The Motion of the Moon

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1 Remarks about the Accuracy of the Solution

The distance from the Earth to the Moon is on the average 384,400 km. The distance from the Earth to the Sun is about 1.49×10^8 km. The mass of the Earth is 5.98×10^{27} g, the Moon weighs 7.34×10^{25} g, and the Sun 1.993×10^{33} g. It is worthwhile to visualize the solar system in a model and actually to construct and lay out this model in order to appreciate these values. The scale factor we will use is 1:500,000,000, as then the Sun is 27.85 cm in diameter – roughly the size of a basketball. The Earth shrinks to a diameter of 2.5 mm, the size of the sphere found in some ball bearings. The Moon is then 0.7 mm, the size of the sphere found in a medium ballpoint pen. The distance from Earth to Sun is around 30 m, whereas the distance from Earth to Moon is just near 8 cm. The Earth will fit about 1.3 million times into the Sun. When we consider masses, the Sun is 300,000 times heavier than the Earth, whereas the Moon has only 1/80 of the Earth's mass.

When one sees this model it is not hard to be convinced that in a first approximation the three bodies can be treated like point masses and that any deviation caused by the oblateness of the Earth or the uneven mass distribution on the Moon can be considered later on with other perturbations.

The planets will also cause perturbations, which can be treated later on. For example, Venus, roughly the same size as the Earth in the above model, is about 20 m away from the Sun. Jupiter will be 150 m away from the Sun, and due to its mass will still cause some noticeable perturbations.

The magnitudes of the different gravitational attractions follow from Newton's law. Due to the enormous size of the Sun it turns out that the gravitational attraction of the Sun on the Moon is twice as large as that of the Earth on the Moon. More precisely, this value is $(1.993 \times 10^{33}/5.98 \times 10^{27})(3.84 \times 10^5/1.49 \times 10^8)^2 = 2.2$. Although we can assume that the Moon will stay close to Earth for all times, the above estimate shows that it is not a good choice to start out with the Moon on an elliptical orbit around the Earth and to treat the influence of the Sun as a perturbing factor. Delaunay and others before him had tried this approach, and the limited accuracy of these theories is due in part to the choice of an elliptic orbit as the starting point for the calculations.

It was Hill [7] who recognized how to account for the effect of the Sun. In essence he proposed to move the Sun infinitely far away, but he kept its gravitational attraction on the Earth–Moon system constant. It was Brown [1] who carried out Hill's idea and calculated the motion of the Moon to an accuracy that satisfied the needs of his time.

Today the measurement of the distance from Earth to Moon is done by laser ranging. From an observatory on Earth to reflectors left on the Moon by astronauts, this distance is measured to within a few centimeters. Also the position of the Moon in its orbit is found with unprecedented accuracy via occultations of stars. The Moon moves 1 second of arc in the sky for every 2 seconds of time. At a distance of 400,000 km this represents a velocity of 2 km/sec. Occultations can be measured within 10^{-4} sec or better, so that the Moon can be fixed in its orbital path to within 20 cm. An analytic theory for the motion of the Moon should therefore have an accuracy of 11 decimal digits. Since the solution for longitude, latitude, and sine parallax will be found in the form

$$\sum A_{k,l} r_1^{k_1} \cdots r_m^{k_m} {\sin \atop \cos} (l_1 \zeta_1 + \cdots + l_n \zeta_n), \tag{1}$$

it will be necessary to increase the accuracy with which each coefficient is found by several orders of magnitudes in order to offset the loss of accuracy through truncation and roundoff; see [4] and [9].

Assume that the coefficients in (1) are normalized so that the largest is 1, and there are N terms of (1) that have been computed to some accuracy. There is no way to give an estimate of the error that one makes by truncating the infinite series (1) to N terms. The only question we can answer is, How accurate is the sum of these N terms for a given set of values for the radial and angular variables? A uniform error estimate might be too pessimistic, as some combination of angles may cause errors in the individual terms to add up. Instead we will look at the mean square error. The radial variables are all less than 1, and we will assume that they are known to any desired accuracy. It means that it suffices to require that the evaluated term $A_{k,l} r_1^{k_1} \cdots r_m^{k_m}$ has to be $< \epsilon$ in absolute value. The angular variables will be linear functions of time and they too are given to any needed accuracy. If the mean error should be $< 10^{-11}$ then we need $\sqrt{N}\epsilon < 10^{-11}$. If the series has around 100,000 terms it is advisable to compute the evaluated terms to an accuracy of 10^{-14} or better.

Actually, there is a very pragmatic answer to the question of how accurately each coefficient should be found. All computers have floating point arithmetic with just two or three different precisions. An accuracy of slightly better than 14 decimal digits is given by floating point numbers represented in eight bytes. The extended precision would give 30 decimal digits, but the extra space and time required to do all of the calculations in extended precision makes this choice impractical.

For a complete solution to the main problem of lunar theory in floating point arithmetic we refer the reader to [3]. In the following sections we follow the presentation given there, but then restrict ourselves to solving the first-order terms in rational arithmetic with the help of a computer algebra program. We have used MACSYMA, and give the corresponding programs in the appendix. We hope that readers will be able to translate these programs into programs for their favored computer algebra system.

2 The Equations of Motion

In a fixed but otherwise unspecified coordinate system, a three-body problem is given by the Hamiltonian function

$$H = \sum_{j=1}^{3} \frac{|p_j|^2}{2m_j} + G \sum_{1 \le i < j \le 3} \frac{m_i m_j}{|q_i - q_j|},$$

where we will use the convention that index 1 refers to the Earth, 2 to the Moon, and 3 to the Sun. The gravitational constant G cannot be set to 1, as this is a real-life problem where distance, mass, and time are measured in the centimeter-gram-second system where the units are specified a priori. The position coordinates q_j and the momenta p_j are vectors in $\mathcal{R}^3$ so that the problem has nine degrees of freedom. Measuring the position of the Moon from an Earth-based coordinate system is more natural, and therefore we introduce the Jacobi coordinates, with Q_1 the vector from the Earth pointing to the Moon, Q_2 the vector from the center of mass of the Earth–Moon system to the Sun, and Q_3 fixes the center of mass of the whole system. Let P_1, P_2, and P_3 be the corresponding momenta. Since the center of mass Q_3 is an integral for the system, the transformed Hamiltonian will not depend on P_3 and the dependency on Q_3 will be ignored in the transformed Hamiltonian.

An algorithm that generates the transformation to Jacobi coordinates is given in Appendix A. The case here is studied well enough and we will list the symplectic transformation of $\mathcal{R}^{18} \to \mathcal{R}^{18}$ in terms of block matrices for the position and momenta vectors:

$$\begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 \\ \frac{-m_1}{m_1+m_2} & \frac{-m_2}{m_1+m_2} & 1 \\ \frac{m_1}{m_1+m_2+m_3} & \frac{m_2}{m_1+m_2+m_3} & \frac{m_3}{m_1+m_2+m_3} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix},$$

$$\begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix} = \begin{pmatrix} \frac{-m_2}{m_1+m_2} & \frac{m_1}{m_1+m_2} & 0 \\ \frac{-m_3}{m_1+m_2+m_3} & \frac{-m_3}{m_1+m_2+m_3} & \frac{m_1+m_2}{m_1+m_2+m_3} \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}.$$

The transformed Hamiltonian has thus six degrees of freedom and is

$$\begin{aligned} H = {} & \frac{m_1+m_2+m_3}{2m_3(m_1+m_2)}|P_2|^2 + \frac{m_1+m_2}{2m_1m_2}|P_1|^2 \\ & - \frac{Gm_1m_2}{|Q_1|} - \frac{Gm_1m_3}{|Q_2 + \frac{m_2}{m_1+m_2}Q_1|} - \frac{Gm_2m_3}{|Q_2 - \frac{m_1}{m_1+m_2}Q_1|}. \end{aligned} \tag{2}$$

Since $\lambda = |Q_1|/|Q_2|$ is approximately 1/400, one is led to an expansion of the last two terms. This is accomplished with the help of the Legendre polynomials $P_j(\cos\gamma)$, whose generating function gives the following defining relation:

$$(1 - 2\lambda\cos\gamma + \lambda^2)^{-1/2} = \sum \lambda^j P_j(\cos\gamma).$$

Let γ be the angle between the two vectors Q_1 and Q_2, so that the expansion of the last two terms in (2) leads to

$$\begin{aligned} H = & \frac{m_1+m_2+m_3}{2m_3(m_1+m_2)}|P_2|^2 - \frac{Gm_3(m_1+m_2)}{|Q_2|} \\ & + \frac{m_1+m_2}{2m_1m_2}|P_1|^2 - \frac{Gm_1m_2}{|Q_1|} - \frac{Gm_1m_2m_3}{m_1+m_2}\sum_{j=2} c_j \frac{|Q_1|^j}{|Q_2|^{j+1}} P_j(\cos\gamma). \end{aligned} \tag{3}$$

The coefficients c_j are given by

$$c_j = \frac{m_1^{j-1} - (-m_2)^{j-1}}{(m_1+m_2)^{j-1}}, \qquad j = 2, 3, \dots \ ,$$

and thus depend only on the mass ratio $\mu = m_2/m_1$. With $c_2 = 1$, $c_3 = 1 - \mu$, the dependency of the solution on this mass ratio is not very pronounced, and it is for this reason that μ is seldom treated as an independent parameter. Instead, its numerical value of 1/81.30068 is used from the beginning.

The first line in (4) is the Hamiltonian for the two-body problem of the Earth–Moon system and the Sun. The second line describes the motion of the Moon around the Earth under the influence of the Sun. Looking at the corresponding differential equations one sees that they can be decoupled and that higher order dependencies can be dealt with later on among the perturbations.

The Hamiltonian function describing the motion of the Moon is therefore

$$H = \frac{m_1+m_2}{2m_1m_2}|P_1|^2 - \frac{Gm_1m_2}{|Q_1|} - \frac{Gm_1m_2m_3}{m_1+m_2}\sum_{j=2} c_j \frac{|Q_1|^j}{|Q_2|^{j+1}} P_j(\cos\gamma).$$

It has three degrees of freedom but it is time dependent, due to the elliptic motion of the center of mass of the Earth–Moon system around the Sun.

The motion of the Sun around the Earth–Moon system in polar coordinates is given by

$$\begin{aligned} r' = |Q_2| &= a'(1 - e'\cos l' + \cdots), \\ \Phi' &= g' + l' + 2e'\sin l' + \cdots \ , \\ l' &= n'(t - t_0) + l'_0. \end{aligned}$$

It is customary in lunar theory to denote everything with a "$'$" that refers to the Sun, so that we have

a' = semimajor axis,
e' = eccentricity of Sun's orbit,
l' = mean anomaly,
n' = mean motion of Sun around Earth–Moon system,
g' = argument of perihelion,
l'_0 = mean anomaly at a given epoch t_0, that is, Jan. 1, 2000.

Kepler's law gives

$$n'^2 a'^3 = G(m_1 + m_2 + m_3).$$

In selecting the coordinate system that describes the Earth-based vector Q_1, we still have some freedom. The orbital plane of our two-body problem, the so-called ecliptic, is the natural choice. The first two components of Q_1 will be parallel to this plane, and the third one perpendicular to it.

If the Sun's motion could be circular, then it would be clear to choose a uniformly rotating coordinate system for Q_1 so that the x-component of this coordinate system always points to the Sun. In this case the Hamiltonian (4) in rotating coordinates would also be time independent. Since the Sun moves on an elliptic orbit, we do the next best thing. We use a uniformly rotating coordinate system, whose x-axis always points to the mean position of the Sun. With $\phi = g' + l'$ the mean longitude of the Sun, the transformation reads

$$Q_1 = b \begin{pmatrix} x\cos\phi - y\sin\phi \\ x\sin\phi + y\cos\phi \\ z \end{pmatrix},$$

and for the momenta

$$P_1 = b(n-n')\frac{m_1 m_2}{m_1+m_2} \begin{pmatrix} X\cos\phi - Y\sin\phi \\ X\sin\phi + Y\cos\phi \\ Z \end{pmatrix}.$$

The factors have been selected so that the new Hamiltonian will depend only on essential parameters. The scale factor b will be determined shortly. With n the mean motion of the Moon around the Earth we also introduce a new time by

$$\tau = (n-n')(t-t_0)$$

so that the basic motion is 2π–periodic. The above time-dependent transformation remains symplectic, provided that the Hamiltonian is multiplied with $(m_1+m_2)/(m_1 m_2 (n-n')b^2)$. With all of these changes the Hamiltonian in the uniformly rotating coordinates becomes

$$\begin{aligned} H = {} & \frac{1}{2}(X^2+Y^2+Z^2) - m(xY - yX) - \frac{G(m_1+m_2)}{b^3(n-n')^2}\frac{1}{r} \\ & - \frac{Gm_3}{a'^3(n-n')^2}\sum c_j \left(\frac{b}{a'}\right)^{j-2}\left(\frac{a'}{r'}\right)^{j+1} r^j P_j(\cos\gamma). \end{aligned} \qquad (4)$$

We have set $r = |Q_1| = \sqrt{x^2+y^2+z^2}$, and the parameter m is the ratio of the mean motions and defined by

$$m = \frac{n'}{n-n'}.$$

In analogy to Kepler's law a quantity a is defined by

$$n^2 a^3 = G(m_1+m_2).$$

It allows us to write the coefficient of the third term in (4) as

$$\frac{G(m_1+m_2)}{b^3(n-n')^2} = \left(\frac{a}{b}\right)^3 (1+m)^2,$$

whereas the coefficient in front of the sum becomes

$$\frac{Gm_3}{a'^3(n-n')^2} = \frac{n'^2}{(n-n')^2}\frac{m_3}{m_1+m_2+m_3} = m^2\left(1-\frac{m_1+m_2}{m_1+m_2+m_3}\right).$$

At this point our choice of parameters deviates from that of Brown, who sets $b = a$ and works with a parameter $\alpha = a/a'$. It leaves the ratio of masses in the above formula as another parameter for which Brown substitutes the numerical value from the beginning. Since these two parameters are related we find it more natural to select

$$\beta = \left(\frac{m_1+m_2}{m_1+m_2+m_3}\right)^{\frac{1}{3}}$$

as the independent parameter instead of α. We also set

$$b = a(1+m)^{2/3},$$

which makes the coefficient of $1/r$ into 1. This in turn gives

$$\frac{Gm_3}{a'^3(n-n')^2} = m^2(1-\beta^3)$$

for the coefficient in front of the sum in (4), and inside this sum we find

$$\frac{b}{a'} = \beta m^{2/3}.$$

One advantage of this choice is that β and the other parameters will have numerically the same order of magnitude. A disadvantage could be that a completely analytical solution would have to be developed in powers of $m^{1/3}$. Since the solution found by the method of Brown is semianalytical, this is actually of no concern because m is replaced by its numerical value from the beginning. Another insignificant drawback is that with Brown's choice of the parameter α, one is able to determine if a given term belongs to the x-y coordinates or to the z coordinates simply by knowing if the power of α is even or odd. With the appearance of β^3 in (4), this rule, which follows from d'Alembert's characteristic, does not apply to powers of β beyond the second.

The Hamiltonian for the lunar problem in real coordinates is therefore

$$H = \frac{1}{2}(X^2+Y^2+Z^2) + m(yX - xY) - \Omega$$

with

$$\begin{aligned}\Omega &= \Omega(\tau, x, y, z; m, \beta, e', \mu) \\ &= m^2(1-\beta^3)\sum c_j m^{2(j-2)/3}\beta^{j-2}\left(\frac{a'}{r'}\right)^{j+1} P_j(\cos\gamma).\end{aligned}$$

The Legendre polynomials can be constructed iteratively by

$$\begin{aligned} P_0(t) &= 1 \\ P_1(t) &= t \\ P_{j+1}(t) &= \frac{2j+1}{j+1} t P_j(t) - \frac{j}{j+1} P_{j-1}(t), \qquad j = 1, 2, \dots\,, \end{aligned}$$

and their argument follows from the dot product of the vectors Q_1 and Q_2 and is given by

$$\cos\gamma = \frac{xx' + yy'}{rr'}.$$

It was Hill who recognized that not all terms of Ω should be considered as a perturbation. He devised a way to account for a significant portion of the Sun's force when he constructed the intermediate orbit. In essence he extracted from Ω the most significant term and wrote

$$\Omega = m^2 r^2 P_2\left(\frac{x}{r}\right) + \tilde{\Omega}$$

and included the first term in the equation for the intermediate orbit. Physically this can be interpreted by saying that the Sun has been moved infinitely away, but its gravitational attraction on the Moon has been kept constant.

The differential equations are therefore

$$\begin{array}{llllll} \ddot{x} & -2m\dot{y} & -m^2 x & -2m^2 x & +x/r^3 & = \partial\tilde{\Omega}/\partial x, \\ \ddot{y} & +2m\dot{x} & -m^2 y & +m^2 y & +y/r^3 & = \partial\tilde{\Omega}/\partial y, \\ \ddot{z} & & & +m^2 z & +z/r^3 & = \partial\tilde{\Omega}/\partial z. \end{array}$$

They have been written so that each column can be identified with one of the forces acting on the Moon. They are, from left to right: inertial, Coriolis, centrifugal, Sun's gravitation reduced to a quadrupole, Earth's gravitation, and perturbations.

The motion of the Moon has to be viewed in a six-dimensional phase space. Its behavior is more easily understood for a time-independent system, that is, when $e' = 0$ and $r' = a'$, as then the trajectories lie on a five-dimensional manifold given by $H =$ const. Since no additional integrals exist, one has to expect that the solution behaves ergodically on this manifold. This is not the case for the intermediate orbit of Hill, which is a periodic orbit. This intermediate orbit will be surrounded by three-dimensional tori, which are specified by the two radial variables e and k. The solution curve on these tori is parametrized by the three angular variables l, F, and τ of Delaunay.

For small e' these 3-tori are not destroyed but they change their size synchronously with the mean motion of the Sun. The tori change their size slowly when compared to the motion of the Moon. Since there is no loss of energy this change is called adiabatic. These heuristic arguments give the justification for the form of the solution as Poisson series:

$$x = \sum A_j \cos\left(j_1 l + j_2 F + j_3 l' + (j_4 + 1)\tau\right),$$

$$y = \sum B_j \sin(j_1 l + j_2 F + j_3 l' + (j_4+1)\tau),$$
$$z = \sum C_j \sin(j_1 l + j_2 F + j_3 l' + j_4 \tau),$$

where each coefficient is itself a series in the radial variables, that is,

$$A_j = \sum_i A_j^i e^{i_1} k^{i_2} e'^{i_3} \beta^{i_4}.$$

The coefficients A_j^i are functions of m and of μ. The characteristics of d'Alembert give some restriction on which indices can occur:

$$i_1 \geq |j_1|, \qquad i_2 \geq |j_2| \qquad \text{and} \qquad i_3 \geq |j_3|.$$

There is no restriction involving i_4. The angles are

$$\begin{aligned} l &= c\tau + l_0, \\ F &= g\tau + F_0, \\ l' &= m\tau + l_0', \\ \tau &= (n-n')(t-t_0), \end{aligned}$$

where the mean motion of the perigee c and the mean motion of the node g have to be determined along with the rest of the solution in the form of the following two series:

$$c = c_0 + \sum c_i e^{i_1} k^{i_2} e'^{i_3} \beta^{i_4} \tag{5}$$

and

$$g = g_0 + \sum g_i e^{i_1} k^{i_2} e'^{i_3} \beta^{i_4}. \tag{6}$$

3 The Solution Method

Already Hill and Brown used complex variables in order to avoid the use of trigonometric identities when one has to manipulate Poisson series. For this reason they introduced the variable

$$\zeta = \mathrm{e}^{\mathrm{i}\tau}.$$

When we write

$$\begin{aligned} &\zeta^c \text{ it will mean } \mathrm{e}^{\mathrm{i}(c\tau+l_0)}, \\ &\zeta^g \text{ it will mean } \mathrm{e}^{\mathrm{i}(g\tau+g_0)}, \\ &\zeta^m \text{ it will mean } \mathrm{e}^{\mathrm{i}(m\tau+l_0')}. \end{aligned}$$

Therefore $\zeta^{j_1 c + j_2 g + j_3 m + j_4}$ really stands for $\mathrm{e}^{\mathrm{i}(j_1 l + j_2 F + j_3 l' + j_4 \tau)}$. Since we will perform only the basic arithmetic operations, like adding and multiplying of series, we can with the above convention simply add the exponents of ζ when we multiply the individual terms together. Also differentiation can be performed within the above convention, and for this it is beneficial to introduce the differential operator D by

$$D = \zeta \frac{d}{d\zeta} = -\mathrm{i} \frac{d}{d\tau}$$

so that

$$D\zeta^n = n\zeta^n.$$

The solution in complex coordinates will be denoted by u and v. Simultaneously, we take into account that we are working in a rotating coordinate system and set

$$\begin{aligned} \zeta u &= x + \mathrm{i}y, \\ \zeta^{-1} v &= x - \mathrm{i}y. \end{aligned}$$

The transformed differential equations then read

$$\begin{aligned} (D+m+1)^2 u + \tfrac{1}{2} m^2 u + \tfrac{3}{2} m^2 \zeta^{-2} v - u/r^3 &= -2\partial\tilde{\Omega}/\partial v, \\ (D-m-1)^2 v + \tfrac{3}{2} m^2 \zeta^{2} u + \tfrac{1}{2} m^2 v - v/r^3 &= -2\partial\tilde{\Omega}/\partial u, \\ D^2 z - m^2 z - z/r^3 &= -\partial\tilde{\Omega}/\partial z. \end{aligned} \tag{7}$$

Since $v = \overline{u}$, the second equation does not need to be considered any longer. Brown called $\lambda = e^{i_1} k^{i_2} e'^{i_3} \beta^{i_4}$ the characteristic of a term. Its degree is $|\lambda| = i_1 + i_2 + i_3 + i_4$. Let u_0 stand for the terms of degree 0, and u_λ and z_λ be the terms with characteristic λ. Then the solution to the main problem of lunar theory will be found in the form

$$u = u_0 + \sum_{|\lambda| \geq 1} u_\lambda \qquad \text{and} \qquad z = \sum_{|\lambda| \geq 1} z_\lambda.$$

4 The Intermediate Orbit

Hill discovered that the Moon moves on a trajectory that stays close to a periodic orbit of (7) with $\tilde{\Omega} = 0$. This periodic orbit is also called the variational orbit, but this name has nothing to do with the variational principle of mechanics or the method of variation of constants. Instead, "variation" is one of the inequalities that have been observed in the motion of the Moon. Inequality is the traditional name given to the deviation from a true uniform motion, that is, a linear increase with time of the angle for longitude or latitude. Several inequalities have a very strong sinusoidal dependence on time, and this has been observed and measured for a very long time.

The largest inequality is caused by the motion of the Moon towards and away from the Earth. Its period is known as the anomalistic month and depends on the mean anomaly l. The inequality will show up as a term $\sin l$ in longitude. Its coefficient is called the principal term in longitude. Its value, as found from observation and agreed upon by the International Astronomical Union (IAU) for epoch 2000, is $22639.55''$.

The second largest inequality is due to the motion perpendicular to the ecliptic. Its period is known as the draconitic month, which depends on the argument of latitude F. The inequality will show up as the term with $\sin F$ in the latitude. Its coefficient is called the principal term in latitude and its value as found from observations for epoch 2000 is given as $18461.40''$. The two principal terms will be used to determine the values of the parameters e and k, which are used in the solution for Cartesian coordinates. The method consists of transforming the Cartesian coordinates into spherical coordinates and then comparing coefficients.

The third largest inequality was observed by Tycho Brahe and called "the variation" by him. It depends on the elongation τ and shows up as the term $\sin 2\tau$ in the longitude with a coefficient of about $2370''$. Newton already pointed out how this inequality could be explained if one assumes that the Moon starts out on a circular orbit around the Earth. It was Hill's contribution to lunar theory to account for most of this inequality already at terms of order 0 and thus to achieve a more rapid convergence of the series solution in Cartesian coordinates. Because of this connection, Hill also called his intermediate orbit the variational orbit. It is a periodic solution of

$$(D+1+m)^2 u + \frac{1}{2}m^2 u + \frac{3}{2}m^2\zeta^{-2}v - \frac{u}{r^3} = 0. \tag{8}$$

There are different ways for finding solutions to this equation. Most of them keep m as a formal parameter and calculate u as a series in m. The speed of convergence for the series describing the intermediate orbit is acceptable, and it is even possible to give estimates for the radius of convergence of this series.

When the calculations are carried out by machine, it is sometimes easiest to compute the terms of the series directly by comparing coefficients. This is the approach to be used here. One checks easily that for $m = 0$ the solution to (8) is $u = 1$. Under the assumption that all terms with degree $< k$ in m have been computed, we write

$$u = 1 + \sum_{j=1}^{k} m^j u_j,$$
$$v = 1 + \sum_{j=1}^{k} m^j v_j,$$

and find for the terms of order k

$$\begin{aligned}(D+1)^2 u_k - \frac{1}{2}u_k - \frac{3}{2}v_k &= -2(D+1)u_{k-1} - \frac{3}{2}u_{k-2} \\ &\quad -\frac{3}{2}\zeta^{-2}v_{k-2} + (u^{-1/2}v^{-3/2})_k.\end{aligned} \tag{9}$$

Here $(u^{-1/2}v^{-3/2})_k$ stands for the terms at order k in $u^{-1/2}v^{-3/2}$ which depend on known terms, that is, on u_j and v_j with $j < k$.

The right-hand side of (9) is of the form

$$\sum_{l=-k/2}^{k/2} \alpha_{kl}\zeta^{2l},$$

and we therefore find

$$u_k = \sum_{l=-k/2}^{k/2} A_{kl}\zeta^{2l} \qquad \text{and} \qquad v_k = \sum_{l=-k/2}^{k/2} A_{kl}\zeta^{-2l}$$

from the following set of linear equations for $l > 0$:

$$\left((1+2l)^2 + \frac{1}{2}\right) A_{kl} + \frac{3}{2}A_{k,-l} = \alpha_{k,l},$$
$$\frac{3}{2}A_{kl} + \left((1-2l)^2 + \frac{1}{2}\right) A_{k,-l} = \alpha_{k,-l}.$$

For $l = 0$ there is only one equation that gives

$$A_{k0} = \frac{1}{3}\alpha_{k0},$$

and for $l > 0$ we obtain from the above equations

$$A_{kl} = \frac{((2l-1)^2 + 1/2)\alpha_{kl} - 3/2\alpha_{k,-l}}{4l^2(4l^2-1)}, \qquad -k/2 \le l \le k/2.$$

A MACSYMA program implementing these calculations is given in Appendix B.

5 The Terms of First Order in the Inclination

After computing the variational orbit, Hill [6] tried to find periodic planar orbits nearby. To accomplish this he developed the theory of infinite determinants. This tool was then also used by Cowell [2] to compute a three-dimensional orbit. Since this case is simpler than the planar one, we will treat it first. We will repeat Cowell's calculations but do not need to introduce any labor-saving shortcuts since our calculations are carried out by machine.

The parameter for the inclination is k. In terms of the notation for the characteristic $\lambda = k$ we want to compute kz_k. It has to satisfy the following second-order equation:

$$D^2 z_k - m^2 z_k - z_k/(u_0 v_0)^{3/2} = 0,$$

which is the linearized version of the third equation in (7). Let

$$M(\zeta) = \frac{1}{2}(m^2 + (u_0 v_0)^{-3/2}) = \sum_{j=-\infty}^{\infty} M_j \zeta^{2j} \tag{10}$$

be the series that is already available from computing the variational orbit. The coefficients satisfy $M_k = M_{-k}$ and the first few terms are given by

$$\begin{aligned}
M_0 &= \frac{1}{2} + m + \frac{5}{4}m^2 - \frac{9}{64}m^4 + 2m^5 + \frac{17}{3}m^6 + \cdots\ , \\
M_1 &= \frac{3}{4}m^2 + \frac{19}{8}m^3 + \frac{10}{3}m^4 + \frac{43}{18}m^5 + \frac{18709}{27648}m^6 + \cdots\ , \\
M_2 &= \frac{33}{32}m^4 + \frac{2937}{640}m^5 + \frac{23051}{2400}m^6 + \cdots\ , \\
M_3 &= \frac{1393}{1024}m^6 + \cdots .
\end{aligned}$$

The following second order linear system therefore has to be solved:

$$D^2 z_k - 2M(\zeta) z_k = 0. \tag{11}$$

Due to the symmetry in the function $M(\zeta)$, the solution can be found as

$$z_k = \mathrm{i}(\zeta^g w - \zeta^{-g}\overline{w}),$$

where w is a series in ζ with real coefficients that have to be computed along with the term g_0 for the mean motion of the node. We will drop the subscript 0 for this section. Substituting the form of the solution into (11) leads to

$$(D+g)^2 w - 2M(\zeta) w = 0. \tag{12}$$

If w is assumed to be

$$w = \sum_{j=-\infty}^{\infty} \kappa_j \zeta^{2j},$$

then by comparing coefficients in (12) leads to the following infinite system of linear equations

$$\vartheta(g)\kappa = 0, \tag{13}$$

where the infinite symmetric matrix $\vartheta(g)$ is given by

$$\vartheta(g) = \begin{pmatrix}
\ddots & \vdots & \vdots & \vdots & \\
\cdots & (-2+g)^2 - 2M_0 & -2M_1 & -2M_2 & \cdots \\
\cdots & -2M_{-1} & g^2 - 2M_0 & -2M_1 & \cdots \\
\cdots & -2M_{-2} & -2M_{-1} & (2+g)^2 - 2M_0 & \cdots \\
 & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}$$

and the column vector of unknowns is

$$\kappa = (\cdots, \kappa_{-1}, \kappa_0, \kappa_1, \cdots)^T.$$

Since (13) is a homogeneous system of linear equations, a nontrivial solution exists only when g is determined such that $\vartheta(g)$ is singular.

Here we will not follow the standard method by approximating the infinite matrix with a finite one; instead, we will develop the solution directly in terms of powers of m. Due to the special form of the function $M(\zeta)$, we can write

$$\begin{aligned}M(\zeta) &= \sum_{j=-\infty}^{\infty} \sum_{l \geq |j|} M_{jl} m^l \zeta^{2j} \\ &= \sum_{l=0}^{\infty} \sum_{j=-l/2}^{l/2} M_{jl} \zeta^{2j} m^l \\ &= \sum_{l=0}^{\infty} \tilde{M}_l m^l\end{aligned}$$

and assume that the solution has the same form:

$$w = \sum_{l=0}^{\infty} w_l m^l = \sum_{l=0}^{\infty} \sum_{j=-l/2}^{l/2} \kappa_{jl} \zeta^{2j} m^l. \tag{14}$$

Finally, set

$$g = \sum_{l=0}^{\infty} g_l m^l.$$

For terms at order 0 in m, we find from (12) that $g_0 = 1$ and $\kappa_{00} = 1$. For the coefficients of m^l in (12) we obtain

$$\sum_{j=0}^{l} \left(\sum_{i=0}^{j} \tilde{g}_j \tilde{g}_{j-i} - 2\tilde{M}_j \right) w_{l-j} = 0, \tag{15}$$

where $\tilde{g}_i = g_i$ for $i > 0$ but $\tilde{g}_0 = 1 + D$. The unknown terms w_l and g_l appear in (15) as

$$((D+1)^2 - 1) w_l + g_l + \alpha_l = 0, \tag{16}$$

and α_l depends on lower order terms. It is given by

$$\alpha_l = \sum_{j=1}^{l-1} \left(2g_j(1+D) + \sum_{i=1}^{j-1} g_i g_{j-i} - 2\tilde{M}_j \right) w_{l-j} - 2\tilde{M}_l + \sum_{j=1}^{l-1} g_j g_{l-j}$$

and has the form

$$\alpha_l = \sum_{i=-l}^{l} \alpha_{il} \zeta^{2i}.$$

With the form of the solution given in (14), we then get

$$\sum_{i=-l}^{l} (4i(i+1)\kappa_{il} + \alpha_{il}) \zeta^{2i} + 2g_l = 0,$$

so that it follows easily that

$$\kappa_{il} = -\frac{\alpha_{il}}{4i(i+1)} \qquad \text{for } i \neq 0, -1,$$

and if we set $\kappa_{0l} = 0$, then

$$g_l = -\frac{\alpha_{0l}}{2}.$$

The coefficient that is not so easily found is that of ζ^{-2}, since it is in the kernel of (16). This also means that $\kappa_{-1,l-1}$ was not determined at the previous order but can now be used to eliminate the terms $\tilde{\alpha}_{-1,l}$ in $\alpha_{-1,l}$ which do not depend on it. One finds

$$\kappa_{-1,l-1} = \tilde{\alpha}_{-1,l}/(2+2g_1).$$

A MACSYMA program that implements the above formula is given in Appendix C and the results of a run are

$$\begin{aligned} w = 1 &+ \frac{3}{8}\zeta^{-2}m + \left(\frac{3}{16}\zeta^2 - \frac{29}{32}\zeta^{-2}\right)m^2 \\ &+ \left(\frac{1}{2}\zeta^2 - \frac{2029}{1536}\zeta^{-2} - \frac{9}{128}\zeta^{-4}\right)m^3 \\ &+ \left(\frac{25}{256}\zeta^4 + \frac{197}{384}\zeta^2 - \frac{18875}{18432}\zeta^{-2} - \frac{105}{512}\zeta^{-4}\right)m^4 \cdots \end{aligned}$$

and

$$\begin{aligned} g = 1 + m &+ \frac{3}{4}m^2 - \frac{33}{32}m^3 - \frac{105}{128}m^4 + \frac{43}{2048}m^5 + \frac{2567}{24576}m^6 \\ &+ \frac{3\,47699}{5\,89824}m^7 + \frac{64\,42309}{70\,77888}m^8 + \frac{17118\,51619}{6794\,77248}m^9 \\ &+ \frac{30\,03648\,19183}{4\,07686\,34880}m^{10} + \frac{3355\,25486\,05553}{489\,22361\,85600}m^{11} + \cdots. \end{aligned}$$

6 Terms at First Order in *e*

In order to find planar orbits in the vicinity of the variational orbit we set

$$u = u_0 + eu_e$$

so that the linearized system (7) reads

$$(D+1+m)^2 u_e + M(\zeta)u_e + N(\zeta)v_e = 0,$$

where

$$M(\zeta) = \frac{1}{2}(m^2 + (u_0 v_0)^{-3/2})$$

was already given in section 5 and

$$N(\zeta) = \frac{3}{2}(\zeta^{-2}m^2 + u_0^2/(u_0 v_0)^{5/2})$$
$$= \sum N_i \zeta^{2i}$$

through order 6 is given by

$$N_{-3} = \frac{1}{4}m^6 + \cdots ,$$

$$N_{-2} = \frac{123}{512}m^4 + \frac{823}{1280}m^5 + \frac{27899}{76800}m^6 + \cdots ,$$

$$N_{-1} = \frac{27}{16}m^2 - \frac{1}{4}m^3 - \frac{217}{96}m^4 - \frac{77}{18}m^5 - \frac{143911}{55296}m^6 + \cdots ,$$

$$N_0 = \frac{3}{2} + 3m + \frac{9}{4}m^2 - \frac{417}{128}m^4 - \frac{551}{64}m^5 - \frac{4993}{256}m^6 + \cdots ,$$

$$N_1 = \frac{69}{16}m^2 + \frac{29}{2}m^3 + \frac{2137}{96}m^4 + \frac{335}{18}m^5 - \frac{5737}{6912}m^6 + \cdots ,$$

$$N_2 = \frac{4497}{512}m^4 + \frac{53121}{1280}m^5 + \frac{7201393}{76800}m^6 + \cdots ,$$

$$N_3 = \frac{31549}{2048}m^6 + \cdots .$$

The form of the solution is

$$u_e = \zeta^c x + \zeta^{-c} y,$$

where c is the zero-order term in the motion of the perigee, which was denoted by c_0 in (5). The resulting equations for x and y are

$$(c + 1 + m + D)^2 x + M(\zeta)x + N(\zeta)\overline{y} = 0,$$
$$(c - 1 - m - D)^2 y + N(\zeta)\overline{x} + M(\zeta)y = 0.$$

Let

$$x = \sum \xi_j \zeta^{2j} \qquad \text{and} \qquad y = \sum \eta_j \zeta^{2j}$$

so that the coefficients ξ_j and η_j follow from the following infinite set of linear equations:

$$(c + 1 + m + 2j)^2 \xi_j + \sum_k M_{j-k}\xi_k + \sum_k N_{j-k}\overline{\eta}_{-k} = 0, \tag{17}$$
$$(c - 1 - m + 2j)^2 \eta_{-j} + \sum_k N_{k-j}\overline{\xi}_k + \sum_k M_{k-j}\eta_{-k} = 0.$$

With the help of the 2 by 2 block matrices

$$\Delta(c) = \begin{pmatrix} (1+2j+c+m)^2 & 0 \\ 0 & (c-1-m+2j)^2 \end{pmatrix}$$

and

$$L_j = \begin{pmatrix} M_j & N_{-j} \\ N_j & M_j \end{pmatrix},$$

the above system can be given in matrix form as

$$\Theta(c)\varepsilon = 0,$$

where

$$\Theta(c) = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \\ \cdots & \Delta_{-1} + L_0 & L_1 & L_2 & \cdots \\ \cdots & L_{-1} & \Delta_0 + L_0 & L_1 & \cdots \\ \cdots & L_{-2} & L_{-1} & \Delta_1 + L_0 & \cdots \\ & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and the vector of unknowns is

$$\varepsilon = (\cdots, \xi_{-1}, \eta_1, \xi_0, \eta_0, \xi_1, \eta_{-1}, \cdots)^T.$$

Since the above system of equations is homogeneous, it is at once clear that a nontrivial solution only exists when c is determined such that $\Theta(c)$ is singular.

Again we will not follow the standard approach of approximating this infinite determinant by a finite one. Instead we will compute the series for x and y directly order by order in m, and with it the series for c. Hill [5] had computed the series for c already, but since he had to do the calculations by hand his method was more sophisticated than what we need when we use the machine.

The form for the series $M(\zeta)$ was already given in section 5 and $N(\zeta)$ has the same form, that is,

$$N(\zeta) = \sum_{l=0}^{\infty} \sum_{j=-l/2}^{l/2} N_{jl}\zeta^{2j}m^l = \sum_{l=0}^{\infty} \tilde{N}_l m^l.$$

We can assume the same form for x and y except that we allow for a slightly larger range for the different powers of ζ at a given order in m:

$$x = \sum_{l=0}^{\infty} x_l m^l = \sum_{l=0}^{\infty} \sum_{j=-(l+1)/2}^{l/2} A_{jl}\zeta^{2j}m^l,$$

$$y = \sum_{l=0}^{\infty} y_l m^l = \sum_{l=0}^{\infty} \sum_{j=-l/2}^{(l+1)/2} B_{jl}\zeta^{2j}m^l,$$

and

$$c = \sum_{l=0}^{\infty} c_l m^l.$$

The reason for the different limits for the summation will become apparent later in (18), when it will be seen that terms quadratic in m determine the coefficient of ζ^{-2} in x_1 and the coefficient of ζ^2 in y_1.

For terms at order 0 in m the two equations (18) are

$$\left((1+c_0)^2 + \frac{1}{2}\right) x_0 + \frac{3}{2} y_0 = 0,$$

$$\frac{3}{2} x_0 + \frac{1}{2} y_0 = 0.$$

They are linearly dependent when $c_0 = 1$. The standard choice is to impose the following normalizing condition for the constant terms:

$$x_0 - y_0 = 1 \qquad \text{and} \quad A_{0l} - B_{0l} = 0 \qquad \text{for } l = 1, 2, \ldots$$

so that

$$x_0 = \frac{1}{4} \qquad \text{and} \qquad y_0 = -\frac{3}{4}.$$

Next consider the coefficient of m in (18):

$$c_1 - 1 + \frac{9}{2} x_1 + \frac{3}{2} y_1 = 0$$

$$\frac{3}{2} x_1 + \frac{1}{2} y_1 = 0.$$

Also, these terms are independent of time, and with $c_1 = 1$ and the normalizing condition we find

$$x_1 = 0 \qquad \text{and} \qquad y_1 = 0.$$

In order to express the terms of m^l with $l > 1$ it is convenient to introduce the notation

$$c + 1 + m + D = \sum_{l=0}^{\infty} \tilde{c}_l m^l,$$

$$c - 1 - m - D = \sum_{l=0}^{\infty} \hat{c}_l m^l,$$

so that

$$c_i = \tilde{c}_i = \hat{c}_i \qquad \text{for} \quad i > 1$$

but $\tilde{c}_0 = D + 2$, $\hat{c}_0 = -D$, $\tilde{c}_1 = 2$, and $\hat{c}_1 = 0$. The square of these series are

$$(c + 1 + m + D)^2 = \sum \tilde{d}_i m^i \qquad \text{with} \quad \tilde{d}_i = \sum_{j=0}^{i} \tilde{c}_j \tilde{c}_{i-j}$$

and

$$(c-1-m-D)^2 = \sum \hat{d}_i m^i \qquad \text{with} \qquad \hat{d}_i = \sum_{j=0}^{i} \hat{c}_j \hat{c}_{i-j}$$

so that the terms at order l are

$$\sum_{i=0}^{l} (\tilde{d}_i x_{l-i} + \tilde{M}_i x_{l-i} + \tilde{N}_i \overline{y}_{l-i}) = 0,$$
$$\sum_{i=0}^{l} (\hat{d}_i y_{l-i} + \tilde{N}_i \overline{x}_{l-i} + \tilde{M}_i y_{l-i}) = 0.$$

Extracting from it the unknown terms x_l, y_l, and c_l we obtain

$$c_l + \left((2+D)^2 + \frac{1}{2}\right) x_l + \frac{3}{2}\overline{y}_l + \alpha_l = 0,$$

$$\frac{3}{2}\overline{x}_l + \left(D^2 + \frac{1}{2}\right) y_l + \beta_l = 0.$$

Terms that depend on those of lower order are α_l and β_l. They have the form

$$\alpha_l = \sum_{k=-(l+1)/2}^{l/2} \alpha_{kl}\zeta^{2k}, \qquad \beta_l = \sum_{k=-l/2}^{(l+1)/2} \beta_{kl}\zeta^{2k}$$

and are given by

$$\alpha_l = \frac{1}{4}\tilde{M}_l - \frac{3}{4}\tilde{N}_l + \frac{1}{4}\sum_{j=1}^{l-1} \tilde{c}_j \tilde{c}_{l-j} + \sum_{i=1}^{l-1} (\tilde{d}_i x_{l-i} + \tilde{M}_i x_{l-i} + \tilde{N}_i \overline{y}_{l-i}),$$
$$\beta_l = -\frac{3}{4}\tilde{M}_l + \frac{1}{4}\tilde{N}_l - \frac{3}{4}\sum_{j=2}^{l-2} c_j c_{l-j} + \sum_{i=l}^{l-1} (\hat{d}_i y_{l-i} + \tilde{N}_i \overline{x}_{l-i} + \tilde{M}_i y_{l-i}).$$

With

$$x_l = \sum_{k=-(l+1)/2}^{l/2} A_{kl}\zeta^{2k}, \qquad y_l = \sum_{k=-l/2}^{(l+1)/2} B_{kl}\zeta^{2k},$$

we find

$$c_l + \sum_{k=-l/2}^{l/2} \left(\left((2+2k)^2 + \frac{1}{2}\right) A_{kl} + \frac{3}{2} B_{-kl} + \alpha_{kl}\right) \zeta^{2k} = 0,$$
$$\sum_{k=-l/2}^{l/2} \left(\frac{3}{2} A_{kl} + \left(4k^2 + \frac{1}{2}\right) B_{-kl} + \beta_{-kl}\right) \zeta^{-2k} = 0.$$

The general solution for the individual system of equations is

$$A_{kl} = \frac{\frac{3}{2}\beta_{-kl} - (4k^2 + \frac{1}{2})\alpha_{kl}}{4k(k+1)(2k+1)^2},$$

$$B_{-kl} = \frac{\frac{3}{2}\alpha_{kl} - ((2+2k)^2 + \frac{1}{2})\beta_{-kl}}{4k(k+1)(2k+1)^2},$$

which is valid when $k \neq 0, -1$. For the case $k = 0$ we find that the two equations are linearly dependent when

$$c_l = 3\beta_{0l} - \alpha_{0l},$$

so that together with the normalizing condition we have

$$A_{0l} = -\beta_{0l}/2, \qquad B_{0l} = -\beta_{0l}/2.$$

When $k = -1$ the terms are more difficult to determine, as then the corresponding equations are linearly dependent. It means that terms at order $l-1$ have to be used to satisfy the condition at order l. The resulting equations are

$$\frac{1}{2}A_{-1,l} + \frac{3}{2}B_{1l} + A_{-1,l-1} + 3B_{1,l-1} = -\tilde{\alpha}_{-1,l},$$

$$\frac{3}{2}A_{-1,l} + \frac{9}{2}B_{1l} + 3A_{-1,l-1} + B_{1,l-1} = -\tilde{\beta}_{1l}$$

with $\tilde{\alpha}_{-1,l}$ and $\tilde{\beta}_{1l}$ now denoting the remaining known terms. These equations show that these terms at order l are determined except for a solution to the homogeneous equation $A_{-1,l} + 3B_{1l} = 0$. We will set $B_{1l} = 0$ so that we have the following set of equations:

$$\begin{aligned} \frac{1}{2}A_{-1,l} + A_{-1,l-1} + 3B_{1,l-1} &= -\tilde{\alpha}_{-1,l}, \\ \frac{3}{2}A_{-1,l} + 3A_{-1,l-1} + B_{1,l-1} &= -\tilde{\beta}_{1l}, \\ A_{-1,l-1} + 3B_{1,l-1} &= 0, \end{aligned} \tag{18}$$

with the solutions

$$A_{-1,l} = -2\tilde{\alpha}_{-1,l}, \quad A_{-1,l-1} = \frac{9\tilde{\alpha}_{-1,l} - 3\tilde{\beta}_{1l}}{8}, \quad B_{1,l-1} = \frac{\tilde{\beta}_{1l} - 3\tilde{\alpha}_{-1,l}}{8}.$$

The MACSYMA program that implements the above formulas is given in Appendix D, and it produced the following results for the solution in Cartesian coordinates:

$$x = \frac{1}{4} - \frac{45}{32}\zeta^{-2}m + \left(\frac{3}{16}\zeta^2 + \frac{3}{16} - \frac{555}{128}\zeta^{-2}\right)m^2$$

$$+\left(\frac{27}{64}\zeta^2+\frac{651}{1024}-\frac{22441}{2048}\zeta^{-2}-\frac{15}{256}\zeta-4\right)m^3$$

$$+\left(\frac{177}{1024}\zeta^4+\frac{697}{1536}\zeta^2+\frac{2817}{4096}-\frac{63583}{3072}\zeta^{-2}-\frac{193}{512}\zeta^{-4}\right)m^4+\cdots\ ,$$

$$y=-\frac{3}{4}+\frac{15}{32}\zeta^2 m+\left(\frac{277}{128}\zeta^2+\frac{3}{16}-\frac{1}{32}\zeta^{-2}\right)m^2$$

$$+\left(\frac{45}{128}\zeta^4+\frac{36329}{6144}\zeta^2+\frac{651}{1024}+\frac{29}{192}\zeta^{-2}\right)m^3$$

$$+\left(\frac{137}{64}\zeta^4-\frac{87599}{9216}\zeta^2+\frac{2817}{4096}+\frac{35}{4608}\zeta^{-2}+\frac{5}{512}\zeta^{-4}\right)m^4+\cdots\ ,$$

and for the mean motion of the perigee

$$c=1+m-\frac{3}{4}m^2-\frac{201}{32}m^3-\frac{2367}{128}m^4-\frac{1\,11749}{2048}m^5$$

$$-\frac{40\,95991}{24576}m^6-\frac{3325\,32037}{5\,89824}m^7-\frac{1\,51062\,11789}{70\,77888}m^8$$

$$-\frac{597\,53329\,16861}{6794\,77248}m^9-\frac{1\,54777\,54421\,75567}{4\,07686\,34880}m^{10}$$

$$-\frac{818\,42933\,65560\,24967}{489\,22361\,85600}m^{11}$$

$$-\frac{2\,18559\,43284\,86055\,04951}{29353\,41711\,36000}m^{12}+\cdots.$$

Of interest is the growth of the coefficients. One has to go fairly far in this expansion to get the desired accuracy of 14 decimal digits. Performing these calculations in floating point arithmetic shows that for $m=0.08084\,89375\,3667$, one has to go to order 30 in m.

Appendix A: Canonical Transformation to Jacobi Coordinates

Jacobi coordinates for the N-body problem are related to binary trees with N leaves in a very natural way. As shown in [8] the transformation matrix can then be generated in a straightforward manner. The following iterative method describes the construction of this binary tree from the bottom up. The tree will contain all the information needed to write down the transformation to the desired set of Jacobi coordinates.

Let m_j and m_k be the masses of the two bodies that at least conceptually will be replaced by a new virtual body of mass $m_j + m_k$. In terms of coordinate transformations, the position coordinates q_j and q_k are to be replaced by the vector giving their relative position $q_j - q_k$ and by the vector to their center of mass $v = (m_j q_j + m_k q_k)/(m_j + m_k)$. The node in the binary tree corresponding to this virtual body will have m_j as its left child and m_k as its right child. The order of the children indicates that the relative coordinates are from q_j to q_k and not in the opposite direction. Repeat the above step for $N-1$ bodies, that is, the $N-2$ original bodies plus the new virtual body.

Since the leaves of the binary tree corresponding to the N bodies will be numbered from 1 to N, we will number the $N-1$ internal nodes from $N+1$ to $2N-1$ in the order in which they were generated. The root of the binary tree thus has the number $2N-1$. Next construct an N by N matrix H whose N columns correspond to the N bodies and whose rows 1 through $N-1$ correspond to the internal nodes $N+1$ through $2N-1$.
The entries h_{ij} for $1 \le i \le N$, $1 \le j \le N-1$ of this matrix are defined by

$$h_{ij} = \begin{cases} -1, & \text{if } m_j \text{ is in the left subtree of the node } N+i, \\ +1, & \text{if } m_j \text{ is in the right subtree of the node } N+i, \\ 0, & \text{otherwise.} \end{cases}$$

The matrix is completed by setting $h_{Nj} = +1$ for $1 \le j \le N$. It indicates that one of the coordinates is the vector to the center of all masses.

Let $A = (a_{ij})$ be the matrix for the transformation from the old position coordinates q_i to the new Jacobi coordinates Q_i. Each entry a_{ij} is a 3 by 3 diagonal matrix, as the transformation is from $\mathbf{R}^{3N}$ to $\mathbf{R}^{3N}$. Similarly, $B = (b_{ij})$ is the matrix for the transformation of the old momenta to the new ones. In order to simplify the discussion we will only write down the entries of these matrices as scalars and not as diagonal block matrices.

For each node i of the binary tree let M_1 be the sum of the masses in the left subtree and M_2 the sum of all masses in the right subtree, that is,

$$M_1 = \sum_{h_{ij}=-1} m_j, \qquad M_2 = \sum_{h_{ij}=1} m_j.$$

Then the entries for the transformation matrices are

$$a_{ij} = \begin{cases} -m_j/M_1, & \text{if } h_{ij} = -1, \\ m_j/M_2, & \text{if } h_{ij} = +1, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$b_{ij} = \begin{cases} -M_2/(M_1+M_2), & \text{if } h_{ij} = -1, \\ M_1/(M_1+M_2), & \text{if } h_{ij} = +1, \\ 0, & \text{otherwise.} \end{cases}$$

The above formulas hold for all rows $i = 1, \ldots, N$ with M_1 and M_2 recomputed for each row. The formulas hold for the last row since the root of the binary tree, node $2N-1$, should be right child of an additional node $2N$, indicating the fact that the center of mass of the N bodies is one of the coordinates. Actually, the above formulas for the last row simplify and are more easily written down from

$$a_{Nj} = m_j/M_2, \qquad b_{Nj} = 1, \qquad 1 \le j \le N.$$

The fact that the above construction leads to a canonical transformation is proven in [8] so we restrict ourselves here to present a MACSYMA program that implements the above ideas. The program is to be called via `Jacob(N)` with `N` the number of bodies. The program will then request in an interactive manner the construction of the binary tree. In the end the transformation matrices are available in the MACSYMA variables `A` and `B`

```
Jacob(n) := block([b1,l,l1,l2,i,j,s],
   h : zeromatrix(n-1,n),
   a : zeromatrix(n,n),
   b : zeromatrix(n,n),
   s : [],
   for i thru n do s : endcons(i,s),
   print("1 thru ",n," represent the ",n," masses,",n+1, " thru ",
         2*n-1," will be the internal nodes of the binary tree"),
   print("The transformation matrix for the coordinates will be A,"),
   print("for the momenta is B"),
   print("Select a pair of numbers from the displayed list"),
   print("Enter the two number as a list, i.e. [1,2]"),
   for i from 1 thru n-1 do
      (b1 : true,
      while(b1) do
         (l:read("Select two values from ",s),
         l1 : first(l),
         l2 : last(l),
         if member(l1,s) and member(l2,s) and not l1=l2 then
            (b1 : false,
            s : delete(l2,delete(l1,s)),
            s : endcons(i+n,s),
            if l1<= n then h[i,l1]:-1
            else (l1: l1 - n,
               for j thru n do h[i,j]:- abs(h[l1,j])),
                     if l2<= n then h[i,l2]: 1
                     else (l2 : l2 - n,
                        for j thru n do
                           h[i,j]: h[i,j] + abs(h[l2,j]))))),

   for i thru n-1 do
      ( m1:m2:0,
      for j thru n do
         (if h[i,j]<0 then m1:m1 + m[j]
         else if h[i,j]>0 then m2:m2 + m[j]),
         m3 : m1 + m2,
         for j thru n do
```

```
          (if h[i,j]<0 then
             (a[i,j]:-m[j]/m1,
             b[i,j]:-m2/m3)
          else if h[i,j]>0 then
             (a[i,j]: m[j]/m2,
             b[i,j]: m1/m3))),

   for j thru n do (a[n,j] : m[j]/m3, b[n,j]:1) ) ;
```

Appendix B: MACSYMA Program for the Intermediate Orbit

```
Hill(n) := block(local(a),
    array([u,v,r2,r32],n),
    globalsolve : true ,
    ratmx : true,           /* use canonical form */
    u[0]:v[0]:r2[0]:r32[0] : 1 ,
    for k : 1 thru n do
      (
      n2 : entier(k/2),

      r2[k] : sum(u[k-l]*v[l],l,1,k-1),   /* u*v  */
                                          /* (u*v)^(-3/2) */
      r32[k] : -3/2*r2[k]
             + sum((-3/2*(k-l)-l)/k*r2[k-l]*r32[l],l,1,k-1),
      eq : - 2*diff(u[k-1]*zeta,zeta)
           + sum(r32[k-l]*u[l],l,0,k-1),

      if k > 1 then
         eq : eq - 3/2*(u[k-2] + zeta^-2 * v[k-2]) ,

      /* compare coefficients of zeta^l and zeta^-l l=0,-2,+2,... */

      a[k,0]:coeff(eq,zeta,0)/3,
      for l : 1 thru n2  do
        (
        a1 :ratcoef(eq,zeta,2*l),
        a2 :ratcoef(eq,zeta,-2*l),
        a[k,l]:(((2*l-1)^2+1/2)*a1-3/2*a2)/(4*l^2*(4*l^2-1)),
        a[k,-l]:(((2*l+1)^2+1/2)*a2-3/2*a1)/(4*l^2*(4*l^2-1))
        ),

      u[k] : sum(a[k,l]*zeta^(2*l),l,-n2,n2),
      v[k] : sum(a[k,l]*zeta^(-2*l),l,-n2,n2),
      print("u[",k,"]=",expand(u[k])) ,
      r2[k] : rat(r2[k]+u[k]+v[k]),   /*update r2[k] and r32[k] */
      r32[k] : rat(r32[k]-3/2*(u[k]+v[k]))
      ) );
```

Once the above program has been compiled it is called by `Hill(n)` with `n` the order to which the intermediate orbit is desired. In designing a first program one might start out with u_k having undetermined coefficients, then evaluate the corresponding differential equation (8) and let MACSYMA solve the resulting linear equations by calls to its built-in routines. One finds out soon that it is very time consuming to work with undetermined coefficients. It is much faster to calculate the solution as given by the formulas at the end of section 4.

The evaluation of $(u^{-1/2}v^{-3/2})_k$ remains the time-consuming aspect of the entire computation, even when we initially set $u_k = v_k = 0$. The method that we use to find the terms of order k for the series

$$y = \sum y_j m^j \qquad \text{so that} \qquad y = x^\alpha \quad \text{when} \quad x = \sum x_j m^j$$

is based on the trick of differentiating the defining relation with respect to m. By comparing coefficients of order k in

$$x \frac{\mathrm{d}y}{\mathrm{d}m} = \alpha y \frac{\mathrm{d}x}{\mathrm{d}m},$$

one obtains

$$y_k = \frac{\alpha x_k y_0}{x_0} + \sum_{l=1}^{k-1} \frac{(\alpha(k-l) - l)x_{k-l} y_l}{k x_0}.$$

Appendix C: MACSYMA Program for Inclination

The program below is a direct coding of the formulas given in section 5. The program uses the array `MM`, which equals the function $2M(\zeta)$ of the text. Once the program is compiled it is called via `Cowell(m)` with `m` not greater than the argument used in the previous program.

```
cowell(n):=    /* compute first order terms for inclination */
  ( array([g,w,MM],n),
  for i from 0 thru n do mm[i] : r32[i],
        /* factor 1/2 still missing  */
  mm[2] : mm[2] + 1,    /* add m^2   */
  g[0]:w[0]:1,
  for l thru n do
    (
    temp : sum(g[j]*g[l-j],j,1,l-1)
         + sum( 2*g[j]*diff(zeta*w[l-j],zeta)
             +(sum(g[i]*g[j-i],i,1,j-1)-MM[j])*w[l-j],j,1,l-1)
         - MM[l] ,
    temp : rat(temp) ,
    w[l] : 0 ,
    for i from -l thru l  do  if (i=0 ) then
        g[l] : - ratcoef( temp,zeta,0 )/2
      else if (i=-1) then
        w[l-1] : w[l-1]
```

```
              + ratcoef(temp,zeta,-2)/(zeta^2*(2+2*g[1]))
       else
         w[l] : w[l]
              - ratcoef(temp,zeta,2*i)/(4*i*(i+1)) * zeta^(2*i),
   print ("g[",l,"]=",g[l]) ) ) ;
```

Appendix D: MACSYMA Program for First-Order Terms in *e*

The program below is a direct coding of the formulas given in section 6. The program uses the array `M2`, which corresponds to the function $M(\zeta)$, and the array `NN` corresponds to the function $N(\zeta)$. These functions are computed from the series found in `Hill(n)`. To call the MACSYMA program after it is compiled, enter `Adams(m)` with the value of the parameter not greater than what was used for the intermediate orbit.

```
Adams(n) :=  /* compute first order terms in e */
  ( array( [c,M2,NN,R25,RT,x,y],n ) ,
  for i from 0 thru n do M2[i] : R32[i]/2,
  M2[2] : M2[2] + 1/2,
  R52[0] : RT[0] : 1 ,
  NN[0] : 3/2 ,
  for k : 1 thru n do
    (
    R52[k] : -5/2*r2[k]
           + sum((-5/2*(k-l)-l)/k*r2[k-l]*R52[l],l,1,k-1),
    RT[k] : sum(u[l]*R52[k-l],l,0,k) ,
    NN[k] : 3/2*sum(u[l]*RT[k-l],l,0,k)
    ),
  NN[2] : NN[2] + 3/2 *zeta^-2,
  for l : 0 thru n do M2[l] : MM[l]/2,
  c[0] : c[1] : 1 ,
  x[0] : 1/4 ,
  y[0] :-3/4 ,
  x[1] : y[1] : 0 ,
  for l from 2 thru n do
    (
    l2 : entier((l+1)/2),
    c[0] : c[1] : 2 ,
    alpha : ( M2[l] - 3* NN[l] + sum( c[j]*c[l-j],j,1,l-1 ) )/4
          + sum(sum(c[j]*c[i-j],j,0,i)*x[l-i]
               + 2*c[i]*diff(x[l-i],zeta)*zeta
          + M2[i]*x[l-i]+NN[i]*ev(y[l-i],zeta=1/zeta),i,1,l-1) ,
    c[0] : c[1] : 0 ,
    beta : 1/4*(-3*M2[l]+NN[l]  - 3* sum(c[j]*c[l-j],j,1,l-1))
         + sum(sum(c[j]*c[i-j],j,1,i-1)*y[l-i]
              - 2*c[i]*diff(y[l-i],zeta)*zeta
```

```
          + NN[i]*ev(x[l-i],zeta=1/zeta)+M2[i]*y[l-i],i,1,l-1) ,

  x[l] : y[l] : 0 ,
  for k : -l2 thru l2 do
    (
    alphakl : ratcoef( alpha,zeta,2*k ),
    betamkl : ratcoef( beta,zeta,-2*k ) ,
    if (k=0) then
      (
      x[l] : x[l] - betamkl/2,
      y[l] : y[l] - betamkl/2,
      c[l] : 3 * betamkl - alphakl,
      print("c[",l,"]=",c[l])
      )
    else if (k=-1) then
      (
      x[l] : x[l] - 2*alphakl * zeta^(-2),
      x[l-1] : x[l-1] + (3*alphakl-betamkl)*3/8*zeta^(-2),
      y[l-1] : y[l-1] + (betamkl-3*alphakl)/8*zeta^2
      )
      else /* general case */
        (
        x[l]:x[l]+(3/2* betamkl-(4*k*k+1/2)*alphakl)*zeta^(2*k)
                 /(4*k*(k+1)*(2*k+1)^2),
        y[l]:y[l]+(3/2*alphakl-((2+2*k)^2+1/2)*betamkl)*zeta^(-2*k)
                 /(4*k*(k+1)*(2*k+1)^2)
        )
    )
) ) ;
```

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