Ch. 5: Addition of angular momentum
(spin-1/2 case)
or
Tensor products of Hilbert spaces

Consider 2 spin-1/2 particles (e.g. proton & electron of Hydrogen atom).
Call the angular momentum operators of particles #1 & 2:

$$\hat{\vec{J}}_1 = (\hat{J}_{x1}, \hat{J}_{y1}, \hat{J}_{z1})$$

$$\hat{\vec{J}}_2 = (\hat{J}_{x2}, \hat{J}_{y2}, \hat{J}_{z2})$$

labels particle

- If particles are far apart, should be able to think of each as being described by its own Hilbert space:

  particle 1 $\leftrightarrow \mathcal{H}_1$
  particle 2 $\leftrightarrow \mathcal{H}_2$

- Question: what is $\mathcal{H}_{1+2}$? I.e. what is Hilbert space describing both particles at once?

- Clue to answer: think about total angular momentum, $\vec{J}$, of the 2-particle system.

Classically: $\vec{J} = \vec{J}_1 + \vec{J}_2$

i.e. just add the 1-particle angular momenta.

Quantumly $\hat{\vec{J}}_1, \hat{\vec{J}}_2$ are operators in 2 different Hilbert spaces $\mathcal{H}_1$ & $\mathcal{H}_2$, and $\hat{\vec{J}}$ is operator in 2-particle H.s. $\mathcal{H}_{1+2}$.

Try naive guess: $\boxed{\text{“}\hat{J}_x = \hat{J}_{x1} + \hat{J}_{x2}\text{”}}$ (& y, z)

This only makes sense if we think of $\hat{J}_{x1}$ & $\hat{J}_{x2}$ now also as operators in $\mathcal{H}_{1+2}$.

Question: What properties do $\hat{\vec{J}}_1$ & $\hat{\vec{J}}_2$ need to have on $\mathcal{H}_{1+2}$ so that $\hat{\vec{J}}$ is an angular momentum operator, i.e., so that

$$[\hat{J}_x, \hat{J}_y] = i\hbar \hat{J}_z \quad (\& \text{ cyclic}) \; ?$$

$$[J_x, J_y] = [J_{x1} + J_{x2}, J_{y1} + J_{y2}]$$

$$= [J_{x1}, J_{y1}] + [J_{x1}, J_{y2}] + [J_{x2}, J_{y1}] + [J_{x2}, J_{y2}]$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$= i\hbar J_{z1} \quad + \quad ? \quad + \quad ? \quad + i\hbar J_{z2}$

$= i\hbar J_z \quad + \quad ?$

$\Rightarrow$ so want: $[J_{x1}, J_{y2}] = [J_{x2}, J_{y1}] = 0.$

Generalize . . .

<u>Rule 1</u>

if $\hat{M}_1, \hat{N}_1, \hat{P}_1$ satisfy $\hat{M}_1\hat{N}_1 = \hat{P}_1$ on $\mathcal{H}_1$

& if $\hat{Q}_2, \hat{R}_2, \hat{S}_2$ " $\hat{Q}_2\hat{R}_2 = \hat{S}_2$ on $\mathcal{H}_2$

then $\hat{M}_1\hat{N}_1 = \hat{P}_1$ & $\hat{Q}_2\hat{R}_2 = \hat{S}_2$ still true on $\mathcal{H}_{1+2}$

and $[\hat{M}_1, \hat{Q}_2] = \ldots = 0$ on $\mathcal{H}_{1+2}$

i.e. all "1" operators commute with all "2" operators.

Also, we can figure out how big $\mathcal{H}_{1+2}$ should be. Say

$\dim \mathcal{H}_1 = d_1$ and $\dim \mathcal{H}_2 = d_2$

So there is a basis of d_1 states describing particle 1 & similarly a basis of d_2 states describing particle 2.

So there should be a basis of states of $\mathcal{H}_{1+2}$ for each state of particle 1 (d_1)

and particle 2 (d_2) at the same time:

Rule 2:

$$\boxed{(\dim \mathcal{H}_{1+2}) = (\dim \mathcal{H}_1)\cdot(\dim \mathcal{H}_2)}$$

This are the rules ("axioms") for how to combine subsystems in quantum mechanics.

The mathematical construction which realizes this rule is called
Tensor product of vector spaces

(Townsend uses "direct product")

2-particle Hilbert space

$$\boxed{\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2}$$

1-particle Hilbert space

tensor product operation

So tensor product is a way to make a new vector space from other vector spaces.

We have already seen (in the spectral theorem and discussion of angular momentum eigenbasis) a way of combining vector spaces, called **direct sum**:

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$$

$\oplus$: If $\{ |a\rangle, a = 1 \cdots d_1 \}$ o-n basis $\mathcal{H}_1$
& $\{ |j\rangle, j = 1 \cdots d_2 \}$ " " $\mathcal{H}_2$
then $\{ |a\rangle, a=1\cdots d_1 \} \cup \{ |j\rangle, j = 1 \cdots d_2 \}$ o-n basis $\mathcal{H}$

$$\Rightarrow \quad \dim \mathcal{H} = \dim \mathcal{H}_1 + \dim \mathcal{H}_2$$

So, direct sum does not obey rule 2.

Definition: **tensor product**

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$$

$\otimes$: If $\{ |a\rangle, a = 1 \cdots d_1 \}$ o-n basis $\mathcal{H}_1$
& $\{ |j\rangle, j = 1 \cdots d_2 \}$ " " $\mathcal{H}_2$
then
$\{ |a, j\rangle, a = 1 \cdots d_1, j = 1 \cdots d_2 \}$ " " $\mathcal{H}$

$$\Rightarrow \quad \dim \mathcal{H} = d_1 \cdot d_2 = \dim \mathcal{H}_1 \cdot \dim \mathcal{H}_2 . \checkmark$$

$|a,j\rangle$ basis vectors have many notations:

$$|a,j\rangle = |a\rangle|j\rangle = |a\rangle_1|j\rangle_2 = |a\rangle\otimes|j\rangle = |a\rangle_1\otimes|j\rangle_2$$

- $\otimes$ should seem stupidly simple: make the new basis by just making a list of all possible subsystem basis vectors in order ...

- But: $\mathcal{H}_1\otimes\mathcal{H}_2 \neq \mathcal{H}_2\otimes\mathcal{H}_1$ the order matters!

  $|a,j\rangle \neq |j,a\rangle$

  So when taking tensor products: pick an order & stick with it!

- $\otimes$ preserves linearity and adjoints:

i.e. $\otimes$ distributes over addition & mult. by scalars:

$$(a|v\rangle + b|w\rangle)\otimes(c|u\rangle) = ac|v,u\rangle + bc|w,u\rangle$$

$$(a|\psi\rangle)(|\chi\rangle) = a|\psi,\chi\rangle = |\psi\rangle(a|\chi\rangle)$$

and adjoint distributes over $\otimes$:

$$(a|\psi,\chi\rangle)^\dagger = [(a|\psi\rangle)\otimes|\chi\rangle]^\dagger = (a|\psi\rangle)^\dagger\otimes|\chi\rangle^\dagger$$

$$= a^*\langle\psi|\otimes\langle\chi| = a^*\langle\psi,\chi|$$

Note: $\dagger$ does not reverse order of tensor products!

- <u>General state</u> in $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ is

$$|\psi\rangle = \sum_{a,j} c_{a,j} \, |a,j\rangle$$

Note: in general $|\psi\rangle \neq |\psi_1\rangle \otimes |\psi_2\rangle$!
Only true in special case that $\exists$ d_a & e_j s.t.

$$c_{a,j} = d_a e_j$$

- <u>Example</u>: 2 spin-$\frac{1}{2}$ particles in J_z-basis

$\mathcal{H}_1$ basis $\{ |\tfrac{1}{2}, \pm\tfrac{1}{2}\rangle \doteq |\pm\rangle \}$

$\mathcal{H}_2$ basis $\{ |\tfrac{1}{2}, \pm\tfrac{1}{2}\rangle \doteq |\pm\rangle \}$

$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ basis:

$|j_1 m_1 ; j_2 m_2\rangle$ book

$$\left\{ \begin{aligned} &|\tfrac{1}{2},\tfrac{1}{2}\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle \equiv |\tfrac{1}{2},\tfrac{1}{2};\tfrac{1}{2},\tfrac{1}{2}\rangle \equiv |++\rangle \\ &|\tfrac{1}{2},\tfrac{1}{2}\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle \equiv |\tfrac{1}{2}\,\tfrac{1}{2};\tfrac{1}{2}\,-\tfrac{1}{2}\rangle \equiv |+-\rangle \\ &|\tfrac{1}{2},-\tfrac{1}{2}\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle \equiv |\tfrac{1}{2}\,-\tfrac{1}{2};\tfrac{1}{2}\,\tfrac{1}{2}\rangle \equiv |-+\rangle \\ &|\tfrac{1}{2},-\tfrac{1}{2}\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle \equiv |\tfrac{1}{2}\,-\tfrac{1}{2};\tfrac{1}{2}\,-\tfrac{1}{2}\rangle \equiv |--\rangle \end{aligned} \right.$$

Note: In $|+-\rangle$ state: $\begin{cases} \text{particle 1 has } J_z = +\frac{\hbar}{2} \\ \text{particle 2 has } J_z = -\frac{\hbar}{2} \end{cases}$

$\neq |-+\rangle$ state!

- **Operators**: If $\hat{M}$ is an operator on $\mathcal{H}_1$
  & $\hat{N}$ " " " " $\mathcal{H}_2$
  then $\hat{M} \otimes \hat{N}$ " " " " $\mathcal{H}_1 \otimes \mathcal{H}_2$
  defined to act linearly and by:

$$\hat{M} \otimes \hat{N} \, |a,j\rangle \equiv \hat{M} \otimes \hat{N} \, |a\rangle \otimes |j\rangle$$

$$\doteq \left(\hat{M}|a\rangle\right) \otimes \left(\hat{N}|j\rangle\right).$$

$\therefore$ the operators:

$\hat{M} \otimes \hat{I} \equiv \hat{M}_1$ acts only on the $\mathcal{H}_1$ factor

$\hat{I} \otimes \hat{N} \equiv \hat{N}_2$ " " " " $\mathcal{H}_2$ ".

E.g. $\hat{M} \otimes \hat{I} \, |a,j\rangle = \left(\hat{M}|a\rangle\right) \otimes \left(\hat{I}|j\rangle\right) = \left(\hat{M}|a\rangle\right)|j\rangle$

- Products of operators (check!)

$$(\hat{M} \otimes \hat{N})(\hat{A} \otimes \hat{B}) = (\hat{M}\hat{A}) \otimes (\hat{N}\hat{B})$$

- General operators on $\mathcal{H}_1 \otimes \mathcal{H}_2$ can be written

$$\hat{O} = \sum_{\alpha,\beta} \hat{M}_\alpha \otimes \hat{N}_\beta \quad \text{for some } \hat{M}_\alpha, \hat{N}_\beta.$$

$\Rightarrow$

$$\hat{O}|\psi\rangle = \Big(\sum_{\alpha,\beta} \hat{M}_\alpha \otimes \hat{N}_\beta\Big)\Big(\sum_{aj} c_{aj}\, |a\rangle \otimes |j\rangle\Big)$$

$$= \sum_{\alpha,\beta,a,j} c_{aj} \left(\hat{M}_\alpha |a\rangle\right) \otimes \left(\hat{N}_\beta |j\rangle\right).$$

- <u>Example:</u> $\hat{J}_{z1}$ = J_z for particle 1
  $\hat{J}_{z2}$ = " " " 2

$$\Rightarrow \hat{J}_{z1} = \hat{J}_z \otimes \hat{I} \qquad \hat{J}_{z2} = \hat{I} \otimes \hat{J}_z$$

$$\begin{aligned}
\hat{J}_{z1}|-+\rangle &= (\hat{J}_z \otimes \hat{I})(|-\rangle|+\rangle) \\
&= (\hat{J}_z|-\rangle)(\hat{I}|+\rangle) \\
&= (-\tfrac{\hbar}{2}|-\rangle)|+\rangle \\
&= -\tfrac{\hbar}{2}|-+\rangle.
\end{aligned}$$

$$\hat{J}_{z2}|-+\rangle = \ldots = +\tfrac{\hbar}{2}|-+\rangle.$$

- $\underline{[\hat{M}_1, \hat{N}_2] = 0} \qquad \forall\ \hat{M}_1, \hat{N}_2$

b/c
$$\begin{aligned}
[\hat{M}_1, \hat{N}_2] &= [\hat{M}\otimes\hat{I}, \hat{I}\otimes\hat{N}] \\
&= (\hat{M}\otimes\hat{I})(\hat{I}\otimes\hat{N}) - (\hat{I}\otimes\hat{N})(\hat{M}\otimes\hat{I}) \\
&= \hat{M}\hat{I}\otimes\hat{I}\hat{N} - \hat{I}\hat{M}\otimes\hat{N}\hat{I} \\
&= \hat{M}\otimes\hat{N} - \hat{M}\otimes\hat{N} = 0. \quad \checkmark
\end{aligned}$$

• Example Define

$$\begin{cases} \hat{J}_x \doteq \hat{J}_{x1} + \hat{J}_{x2} = \hat{J}_x \otimes \hat{I} + \hat{I} \otimes \hat{J}_x \\ \hat{J}_y \doteq \hat{J}_{y1} + \hat{J}_{y2} = \hat{J}_y \otimes \hat{I} + \hat{I} \otimes \hat{J}_y \\ \hat{J}_z \doteq \hat{J}_{z1} + \hat{J}_{z2} = \hat{J}_z \otimes \hat{I} + \hat{I} \otimes \hat{J}_z \end{cases}$$

Check: $[\hat{J}_x, \hat{J}_y] = i\hbar \hat{J}_z$ & cyclic in xyz

"Addition" of angular momentum

Since total angular momentum

$$\hat{\vec{J}} = \hat{\vec{J}}_1 + \hat{\vec{J}}_2$$

satisfies the ang. mom. algebra, it must give a $\{|j,m\rangle\}$ eigenbasis of

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$$

in usual way, i.e.,

$$\hat{\vec{J}}^2 |j,m\rangle = \hbar^2 j(j+1) |j,m\rangle \qquad j \in \{0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \dots\}$$

$$\hat{J}_z |j,m\rangle = \hbar m |j,m\rangle \qquad m \in \{-j, -j+1, \dots, j\}$$

& $\langle j,m|j',m'\rangle = \delta_{jj'}\delta_{mm'}$.

Question: How is this J_z basis for $\mathcal{H}$ related to the J_{z1} & J_{z2} bases of $\mathcal{H}_1, \mathcal{H}_2$?

- Consider this question for two spin-1/2 particles:

spin-1/2 $\mathcal{H}_1 = \{|j,m\rangle \ j=\tfrac{1}{2}, m=\pm\tfrac{1}{2}\} = \{|\pm\rangle\}$

" $\mathcal{H}_2$ = same $= \{|\pm\rangle\}$

$$\Rightarrow \dim\mathcal{H} = \dim(\mathcal{H}_1\otimes\mathcal{H}_2) = (\dim\mathcal{H}_1)(\dim\mathcal{H}_2) = 2\cdot 2 = 4$$

So what can $\{|j,m\rangle\}$ basis of $\mathcal{H}=\mathcal{H}_1\otimes\mathcal{H}_2$ be?

① $\{|j=\tfrac{3}{2}, m=\pm\tfrac{3}{2},\pm\tfrac{1}{2}\rangle\}$ $\Rightarrow \dim = 4$

② $\{|j=\tfrac{1}{2}, m=\pm\tfrac{1}{2}\rangle\} \oplus \{|j=\tfrac{1}{2}, m=\pm\tfrac{1}{2}\rangle\}$ $\Rightarrow \dim = 2+2=4$

③ $\{|j=1, m=\pm1, 0\rangle\} \oplus \{|j=0, m=0\rangle\}$ $\Rightarrow \dim = 3+1=4$

④ $\{|j=\tfrac{1}{2}, m=\pm\tfrac{1}{2}\rangle\} \oplus \{|j=0, m=0\rangle\} \oplus \{|j=0, m=0\rangle\}$ $\Rightarrow \dim = 2+1+1=4$

⑤ $\{|j=0,m=0\rangle\} \oplus \{|j=0,m=0\rangle\} \oplus \{|j=0,m=0\rangle\} \oplus \{|j=0,m=0\rangle\}$ $\Rightarrow \dim = 1+1+1+1=4$

direct sum

i.e., 5 possibilities:

$$\mathcal{H}_{1/2}\otimes\mathcal{H}_{1/2} = \mathcal{H}_{3/2}$$

j

$$= \mathcal{H}_{1/2}\oplus\mathcal{H}_{1/2}$$

$= \mathcal{H}_1 \oplus \mathcal{H}_0$

$= \mathcal{H}_{1/2} \oplus \mathcal{H}_0 \oplus \mathcal{H}_0$

$= \mathcal{H}_0 \oplus \mathcal{H}_0 \oplus \mathcal{H}_0 \oplus \mathcal{H}_0$

One is right, rest are wrong!

- Write basis of $\mathcal{H}_{1/2} \otimes \mathcal{H}_{1/2} = \{ |\pm \pm\rangle \}$

$\hat{J}_{z1} = \pm \frac{\hbar}{2}$ $\quad$ $\hat{J}_{z2} = \pm \frac{\hbar}{2}$

Then $\hat{J}_z |++\rangle = (\hat{J}_{z1} + \hat{J}_{z2}) |++\rangle$

$= (+\frac{\hbar}{2} + \frac{\hbar}{2}) |++\rangle = \hbar |++\rangle$

But $\hat{J}_z |jm\rangle = \hbar m |jm\rangle$

$\therefore$ learn that $|++\rangle = |j, m=1\rangle$ for some j (or j's)

Since $j \geq |m| = 1 \Rightarrow j$ must be ≥ 1.

So looking at our list of 5 possible answers:

$\mathcal{H}_{1/2} \otimes \mathcal{H}_{1/2} = \mathcal{H}_{3/2}$

$= \mathcal{H}_{1/2} \oplus \mathcal{H}_{1/2}$ ✗

$= \mathcal{H}_1 \oplus \mathcal{H}_0$

$= \mathcal{H}_{1/2} \oplus \mathcal{H}_0 \oplus \mathcal{H}_0$ ✗

$= \mathcal{H}_0 \oplus \mathcal{H}_0 \oplus \mathcal{H}_0 \oplus \mathcal{H}_0$ ✗

all $j < 1$

But for $\mathcal{H}_{3/2}$, basis $\{|j=3/2, m=\pm\frac{3}{2}, \pm\frac{1}{2}\rangle\}$ has no states with $m=1$. ✗

Therefore answer must be

$$\boxed{\begin{gathered}\mathcal{H}_{1/2} \otimes \mathcal{H}_{1/2} = \mathcal{H}_1 \oplus \mathcal{H}_0 \\ \text{“} \quad \tfrac{1}{2} \otimes \tfrac{1}{2} = 0 \oplus 1 \quad \text{”}\end{gathered}}$$

- Check this by finding explicit relationship between $\mathcal{H}_{1/2} \otimes \mathcal{H}_{1/2}$ basis $\{|\pm,\pm\rangle\}$ and $\mathcal{H}_0 \oplus \mathcal{H}_1$ basis $\{|\overset{j}{1}, \overset{m}{\pm 1}\rangle, |1,0\rangle, |0,0\rangle\}$

Compute:

$$J_z|++\rangle = +\hbar\,|++\rangle \quad \Rightarrow |++\rangle \propto |j, m=1\rangle$$

$$J_z|+-\rangle = 0 \quad \Rightarrow |+-\rangle \propto |j, m=0\rangle$$

$$J_z|-+\rangle = 0 \quad \Rightarrow |-+\rangle \propto |j, m=0\rangle$$

$$J_z|--\rangle = -\hbar\,|--\rangle \quad \Rightarrow |--\rangle \propto |j, m=-1\rangle$$

∴ must have:

$$|++\rangle = |1,1\rangle$$

$$|+-\rangle = a|1,0\rangle + b|0,0\rangle \qquad |a|^2+|b|^2=1$$

$|-+\rangle = c|1,0\rangle + d|0,0\rangle \qquad |c|^2+|d|^2=1$

$|--\rangle = |1,-1\rangle$

for some $a,b,c,d \in \mathbb{C}$.

But $0 = \langle -+|+-\rangle = \left(c^*\langle 10| + d^*\langle 00|\right)\left(a|10\rangle + b|00\rangle\right)$

$\Rightarrow \quad 0 = c^*a + d^*b$

$\Rightarrow \quad \boxed{c = b^* \quad \& \quad d = -a^*}$. (up to overall phase)

To figure out a, b, recall

$$\hat{J}_-|j=1,m=1\rangle = \hbar\sqrt{j(j+1)-m(m-1)}\,|j,m-1\rangle = \hbar\sqrt{2}\,|1,0\rangle$$

$$\parallel$$

$$(\hat{J}_{-1}+\hat{J}_{-2})|++\rangle = (\hat{J}_-|+\rangle)|+\rangle + |+\rangle(\hat{J}_-|+\rangle)$$

$$= (\hbar|-\rangle)|+\rangle + |+\rangle(\hbar|-\rangle)$$

$$= \hbar\left(|-+\rangle + |+-\rangle\right)$$

$$\therefore \quad |j=1,m=0\rangle = \frac{1}{\sqrt{2}}\left(|+-\rangle + |-+\rangle\right).$$

$\Rightarrow$

$\{\|j,m\rangle\}$	$\longleftrightarrow$ $\{\|\pm\pm\rangle\}$
$\|1,1\rangle$	$= \|++\rangle$
$\|1,0\rangle$	$= \frac{1}{\sqrt{2}}(\|+-\rangle + \|-+\rangle)$
$\|1,-1\rangle$	$= \|--\rangle$
$\|0,0\rangle$	$= \frac{1}{\sqrt{2}}(\|+-\rangle - \|-+\rangle)$

Memorize!

"triplet"

"singlet"

- **General "addition" of angular momentum answer:**

j-values

$$j_1 \otimes j_2 = |j_1-j_2| \oplus (|j_1-j_2|+1) \oplus \cdots \oplus (j_1+j_2)$$

Check: dimensions match:

$$(2j_1+1)(2j_2+1) = \sum_{j=|j_1-j_2|}^{j_1+j_2} (2j+1)$$

Example: $j_1 = 1 \quad j_2 = 3/2 \quad \Rightarrow$

$$1 \otimes \tfrac{3}{2} = |1-\tfrac{3}{2}| \oplus (|1-\tfrac{3}{2}|+1) \oplus \cdots \oplus (1+\tfrac{3}{2})$$
$$= \tfrac{1}{2} \oplus \tfrac{3}{2} \oplus \tfrac{5}{2}$$

dimensions: $(2\cdot 1+1)(2\cdot\tfrac{3}{2}+1) \overset{?}{=} (2\cdot\tfrac{1}{2}+1) + (2\cdot\tfrac{3}{2}+1) + (2\cdot\tfrac{5}{2}+1)$

$$3 \cdot 4 = 2 + 4 + 6 \quad \checkmark$$

General formula relating bases:

$$|j,m\rangle = \sum_{m_1,m_2} \underbrace{C^{jm}_{j_1m_1,j_2m_2}}_{} |j_1m_1,j_2m_2\rangle$$

"Clebsch-Gordon coefficients"
(complicated: look them up!)

EPR & Bell's Inequality

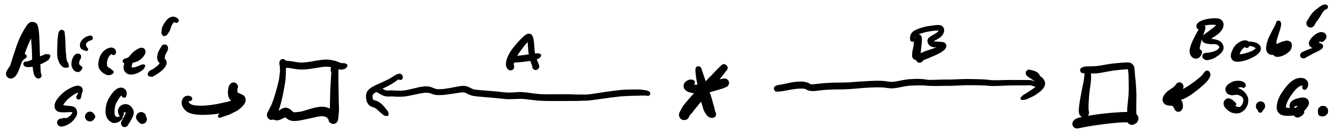
- Einstein, Podolsky, Rosen 1935
- Schrödinger 1935
- Bohm 1951
- Bell 1964

Thought experiments involving entangled state

a) Prepare entangled state of systems A & B

b) Separate systems A & B in space

c) Make independent measurements on A & B



a) Preparation

$$\mathcal{H}_A = \{ |\pm z\rangle_A \} \qquad \left(\text{e.g. } J_z = \pm \hbar/2 \text{ states} \right)$$
$$\mathcal{H}_B = \{ |\pm z\rangle_B \}$$

Put in "EPR state"

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|+z\rangle_A |-z\rangle_B - |-z\rangle_A |+z\rangle_B \right)$$

(e.g. total $j=0$ state)

b) Separate

let fly apart...

c) Measure:

Alice measures $J_{A,\theta} \doteq \vec{J} \cdot \hat{n}_\theta$

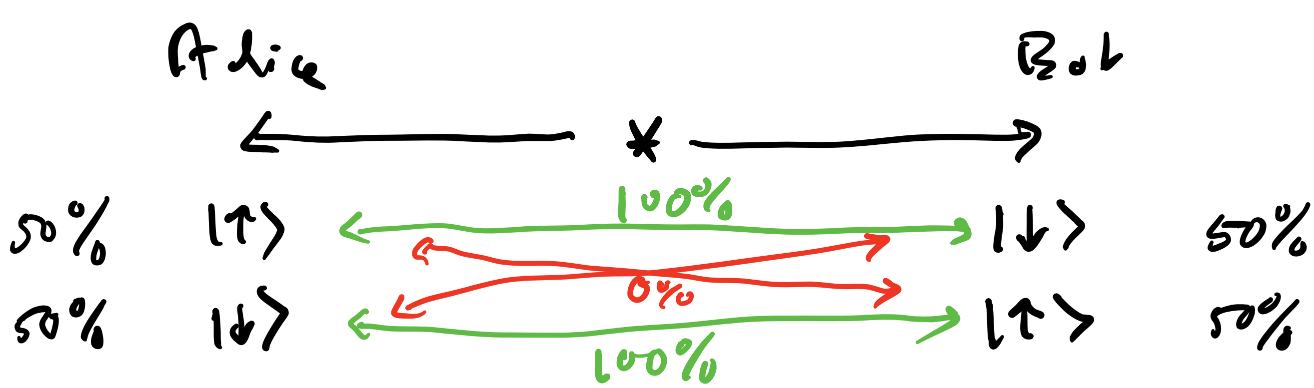
Bob " $J_{B,\theta} \doteq \vec{J} \cdot \hat{n}_\theta$

w/ $\hat{n}_\theta \doteq \cos\theta \, \hat{z} + \sin\theta \, \hat{x}$

Claim: Alice & Bob's measurements are <u>correlated</u>:

$$\begin{cases} \text{Prob}\left(J_{A,\theta} = \frac{\hbar}{2}\right) = \text{Prob}\left(J_{B,\theta} = \frac{\hbar}{2}\right) = \frac{1}{2} \\ \text{Prob}\left(J_{A,\theta} = \frac{\hbar}{2} \;\&\; J_{B,\theta} = \frac{\hbar}{2}\right) = 0 \\ \text{Prob}\left(J_{A,\theta} = \frac{\hbar}{2} \;\&\; J_{B,\theta} = -\frac{\hbar}{2}\right) = \frac{1}{2} \end{cases}$$

- Clear for $\theta = \frac{\pi}{2} \Rightarrow \begin{cases} J_{A,\theta} = J_z \\ J_{B,\theta} = J_z \end{cases}$



- But also true for *any* θ!

$J_\theta = +\frac{\hbar}{2}$ eigenstate $\doteq |\theta+\rangle = \cos\frac{\theta}{2}|+z\rangle + \sin\frac{\theta}{2}|-z\rangle$

$J_\theta = -\frac{\hbar}{2}$ eigenstate $\doteq |\theta-\rangle = -\sin\frac{\theta}{2}|+z\rangle + \cos\frac{\theta}{2}|-z\rangle$

<u>Check</u> for <u>any</u> θ:

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\theta+\rangle_A|\theta-\rangle_B - |\theta-\rangle_A|\theta+\rangle_B\right) \qquad !$$

c') <u>Measure</u> (new experiment)

Alice measures J_z } & perform measurements simultaneously, so no causal influence

Bob measures J_x

$\Rightarrow$ Alice <u>predicts</u> Bob's J_z w/ 100% certainty
& Bob <u>predicts</u> Alice's J_x w/ 100% certainty

??!? QM: $[J_z, J_x] \neq 0$, $\therefore$ uncertainty principle implies $\Delta J_z \Delta J_x \gtrsim \hbar$
$\therefore$ <u>Can't</u> know J_x & J_z simultaneously with 100% certainty.

Compute using rules of QM:

$$\hat{P}_{(J_{zA}=+,\,J_{xB}=+)} = (|z+\rangle\langle z+|)_A \otimes (|x+\rangle\langle +x|)_B$$

$$= (|z+\rangle\langle z+|)_A \otimes \frac{1}{2}(|z+\rangle + |z-\rangle)_B(\langle z+| + \langle z-|)_B$$

$$= \frac{1}{2}\, |z+\rangle_A (|z+\rangle + |z-\rangle)_B \;\langle z+|_A\, (\langle z+| + \langle z-|)_B$$

$$= \frac{1}{2}\left(|++\rangle + |+-\rangle\right)\left(\langle ++| + \langle +-|\right)$$

$\therefore$ If measure $J_{zA} = +\frac{\hbar}{2}$ & $J_{xB} = +\frac{\hbar}{2}$

$$|\psi\rangle \to |\psi'\rangle \to \frac{\hat{P}|\psi\rangle}{\|\cdot\|}$$

$$\propto \underbrace{\frac{1}{2}(|++\rangle + |+-\rangle)(\langle ++| + \langle +-|)}_{\hat{P}}\;\underbrace{\frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle)}_{|\psi\rangle}$$

$$\propto (|++\rangle + |+-\rangle) = |+z\rangle(|+z\rangle + |-z\rangle)$$

$$\propto |+z\rangle_A |+x\rangle_B \;. \checkmark$$

No paradox: though $[J_{zA}, J_{xA}] \neq 0$,

$[J_{zA}, J_{xB}] = [J_z \otimes I, I \otimes J_x] = 0$, $\therefore$

$\Delta J_{zA}\, \Delta J_{xB} \geq 0$ so 100% certainty allowed!

E.P.R. argument:

- Since *can* predict J_{xA} or J_{zB} with certainty (by measuring J_{xB} or J_{zA})

- And because locality means B's measurement can't affect A's & vice versa

- $\therefore$ J_{Az} & J_{Ax} are "definite properties" ("elements of reality", "objective properties of nature") of particle A.

- But QM does not have any object with definite values of J_{Az} & J_{Ax} ($|\psi\rangle$ can only have definite value of one or the other)

$\Rightarrow$ "QM is incomplete": It is missing "hidden variables".

30 years later: J. Bell

- Main point of EPR are "correlators" of A & B's measurements.

Correlators = expectation value

$$C(\theta,\varphi) \equiv \langle \hat{J}_{A,\theta}\,\hat{J}_{B,\varphi}\rangle = \langle \hat{J}_\theta \otimes \hat{J}_\varphi \rangle$$

in state $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right)$, with

$$\hat{J}_\theta \doteq \cos\theta\, \hat{J}_z + \sin\theta\, \hat{J}_x \qquad \ldots \pm\tfrac{\hbar}{2} \text{ eigenvalues}$$

- E.g. $\theta = \varphi = 0$ $\quad C(0,0) = \langle \hat{J}_z \otimes \hat{J}_z \rangle = -\left(\frac{\hbar}{2}\right)^2$

  $\theta = \varphi$ $\quad C(\theta,\theta) = -\left(\frac{\hbar}{2}\right)^2$ (rot'l inv.)

  $\theta = 0,\ \varphi = \frac{\pi}{2}$ $\quad C(0,\frac{\pi}{2}) = \langle \hat{J}_z \otimes \hat{J}_x \rangle = 0$

Compute ... (good exercize!)

$$C(\theta,\varphi) = \langle \hat{J}_\theta \otimes \hat{J}_\varphi \rangle = \langle \psi | \hat{J}_\theta \otimes \hat{J}_\varphi | \psi \rangle = \ldots$$

$$\boxed{C(\theta,\varphi) = -\left(\frac{\hbar}{2}\right)^2 \cos(\theta-\varphi)} \qquad (*)$$

→ QM prediction

Checks:
- rot'l invariance $\Leftrightarrow$ depends only on $\theta-\varphi$ ✓
- $\theta-\varphi = 0 \Rightarrow = -(\hbar/2)^2 \Rightarrow$ 100% anticorrelated ✓
- $\theta-\varphi = \pi/2 \Rightarrow = 0 \Rightarrow$ uncorrelated ✓

Bell's inequality (not a statement about QM!)

- Following EPR, assume exists local hidden variable theory underlying (completing) QM such that values of $J_{Ax}, J_{Az}, J_{Bx}, J_{Bz}$ are all objective properties of "state" $|\psi\rangle_{h.v.}$ before measurement.

E.g., if:

Alice measures $\hat{J}_\theta = a$
& Bob measures $\hat{J}_\varphi = b$ } then system has definite "hidden" values (a, b),

- Any probabilities only enter in our preparation step from lack of experimental control over hidden variables.

$\Rightarrow$ System prepared with values (a,b) with probability $\mathcal{P}(a,b)$.

- Now consider correlators of various measurements

$\mathcal{A}_1 \doteq \hat{J}_{A\theta_1}$ $\quad\quad$ $\mathcal{B}_1 \doteq \hat{J}_{B\varphi_1}$

$\mathcal{A}_2 \doteq \hat{J}_{A\theta_2}$ $\quad\quad$ $\mathcal{B}_2 \doteq \hat{J}_{B\varphi_2}$

with results a_1, a_2, b_1, b_2 respectively all either $\pm \hbar/2$.

• Then (Clauser, Horne, Shimony, Holt):

$$\boxed{\langle A_1 B_1\rangle + \langle A_2 B_1\rangle + \langle A_2 B_2\rangle - \langle A_1 B_2\rangle \leq \frac{\hbar^2}{2}} \quad (**)$$

hidden variable prediction

Proof: $=\langle A_1 B_1 + A_2 B_1 + A_2 B_2 - A_1 B_2\rangle$

$$= \sum_{a_i, b_i} P(a_i, b_i)\,(a_1 b_1 + a_2 b_1 + a_2 b_2 - a_1 b_2)$$

$$= \sum_{a_i\, b_i} P(a_i, b_i)\,\left[(a_1 + a_2) b_1 + (a_2 - a_1) b_2\right]$$

Since $a_i = \pm\frac{\hbar}{2} \Rightarrow$ either $\begin{pmatrix} a_1 + a_2 = \pm\hbar \\ a_2 - a_1 = 0 \end{pmatrix}$ or $\begin{pmatrix} a_1 + a_2 = 0 \\ a_2 - a_1 = \pm\hbar \end{pmatrix}$

& since $b_i = \pm\frac{\hbar}{2} \Rightarrow \left[(a_1 + a_2) b_1 + (a_2 - a_1) b_2\right] = \pm\frac{\hbar^2}{2} \leq \frac{\hbar^2}{2}$

$$\Rightarrow \leq \sum_{a_i\, b_i} P(a_i, b_i)\,\frac{\hbar^2}{2} = \frac{\hbar^2}{2}\sum_{a_i\, b_i} P(a_i, b_i) = \frac{\hbar^2}{2}. \quad \checkmark$$

• Now compare $(**)$ (local hidden variable theory)
to $(*)$ (quantum mechanics)

$$(*) \Rightarrow \langle A_1 B_1 + A_2 B_1 + A_2 B_2 - A_1 B_2\rangle$$

$$= -\left(\frac{\hbar}{2}\right)^2 \left[\cos(\theta_1 - \varphi_1) + \cos(\theta_2 - \varphi_1) + \cos(\theta_2 - \varphi_2) - \cos(\theta_1 - \varphi_2)\right]$$

Does this function satisfy $(**)$? No!

E.g. (maximal violation), when

$$\theta_1 = 0, \quad \theta_2 = \frac{\pi}{2}, \quad \varphi_1 = -\frac{3\pi}{4}, \quad \varphi_2 = -\frac{\pi}{4}$$

Alice Bob

$$(*) = -\left(\frac{\hbar}{2}\right)^2\left[\cos\left(\frac{3\pi}{4}\right) + \cos\left(\frac{5\pi}{4}\right) + \cos\left(\frac{3\pi}{4}\right) - \cos\left(\frac{\pi}{4}\right)\right]$$

$$= -\left(\frac{\hbar}{2}\right)^2\left[-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right]$$

$$= \frac{\hbar^2}{2}\sqrt{2} \quad > \quad \frac{\hbar^2}{2} \Rightarrow \text{violates } (**) \;!!$$

Therefore

QM is not a local hidden variable theory.

Have to give up either or both:

"realism": $\left\{\begin{array}{l}\text{physical properties have definite} \\ \text{values independent of observer}\end{array}\right.$

"locality": $\begin{cases} \text{simultaneous separated measurements} \\ \text{do not influence each other} \end{cases}$

- The experiment was actually done in the 1970's & QM prediction was confirmed.

- If give up locality, get "spooky action at a distance".
  But QM does <u>not</u> allow

"acausality": $\begin{cases} \text{transfer of information or energy} \\ \text{faster than light or backwards} \\ \text{in time.} \end{cases}$