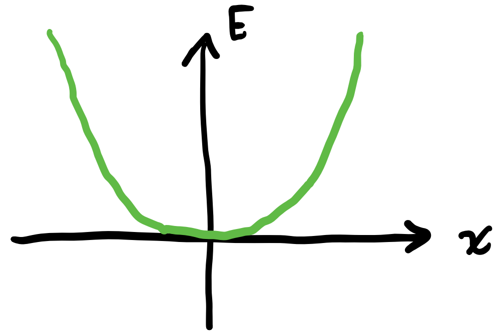
Chapter 7 Simple Harmonic Oscillator

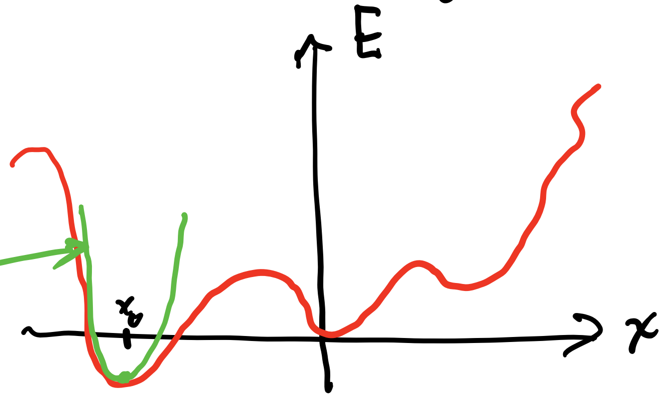
$$\hat{H} = \frac{1}{2m}\hat{P}^2 + \frac{m\omega^2}{2}\hat{X}^2$$



Why important?

- exactly solvable
- basic example of all quantum field theories (e.g. photons, phonons, conduction electrons...)
- leading approximation to low-energy physics:

$$V(x) = \overbrace{V(x_0)}^{\text{irrelevant constant}} + (x-x_0)\cancelto{0}{V'(x_0)}$$
$$+ \underbrace{\tfrac{1}{2}(x-x_0)^2 V''(x_0)} + \cdots$$



(Classically: $E = \frac{m}{2}\dot{x}^2 + \frac{k}{2}x^2$

$\Rightarrow m\ddot{x} = -kx$

$\Rightarrow x = Ae^{i\omega t} + Be^{-i\omega t}$ w/ $\omega = \sqrt{\frac{k}{m}}$.)

Can solve directly from operator algebra without solving a differential equation. (Analogously to way used angular momentum algebra $[\hat{J}_x, \hat{J}_y] = i\hbar\hat{J}_z$ to deduce $|j,m\rangle$ states.)

$[\hat{X}, \hat{P}] = i\hbar$

$\Rightarrow$ $[\hat{X}, \hat{H}] = i\frac{\hbar}{m}\hat{P}$ & $[\hat{P}, \hat{H}] = -i\hbar m\omega^2 \hat{X}$

(Use: $[\hat{A}, \hat{B}\hat{C}] = [\hat{A}, \hat{B}]\hat{C} + \hat{B}[\hat{A}, \hat{C}]$
= Leibniz rule for commutators.)

Find analog of $\hat{J}_\pm \doteq \hat{J}_x \pm i\hat{J}_y$ ...

$$\boxed{\begin{aligned} \hat{a} &\doteq \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{X} + \frac{i}{m\omega}\hat{P}\right) \\ \hat{a}^\dagger &\doteq \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{X} - \frac{i}{m\omega}\hat{P}\right) \end{aligned}}$$

"lowering" or "annihilation" op.

"raising" or "creation" op.

Check!

$$\boxed{\begin{aligned} \hat{X} &= \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a} + \hat{a}^\dagger\right) \\ \hat{P} &= -i\sqrt{\frac{m\omega\hbar}{2}}\left(\hat{a} - \hat{a}^\dagger\right) \\ [\hat{a}, \hat{a}^\dagger] &= 1 \qquad (*) \\ \hat{H} &= \hbar\omega\left(\hat{a}^\dagger\hat{a} + \tfrac{1}{2}\right) \end{aligned}}$$

- Number operator:

$$\hat{N} \doteq \hat{a}^\dagger \hat{a} \qquad \hat{N}^\dagger = \hat{N}$$

Since $\hat{H} = \hbar\omega(\hat{N} + \frac{1}{2})$, $\Rightarrow$

$\hat{H}$ eigenstate = $\hat{N}$ eigenstate:

$$\boxed{\begin{array}{l} \hat{N}|n\rangle = n|n\rangle \qquad n \in \mathbb{R} \\ \Rightarrow \hat{H}|n\rangle = E_n|n\rangle \quad \text{w/} \quad E_n \doteq \hbar\omega(n+\frac{1}{2}) \end{array}}$$

- What is $\text{spec}(\hat{N})$? i.e. which n are eigenvalues?

Let $|\psi\rangle = \hat{a}|n\rangle$.

$$0 \leq \langle\psi|\psi\rangle = \langle n|\hat{a}^\dagger \hat{a}|n\rangle = \langle n|\hat{N}|n\rangle = n\langle n|n\rangle = n$$

$\therefore \; n \geq 0$.

- Check! $\boxed{\begin{array}{l} [\hat{N}, \hat{a}] = -\hat{a} \\ [\hat{N}, \hat{a}^\dagger] = +\hat{a}^\dagger \end{array}}$

$$\Rightarrow \hat{N}\hat{a}^\dagger|n\rangle = \hat{a}^\dagger(\hat{N}+1)|n\rangle = (n+1)\hat{a}^\dagger|n\rangle$$

$$\Rightarrow \quad \hat{a}^{+}|n\rangle = C_{+}|n+1\rangle \qquad \text{"raising"}$$
$$\hat{a}\,|n\rangle = C_{-}|n-1\rangle \qquad \text{"lowering"}$$

- There is a minimum value of n, $n \doteq n_{min}$

$$\Rightarrow \quad \hat{a}\,|n_{min}\rangle = 0$$

$$\Rightarrow \quad \hat{N}\,|n_{min}\rangle = \hat{a}^{+}\hat{a}\,|n_{min}\rangle = 0$$
$$= n_{min}\,|n_{min}\rangle$$

$$\Rightarrow \quad n_{min} = 0.$$

$\therefore$ Apply raising operator $\hat{a}^{+}$

$$\Rightarrow \quad \mathrm{spec}(\hat{N}) = \{n = 0, 1, 2, 3, \cdots\}$$

$$\boxed{\hat{H}|n\rangle = E_n|n\rangle = \hbar\omega\left(n+\tfrac{1}{2}\right)|n\rangle \qquad n = 0, 1, 2, 3, \ldots}$$

- $\| \hat{a}^{+}|n\rangle \|^2 = \| C_{+}|n+1\rangle \|^2 = |C_{+}|^2 \langle n+1|n+1\rangle = |C_{+}|^2$

$$\| \| \qquad$$

$$\langle n|\hat{a}\hat{a}^{+}|n\rangle = \langle n|(\hat{a}^{+}\hat{a}+1)|n\rangle = \langle n|(\hat{N}+1)|n\rangle$$

$$= (n+1)\langle n|n\rangle = n+1.$$

$$\Rightarrow \quad C_{+} = \sqrt{n+1}.$$

$$\Rightarrow \boxed{\begin{aligned} \hat{a}^{+}|n\rangle &= \sqrt{n+1}\,|n+1\rangle \\ \hat{a}\,|n\rangle &= \sqrt{n}\,|n-1\rangle \end{aligned}}$$

check! →

$$\Rightarrow \boxed{|n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^{+})^{n}|0\rangle}$$

← check!

$$\langle n|m\rangle = \delta_{n,m} \qquad \sum_{n=0}^{\infty}|n\rangle\langle n| = \mathbb{1}$$

Position basis

$$\hat{a}|0\rangle = 0 \quad \Rightarrow \quad \langle x|\hat{a}|0\rangle = 0$$

$$\therefore \quad 0 = \sqrt{\frac{m\omega}{2\hbar}}\langle x|\left(\hat{X} + \frac{i}{m\omega}\hat{P}\right)|0\rangle$$

$$= \sqrt{\frac{m\omega}{2\hbar}}\left(x\langle x|0\rangle + \frac{i}{m\omega}\left(-i\hbar\frac{d}{dx}\right)\langle x|0\rangle\right)$$

Denote $\varphi_n(x) \doteq \langle x|n\rangle$ energy eigenfunctions

$$\therefore \quad 0 = \sqrt{\frac{m\omega}{2\hbar}}\left(x\,\varphi_0(x) + \frac{\hbar}{m\omega}\varphi_0'(x)\right)$$

Easy 1^{st}-order differential equation:

$$\Rightarrow \quad \boxed{\underbrace{\varphi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}}_{} e^{-m\omega x^2/2\hbar}} \quad \text{(Gaussian)}$$

chosen so $1 = \langle 0|0\rangle = \int dx\, |\varphi_0(x)|^2$.

- $\varphi_n(x)$?

$$\varphi_n(x) = \langle x|n\rangle = \frac{1}{\sqrt{n!}} \langle x|(\hat{a}^\dagger)^n|0\rangle$$

$$= \frac{1}{\sqrt{n!}} \left(\sqrt{\frac{m\omega}{2\hbar}}\right)^n \left(x - \frac{\hbar}{m\omega}\frac{d}{dx}\right)^n \langle x|0\rangle$$

$$= \frac{1}{\sqrt{n!}} \left(\frac{m\omega}{2\hbar}\right)^{n/2} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \left(x - \frac{\hbar}{m\omega}\frac{d}{dx}\right)^n e^{-\frac{m\omega x^2}{2\hbar}}$$

$$= \frac{1}{\sqrt{n!}} \frac{1}{2^{n/2}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \cdot \underbrace{H_n\left(\sqrt{\frac{m\omega}{\hbar}}\, x\right)}_{\text{"Hermite polynomials"}} \cdot e^{-\frac{m\omega x^2}{2\hbar}}$$

$H_0(z) = 1$

$H_1(z) = 2z$

$H_2(z) = 4z^2 - 2$

$H_3(z) = 8z^3 - 12z$

$\vdots$

$H_n(-z) = (-)^n H_n(z)$

$H_{n+1}(z) = 2z H_n(z) - H_n'(z)$

- Time dependence

$$|\psi(t)\rangle = \sum_{n=0}^{\infty} e^{-iE_n t/\hbar} |n\rangle \langle n|\psi(0)\rangle$$

Example $\quad |\psi(0)\rangle = C\left(3|2\rangle + 4i|4\rangle\right)$

What is C?

What is $|\psi(t)\rangle$?

What is $\langle x\rangle(t)$?

What is $\langle x^2\rangle(t)$?

What is $\langle E\rangle(t)$?

What is $\mathrm{Prob}(1<x<2, t)$?

Quantization of electromagnetic (EM) field

(≡ Quantum field theory ≡ relativistic QM)

- Classically: $\vec{E}(\vec{x},t)$ & $\vec{B}(\vec{x},t)$ obey ME (vac)

$$\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot\vec{B} = \vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial}{\partial t}\vec{B} = \vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial}{\partial t}\vec{E} = 0$$

- sol'ns: "plane waves"

≡ complete o-n basis of $\vec{E}$, $\vec{B}$ fncs

$$\vec{E}(\vec{x},t) = \mathrm{Re}\left(\vec{E}_0\, e^{i(\vec{k}\cdot\vec{x} - \omega t)}\right)$$
$$\vec{B}(\vec{x},t) = \hat{k}\times\vec{E}$$

w/

$$\hat{k} \equiv \vec{k}/k \qquad \omega = ck \qquad \hat{k} = \text{direction of light}$$
$$k \equiv |\vec{k}| \qquad \vec{E}_0\cdot\vec{k} = 0 \qquad \omega = \text{frequency } "$$
$$\lambda \doteq \frac{2\pi}{k} = \text{wavelength } "$$

$\vec{E}_0 \doteq$ cplx polarization vector

$$\frac{\text{energy}}{\text{vol}} = |E_0|^2 \qquad \frac{\text{momentum}}{\text{vol}} = \frac{\hat{k}}{c}|E_0|^2$$

- E.g., take $\hat{k} = \hat{z}$, $\vec{E}_0 = E_{0x}\hat{x} + E_{0y}\hat{y}$

Define $\begin{cases} \hat{R} \equiv \frac{1}{\sqrt{2}}(\hat{x} + i\hat{y}) \\ \hat{L} \equiv \frac{1}{\sqrt{2}}(\hat{x} - i\hat{y}) \end{cases} \Rightarrow \vec{E}_0 = E_R\hat{R} + E_L\hat{L}$

"Right- & Left-circularly polarized"

$$\frac{J_z}{\text{vol}} = + \frac{1}{\omega}\left(|E_R|^2 - |E_L|^2\right)$$ = angular momentum of light

Quantumly:

Each plane wave mode $(\vec{k}, {}^{R}_{L}) \doteq (\vec{k}, \pm)$

is effectively described by a harmonic oscillator!

I.e., mode has quantized energies

$$\boxed{E_{\vec{k},\pm} = \hbar\omega\left(\hat{N}_{\vec{k}\pm} + \tfrac{1}{2}\right)}$$

with interpretation

$\hat{N}_{\vec{k}\pm}$ = counts number of "light particles"

= "photons" (Einstein, 1905)

corresponding to mode $(\vec{k}, \pm)$.

$$\Rightarrow \hat{H}_{EM} = \int d^3k\, \hbar\omega\left(\hat{N}_{\vec{k}+} + \tfrac{1}{2} + \hat{N}_{\vec{k}-} + \tfrac{1}{2}\right)$$

$$= \int d^3k\, \hbar\omega\left(\hat{a}^\dagger_{\vec{k}+}\hat{a}_{\vec{k}+} + \hat{a}^\dagger_{\vec{k}-}\hat{a}_{\vec{k}-} + \cancelto{0}{1}\right)$$

(infinite) constant, drop

$\boxed{\Rightarrow \text{single photon has } H_{EM} = \hbar\omega \qquad (\omega \doteq c|\vec{k}|)}$

$\hat{a}^\dagger_{\vec{k}+}$ creation operator of $(\vec{k}+)$ photon

$\hat{a}_{\vec{k}+}$ annihilation op. of $(\vec{k}+)$ photon

...

- E&M Hilbert space $\mathcal{H} = \bigoplus_{\vec{k}} \mathcal{H}_{\vec{k}}$

and o-n basis $\mathcal{H}_{\vec{k}} = \{ |n, \pm\rangle, \; n = 0, 1, 2, \ldots \}$

$$= \mathcal{H}_{S.H.O.} \otimes \mathcal{H}_{helicity}$$

$\{ |n\rangle, n = 0, 1, 2 .. \}$ $\qquad$ $\{ |+\rangle, |-\rangle \}$

dim $\mathcal{H}_{helicity} = 2$: like spin $j = 1/2$ H.S.

but is called "helicity = 1" H.S.

- From angular momentum in E&M, find ...

$$\hat{J}_{\hat{k}} = \int d^3k \cdot \hbar \cdot (\hat{a}^\dagger_{\vec{k}+} \hat{a}_{\vec{k}+} - \hat{a}^\dagger_{\vec{k}-} \hat{a}_{\vec{k}-})$$

$$= \int d^3k \cdot \hbar \cdot (\hat{N}_{\vec{k}+} - \hat{N}_{\vec{k}-})$$

component of ang. mom. in $\hat{k}$-direction, is called "helicity" for relativistic particle.

⇒ single photon has $\begin{cases} J_{\vec{k}} = +\hbar & \text{for } |\vec{k}+\rangle \\ J_{\vec{k}} = -\hbar & \text{for } |\vec{k}-\rangle \end{cases}$

So more like spin $j=1$, but no $J_{\vec{k}}=0$ state!

- From (linear) momentum in E&M, find...

$$\hat{\vec{P}} = \int d^3k \cdot \hbar \vec{k} \left(\hat{a}^{\dagger}_{\vec{k}+} \hat{a}_{\vec{k}+} + \hat{a}^{\dagger}_{\vec{k}-} \hat{a}_{\vec{k}-} \right)$$

$$= \int d^3k \cdot \hbar \vec{k} \cdot \left(\hat{N}_{\vec{k}+} + \hat{N}_{\vec{k}-} \right)$$

⇒ single photon has $\vec{P} = \hbar \vec{k}$.

- $\hat{P}_\mu \doteq (\hat{H}, c\hat{\vec{P}})$ $\qquad \mu = \underset{\downarrow\, H}{0}, \underbrace{1,2,3}_{c\vec{P}}$

transforms as a 4-vector under Lorentz group. ($c \doteq$ speed o' light)

Recall from special relativity

$$(mc^2)^2 = p_\mu p^\mu \doteq p_0^2 - p_1^2 - p_2^2 - p_3^2$$

particle mass

$$= H^2 - c^2\, \vec{P}\cdot\vec{P}$$

$\therefore$ For photon $(mc^2)^2 = (\hbar\omega)^2 - c^2\hbar^2\vec{k}\cdot\vec{k}$

$$= \hbar^2c^2k^2 - \hbar^2c^2k^2$$

$$= 0$$

$\Rightarrow$ Photon is massless.

$\Rightarrow$ Photon velocity $= c$ (special relativity). ✓

Generalization to any relativistic particle/field:

$$\mathcal{H} = \bigoplus_{\vec{k}} \mathcal{H}_{\vec{k}}$$

$\hat{a}^{+}_{\vec{k},m}$ = creates particle of momentum $\hbar\vec{k}$ & $\hat{J}_z = m\hbar$

$\hat{a}_{\vec{k},m}$ = annihilates " " " " "

with $m \in \{-j, -j+1, \ldots, j\}$

where $j \in \{0, \frac{1}{2}, 1, \ldots\}$ = spin of particle

$\Rightarrow$ basis $\mathcal{H}_{\vec{k}} = \{|0\rangle, \hat{a}^{+}_{\vec{k},m}|0\rangle, \hat{a}^{+}_{\vec{k},m_1}\hat{a}^{+}_{\vec{k},m_2}|0\rangle, \ldots\}$

"Fock space"

$$\Rightarrow \quad \hat{H} = \int d^3k \sum_{m=-j}^{j} \hbar\omega\, \hat{N}_{\vec{k},m}$$

$$\hat{\vec{P}} = \int d^3k \sum_{m=j}^{j} \hbar \vec{k} \, \hat{N}_{\vec{k},m}$$

$$\hat{J}_z = \int d^3k \sum_{m=-j}^{j} \hbar m \, \hat{N}_{\vec{k},m}$$

($\hat{J}_x, \hat{J}_y$ a little more complicated ...)

w/ $\omega \doteq \sqrt{c^2k^2 + \frac{1}{\hbar^2}c^4 m^2}$ $\quad$ $m \doteq$ mass particle

- This works for any mass $m > 0$
  & spin $j \in \{0, 1, 2, \ldots\}$ "bosons"

- For mass $m = 0$:

  spin j $\rightarrow$ helicity

  $\underbrace{m \in \{-j, -j+1, \cdots, j\}}_{2j+1 \text{ values}}$ $\qquad$ $\underbrace{h \in \{-j, +j\}}_{2 \text{ values}}$

- For spin $j \in \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots\}$ "fermions"

$$(\hat{a}^\dagger_{\vec{k},m}, \hat{a}_{\vec{k},m}) \longrightarrow (\hat{c}^\dagger_{\vec{k},m}, \hat{c}_{\vec{k},m})$$

$$[\hat{a}_{k,m}, \hat{a}_{k',m'}] = 0 \qquad \longrightarrow \qquad \{\hat{c}_{k,m}, \hat{c}_{k',m'}\} = 0$$

$$[\hat{a}_{k,m}, \hat{a}^\dagger_{k',m'}] = \delta_{k,k'}\delta_{m,m'} \qquad \{\hat{c}_{k,m}, \hat{c}^\dagger_{k',m'}\} = \delta_{k,k'}\delta_{m,m'}$$

↑ "Canonical commutation relations"

↑ "Canonical anticommutation relations"

- E.g. electrons: $m = .5\,\mathrm{MeV}/c^2$ $j = 1/2$
  gravitons: $m = 0$ $h = 2$
  ...

- All this is for free particles/fields.

- For interacting particles/fields it is known how to write down a consistent* quantum theory only for spin/helicity ≤ 1 !!??

(* consistent = causal, local)

- In particular gravity (helicity = 2) and supergravity (helicity = 3/2) are not understood as consistent quantum theories (yet) ...