Chapter 4: Time evolution

- We now let our states $|\psi\rangle$ depend on time "t": $|\psi(t)\rangle$, and we want to know the "equation of motion" of $|\psi(t)\rangle$.

Write $\boxed{|\psi(t_2)\rangle \doteq \hat{U}(t_2,t_1)\,|\psi(t_1)\rangle}$

for some operator

$$\boxed{\hat{U}(t_2,t_1) \equiv \text{time evolution operator (from time } t_1 \text{ to time } t_2)}$$

- Want $|\psi(t_2)\rangle$ = state ∴

$$1 = \langle\psi(t_2)|\psi(t_2)\rangle = \langle\psi(t_1)|\hat{U}^\dagger(t_2,t_1)\,\hat{U}(t_2,t_1)|\psi(t_1)\rangle$$

for all $|\psi(t_1)\rangle \Rightarrow$

$$\boxed{\hat{U}^\dagger(t_2,t_1)\,\hat{U}(t_2,t_1) = \hat{1}}$$

Therefore, the time evolution operator is unitary. $\therefore$

$$\hat{U}(t_2,t_1) = e^{-i\hat{S}(t_2,t_1)/\hbar}$$

for some hermitian $\hat{S}(t_2,t_1)$. ("action operator")

- If the system has time-translation symmetry (e.g. any isolated system*) then the origin of time coordinate has no physical significance $\Rightarrow$

- $\hat{U}(t_2,t_1) = \hat{U}(t_2-t_1)$
- $\hat{U}(t_3-t_1) = \hat{U}(t_3-t_2)\,\hat{U}(t_2-t_1)$

("Group law"). $\Rightarrow$ $\hat{S}(t_2,t_1) = (t_2-t_1)\hat{H}$

$$\boxed{\hat{U}(t) = e^{-it\hat{H}/\hbar}}$$

where $\hat{H} = \hat{H}^\dagger$ is t-indep't: $\frac{\partial}{\partial t}\hat{H} = 0$

- $\hat{H}$ is called the Hamiltonian

- $\hat{H}$ generates time translations.

$\Rightarrow$ - $\hat{H}$ measures energy, which should be conserved.

$$|\psi(t)\rangle = \hat{U}(t)\,|\psi(0)\rangle$$

$$= e^{-it\hat{H}/\hbar}\,|\psi(0)\rangle$$

$$\frac{d}{dt} f(t) = \lim_{\varepsilon \to 0} \frac{f(t+\varepsilon) - f(t)}{\varepsilon}$$

$$\therefore \quad \frac{d}{dt}|\psi(t)\rangle = \frac{d}{dt}\left(e^{-it\hat{H}/\hbar}\,|\psi(0)\rangle\right)$$

$$= \frac{d}{dt}\left(e^{-it\hat{H}/\hbar}\right)|\psi(0)\rangle$$

$$\boxed{\begin{aligned}\frac{d}{dt}\left(e^{t\hat{\alpha}}\right) &= \frac{d}{dt}\left(\sum_{l=0}^{\infty} \frac{t^l}{l!}\hat{\alpha}^l\right) = \sum_{l=0}^{\infty} \frac{l\,t^{l-1}}{l!}\hat{\alpha}^l \\ &= \sum_{l=1}^{\infty} \frac{t^{l-1}}{(l-1)!}\hat{\alpha}^l = \hat{\alpha}\left(\sum_{l-1=0}^{\infty} \frac{t^{l-1}}{(l-1)!}\hat{\alpha}^{l-1}\right) \\ &= \hat{\alpha}\, e^{t\hat{\alpha}}\end{aligned}}$$

$$\Rightarrow \quad \frac{d}{dt}|\psi(t)\rangle = -\frac{i}{\hbar}\hat{H}\, e^{-it\hat{H}/\hbar}\,|\psi(0)\rangle$$

$$\Rightarrow \quad \boxed{\frac{d}{dt}|\psi(t)\rangle = -\frac{i}{\hbar}\hat{H}\,|\psi(t)\rangle}$$

Schrödinger's Equation.

- Analog of $\vec{F} = m\vec{a}$ for QM.

- $\frac{d}{dt}\langle H\rangle = \frac{d}{dt}\left\{\langle\psi(t)|\,\hat{H}\,|\psi(t)\rangle\right\}$

$$= \left(\frac{d}{dt}\langle\psi(t)|\right)\hat{H}|\psi(t)\rangle + \langle\psi(t)|\hat{H}\left(\frac{d}{dt}|\psi(t)\rangle\right)$$

$$= \left(+\frac{i}{\hbar}\langle\psi(t)|\hat{H}\right)\hat{H}|\psi(t)\rangle + \langle\psi(t)|\hat{H}\left(\frac{-i}{\hbar}\hat{H}|\psi(t)\rangle\right)$$

$$= \left(\frac{i}{\hbar} - \frac{i}{\hbar}\right)\langle\psi(t)|\hat{H}^2|\psi(t)\rangle = 0.$$

$\Rightarrow$ Average energy is conserved. But even more:

$$\frac{d}{dt}\,\mathrm{Prob}(H = E_n) = 0$$

i.e., energy probabilities conserved too.
So not just <u>average</u>, but whole energy <u>probability distribution</u> is conserved!

Proof:

$$\mathrm{Prob}(H=E_n) = \left| \langle n | \psi(t) \rangle \right|^2$$

when $\hat{H} | n \rangle = E_n | n \rangle$.

Since $\frac{d}{dt} \hat{H} = 0 \Rightarrow \frac{d}{dt} | n \rangle = 0$.

$$\therefore \frac{d}{dt} \mathrm{Prob}(H=E_n) = \frac{d}{dt} \left| \langle n | \psi(t) \rangle \right|^2$$

$$= \frac{d}{dt} \left\{ \langle \psi(t) | n \rangle \langle n | \psi(t) \rangle \right\}$$

$$= \left(\frac{d}{dt} \langle \psi(t) | \right) | n \rangle \langle n | \psi(t) \rangle$$

$$+ \langle \psi(t) | n \rangle \langle n | \left(\frac{d}{dt} | \psi(t) \rangle \right)$$

$$= + \frac{i}{\hbar} \langle \psi(t) | \hat{H} | n \rangle \langle n | \psi(t) \rangle$$
$$- \frac{i}{\hbar} \langle \psi(t) | n \rangle \langle n | \hat{H} | \psi(t) \rangle$$

$$= + \frac{i}{\hbar} \langle \psi(t) | E_n | n \rangle \langle n | \psi(t) \rangle$$
$$- \frac{i}{\hbar} \langle \psi(t) | n \rangle \langle n | E_n | \psi(t) \rangle$$

$$= \left(\frac{i}{\hbar} - \frac{i}{\hbar} \right) E_n \left| \langle n | \psi(t) \rangle \right|^2 = 0.$$

- What if system does not have time translation symmetry?

* E.g. say your system is <u>not</u> isolated from outside world, so feels some time-dependent force from outside. For example, could be an atom in an external magnetic field

$$\vec{B} = \vec{B}(t)$$

that varies with time.

Idea: for <u>very</u> <u>short</u> <u>times</u> energy is approximately conserved since external influence approx. constant over those short times.
So should have Schrö Eqn:

$$\boxed{\begin{gathered}\frac{d}{dt}|\psi(t)\rangle = -\frac{i}{\hbar}\hat{H}(t)\,|\psi(t)\rangle \\ \text{for some "instantaneous Hamiltonian"} \\ \hat{H}(t) = \hat{H}(t)^{\dagger}\end{gathered}}$$

*

which measures a time-varying energy (so, no longer conserved).

- **Note:** in case $\frac{d}{dt}\hat{H}(t) \neq 0$, hard to integrate (*)

$$|\psi(t_2)\rangle = \hat{U}(t_2,t_1)\,|\psi(t_1)\rangle$$

with $$\hat{U}(t_2,t_1) = \underbrace{P\exp}_{\text{"path-ordered exponential"}}\Big\{-\frac{i}{\hbar}\underbrace{\int_{t_1}^{t_2} dt\,\hat{H}(t)}_{\text{"action"}}\Big\}$$

Solving Schrö's Eqn

(when energy is conserved $\Rightarrow \frac{d}{dt}\hat{H} = 0$)

$$\frac{d}{dt}|\psi(t)\rangle = -\frac{i}{\hbar}\hat{H}|\psi(t)\rangle$$

$$\Downarrow$$

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}t\hat{H}}\,|\psi(0)\rangle$$

So, how to evaluate $e^{-\frac{i}{\hbar}t\hat{H}}$?

Look at energy eigenbasis:

$$\boxed{\hat{H}|n\rangle = E_n |n\rangle}$$ "time-indep't Schrö. Eqn" (A)

$\{E_n\}$ = energy eigenvalues
$\{|n\rangle\}$ = " eigenbasis

$\Rightarrow \langle n|m\rangle = \delta_{n,m}\ , \ \sum_n |n\rangle\langle n| = 1$

If you can solve (A), then

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}t\hat{H}}|\psi(0)\rangle$$

$$= \sum_n e^{-\frac{i}{\hbar}t\hat{H}}|n\rangle\langle n|\psi(0)\rangle$$

$$= \sum_n e^{-\frac{i}{\hbar}tE_n}|n\rangle\langle n|\psi(0)\rangle$$

$$\boxed{\begin{gathered}|\psi(t)\rangle = \sum_n \psi_n(0)\, e^{-iE_nt/\hbar}\,|n\rangle \\ \text{w/}\ \ \psi_n(0) \doteq \langle n|\psi(0)\rangle\end{gathered}}$$ (AA)

So:
- compute $\{E_n\}, \{|n\rangle\}$ "energy eigenbasis"
- compute $\psi_n(0) = \langle n|\psi(0)\rangle$ "initial conditions"
  (= components of $|\psi(0)\rangle$ in energy eigenbasis)
- then plug into (AA).

Conservation laws

If observable $\hat{A}$ commutes w/ $\hat{H}$, it is conserved.

$$\boxed{[\hat{A},\hat{H}]=0 \iff \hat{A} \text{ is conserved}}$$

Because, if $[\hat{A},\hat{H}]=0 \Rightarrow$ simultaneously diagonalizable, i.e., have common o-n eigenbasis $\{|n\rangle\}$ s.t.

$$\begin{cases} \hat{H}\,|n\rangle = E_n\,|n\rangle \\ \hat{A}\,|n\rangle = A_n\,|n\rangle \end{cases}.$$

$$\begin{aligned}
\Rightarrow \quad \text{Prob}(A=A_n \text{ @ } t) &= |\langle n|\psi(t)\rangle|^2 \\
&= \Big|\langle n|\Big(\sum_m e^{-iE_mt/\hbar}|m\rangle\langle m|\psi(0)\rangle\Big)\Big|^2 \\
&= \Big|\sum_m e^{-iE_mt/\hbar}\underbrace{\langle n|m\rangle}_{\delta_{nm}}\langle m|\psi(0)\rangle\Big|^2 \\
&= \Big|e^{-iE_nt/\hbar}\langle n|\psi(0)\rangle\Big|^2 \\
&= |\langle n|\psi(0)\rangle|^2 = \text{Prob}(A=A_n \text{ @ } t=0)
\end{aligned}$$

So probabilities for measuring $\hat{A}$ are time-independent = conserved.

Equation of motion for averages:

$$\frac{d}{dt}\langle A(t)\rangle = \frac{d}{dt}\left(\langle\psi(t)|\hat{A}|\psi(t)\rangle\right)$$
$$= \left(\frac{d}{dt}\langle\psi(t)|\right)\hat{A}|\psi(t)\rangle + \langle\psi(t)|\hat{A}\frac{d}{dt}\left(|\psi(t)\rangle\right)$$
$$= +\frac{i}{\hbar}\langle\psi(t)|\hat{H}\hat{A}|\psi(t)\rangle - \frac{i}{\hbar}\langle\psi(t)|\hat{A}\hat{H}|\psi(t)\rangle$$
$$= \frac{i}{\hbar}\langle\psi(t)|[\hat{H},\hat{A}]|\psi(t)\rangle$$

$$\Rightarrow \boxed{\frac{d}{dt}\langle A\rangle = \frac{i}{\hbar}\langle[H,A]\rangle}$$

So if $[\hat{H},\hat{A}]=0 \Rightarrow \frac{d}{dt}\langle A\rangle = 0$. ✓.

Uncertainty Relations

$$\boxed{\Delta A\cdot\Delta B \geq \frac{1}{2}\left|\langle[A,B]\rangle\right|}$$ U.R.

true for any hermitian op.s $\hat{A},\hat{B}$.

Constrains how "wide" the probability distributions of $\hat{A}, \hat{B}$ measurements must be in any state.

Recall, if $|\psi\rangle$ is eigenstate of $\hat{A}$, $\hat{A}|\psi\rangle = \alpha|\psi\rangle$,

$$\Rightarrow (\Delta A)^2 = \langle A^2\rangle - \langle A\rangle^2 = \langle\psi|\hat{A}^2|\psi\rangle - \langle\psi|\hat{A}|\psi\rangle^2$$
$$= \langle\psi|\alpha^2|\psi\rangle - \langle\psi|\alpha|\psi\rangle^2 = \alpha^2\langle\psi|\psi\rangle - \alpha^2\langle\psi|\psi\rangle^2$$
$$= 0.$$

So, if $\Delta A = 0$ & $\langle\psi|[\hat{A},\hat{B}]|\psi\rangle \neq 0$

then U.R. $\Rightarrow$ $\Delta B = \infty$!

i.e. exact quantum info about $\hat{A}$ $\Rightarrow$ no info. about $\hat{B}$!

Proof:

1. $\langle \alpha\hat{A} + \beta\hat{B}\rangle = \alpha\langle\hat{A}\rangle + \beta\langle\hat{B}\rangle$ $\quad \alpha,\beta\in\mathbb{C}$
by linearity. $\Rightarrow$

$$(\Delta A)^2 = \langle A^2\rangle - \langle A\rangle^2 = \langle A^2 - \langle A\rangle^2 1\rangle$$

$$= \langle A^2 - 2\langle A\rangle A + \langle A\rangle^2\rangle$$

$$= \langle (A - \langle A\rangle)^2\rangle$$

So define $\hat{A}_0 \doteq \hat{A} - \langle A\rangle\hat{1}$, then

$$(\Delta A)^2 = \langle\hat{A}_0^2\rangle = \langle\psi|\hat{A}_0\hat{A}_0|\psi\rangle = \|\hat{A}_0|\psi\rangle\|^2$$

② $(\Delta A)^2(\Delta B)^2 = \langle \hat{A}_0^2\rangle\langle \hat{B}_0^2\rangle$

$$= \langle\psi|\hat{A}_0\hat{A}_0|\psi\rangle\langle\psi|\hat{B}_0\hat{B}_0|\psi\rangle$$

$$= \| \hat{A}_0|\psi\rangle\|^2 \cdot \|\hat{B}_0|\psi\rangle\|^2$$

Schwarz's inequality:

$$\boxed{\| |\psi\rangle\| \cdot \| |\chi\rangle\| \geq |\langle\psi|\chi\rangle| \quad \text{all } |\psi\rangle, |\chi\rangle}$$

A result from linear algebra...

$$\Rightarrow (\Delta A)^2(\Delta B)^2 \geq |\langle\psi|\hat{A}_0\hat{B}_0|\psi\rangle|^2 = |\langle\hat{A}_0\hat{B}_0\rangle|^2$$

③ $\hat{A}_0\hat{B}_0 = \frac{1}{2}(\hat{A}_0\hat{B}_0 - \hat{B}_0\hat{A}_0) + \frac{1}{2}(\hat{A}_0\hat{B}_0 + \hat{B}_0\hat{A}_0)$

$$= \frac{1}{2}[\hat{A}_0,\hat{B}_0] + \frac{1}{2}\{\hat{A}_0,\hat{B}_0\}$$

"anticommutator"

$\hat{A}_0$ + $\hat{B}_0$ hermitian $\Rightarrow \begin{cases} \{\hat{A}_0,\hat{B}_0\} \text{ is hermitian} \\ i[\hat{A}_0,\hat{B}_0] \text{ is hermitian} \end{cases}$

$$\therefore \hat{A}_0\hat{B}_0 = -\frac{i}{2}\underbrace{\left(i[\hat{A}_0,\hat{B}_0]\right)}_{\substack{\text{hermitian}\\ \Downarrow \\ \text{real}}} + \frac{1}{2}\underbrace{\left(\{\hat{A}_0,\hat{B}_0\}\right)}_{\substack{\text{hermitian}\\ \Downarrow \\ \text{real}}}$$

$$\therefore \langle\hat{A}_0\hat{B}_0\rangle = -\frac{i}{2}\overbrace{\langle i[\hat{A}_0,\hat{B}_0]\rangle} + \frac{1}{2}\overbrace{\langle\{\hat{A}_0,\hat{B}_0\}\rangle}$$

$$\therefore\quad |\langle \hat{A}_0\hat{B}_0\rangle|^2 = \frac{1}{4}\,\underbrace{|\langle i[\hat{A}_0,\hat{B}_0]\rangle|^2}_{\geq 0} + \frac{1}{4}\,\underbrace{|\langle \{\hat{A}_0,\hat{B}_0\}\rangle|^2}_{\geq 0}$$

$$\geq \frac{1}{4}|\langle [\hat{A}_0,\hat{B}_0]\rangle|^2$$

4 $\therefore\quad (\Delta A)^2(\Delta B)^2 \geq \frac{1}{4}|\langle [\hat{A}_0,\hat{B}_0]\rangle|^2$

$$\Rightarrow\quad \Delta A\cdot\Delta B \geq \frac{1}{2}|\langle [\hat{A}_0,\hat{B}_0]\rangle|\,.$$

But $[\hat{A}_0,\hat{B}_0] = [\hat{A}-\langle A\rangle\hat{1},\ \hat{B}-\langle B\rangle\hat{1}]$

$$= [\hat{A},\hat{B}] - \langle A\rangle\cancel{[\hat{1},\hat{B}]}^{\,0} - \langle B\rangle\cancel{[\hat{A},\hat{1}]}^{\,0} + \langle A\rangle\langle B\rangle\cancel{[\hat{1},\hat{1}]}^{\,0}$$

$$\therefore\quad \Delta A\cdot\Delta B \geq \frac{1}{2}|\langle [\hat{A},\hat{B}]\rangle|\,.$$

Apply to $\hat{H}$:

$$\Delta A\cdot\Delta E \geq \frac{1}{2}|\langle [\hat{A},\hat{H}]\rangle|$$

$$= \frac{1}{2}\left|i\hbar\frac{d}{dt}\langle A\rangle\right|$$

$$\therefore\quad \boxed{\Delta A\cdot\Delta E \geq \frac{\hbar}{2}\left|\frac{d\langle A\rangle}{dt}\right|}$$

Define $\Delta T \doteq \dfrac{\Delta A}{|d\langle A\rangle/dt|}$

$\Rightarrow$ " $\Delta E \cdot \Delta T \geq \frac{\hbar}{2}$ " (E-T uncert. rel'tn.)

Meaning unclear, because ΔT depends on the observable A:

$$\Delta T \sim \delta t \cdot \frac{\Delta A}{\delta \langle A \rangle} \approx \text{time for } \langle A \rangle \text{ to change by } \Delta A$$

Can define $\Delta T = \min_A \left\{ \frac{\Delta A}{|d\langle A \rangle / dt|} \right\}$

$\doteq$ shortest time for any property of system to change & E-T u.r. $\Rightarrow$

$$\Delta E \geq \frac{\hbar}{2 \Delta T}$$

$\Rightarrow$ minimum energy spread of state of system, i.e., estimate of how far from energy eigenstate system must be.

Spin-1/2 precession in B-field

(§4.3 Townsend)

- $\hat{H} = -\vec{\mu}\cdot\vec{B} = -\frac{gq}{2mc}\hat{\vec{S}}\cdot\vec{B}$

  Take $\vec{B} = B_0\hat{z}$, $q = -e$ $\Big\}$ e^- in uniform B-field

  $\Rightarrow \hat{H} = \omega_0 \hat{S}_z$ with $\omega_0 \equiv \frac{ge}{2mc}B_0$.

- $\hat{S}_z$ eigenbasis: $\{ |s,m\rangle,\ s=\frac{1}{2},\ m\in\{-\frac{1}{2},\frac{1}{2}\}\}$ is $\hat{H}$ eigenbasis.

  Re-name: $|\frac{1}{2},\frac{1}{2}\rangle \doteq |+\rangle \quad |\frac{1}{2},-\frac{1}{2}\rangle \doteq |-\rangle$

  Then $\hat{H}|\pm\rangle = E_\pm|\pm\rangle = \pm\frac{\hbar\omega_0}{2}|\pm\rangle$.

- Time evolution operator

$$\hat{U}(t) = e^{-it\hat{H}/\hbar} = e^{-i\hat{S}_z\omega_0 t/\hbar}$$

$$= e^{-i\phi\hat{S}_z/\hbar} \quad \text{with } \phi = \omega_0 t.$$

= Rotation operator around $\hat{z}$-axis by angle $\phi = \omega_0 t$.

$\therefore$ Time evolution = precession of spin around direction of B-field with angular frequency ω_0

This is the same as the classical answer.

- Example: assume initial condition at $t=0$:

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right) \quad \left(\hat{S}_x = +\frac{\hbar}{2} \text{ eigenstate}\right)$$

$$\Rightarrow \quad |\psi(t)\rangle = \sum_n e^{-itE_n/\hbar} |n\rangle\langle n|\psi(0)\rangle$$

$$= \sum_{\pm} e^{-itE_\pm/\hbar} |\pm\rangle\langle \pm|\psi(0)\rangle$$

$$= \sum_{\pm} e^{-it(\pm\frac{\hbar\omega_0}{2})/\hbar} |\pm\rangle \frac{1}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}}\left(e^{-i\omega_0 t/2}|+\rangle + e^{+i\omega_0 t/2}|-\rangle\right)$$

$$= \underbrace{e^{-i\omega_0 t/2}}_{\text{(unobservable)}} \frac{1}{\sqrt{2}}\left(|+\rangle + \underbrace{e^{i\omega_0 t}}_{\text{observable!}}|-\rangle\right)$$

- $\text{Prob}(S_z = -\frac{\hbar}{2}) = |\langle -|\psi(t)\rangle|^2 = \left|\frac{1}{\sqrt{2}}e^{i\omega_0 t}\right|^2 = \frac{1}{2}$

Time-indep't: expected since $S_z = -\hbar/2$ is

$\hat{H}$ eigenstate, & energy is conserved.

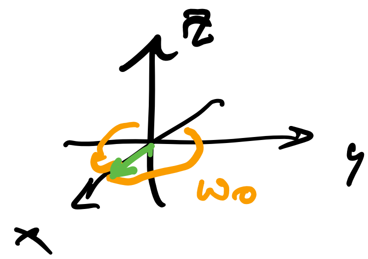
- $\text{Prob}(S_x = \frac{\hbar}{2}) = \left| \frac{1}{\sqrt{2}} (\langle + | + \langle - |) | \psi(t) \rangle \right|^2$

$$= \left| \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}} e^{-i\omega_0 t/2} + \frac{1}{\sqrt{2}} e^{-i\omega_0 t/2} \right) \right|^2$$

$$= \frac{1}{4} \left(2 + e^{i\omega_0 t} + e^{-i\omega_0 t} \right) = \frac{1}{2} (1 + \cos\omega_0 t)$$

$$= \cos^2\left(\frac{\omega_0 t}{2}\right).$$

<u>Note</u>:
- Time-dependent, b/c $\hat{S}_x$ not conserved.
- Prob $\in [0,1]$.
- Oscillates like expected classical precession

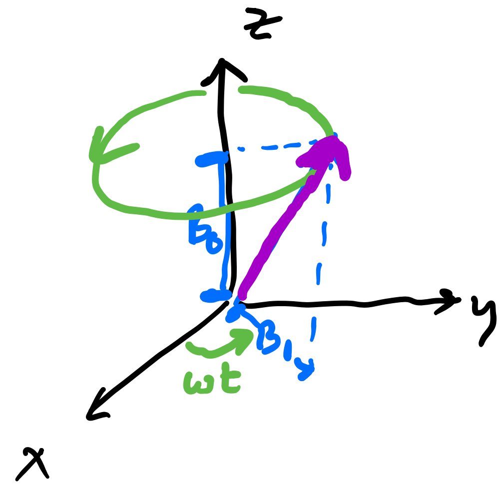


- But in <u>probabilistic</u> sense, b/c QM.

Magnetic Resonance (§4.4 Townsend)

Spin-1/2 in time-dependent mag'n field:

$$\vec{B} = B_0 \hat{z} + B_1 \cos(\omega t)\, \hat{x}$$



(Will be interested in situation where $|B_1| \ll |B_0|$.)

So instantaneous (t-dependent) energy operator

$$\hat{H} = \omega_0 \hat{J}_z + \omega_1 \hat{J}_x \cos \omega t$$

w/ $\omega_0 \doteq \frac{gq}{2mc} B_0 \quad \omega_1 \doteq \frac{gq}{2mc} B_1$

Since $\hat{H}$ is t-dependent, must solve Schrö eqn

$$-\frac{i}{\hbar} \hat{H} |\psi(t)\rangle = \frac{d}{dt} |\psi(t)\rangle.$$

(Can't use t-indep't formula b/c energy eigenvectors now change with time!)

- Write $\hat{J}_z$ basis

$$|\tfrac{1}{2},\tfrac{1}{2}\rangle = \begin{pmatrix}1\\0\end{pmatrix} \qquad |\tfrac{1}{2},-\tfrac{1}{2}\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$

and write

$$|\psi(t)\rangle \doteq \begin{pmatrix}a(t)\\b(t)\end{pmatrix}.$$

Then (recall):

$$\hat{J}_z = \frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix} \qquad \hat{J}_x = \frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix} \qquad \hat{J}_y = \frac{\hbar}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix}$$

So S.E. becomes $\qquad \left(\dot{x} \equiv \frac{dx}{dt}\right)$

$$-\frac{i}{\hbar}\cdot\frac{\hbar}{2}\begin{pmatrix}\omega_0 & \omega_1\cos\omega t\\ \omega_1\cos\omega t & -\omega_0\end{pmatrix}\begin{pmatrix}a(t)\\b(t)\end{pmatrix} = \begin{pmatrix}\dot{a}(t)\\\dot{b}(t)\end{pmatrix}$$

$$\Rightarrow \left\{\boxed{\begin{aligned}\dot{a} &= -\frac{i}{2}\left(\omega_0\, a + \omega_1\cos\omega t\, b\right)\\ \dot{b} &= -\frac{i}{2}\left(\omega_1\cos\omega t\, a - \omega_0\, b\right)\end{aligned}}\right. \qquad \text{(SE)}$$

This is coupled system of diff. eq.ns ≡ hard to solve exactly...

But can simplify a bit: Note that if $\omega_1 = 0$ then SE becomes

$$\begin{cases} \dot{a} = -\frac{i}{2}\omega_0 a \\ \dot{b} = +\frac{i}{2}\omega_0 b \end{cases} \Rightarrow \begin{cases} a(t) = a(0)\, e^{-i\omega_0 t/2} \\ b(t) = b(0)\, e^{+i\omega_0 t/2} \end{cases}$$

Which is just the sol'n found before for spin-1/2 precession in constant B-field.

If $|B_0| \gg |B_1|$, then $\omega_0 \gg \omega_1$, so above sol'n is "main" t-dependence.

Suggests to define new variables c, d:

$$\boxed{\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} \equiv \begin{pmatrix} c(t)\, e^{-i\omega_0 t/2} \\ d(t)\, e^{+i\omega_0 t/2} \end{pmatrix}} \qquad (*)$$

(∴ Expect c(t) & d(t) will vary with time more slowly than $e^{\pm i\omega_0 t/2}$.)

Plug (*) into (SE), gives ...

$$\boxed{\begin{aligned} \dot{c} &= -\frac{i\omega_1}{4}\left(e^{i(\omega_0+\omega)t} + e^{i(\omega_0-\omega)t}\right) d \\ \dot{d} &= -\frac{i\omega_1}{4}\left(e^{-i(\omega_0+\omega)t} + e^{-i(\omega_0-\omega)t}\right) c \end{aligned}} \qquad (**)$$

(where used: $\cos\omega t = \frac{1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right)$.)

- Approximate analysis of (**):

- If: $|\omega_0 - \omega| \gg \omega_1$

(i.e. if ω = rate of rotation of B-field is not too close to ω_0 = precession rate of spin)

then $e^{\pm i(\omega_0 \pm \omega)t}$ terms are oscillating quickly compared to ω_1, so average to zero, so

(**) $\Rightarrow$ $\dot{c} \approx 0$ & $\dot{d} \approx 0$ (small).

- If $|\omega_0 - \omega| \ll \omega_1$

then $e^{\pm i(\omega_0 + \omega)t}$ terms oscill. quickly, but $e^{\pm i(\omega_0 - \omega)t} \approx e^{\pm i \cdot 0 \cdot t} = 1$, so

$$(**) \Rightarrow \begin{cases} \dot{c} = -\dfrac{i\omega_1}{4} d \\ \dot{d} = -\dfrac{i\omega_1}{4} c \end{cases}$$

$$\Rightarrow \begin{cases} \ddot{c} = -\dfrac{i\omega_1}{4}\dot{d} = -\dfrac{i\omega_1}{4}\left(-\dfrac{i\omega_1}{4}c\right) = -\left(\dfrac{\omega_1}{4}\right)^2 c \\ \ddot{d} = -\dfrac{i\omega_1}{4}\dot{c} = -\dfrac{i\omega_1}{4}\left(-\dfrac{i\omega_1}{4}d\right) = -\left(\dfrac{\omega_1}{4}\right)^2 d \end{cases}$$

$$\Rightarrow \begin{cases} c(t) = \alpha \cos\left(\frac{\omega_1}{4}t\right) + \beta \sin\left(\frac{\omega_1}{4}t\right) \\ d(t) = \frac{4i}{\omega_1}\dot{c}(t) = -i\alpha \sin\left(\frac{\omega_1}{4}t\right) + i\beta\cos\left(\frac{\omega_1}{4}t\right) \end{cases}$$

From (*) $\Rightarrow \begin{cases} a(0) = c(0) = \alpha \\ b(0) = d(0) = i\beta \end{cases}$, $\therefore$

This is called the solution "at resonance" (i.e. when $\omega = \omega_0$).

<u>Ex.</u> Take $a(0)=1$ $b(0)=0$ initial conditions. So $|\psi(0)\rangle = |\frac{1}{2},+\frac{1}{2}\rangle$. Then

$$\text{Prob}\left(J_z = +\frac{\hbar}{2}\right) = \left|\langle\tfrac{1}{2},\tfrac{1}{2}|\psi(t)\rangle\right|^2 = |a(t)|^2$$

$$= |c(t)|^2 = \cos^2\left(\frac{\omega_1}{4}t\right)$$

$$\text{Prob}\left(J_z = -\frac{\hbar}{2}\right) = \cdots = \sin^2\left(\frac{\omega_1}{4}t\right) \quad ✓$$

- So, at resonance

$$\langle J_z\rangle(t) = \frac{\hbar}{2}\left[\cos^2\left(\frac{\omega_1}{4}t\right) - \sin^2\left(\frac{\omega_1}{4}t\right)\right]$$

$$= \frac{\hbar}{2}\cos\left(\frac{\omega_1}{2}t\right)$$

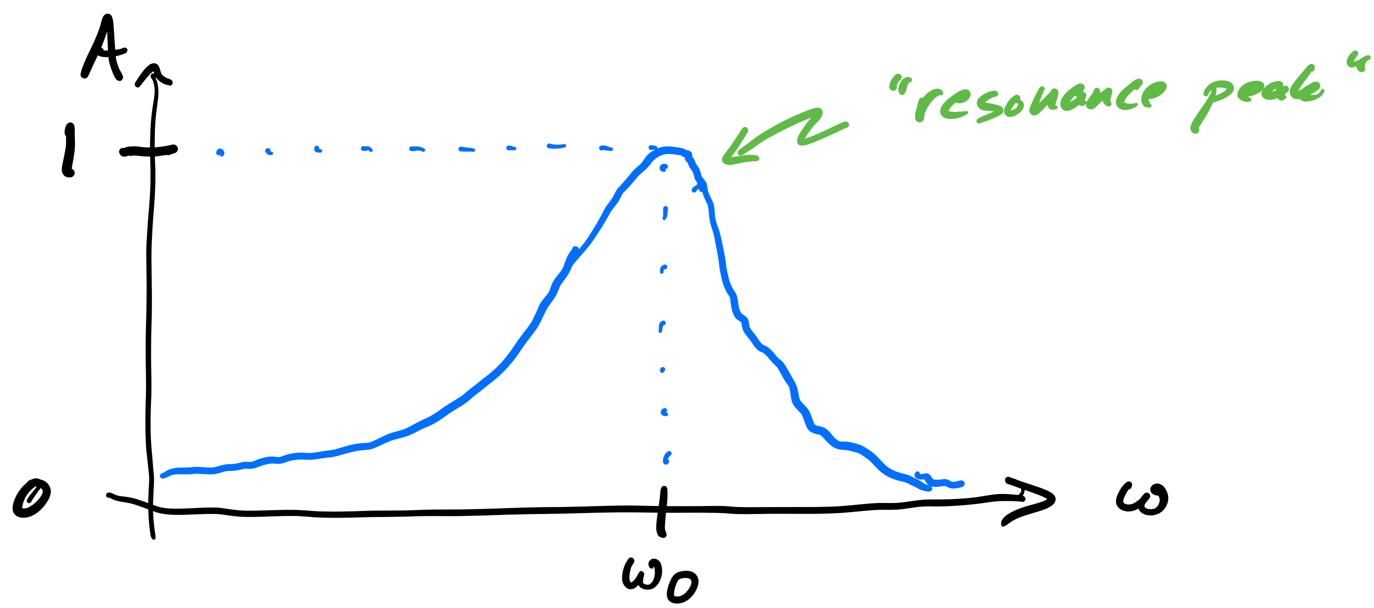
i.e. oscillates w/ frequency $\frac{\omega_1}{2}$ between $\pm \hbar/2$ (max. values).

- What happens if $\omega \approx \omega_0$ but $\omega \neq \omega_0$? I.e., slightly "off resonance"?

Find approx. ...

$$\langle J_z \rangle (t) \approx \frac{\hbar}{2} A(\omega) \cos(\Omega(\omega)\, t)$$

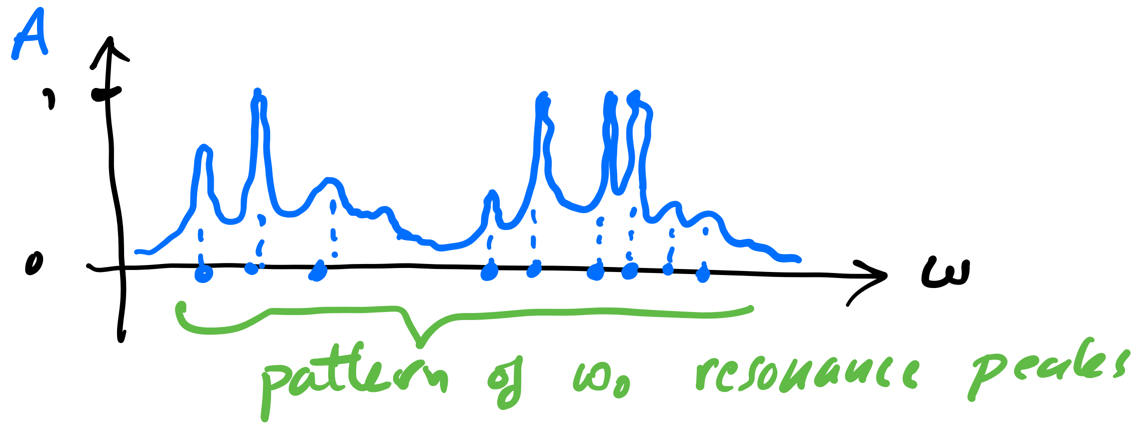
with
$$\begin{cases} A(\omega) \approx \dfrac{\left(\frac{\omega_1}{2}\right)^2}{(\omega-\omega_0)^2 + \left(\frac{\omega_1}{2}\right)^2} \\[2ex] \Omega(\omega) \approx \sqrt{(\omega-\omega_0)^2 + \left(\frac{\omega_1}{2}\right)^2} \end{cases}$$



- See for $|\omega - \omega_0| \gg \omega_1$, $A \to 0$. ✓

Nuclear magnetic resonance

- Put object in $\vec{B}(t)$ and slowly "scan" ω over a whole range.
- When $\omega \approx \omega_0$, pass through a resonance peak where spin flips maximally $\Rightarrow$ energy maximally absorbed.
- So by measuring energy absorption get "map" of resonance peaks:



- Values of $\omega_0 \doteq \frac{gq}{2mc} B_0$ depend on

  g, q, m of spins. For nuclei:

  $g \approx O(1)$, $q \simeq Ze$, $m = N m_p$

$\Rightarrow$ From pattern of ω_0's can identify nuclei
$\Rightarrow$ From amplitudes $\Rightarrow$ deduce concentration "
- Reconstruct map of atom-types in object.