(k) \approx 0$ for all $\ell \gtrsim \ell_{max} \doteq ak$.

- Why?



$$|L_y| = \hbar k b \Rightarrow \ell \approx kb \qquad b \approx \frac{\ell}{k}$$

- If $b > a$ the particle "misses" target, i.e. doesn't scatter (much).

So amplitude ≈ 0 for $\frac{\ell}{k} > a$. ✓

∴ In limit $k \to 0$ ($ka \ll 1$) need only keep the first term in the partial wave expansion. (recall $k \to 0 \Rightarrow E \to 0$).

Total x-section

$$P_\ell = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_{\ell,0}$$

$$\sigma = \int d\Omega \frac{d\sigma}{d\Omega} = \int d\Omega \, |f(\theta)|^2$$

$$= \int d\Omega \left| \sum_\ell (2\ell+1)\, a_\ell P_\ell(\cos\theta) \right|^2$$

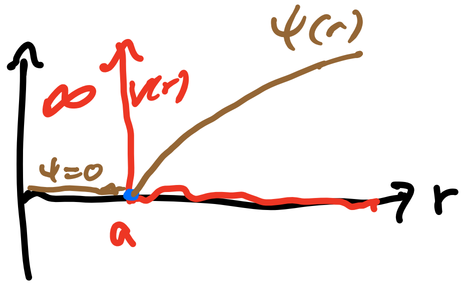
$$= \sum_{\ell\ell'} \int d\Omega \, (2\ell+1)(2\ell'+1)\, a_\ell^* a_{\ell'} \cdot$$

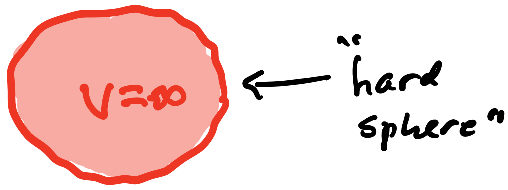
$$\cdot \sqrt{\frac{4\pi}{2\ell+1}} \sqrt{\frac{4\pi}{2\ell'+1}} \underbrace{Y_{\ell,0}^* Y_{\ell',0}}_{\delta_{\ell,\ell'}}$$

$$\boxed{\begin{aligned} \sigma &= \sum_\ell 4\pi (2\ell+1)\, |a_\ell|^2 \\ &= \frac{4\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell+1) \sin^2 \delta_\ell \end{aligned}}$$

Example: Hard-sphere scattering

$$V(r) = \begin{cases} \infty & r < a \\ 0 & r > 0 \end{cases}$$





So boundary conditions on wave function:

$$\psi(a) = 0 \qquad\qquad (\psi(r)=0 \quad r \le a)$$

& $\psi(r)$ free Schrö eqn for $r > a$.

- Compute the $\ell = 0$ partial wave ("s-wave scattering").

Solve the radial (free) equation for $r > a$:

$$\psi = R_\ell(r)\, Y_{\ell m}(\theta,\varphi) \quad \text{with} \quad \ell = 0$$

$$\frac{d^2R_\ell}{dr^2} + \frac{2}{r}\frac{dR_\ell}{dr} - \cancelto{0}{\frac{\ell(\ell+1)}{r^2}} R + k^2 R_\ell = 0 \qquad (\ell = 0)$$

Define $R_0(r) \doteq \frac{1}{r} u_0(r)$ as usual, to get

$$\frac{d^2 u_0}{dr^2} = -k^2 u_0$$

which has general solution

$$u_0 = A_0 \cos kr + B_0 \sin kr = C_0 \sin(kr + \delta_0)$$

- With boundary condition $\psi(a) = 0$, implies $u_0(a) = 0$, therefore $\sin(ka + \delta_0) = 0$,

so find $\boxed{\delta_0 = -ka}$.

$$\therefore \quad \boxed{\sigma_{\ell=0} = \frac{4\pi}{k^2}\sin^2(ka)} \qquad ka \ll 1$$

which is expected to approximate total x-section

$\sigma_{\ell=0} \approx \sigma$ for $ka \ll 1$

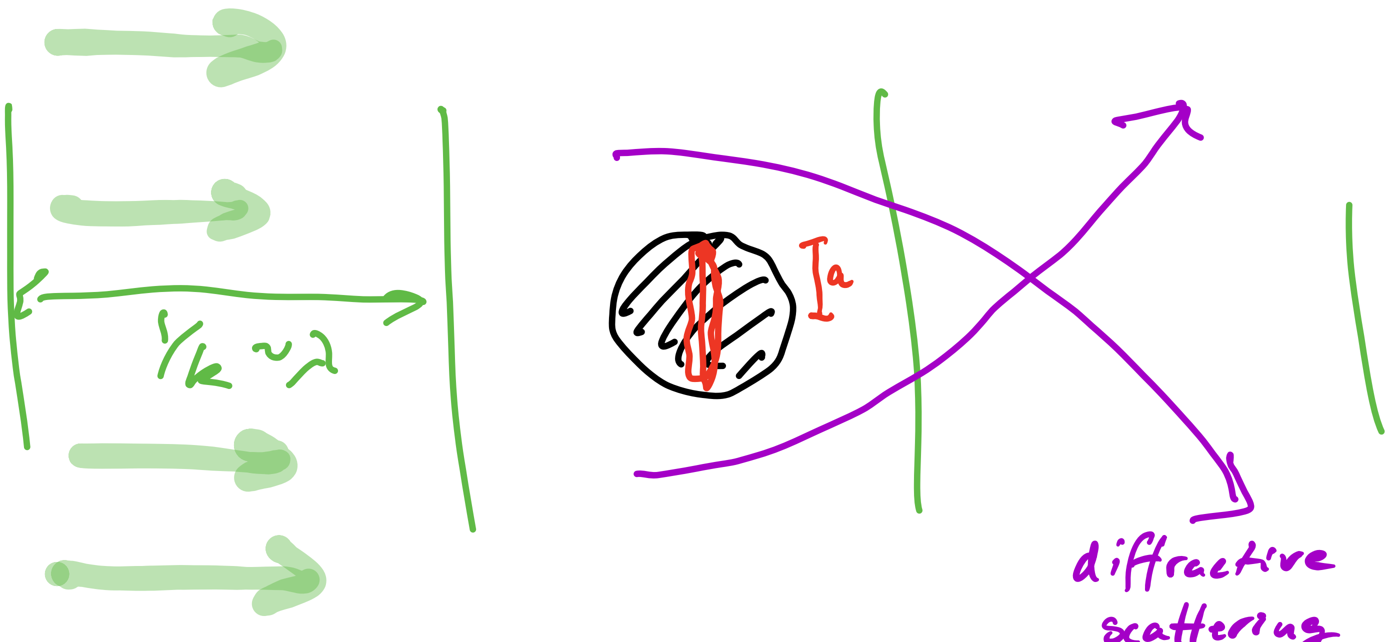
For $ka \ll 1$, $\sin ka \approx ka$, so get

$$\boxed{\lim_{k \to 0} \sigma = 4\pi a^2}$$

- Note, x-sectional area of hard sphere is πa^2, not $4\pi a^2$!

So quantum-mechanical answer is four times the classical answer!

This can be thought of as due to wave-like interference effects (diffraction):



- Can also compute $l=1$ ("p-wave") scattering contribution. Expect

$$\sigma_{l=1} \ll \sigma_{l=0} \quad \text{for} \quad ka \ll 1$$

See problem set.

Resonances

- From partial wave expansion:

$$\sigma = \frac{4\pi}{k^2} \sum_\ell (2\ell+1) \sin^2(\delta_\ell(E))$$

- Suppose there is an energy $E = E_{0,\ell}$ such that

$$\delta_\ell(E_{0,\ell}) = \frac{\pi}{2} \qquad \left(\sin\frac{\pi}{2} = 1 = \max\right)$$

and expand $\delta_\ell(E)$ around this value:

$$\delta_\ell(E) = \delta_\ell(E_{0,\ell}) + (E - E_{0,\ell})\,\delta_\ell'(E_{0,\ell}) + \cdots$$

$$= \frac{\pi}{2} + (E - E_{0,\ell})\frac{2}{\Gamma_\ell} + \mathcal{O}(E - E_{0,\ell})^2$$

(definition of "Γ_ℓ")

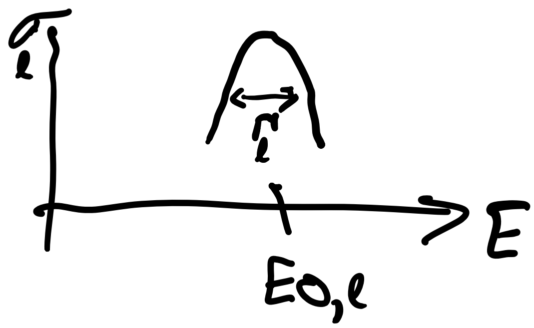
- Then $\sin\delta_\ell \approx \sin\left(\frac{\pi}{2} + \frac{E - E_{0,\ell}}{\Gamma_\ell/2}\right)$

$$= \cos\left(\frac{E - E_{0,\ell}}{\Gamma_\ell/2}\right)$$

so

$$\sigma_\ell \approx \frac{4\pi(2\ell+1)}{k^2} \cos^2\left(\frac{E-E_{0,\ell}}{\Gamma_\ell/2}\right)$$

$$\approx \frac{4\pi(2\ell+1)}{k^2} \cdot \frac{\Gamma_\ell^2/4}{(E-E_{0,\ell})^2 + \Gamma_\ell^2/4}$$ "Briet-Wigner resonance"

$\sim$ 

Sum all partial waves $\ell = 0, 1, 2 \ldots$
get local maxima = "resonances" in σ: