Ch 3. Angular Momentum

- In physics we mainly try to understand the world by pulling it apart:
  we break it into subsystems which we try to isolate in space & time

- So the main operations we perform on systems are:
  spatial translations & rotations
  temporal translation & boosts

- These can all be thought of as symmetry operations on an isolated system:
  → you can't tell the difference between translating a system or leaving it fixed and just changing the origin of your coordinate system

- In QM a symmetry operation is a linear transformation of the Hilbert space which preserves inner products (and therefore probabilities by Born's rule).

$\therefore$ Symmetry operation $\longleftrightarrow$ unitary operator.

- 3 main symmetry operations in this course:

$$\boxed{\begin{array}{lll} (1)\ \underline{\text{Rotations}} & \longleftrightarrow & \hat{R}(\varphi\,\hat{n}) \\ (2)\ \underline{\text{Time translations}} & \longleftrightarrow & \hat{U}(t) \\ (3)\ \underline{\text{Space translations}} & \longleftrightarrow & \hat{T}(\vec{x}) \end{array}}$$

ch.3

ch.4

ch.6

- Symmetry operations form Lie groups

e.g. move along x-axis by amount a_1, then again by amount a_2 = move by amount a_1+a_2:

$$\hat{T}(a_2)\,\hat{T}(a_1)\,|\psi\rangle = \hat{T}(a_1+a_2)\,|\psi\rangle$$

- Unitary op $= e^{i\,(\text{Hermitian})}$. So define

$$\boxed{\hat{T}(a) = e^{ia\hat{P}_x/\hbar}} \quad \text{w/}\ \hat{P}_x = \hat{P}_x^{\dagger}$$

$\hat{P}_x \doteq$ "generator of x-translations".

$$\Rightarrow \begin{cases} \hat{T}(a)^{-1} = \hat{T}(a)^{\dagger} = \hat{T}(-a) \\ \hat{T}(a_1)\hat{T}(a_2) = \hat{T}(a_1+a_2) \\ [\hat{T}(a_1), \hat{T}(a_2)] = 0 \end{cases} .$$

check!

- In classical mechanics generator of translations is momentum:

$$\{p, f(x)\}_{P.B.} = -\frac{df}{dx}$$

"Poisson bracket"

"symplectomorphism"

or "canonical transformation"

$$\Rightarrow e^{-ap} \circ f(x) = e^{a\frac{d}{dx}} f(x) = f(x) + a\frac{df}{dx}(x) + \frac{a^2}{2!}\frac{d^2f}{dx^2} + \ldots$$

$$= f(x+a) \qquad \text{(Taylor exp.)}$$

$$= \text{translation } x \to x+a . \quad \checkmark$$

- In QM, $\{A, B\}_{P.B} \leftrightarrow [\hat{A}, \hat{B}] \cdot \frac{i}{\hbar}$

so $e^{-ap} \leftrightarrow e^{-ia\hat{p}/\hbar}$

"correspondence"

So <u>define</u> generator translations = momentum.

- Similarly for time translations & rotations.
  - Classically,

$$\begin{cases} \text{generator time transl.} = \text{energy} \\ \text{generator rotations} = \text{angular momentum} \end{cases}$$

So define in QM

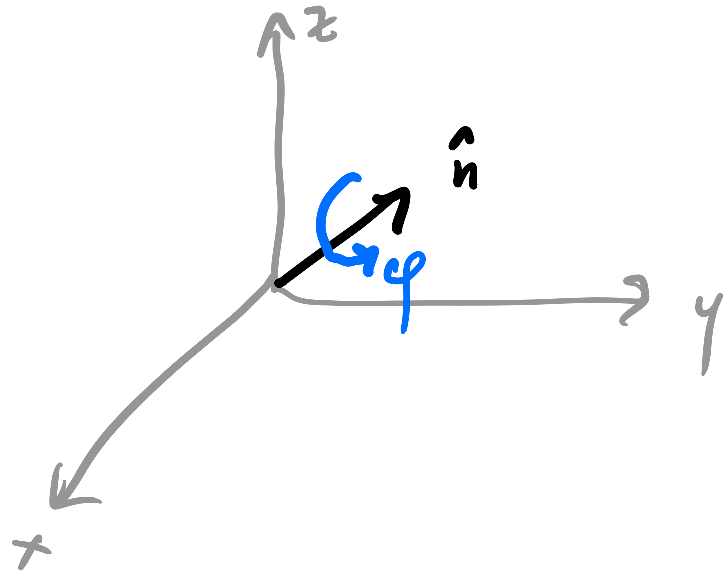
$$\boxed{\begin{aligned} \hat{U}(t) &= e^{-it\hat{H}/\hbar}, \quad \hat{H} = \text{energy op.} \\ \hat{R}(\varphi\hat{n}) &= e^{-i\varphi\hat{n}\cdot\hat{\vec{J}}/\hbar}, \quad \hat{\vec{J}} = \text{ang. mom. op.} \end{aligned}}$$

$$\hat{\vec{J}} \doteq (\hat{J}_x, \hat{J}_y, \hat{J}_z)$$

$\hat{n}$ = unit vector in 3d space

φ = angle

- $\hat{R}(\varphi\hat{n})$ has physical interp. as rotation around axis $\hat{n}$ by angle φ:



- 3d rot'n form a Lie group:
  - product of two rot'ns is another one

$$\hat{R}(\varphi_2\,\hat{n}_2)\,\hat{R}(\varphi_1\,\hat{n}_1) = \hat{R}(\varphi'\,\hat{n}')$$

$$\text{w/}\ \begin{cases} \varphi' = \varphi'(\varphi_1, \varphi_2, \hat{n}_1, \hat{n}_2) \\ \hat{n}' = \hat{n}'(\varphi_1, \varphi_2, \hat{n}_1, \hat{n}_2) \end{cases}$$

(*) ["addition of Euler angles"]

  - inverse is $\hat{R}(\varphi\,\hat{n})^{-1} = \hat{R}(-\varphi\,\hat{n})$

[Various names: "O(3)", "SO(3)", "SU(2)".

orthogonal group, 3d, special, unitary

unitary = cplx " " $UU^\dagger = 1$

orthog = real matrices s.t. $UU^T = 1$ (=real unitary)

special = $\det U = 1$]

• 3d rot'ns do not commute! ("non-abelian")

$$\hat{R}(\varphi_1 \hat{n}_1)\,\hat{R}(\varphi_2 \hat{n}_2) \neq \hat{R}(\varphi_2 \hat{n}_2)\,\hat{R}(\varphi_1 \hat{n}_1)$$

E.g. check: $\hat{R}(\frac{\pi}{2}\hat{x})\hat{R}(\frac{\pi}{2}\hat{y}) \neq \hat{R}(\frac{\pi}{2}\hat{y})\hat{R}(\frac{\pi}{2}\hat{x})$

• $\therefore$ $(\hat{J}_x, \hat{J}_y, \hat{J}_z)$ must not commute.

In fact they must satisfy particular commutation relations ("Lie algebra") to enforce the rot'n group law. (*)

They are:

$$\boxed{\begin{aligned} [\hat{J}_x, \hat{J}_y] &= i\hbar \hat{J}_z \\ [\hat{J}_y, \hat{J}_z] &= i\hbar \hat{J}_x \\ [\hat{J}_z, \hat{J}_x] &= i\hbar \hat{J}_y \end{aligned}}$$

(*)

Angular momentum algebra

(a.k.a. so(3) Lie alg.
or su(2) " ")

Note: $[\hat{A},\hat{B}]^\dagger = -[\hat{A}^\dagger, \hat{B}^\dagger]$.

This operator algebra will be our main focus for the rest of the chapter.

Angular momentum algebra

(1) Rewrite more compactly:

$$\hat{\vec{J}} = (\hat{J}_x, \hat{J}_y, \hat{J}_z) \doteq (\hat{J}_1, \hat{J}_2, \hat{J}_3)$$
$$= (\hat{J}_j,\ j=1,2,3)$$

Algebra:

$$\boxed{[\hat{J}_j, \hat{J}_k] = i\hbar \sum_{l=1}^{3} \epsilon_{jkl} \hat{J}_l}$$

with ϵ_{jkl} : "completely antisymmetric rank-3 tensor"

$$\left.\begin{aligned} \epsilon_{jkl} &= -\epsilon_{kjl} \\ &= -\epsilon_{lkj} \\ &= -\epsilon_{jlk} \end{aligned}\right\} \text{completely antisymmetric}$$

$\Rightarrow$ $\epsilon_{jkl} = 0$ if any 2 of its indices are equal.

$\epsilon_{123} \doteq 1$ (def'n normalization)

$\Rightarrow$ antisymmetry then determines all $\epsilon_{ijk} \in \{\pm 1, 0\}$.

Ex: show $\epsilon_{jk\ell} = + \epsilon_{\ell jk}$.

Ex: list all non-zero components of ϵ_{ijk} (there are only 6).

Ex: check that (A) $\Rightarrow$ (A).

(2) Names: $\hat{J}_j$ or $\hat{S}_j$ or $\hat{L}_j$

"total ang. mom." "spin ang. mom." "orbital ang. mom."

(3) $\hat{J}_j^+ = \hat{J}_j$ are hermitian. What are their eigenvalues = possible outcomes of measuring them?

- They are determined by (A).
- Classic math. argument...

• $\hat{J}_z$ o-n eigenbasis $\{|m\rangle, m = ?\}$

$$\boxed{\hat{J}_z |m\rangle = m\hbar |m\rangle} \quad m \in \mathbb{R}.$$

$$\langle m | n \rangle = \delta_{m,n}$$

$\hat{J}_x, \hat{J}_y$ also have eigenbases but since don't commute, their bases are typically different.

- Trick #1: Define Casimir operator

$$\boxed{\hat{J}^2 \doteq \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2}$$

- "J^2" is name of operator.

A.k.a. "total ang. mom. operator"

Check: - $\hat{J}^2$ is hermitian

- $\boxed{[\hat{J}_j, \hat{J}^2] = 0}$ $\quad j \in \{x, y, z\}$

$\therefore \exists$ simultaneous eigenbasis $\{|\lambda, m\rangle\}$ (o-n)

$$\boxed{\begin{aligned} \hat{J}_z |\lambda, m\rangle &= \hbar m \, |\lambda, m\rangle \\ \hat{J}^2 |\lambda, m\rangle &= \hbar^2 \lambda \, |\lambda, m\rangle \end{aligned}}$$

$\langle \lambda, m | \lambda', m' \rangle = \delta_{\lambda\lambda'} \cdot \delta_{mm'}$

- $$\hat{J}^2 = \sum_{j=1}^{3} \hat{J}_j^2 = \sum_{j=1}^{3} \hat{J}_j \hat{J}_j = \sum_{j=1}^{3} \hat{J}_j^\dagger \hat{J}_j$$

"positive operator"

$$\Rightarrow \hbar^2\lambda = \hbar^2\lambda \langle \lambda, m | \lambda, m\rangle = \langle \lambda, m | \hat{J}^2 | \lambda, m\rangle$$

$$= \langle \lambda, m | \left(\sum_{j=1}^{3} \hat{J}_j^\dagger \hat{J}_j \right) | \lambda, m\rangle$$

$$= \sum_{j=1}^{3} \langle \lambda, m | \hat{J}_j^\dagger \hat{J}_j | \lambda, m\rangle$$

$$= \sum_{j=1}^{3} \| \hat{J}_j | \lambda, m\rangle \|^2 \geq 0$$

$$\therefore \quad \boxed{\lambda \geq 0}.$$

$$0 \leq \| \hat{J}_x | \lambda, m\rangle \|^2 + \| \hat{J}_y | \lambda, m\rangle \|^2$$

$$= \langle \lambda, m | \left(\hat{J}_x^\dagger \hat{J}_x + \hat{J}_y^\dagger \hat{J}_y \right) | \lambda, m\rangle$$

$$= \langle \lambda, m | \left(\hat{J}^2 - \hat{J}_z^2 \right) | \lambda, m\rangle$$

$$= \langle \lambda, m | \left(\lambda \hbar^2 - m^2 \hbar^2 \right) | \lambda, m\rangle$$

$$= \hbar^2 (\lambda - m^2)$$

$$\therefore \quad \boxed{\lambda \geq m^2}.$$

- Trick #2: Define "raising & lowering op.s"

$$\boxed{\begin{aligned}\hat{J}_+ &\doteq \hat{J}_x + i\hat{J}_y \\ \hat{J}_- &\doteq \hat{J}_x - i\hat{J}_y .\end{aligned}}$$

- Note, not hermitian, but

$$\hat{J}_+^\dagger = \hat{J}_-$$

Check: -

$$\boxed{\begin{aligned}[\hat{J}_z, \hat{J}_\pm] &= \pm\hbar\,\hat{J}_\pm \quad (*) \\ [\hat{J}_+, \hat{J}_-] &= 2\hbar\,\hat{J}_z \\ [\hat{J}^2, \hat{J}_\pm] &= 0\end{aligned}}$$

- Note $(*)$ $\Rightarrow$ $\boxed{\hat{J}_z\hat{J}_\pm = \hat{J}_\pm(\hat{J}_z \pm \hbar)}$

- Compute:

$$\begin{aligned}\hat{J}_z\,\hat{J}_\pm|\lambda,m\rangle &= \hat{J}_\pm(\hat{J}_z \pm \hbar)\,|\lambda,m\rangle \\ &= \hat{J}_\pm\,(m\hbar \pm \hbar)\,|\lambda,m\rangle \\ &= (m\pm 1)\hbar\;\hat{J}_\pm\,|\lambda,m\rangle\end{aligned}$$

$$
\begin{aligned}
\hat{J}^2 \hat{J}_\pm |\lambda, m\rangle &= \hat{J}_\pm \hat{J}^2 |\lambda, m\rangle \\
&= \hat{J}_\pm \lambda \hbar^2 |\lambda, m\rangle \\
&= \lambda \hbar^2 \hat{J}_\pm |\lambda, m\rangle
\end{aligned}
$$

$$\therefore \quad \boxed{\hat{J}_\pm |\lambda, m\rangle \propto |\lambda, m\pm 1\rangle} \quad \text{eigenstates!}$$

$$\cdots \leftarrow |\lambda, m-2\rangle \xleftarrow{\hat{J}_-} |\lambda, m-1\rangle \xleftarrow{\hat{J}_-} |\lambda, m\rangle \xrightarrow{\hat{J}_+} |\lambda, m+1\rangle \xrightarrow{\hat{J}_+} |\lambda, m+2\rangle \rightarrow \cdots$$

∞ chain of orthog. states. But what if Hilbert space is finite-dimensional?

Only way out: must exist maximum value of m, $m \doteq j$, such that

$$\hat{J}_+ |\lambda, j\rangle = 0.$$

$$
\begin{aligned}
\therefore \quad 0 = \| \hat{J}_+ |\lambda, j\rangle \|^2 &= \langle \lambda, j | \hat{J}_- \hat{J}_+ | \lambda, j\rangle \\
&= \langle \lambda, j | (\hat{J}_x - i\hat{J}_y)(\hat{J}_x + i\hat{J}_y) | \lambda, j\rangle \\
&= \langle \lambda, j | \left(\hat{J}_x^2 + \hat{J}_y^2 + i[\hat{J}_x, \hat{J}_y]\right) | \lambda, j\rangle \\
&= \langle \lambda, j | \left(\hat{J}^2 - \hat{J}_z^2 - \hbar \hat{J}_z\right) | \lambda, j\rangle
\end{aligned}
$$

$$= \langle \lambda, j | \left(\hbar^2 \lambda - \hbar^2 j^2 - \hbar^2 j \right) | \lambda, j \rangle$$

$$= \hbar^2 (\lambda - j^2 - j)$$

$$\Rightarrow \boxed{\lambda = j(j+1)} \quad \text{(max)}$$

Also, $\exists$ minimum value $m = j'$ s.t.

$$\hat{J}_- |\lambda, j'\rangle = 0$$

$\Rightarrow$ ... check! ...

$$\Rightarrow \boxed{\lambda = j'(j'-1)} \quad \text{(min)}$$

Only solutions to (max) & (min) are

$$j' = -j \checkmark \text{ or } j' = j+1 \; \text{✗}$$

- **Summary** $\exists$ o-n basis $\{|j,m\rangle\}$ with

$$\hat{J}^2 |j,m\rangle = \hbar^2 j(j+1) |j,m\rangle$$
$$\hat{J}_z |j,m\rangle = \hbar m |j,m\rangle$$

$$m \in \{-j, -j+1, -j+2, \ldots, j-2, j-1, j\}$$

$$\Rightarrow j \in \{0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \ldots\}.$$

E.g. • $\underline{j=0}$, only $m=0$ allowed

& $\hat{J}^2 = \hat{J}_z = 0$ "spin 0"

• $\underline{j=\frac{1}{2}}$, $\Rightarrow$ $m=\pm 1/2$ allowed

& $\hat{J}^2 = \hbar^2 \frac{1}{2}(\frac{1}{2}+1) = \frac{3\hbar^2}{4}$ all states

$$\left.\begin{aligned} \hat{J}_z\,|\tfrac{1}{2},+\tfrac{1}{2}\rangle &= +\tfrac{\hbar}{2}\,|\tfrac{1}{2},+\tfrac{1}{2}\rangle \\ \hat{J}_z\,|\tfrac{1}{2},-\tfrac{1}{2}\rangle &= -\tfrac{\hbar}{2}\,|\tfrac{1}{2},-\tfrac{1}{2}\rangle \end{aligned}\right\} \text{"spin 1/2"}$$

$$\text{So}\left\{\begin{aligned} |+\hat{z}\rangle &= |\tfrac{1}{2},\tfrac{1}{2}\rangle \\ |-\hat{z}\rangle &= |\tfrac{1}{2},-\tfrac{1}{2}\rangle \end{aligned}\right.$$

• Finally, can compute matrix elements of $\hat{J}_x$ & $\hat{J}_y$ in this basis.

Know $\hat{J}_+|j,m\rangle = c\,\hbar\,|j,m+1\rangle$ for some cplx no. c. Compute:

$$\hbar^2|c|^2 = \|\hat{J}_+|j,m\rangle\|^2 = \langle j,m|\hat{J}_-\hat{J}_+|j,m\rangle$$

$$= \langle j,m|(\hat{J}^2 - \hat{J}_z^2 - \hbar\hat{J}_z)|j,m\rangle$$

$$= \hbar^2 (j(j+1) - m(m+1))$$

$$\Rightarrow \quad c = \sqrt{j(j+1) - m(m+1)} \qquad \text{(Phase = choice.)}$$

Same for $\hat{J}_-$ .. $\Rightarrow$ ...

$$\boxed{\hat{J}_\pm |j,m\rangle = \hbar \sqrt{j(j+1) - m(m\pm 1)}\, |j, m\pm 1\rangle}$$

- Can find $\hat{J}_x$, $\hat{J}_y$ using

$$\hat{J}_x = \frac{1}{2} (\hat{J}_+ + \hat{J}_-)$$

$$\hat{J}_y = \frac{1}{2i} (\hat{J}_+ - \hat{J}_-) .$$

Summary of angular momentum quantum numbers

$$\left.\begin{aligned} [\hat{J}_x, \hat{J}_y] &= i\hbar \hat{J}_z \\ [\hat{J}_y, \hat{J}_z] &= i\hbar \hat{J}_x \\ [\hat{J}_z, \hat{J}_x] &= i\hbar \hat{J}_y \end{aligned}\right\} \text{Angular momentum algebra}$$

$$\text{define:} \begin{cases} \hat{J}^2 \doteq \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2 \\ \hat{J}_+ \doteq \hat{J}_x + i\hat{J}_y \\ \hat{J}_- \doteq \hat{J}_x - i\hat{J}_y \end{cases} \qquad \hat{J}_+^\dagger = \hat{J}_-$$

$$\Longleftrightarrow \begin{cases} [\hat{J}_j, \hat{J}^2] = 0 \qquad j \in \{x, y, z\} \\ [\hat{J}_z, \hat{J}_\pm] = \pm\hbar \hat{J}_\pm \\ [\hat{J}_+, \hat{J}_-] = 2\hbar \hat{J}_z \\ [\hat{J}^2, \hat{J}_\pm] = 0 \end{cases}$$

Theorem (proven above)

Any Hilbert space with angular momentum operators has an orthogonal direct sum decomposition

$$\mathcal{H} = \bigoplus_j \mathcal{H}_j \qquad j \in \{0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots\}$$

where each $\mathcal{H}_j$ subspace has o-n basis

$$\{|j,m\rangle, \; m \in \{-j, -j+1, -j+2, \ldots, j-2, j-1, j\}\}$$

such that

$$\hat{J}^2 |j,m\rangle = \hbar^2 j(j+1) |j,m\rangle$$
$$\hat{J}_z |j,m\rangle = \hbar m |j,m\rangle$$
$$\hat{J}_\pm |j,m\rangle = \hbar \sqrt{j(j+1) - m(m\pm 1)} \, |j, m\pm 1\rangle.$$

The $\mathcal{H}_j$ subspaces are called the "spin j" subspaces, or the "spin j representations".

Examples

- **Spin-j particle** (ignoring its position & momentum)

→ Hilbert space is $\mathcal{H}_j$

$$\hat{J}^2 = \hbar^2 j(j+1)\,\hat{1}$$

since basis is $|j,m\rangle$ $m \in \{-j, -j+1, \ldots, j\}$ and all have same $\hat{J}^2$ eigenvalue.

→ $\dim(\mathcal{H}_j) = 2j+1$
because thats how many m's there are.

→ $\hat{J}_z |j,m\rangle = \hbar m |j,m\rangle$

⇒ Matrix elements of $\hat{J}_z$ are

$$(J_z)_{mm'} = \langle j, m| \hat{J}_z |j, m'\rangle = \hbar m' \langle j,m|j,m'\rangle$$

$$= \hbar m' \delta_{mm'}$$

$$J_z \leftrightarrow \begin{pmatrix} -\hbar j & & & \\ & \hbar(1-j) & & \\ & & \ddots & \\ & & & \hbar j \end{pmatrix} \quad \begin{matrix} |j,-j\rangle \\ \vdots \\ |j,+j\rangle \end{matrix}$$

$\rightarrow \quad \hat{J}_x |j,m\rangle = ?$

$$= \frac{1}{2}(\hat{J}_+ + \hat{J}_-)|j,m\rangle$$

$$= \frac{1}{2}\hbar\Big(\sqrt{j(j+1)-m(m+1)}\,|j,m+1\rangle + \sqrt{j(j+1)-m(m-1)}\,|j,m-1\rangle\Big)$$

$\rightarrow$ Matrix elements of $\hat{J}_x$?

$$(J_x)_{mm'} = \langle j,m|\hat{J}_x|j,m'\rangle$$

$$= \frac{\hbar}{2}\Big(\sqrt{j(j+1)-m'(m'+1)}\,\langle j,m|j,m'+1\rangle + \sqrt{j(j+1)-m'(m'-1)}\,\langle j,m|j,m'-1\rangle\Big)$$

$$= \frac{\hbar}{2}\Big(\sqrt{j(j+1)-m'(m'+1)}\,\delta_{m,m'+1} + \sqrt{j(j+1)-m'(m'-1)}\,\delta_{m,m'-1}\Big)$$

$$= \frac{\hbar}{2}\Big(\sqrt{j(j+1)-m(m-1)}\,\delta_{m,m'+1} + \sqrt{j(j+1)-m(m+1)}\,\delta_{m,m'-1}\Big)$$

(Note $\delta_{m,m'+1} = \delta_{m-1,m'}$ etc.)

$$J_x \longleftrightarrow \begin{pmatrix} 0 & \times & & & & \\ \times & 0 & \times & & & \\ & \times & 0 & \times & & \\ & & \times & \ddots & \ddots & \\ & & & \ddots & \ddots & \times \\ & & & & \times & 0 \end{pmatrix}$$

→ Spin-j particle can be in any state

$$|\psi\rangle = \sum_{m=-j}^{j} c_m |j, m\rangle \qquad \left(\sum_m |c_m|^2 = 1\right)$$

& calculate probabilities, expectation values, etc. by usual rules.

- <u>Hydrogen atom in $n=2$ energy level</u>

$$\mathcal{H}_{n=2} = \mathcal{H}_{\ell=0} \oplus \mathcal{H}_{\ell=1}$$

$\therefore$ o-n $(\hat{L}^2, \hat{L}_z)$ eigenbasis is

$$\Big\{ \underbrace{|\ell=0, m=0\rangle}_{\text{basis of } \mathcal{H}_{\ell=0}}, \underbrace{|\ell=1, m=-1\rangle, |\ell=1, m=0\rangle, |\ell=1, m=1\rangle}_{\text{basis of } \mathcal{H}_{\ell=1}} \Big\}$$

(Write $J \to L$ & $j \to \ell$ by convention.)

→ General H-atom energy-angular mom. eigenstates denoted "$|n, l, m\rangle$". So above $n=2$ eigenbasis are usually written:

$$\{ |2,0,0\rangle, |2,1,-1\rangle, |2,1,0\rangle, |2,1,1\rangle \}.$$

→ Consider state of H-atom

$$|\psi\rangle = \frac{1}{\sqrt{6}}\left(|2,0,0\rangle + \sqrt{2}|2,1,0\rangle + \sqrt{3}\,|2,1,1\rangle\right)$$

(a) What is $\langle L_x \rangle$?

(b) What is $\langle L^2 \rangle$?

(c) What are poss. values of measuring $\hat{L}_z$ & what are their probabilities?

(d) If measure $L_z = 0$ what is the state right after the measurement?

(a) $\langle L_x \rangle = \langle \psi | \hat{L}_x | \psi \rangle$

$|\psi\rangle = \frac{1}{\sqrt{6}}\left(|2,0,0\rangle + \sqrt{2}|2,1,0\rangle + \sqrt{3}\,|2,1,1\rangle\right)$

$$= \frac{1}{2}\langle \psi | (\hat{L}_+ + \hat{L}_-) | \psi \rangle$$

$$= \frac{1}{2\sqrt{6}}\langle \psi | (\hat{L}_+ + \hat{L}_-) \left\{ |2,0,0\rangle + \sqrt{2}\,|2,1,0\rangle + \sqrt{3}\,|2,1,1\rangle \right\}$$

$$= \frac{1}{2\sqrt{6}} \langle\psi| \Big\{ \big(\# |2,0,1\rangle + \# |2,0,-1\rangle\big) + \sqrt{2}\big(\# |2,1,1\rangle + \# |2,1,-1\rangle\big) + \sqrt{3}\big(0 + \# |2,1,0\rangle\big) \Big\}$$

$$= \frac{\sqrt{2}}{2\sqrt{6}} \langle\psi| \hbar\sqrt{1(1+1)-0(0+1)}\, |2,1,1\rangle + \frac{\sqrt{3}}{2\sqrt{6}} \langle\psi|\hbar\sqrt{1(1+1)-1(1-1)}\, |2,1,0\rangle$$

$$= \frac{\hbar}{\sqrt{6}} \frac{\sqrt{3}}{\sqrt{6}} \langle 2,1,1|2,1,1\rangle + \frac{\hbar}{2}\frac{\sqrt{2}}{\sqrt{6}} \langle 2,1,0|2,1,0\rangle$$

$$= \frac{\hbar}{2\sqrt{3}} + \frac{\hbar}{2\sqrt{3}} = \frac{\hbar}{\sqrt{3}} .$$

$|\psi\rangle = \frac{1}{\sqrt{6}}\left(|2,0,0\rangle + \sqrt{2}|2,1,0\rangle + \sqrt{3}\,|2,1,1\rangle\right)$

(b) $\langle L^2\rangle = \langle\psi|\hat{L}^2|\psi\rangle$

$$= \langle\psi| \frac{1}{\sqrt{6}}\Big(0 + 2\hbar^2[\sqrt{2}|2,1,0\rangle + \sqrt{3}|2,1,1\rangle] \Big)$$

$$= \frac{1}{6} 2\hbar^2 \left(\sqrt{2}^2 + \sqrt{3}^2\right) = \frac{5}{3}\hbar^2 .$$

(c) $\hat{L}_z$ eigenvalues $\in \{-\hbar, 0, +\hbar\}$ for $l=0$ or $l=1$.
But only $m=0,1$ occur in $|\psi\rangle$, so only possible outcomes are

$L_z = 0$ or $L_z = +\hbar$.

Probabilities

$|\psi\rangle = \frac{1}{\sqrt{6}}\left(|2,0,0\rangle + \sqrt{2}|2,1,0\rangle + \sqrt{3}\,|2,1,1\rangle\right)$

$$P(L_z=0) = \langle\psi|\hat{P}_{L_z=0}|\psi\rangle$$

$$\hat{P}_{L_z=0} = |2,0,0\rangle\langle 2,0,0| + |2,1,0\rangle\langle 2,1,0|$$

$$\therefore P(L_z=0) = |\langle 200|\psi\rangle|^2 + |\langle 210|\psi\rangle|^2$$

$$= \frac{1}{6} + \frac{2}{6} = \frac{1}{2}.$$

$$P(L_z=\hbar) = \langle\psi|\hat{P}_{L_z=\hbar}|\psi\rangle$$

$$= \langle\psi|\left(|211\rangle\langle 211|\right)|\psi\rangle$$

$$= |\langle 211|\psi\rangle|^2 = \frac{3}{6} = \frac{1}{2}. \checkmark$$

(d) If measure $L_z = 0$ $\qquad |\psi\rangle = \frac{1}{\sqrt{6}}\left(|2,0,0\rangle + \sqrt{2}|2,1,0\rangle + \sqrt{3}\,|2,1,1\rangle\right)$

$$|\psi\rangle \xrightarrow{L_z=0} |\psi'\rangle = \frac{\hat{P}_{L_z=0}|\psi\rangle}{\|\hat{P}_{L_z=0}|\psi\rangle\|}.$$

$$= \frac{\frac{1}{\sqrt{6}}\left(|200\rangle + \sqrt{2}|210\rangle\right)}{\sqrt{P(L_z=0)}} \quad = \frac{1}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{3}}\left(|200\rangle + \sqrt{2}|210\rangle\right).$$

(normalized ✓)