Ugrad QM (Course: 5020-21)

Braunstein 316, MWF 12:20-1:15

Admin

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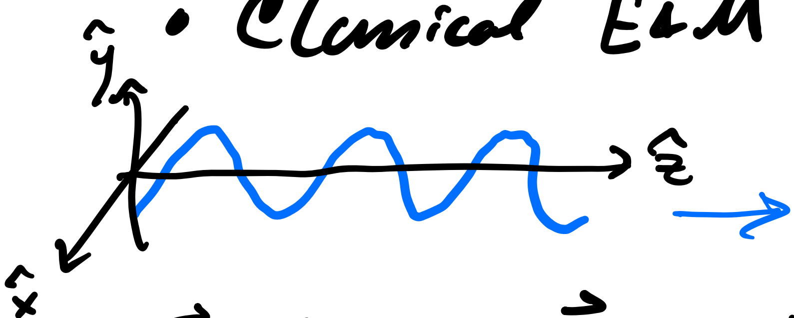
- Townsend "Modern Intro QM" (2^{nd} ed)
- 2^{nd} course in QM!
- Broad outline:
  - F-sem: foundations
  - S-sem: basic applications & methods

Motivation

- Classical E&M (light)



$$e^{i\theta} = \cos\theta + i\sin\theta$$

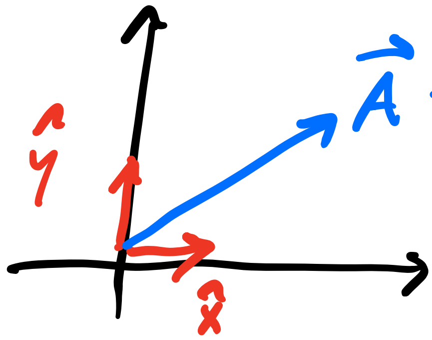
$$\vec{E} = A\,\hat{n}\cos(\vec{k}\cdot\vec{x} - \omega t + \theta)$$

- amp.: $A^2 \propto$ prob. particles
- pol.: $\hat{n}\cdot\hat{k} = 0$ — ang. mom. (spin/helicity)
- wv. vect.: $\vec{k}$ = dir. — momentum
- freq.: $\omega = ck$ — energy
- phase: $\omega = 2\pi\nu$, $k = 2\pi/\lambda$ — QM phase

$$= \mathrm{Re}\big(\underbrace{A\hat{n}e^{i(\theta + \vec{k}\cdot\vec{x} - \omega t)}}\big)$$

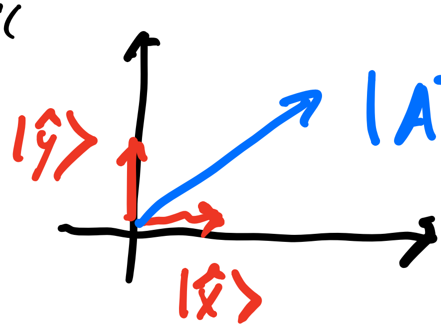
2d complex vector "amplitude" $\rightarrow$ phase responsible for interference!

2d real vectors $\vec{A}$



$$\vec{A} = A_x \hat{x} + A_y \hat{y} \sim \begin{pmatrix} A_x \\ A_y \end{pmatrix} \in \mathbb{R}$$

2d cplx vectors : $\vec{A} \to |A\rangle$ "ket"



$$|A\rangle = A_x |\hat{x}\rangle + A_y |\hat{y}\rangle \sim \begin{pmatrix} A_x \\ A_y \end{pmatrix} \in \mathbb{C}$$

transpose

$$\vec{A} = A\hat{n} \qquad A = |\vec{A}| = \sqrt{\vec{A}\cdot\vec{A}} = \sqrt{\vec{A}^T \vec{A}}$$

$$= \sqrt{(A_x \ A_y)\begin{pmatrix} A_x \\ A_y \end{pmatrix}}.$$ "norm" ≥ 0

hermitian conj.
"dagger"

$$\| |A\rangle \| = \sqrt{\langle A|A\rangle} = \sqrt{(|A\rangle)^\dagger |A\rangle}$$

"bra"

$$= \sqrt{(A_x^* \ A_y^*)\begin{pmatrix} A_x \\ A_y \end{pmatrix}} \geq 0$$

complex conj.

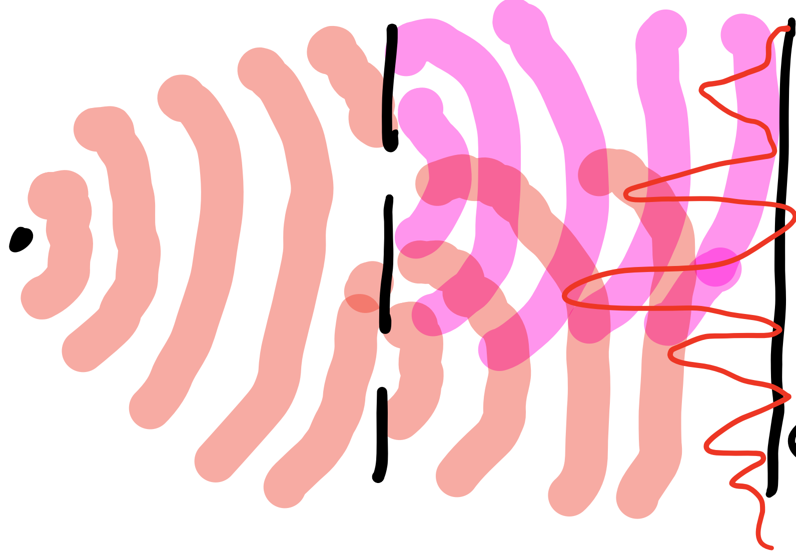
$\vec{A}\cdot\vec{B} \in \mathbb{R}$ $\qquad \langle A|B\rangle \in \mathbb{C}$

"bracket", "inner product".

Cplx vector space w/ inner product = "Hilbert space"

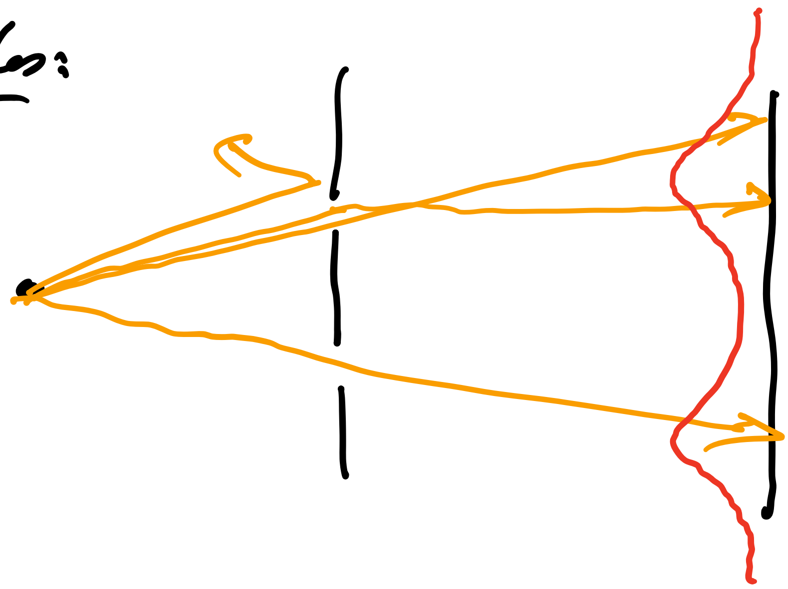
$\langle A| \equiv |A\rangle^\dagger$

Waves:



What happens as $A \to 0$?

Particles:

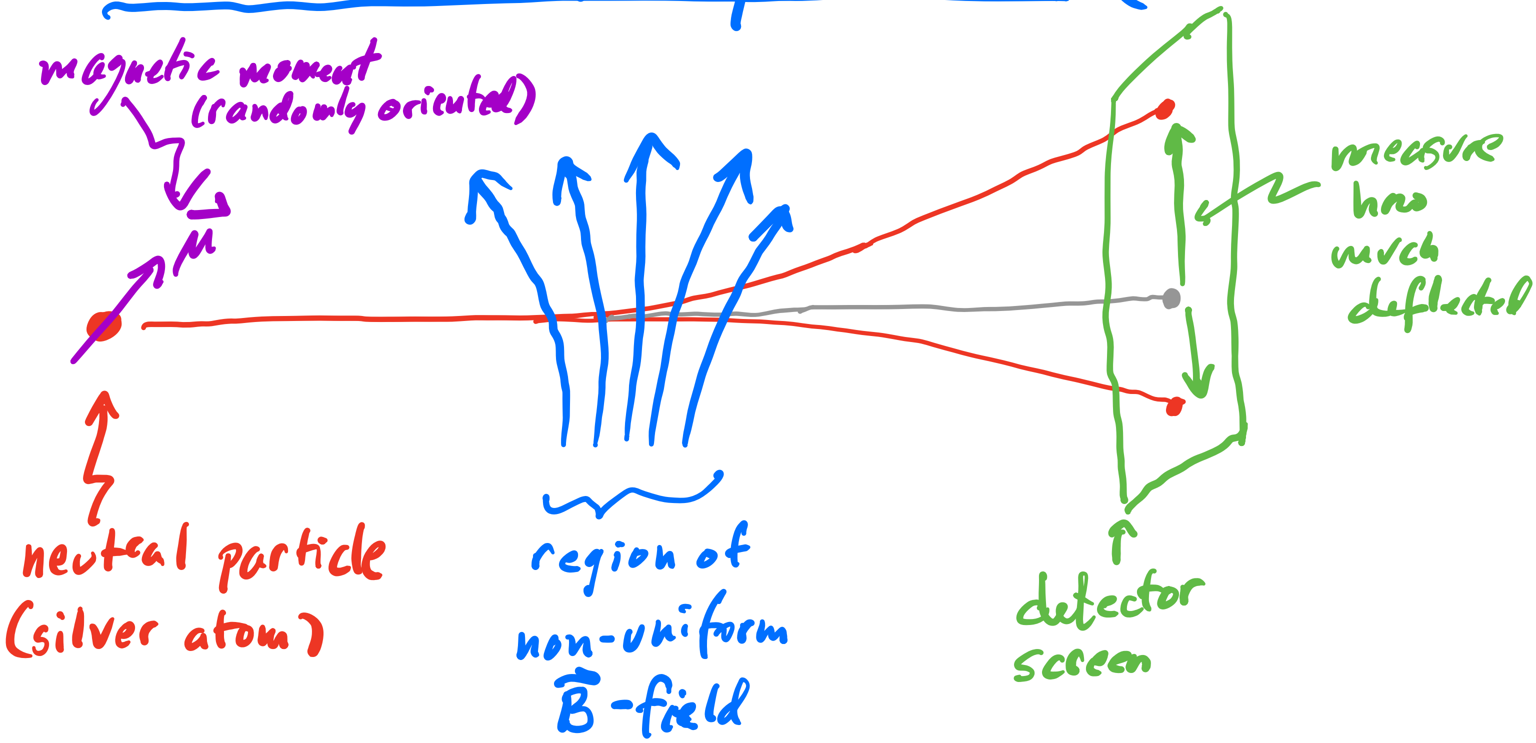


What happens when make holes v. small?

Experiment: same answer in both cases!
- light & e^-: both interfere
- " " : both give prob. dist. of individual "hits" on screen

History: (1900 Planck) & 1905 Einstein:
light discrete $E = n\hbar\omega \quad n \in \mathbb{Z}$
$\hbar \equiv h/2\pi = 10^{-27}$ erg·s

Stern Gerlach experiments (ch. 1)



What do we expect classically?

- Energy of neutral particle in $\vec{B}$-field is
$$V = -\vec{\mu}\cdot\vec{B}$$

$\Rightarrow$ Force is
$$\vec{F} = -\vec{\nabla} V = \sum_{i=x,y,z} \mu_i (\vec{\nabla} B_i)$$

- Arrange $\vec{B}$-field so that
$$\vec{\nabla} B_z \approx \hat{e}_z \frac{\partial B_z}{\partial z} \quad \& \text{ rest } \vec{\nabla} B_i \approx 0$$

$$\Rightarrow \vec{F} = \hat{e}_z \, \mu_z \frac{\partial B_z}{\partial z}, \quad \text{so:}$$

$\boxed{\text{displacement in } z\text{-direction, } \& \propto \mu_z.}$

- $\vec{\mu} = \frac{gq}{2mc} \vec{S}$

w/
- q = charge of particle (e.g. $q = -e$ for e^-)
- m = mass " "
- c = speed of light
- $g/2$ = numerical fudge factor ($g \approx 2.0011\ldots$ for e^-)
- $\vec{S}$ = angular momentum of particle = a.k.a. "spin"

We divide angular momentum into "spin" & "orbital" parts

$\vec{S}$ = spin = ang. mom. of particle coming from spinning on an axis through its center

$\vec{L}$ = orbital = ang. mom. of particle coming from its motion relative to the origin

$\vec{J}$ = Total ang. mom.

$$= \sum_{a = \text{particles}} (\vec{S}_a + \vec{L}_a)$$

sum contributions from all particles making up atom

- Silver atom: particles = 1 nucleus, 47 e^-'s

$$\therefore \quad \vec{J} = \vec{S}_N + \sum_{a=1}^{47} (\vec{S}_a + \vec{L}_a)$$

$$\therefore \quad \vec{\mu}_{Ag} \sim \frac{g_N \cdot (47e)}{2m_N c} \vec{S}_N + \sum_{a=1}^{47} \frac{2(-e)}{2m_e c} (\vec{S}_a + \vec{L}_a)$$

$\underbrace{\qquad\qquad}_{\approx 0}$ $\qquad$ $\frac{-e}{m_e c}\vec{S}$ $\qquad$ $= 0$

($m_N \approx 2000\, m_e$) $\qquad$ (all but one e^- cancels) $\qquad$ (all cancel)

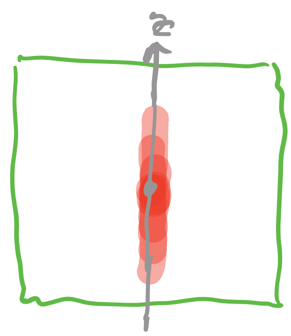
$$\Rightarrow \quad \vec{\mu} \approx \frac{-e}{m_e c}\vec{S}_{\text{electron}}$$

$$\therefore \quad \boxed{\mu_z \propto S_z}$$

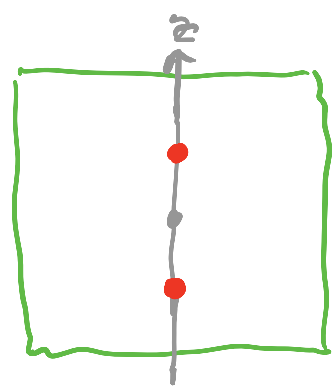
z-component of e^- spin
w/ computable coefficient

$\Rightarrow$ Stern-Gerlach experiment should see a distribution of displacements of the silver atoms proportional to S_z, which is random.

So expect to see on screen:



Instead, see:



Just two spots
w/ equal probability

!!??

This makes no sense: even though the e^- spins were prepared <u>completely</u> <u>randomly</u>, their z-component takes just 2 specific values:

$$S_z = \begin{cases} +\hbar/2 & 50\% \text{ of time} \\ -\hbar/2 & 50\% \text{ of time} \end{cases}$$

$[\hbar]$ = Energy × Time = Mass × Length^2/Time = [ang. mom.]

Will try to model this with vectors that add (to mimic wave interference).

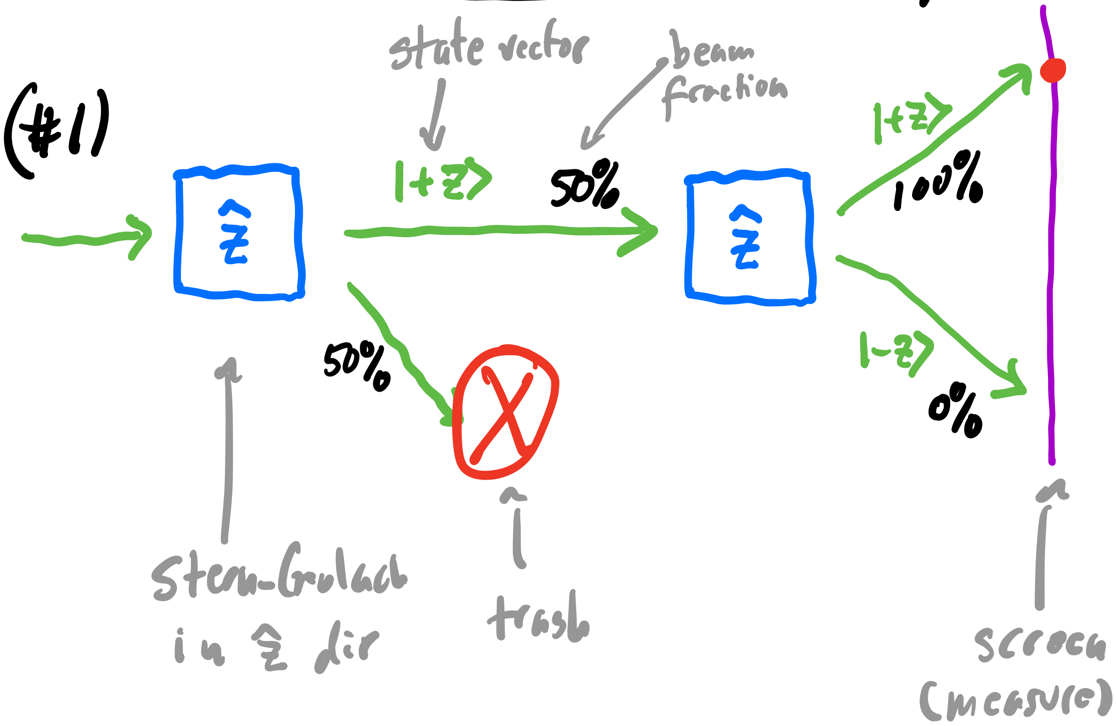
So <u>define</u>

$|+\hat{z}\rangle \doteq$ atom in state w/ $S_z = +\hbar/2$

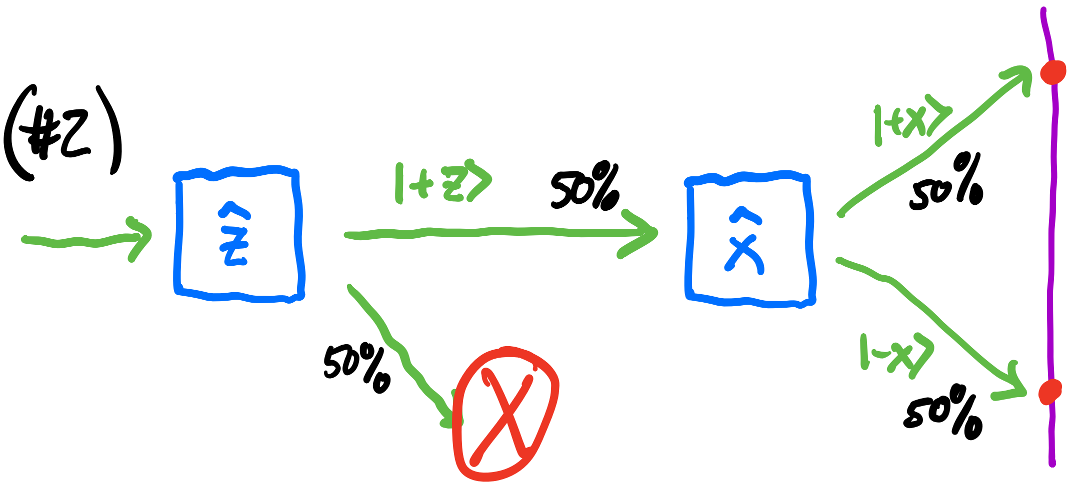
$|-\hat{z}\rangle \doteq$ " " " " $S_z = -\hbar/2$

§1.2 some Stern-Gerlach experiments

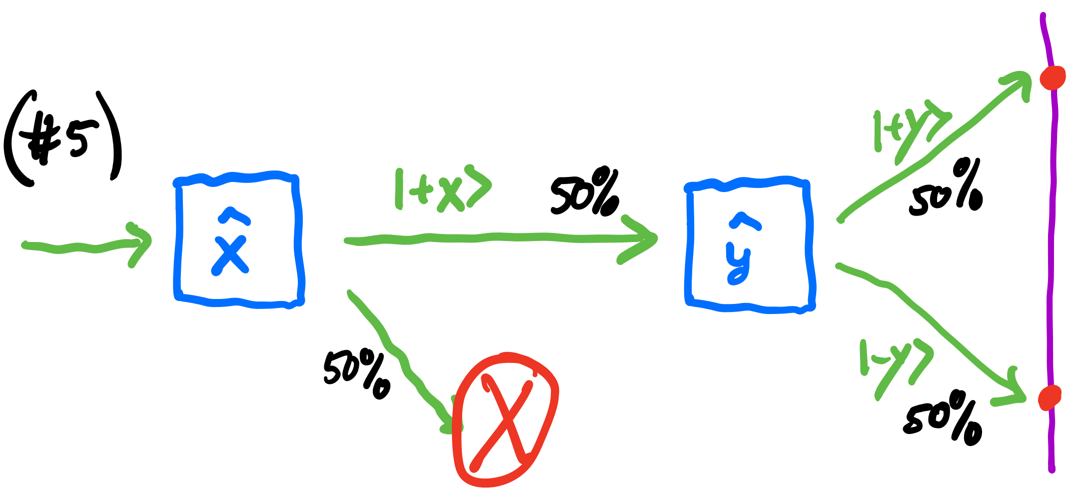
(#1)



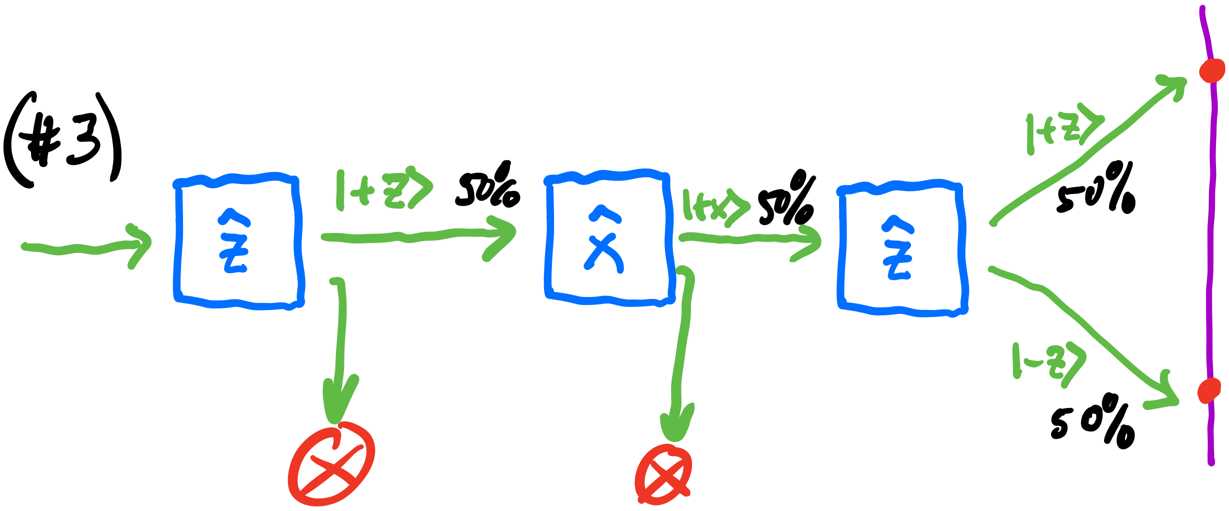
(#2)



(#5)



(#3)



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§1.3) Given vector space rules of quantum mech.

- We motivated using vectors $|+z\rangle, |-z\rangle$, etc to describe spin states by analogy with olarization of waves.

- Intensity of wave $\propto |\text{vector component}|^2$

  E.g. intensity of part of wave $\vec{A}$ polarized in $\hat{n}$ direction is $|\hat{n}\cdot\vec{A}|^2$

- Replace: $\vec{A} \rightarrow |A\rangle$

  Polariz. $\hat{n} \rightarrow$ Stern-Gerlach outcomes $|\pm\hat{n}\rangle$

  $\vec{A}\cdot\vec{B} \rightarrow \langle A|B\rangle$

⇒ So <u>guess</u> rules for spins:

If send particles in spin state $|\psi\rangle$ through a Stern-Gerlach magnet oriented in the $\hat{n}$ direction,

then there are 2 possible outcomes:

- they come out in state $|+\hat{n}\rangle$
- " " " " " $|-\hat{n}\rangle$

M

with fractions (a.k.a. probabilities)

- $\text{Prob}(|\psi\rangle \Rightarrow |+\hat{n}\rangle) = |\langle +\hat{n}|\psi\rangle|^2$
- $\text{Prob}(|\psi\rangle \Rightarrow |-\hat{n}\rangle) = |\langle -\hat{n}|\psi\rangle|^2$

B

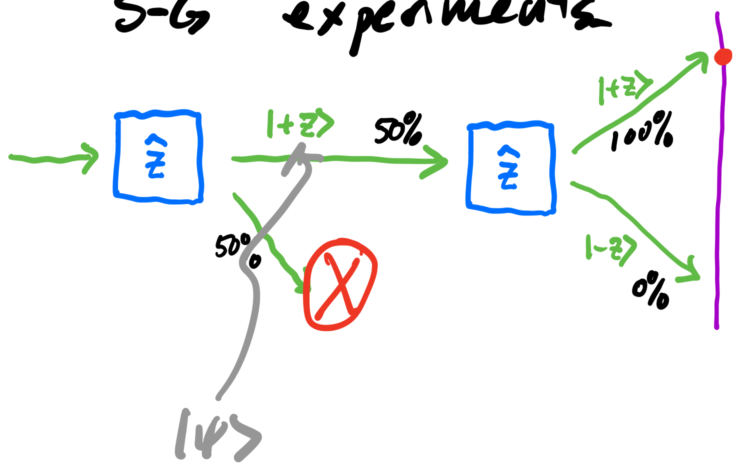
M = measurement rule: measurements have <u>limited set</u> of possible outcomes

B = Born rule: only <u>probabilities</u> of measurement outcomes are described.

Use these rules, together with <u>assumption</u> that

$\boxed{|\psi\rangle \in \text{2-dimensional vector space } V}$ V to model our S-G experiments

Experiment #1)



B $\Rightarrow$

$|\langle +z|\psi\rangle|^2 = |\langle +z|+z\rangle|^2 = 1 \Rightarrow \langle +z|+z\rangle = 1$

$|\langle -z|\psi\rangle|^2 = |\langle -z|+z\rangle|^2 = 0 \Rightarrow \langle -z|+z\rangle = 0$

& similarly w/ $|\psi\rangle = |-z\rangle \Rightarrow \langle -z|-z\rangle = 1$

$\langle +z|-z\rangle = 0$

$\therefore$ $\boxed{\{|+z\rangle, |-z\rangle\} \text{ forms an orthonormal basis of } V.}$

$\Rightarrow$ any state $|\psi\rangle$ can be written

$$|\psi\rangle = c_+|+z\rangle + c_-|-z\rangle$$

$c_\pm \in \mathbb{C}$ are "components of $|\psi\rangle$" = "amplitudes"

$\Rightarrow \langle +z|\psi\rangle = c_+\langle +z|+z\rangle + c_-\langle +z|-z\rangle = c_+\cdot 1 + c_-\cdot 0 = c_+$

$\therefore$ $c_\pm = \langle \pm z|\psi\rangle \Rightarrow \text{Prob}(|\psi\rangle \Rightarrow |\pm z\rangle) = |c_\pm|^2$

$\therefore$ B $\Leftrightarrow$ "probabilities = $|\text{amplitudes}|^2$"

$\Rightarrow$ probabilities ≥ 0 & $\in \mathbb{R}$ ✓

Also want: $\sum$ probabilities $= 1$?

$$1 = \text{Prob}(|\psi\rangle \Rightarrow |+z\rangle) + \text{Prob}(|\psi\rangle \Rightarrow |-z\rangle)$$
$$= |c_+|^2 + |c_-|^2$$

But
$$\langle\psi|\psi\rangle = \left(c_+^* \langle +z| + c_-^* \langle -z|\right)\left(c_+|+z\rangle + c_-|-z\rangle\right)$$
$$= c_+^* c_+ \langle +z|+z\rangle + c_-^* c_+ \langle -z|+z\rangle + c_+^* c_- \langle +z|-z\rangle + c_-^* c_- \langle -z|-z\rangle$$

$$= |c_+|^2 \cdot 1 + c_-^* c_+ \cdot 0 + c_+^* c_- \cdot 0 + |c_-|^2 \cdot 1$$
$$= |c_+|^2 + |c_-|^2.$$

$\therefore$ Physical states are normalized:

$$\boxed{\langle\psi|\psi\rangle = 1} \quad \text{N}$$

Quantum weirdness: if both $c_+ \neq 0$ & $c_- \neq 0$, then particle in state $|\psi\rangle$ is not in state $|+z\rangle$ or $|-z\rangle$ but in a "superposition" of both.

Nevertheless, if measure $|\psi\rangle$ using a Stern-Gerlach magnet in the $\hat{z}$-direction, get only 1 of 2 possible answers $|+z\rangle$ or $|-z\rangle$.

$\therefore$ As a result of measurement, state changes

$$\left\{\boxed{\begin{array}{lll} |\psi\rangle & \Longrightarrow & |\psi'\rangle = |+z\rangle \\ & \text{or} & \\ |\psi\rangle & \Longrightarrow & |\psi'\rangle = |-z\rangle \end{array}}\right.$$

C

"collapse of wv. fn."

§1.4 & §1.5) <u>Analysis of other SG experiments</u>

These rules model other SG experiments consistently if:

$$\begin{cases} |+x\rangle = \frac{1}{\sqrt{2}}\left(|+z\rangle + |-z\rangle\right) \\ |-x\rangle = \frac{1}{\sqrt{2}}\left(|+z\rangle - |-z\rangle\right) \end{cases}$$

and

$$\begin{cases} |+y\rangle = \frac{1}{\sqrt{2}}\left(|+z\rangle + i\,|-z\rangle\right) \\ |-y\rangle = \frac{1}{\sqrt{2}}\left(|+z\rangle - i\,|-z\rangle\right) \end{cases}$$

this "i" is necessary & shows V must be a complex vector space

Check! $\{|\pm x\rangle\}$ & $\{|\pm y\rangle\}$ are also orthonormal bases of V.

§1.6) Generalize our rules to "axioms of QM"

Ⓥ States are vectors $|\psi\rangle \in V$ in a complex Hilbert space V.

Ⓝ States are normalized $\langle\psi|\psi\rangle = 1$.

Ⓜ The outcomes of a measurement "M" is an orthonormal basis of V

$$\{|m_1\rangle, \ldots, |m_d\rangle\} \qquad \langle m_i|m_j\rangle = \delta_{ij}$$

Will call the $|m_i\rangle$ an (orthonormal) "eigenbasis of M"

Ⓒ If an outcome $|m_i\rangle$ is observed upon measuring M, then the state changes to

$$|\psi\rangle \xrightarrow{M=m_i} |\psi'\rangle = |m_i\rangle$$

Ⓑ The probability of observing an outcome $|m_i\rangle$ of a measurement M is

$$\text{Prob}(|\psi\rangle \Rightarrow |m_i\rangle) = |\langle m_i|\psi\rangle|^2$$

- Blue = physical concepts
  Purple = math. concepts

- Axioms essentially define a dictionary between physics and a math. model.

- We will refine these axioms over the next few weeks to get to a (more) complete form.

- The above axioms are missing any mention of time evolution; we have not given yet the quantum analog of the equations of motion ($\vec{F} = m\ddot{\vec{x}}$).

One subtle consequence of these rules is that the overall phase of a state is physically unobservable. I.e., no predictions of quantum mechanics depends on the overall phase.

Predictions are the probabilities computed by the Born rule B:

$$\mathrm{Prob}(|\psi\rangle \Rightarrow |\chi\rangle) = |\langle\chi|\psi\rangle|^2$$

Say we have another state

$$|\psi'\rangle = e^{i\alpha}|\psi\rangle \qquad \alpha \in \mathbb{R}$$

i.e., differing by an overall phase factor.
Then

$$\begin{aligned}
\text{Prob}(|\psi'\rangle \Rightarrow |\chi\rangle) &= |\langle\chi|\psi'\rangle|^2 \\
&= |\langle\chi|e^{i\alpha}|\psi\rangle|^2 \\
&= |e^{i\alpha}\langle\chi|\psi\rangle|^2 \\
&= |e^{i\alpha}|^2\,|\langle\chi|\psi\rangle|^2 \\
&= |\langle\chi|\psi\rangle|^2 = \text{Prob}(|\psi\rangle \Rightarrow |\chi\rangle).
\end{aligned}$$

So there is no way of telling the difference between $|\psi'\rangle$ and $|\psi\rangle$.

(I used: $|e^{i\alpha}|^2 = (e^{i\alpha})(e^{i\alpha})^* = e^{i\alpha}e^{-i\alpha} = e^{i\alpha-i\alpha} = e^0 = 1.$)

<u>**Relative phases matter!**</u>

$i = e^{i\pi/2}$

$e^{i\alpha} = \cos\alpha + i\sin\alpha$

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|+z\rangle + |-z\rangle\right)$$

$$|\psi'\rangle = \frac{1}{\sqrt{2}}\left(|+z\rangle + i|-z\rangle\right)$$

Check: normalized!

$$|\chi\rangle = \frac{1}{\sqrt{2}}\left(|+z\rangle + |-z\rangle\right)$$

$$\text{Prob}(|\psi\rangle \Rightarrow |x\rangle) = |\langle x|\psi\rangle|^2$$

$$= \left| \frac{1}{\sqrt{2}}\left(\langle +z| + \langle -z|\right) \frac{1}{\sqrt{2}}\left(|+z\rangle + |-z\rangle\right)\right|^2$$

$$= \left| \frac{1}{2}\left(\cancelto{1}{\langle +z|+z\rangle} + \cancelto{0}{\langle +z|-z\rangle} + \cancelto{0}{\langle -z|+z\rangle} + \cancelto{1}{\langle -z|-z\rangle} \right)\right|^2$$

$$= 1.$$

$$\text{Prob}(|\psi'\rangle \Rightarrow |x\rangle) = |\langle x|\psi'\rangle|^2$$

$$= \left| \frac{1}{\sqrt{2}}\left(\langle +z| + \langle -z|\right) \frac{1}{\sqrt{2}}\left(|+z\rangle + i\,|-z\rangle\right)\right|^2$$

$$= \left| \frac{1}{2}\left(\cancelto{1}{\langle +z|+z\rangle} + i\,\cancelto{0}{\langle +z|-z\rangle} + \cancelto{0}{\langle -z|+z\rangle} + i\,\cancelto{1}{\langle -z|-z\rangle} \right)\right|^2$$

$$= \left| \frac{1}{2}(1+i)\right|^2 = \frac{1}{2}(1+i)\,\frac{1}{2}(1-i)$$

$$= \frac{1}{4}(1+1) = \frac{1}{2}.$$

Expectation value and uncertainty

- When we do a measurement M on a system, we get a probability distribution of outcomes

$$P(|\psi\rangle \to |m_i\rangle) = |\langle m_i|\psi\rangle|^2$$

$$\doteq P_i$$ ← shorthand notation

This just means that $\{P_1, P_2, \ldots, P_d\}$ are a set of d real numbers satisfying

$$0 \leq P_i \leq 1 \quad \& \quad \sum_{i=1}^{d} P_i = 1$$

- Say "m_i" is the value of "M" for the i-th outcome.

E.g. if $M = S_z$, then on a spin-$1/2$ system we have:

$$m_1 = -\hbar/2 \ , \ m_2 = +\hbar/2 .$$

E.g. if $M = E$ (energy), then on a hydrogen atom we have

"electron volt"

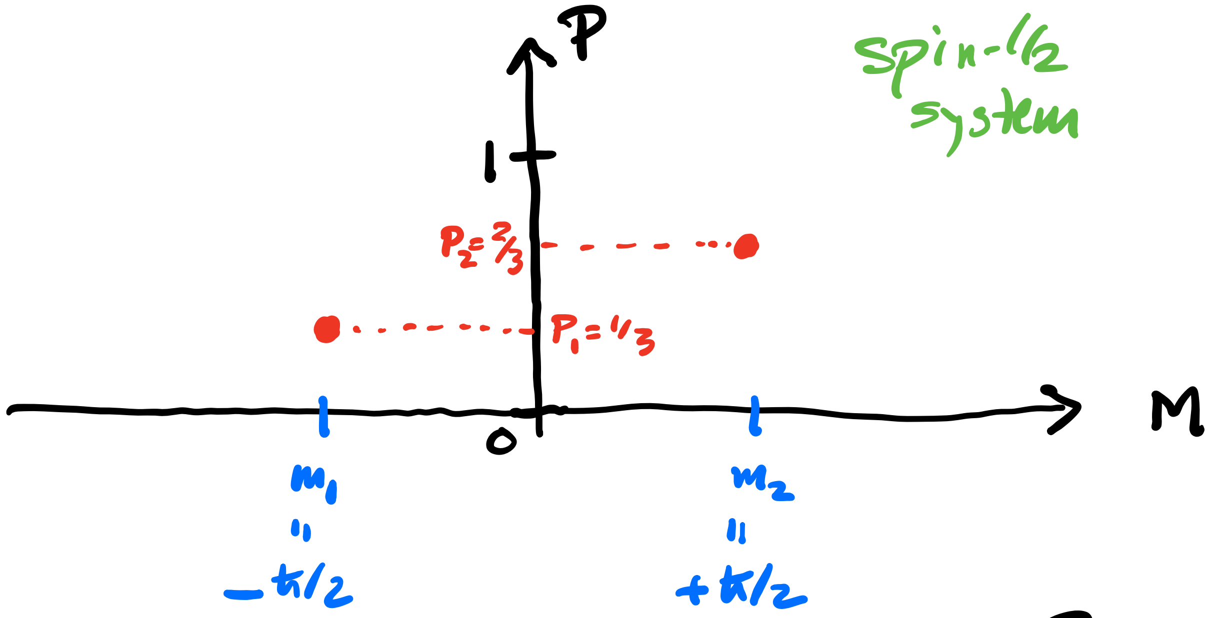
$$m_n = -\frac{13.6\,\text{eV}}{n^2} \qquad n = 1, 2, 3, \ldots \infty$$

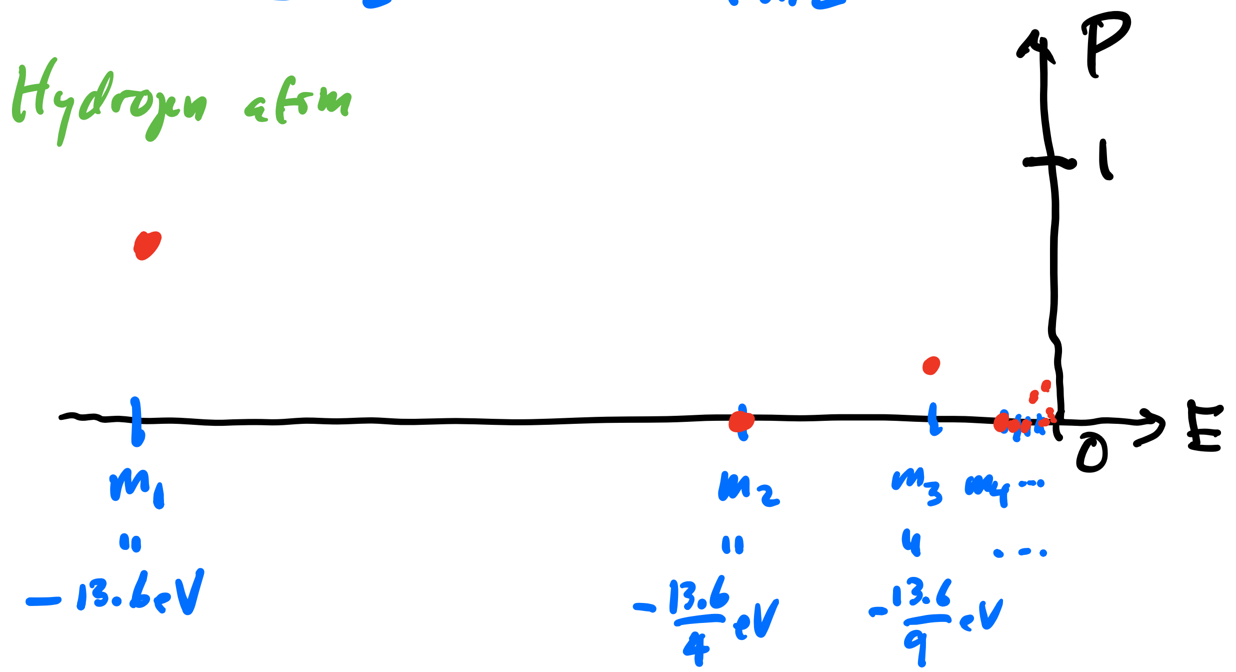
- The numerical value of an outcome of a measurement "m_i" and the probability of observing that outcome "P_i" are completely different things!

  $\{m_i\}$ = property of the <u>system</u>, usually <u>dimensionful</u> quantities

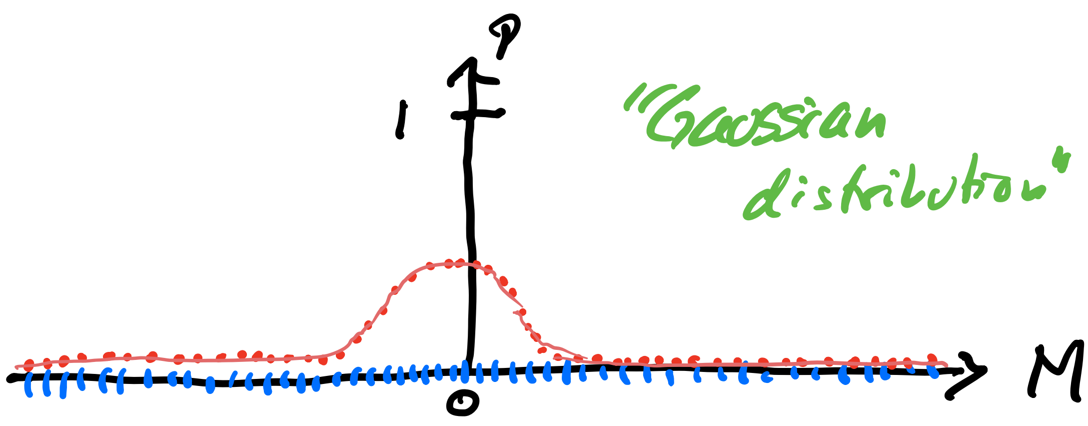
  $\{P_i\}$ = property of the <u>state</u> of the system, always a <u>dimensionless</u> number (between 0 & 1).

- Can visualize a probability distribution by plotting P_i vs. m_i:





- If the m_i are closely spaced, and P_i vary in small increments (i.e. $|P_i - P_{i+1}| \ll 1$) can visualize as a continuous function



- Given a probability distribution two common questions are:

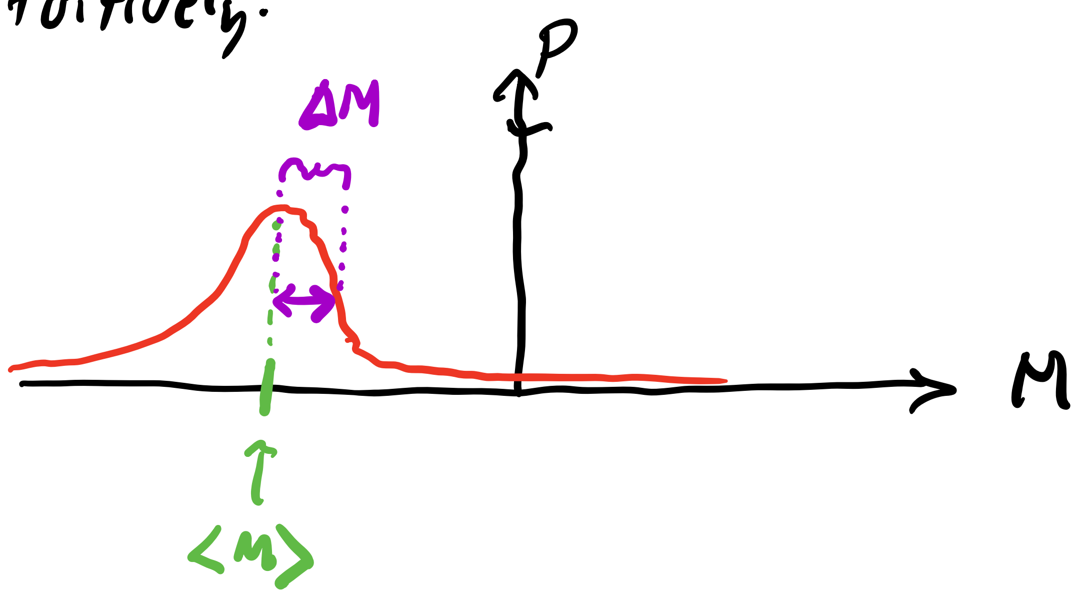
  (1) What is the "average value" of M = "expected value" = "mean value"

  $$\doteq \langle M \rangle \quad \left(= \langle M \rangle_{|\psi\rangle} \right)$$

  (2) What is the "uncertainty" of M = "standard deviation" = "width" = "variance"

  $$\doteq \Delta M$$

Intuitively:



- Math definitions:

$$\boxed{\langle M \rangle \doteq \sum_i m_i P_i} = \sum_i m_i \left| \langle m_i | \psi \rangle \right|^2$$

$$\Rightarrow \langle f(M)\rangle \doteq \sum_i f(m_i)\, P_i \qquad (\text{any function } f)$$

e.g. $\langle M^2\rangle \doteq \sum_i m_i^2 P_i$.

number

$$\langle a\, f(M)\rangle = a\,\langle f(M)\rangle$$

Note: $\langle f(M) + g(M)\rangle = \langle f(M)\rangle + \langle g(M)\rangle$

$$\langle f(M)\cdot g(M)\rangle \neq \langle f(M)\rangle \cdot \langle g(M)\rangle$$

$$\boxed{\Delta M \doteq \sqrt{\langle M^2\rangle - \langle M\rangle^2}}$$

Note: $\Delta M \geq 0$. Proof:

$$0 \leq \langle (M - \langle M\rangle)^2\rangle = \langle M^2 - 2M\langle M\rangle + \langle M\rangle^2\rangle$$

$$= \langle M^2\rangle - \langle 2M\langle M\rangle\rangle + \langle \langle M\rangle^2\rangle$$

$$= \langle M^2\rangle - 2\langle M\rangle\langle M\rangle + \langle M\rangle^2 \cancel{\langle 1\rangle}\ 1$$

$$= \langle M^2\rangle - 2\langle M\rangle^2 + \langle M\rangle^2 = \langle M^2\rangle - \langle M\rangle^2$$

$$= (\Delta M)^2 \quad \lrcorner$$

Exercise: justify every step!