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UNIQUENESS AND EXACT MULTIPLICITY RESULTS FOR TWO CLASSES OF SEMILINEAR PROBLEMS

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1. INTRODUCTION

Our main results deal with exact multiplicity and uniqueness of solutions for problems on annular domains in R^n. On an annulus $\Omega = \{x \mid A < |x| < B\}$ in R^n, $n \geq 2$, we consider the problem

$$\Delta u + \lambda f(u) = 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{1.1}$$

We study positive radially symmetric solutions of (1.1), depending on a positive parameter λ. This problem arises in many applications, and there is large literature on the subject, including Ni and Nussbaum [1], Lin [2], Nagasaki and Suzuki [3]. Recall that the problem (1.1) may also have positive nonradial solutions, in contrast to the case when domain is a ball in R^n, when all positive solutions are necessarily radially symmetric, in view of the well-known results of Gidas, Ni and Nirenberg [4]. To get exact multiplicity results, we shall restrict our attention to the case of "thin" (or "narrow") annulus, which we define next. Set $c_n = (n-1)^{1/(n-2)}$ for $n \geq 3$, and $c_2 = e$. We shall assume

$$B \leq c_n A. \tag{1.2}$$

The special role of "thin" annulus was recognized first by Ni and Nussbaum [1], who introduced the above condition. The same condition appeared later in Lin [2].

The crucial role in our study will be played by the linearized problem

$$\Delta w + \lambda f'(u)w = 0 \quad \text{in } \Omega, \qquad w = 0 \quad \text{on } \partial\Omega. \tag{1.3}$$

In fact, we believe the following theorem to be our main result, even though its applications include extensions of some well-known results.

THEOREM 1.1. Assume that $f(u) \in C^2(\bar{R}^+)$ satisfies $f(u) > 0$ for $u > 0$, and the condition (1.2) holds. Then any nontrivial radial solution of (1.3) is of one sign.

This theorem implies (among other things) that the crucial condition of the Crandall–Rabinowitz bifurcation theorem (which is recalled below) is satisfied, and so the solution set of (1.1) consists of smooth curves. It is often possible to prove that there is only one solution curve. If moreover, one can show that this solution curve admits no turns, one concludes uniqueness of solution; if the solution curve has exactly one turn, one gets an exact multiplicity result. Notice that in contrast to much of the previous work, our approach does not require any "shooting" techniques.

When the nonlinearity $f(u)$ is convex, we show that all solutions of (1.1) lie on one curve, which turns exactly once at some $\lambda = \lambda_0 > 0$. It follows that the problem has exactly two, one or zero solutions, depending on whether $\lambda < \lambda_0$, $\lambda = \lambda_0$ or $\lambda > \lambda_0$. This improves the corresponding result in Lin [2], since we do not require $f(u)$ to be increasing. (Also, the fact that all solutions lie on a unique solution curve is new. This is important for numerical computations, as one can obtain all solutions by efficient continuation algorithms.) For nonlinearities satisfying $uf'(u) > f(u)$ we prove uniqueness of solution, providing an alternative proof of the corresponding result in Ni and Nussbaum [1]. Moreover, our approach provides some additional information. Namely, we show that all positive solutions must lie on a unique curve, bifurcating from infinity at $\lambda = 0$. It is shown in [1] that the above uniqueness result fails without the assumption (1.2). It follows that condition (1.2) cannot be removed in our Theorem 1.1. Moreover, since this condition appears in two different approaches, it seems natural to wonder if it is sharp.

In another direction we use a similar approach to study a class of symmetric boundary-value problems in one dimension, with nonlinearities similar to those studied by Kwong and Zhang [5]. We prove a uniqueness result for a class of "nonpositone" problems. Again we show that solutions lie on smooth curves, which admit no turns. We then continue the curve of solutions for decreasing λ, and show that near $\lambda = 0$ there is only one solution curve, thus proving uniquess. The main technical difficulty here is to show that that solution branch cannot lose its positivity. This section represents continuation of our earlier work with Li and Ouyang [6-9] on symmetric problems in one space dimension.

Next we recall the bifurcation theorem of Crandall and Rabinowitz [10].

THEOREM 1.2 [10]. Let X and Y be Banach spaces. Let $(\bar{\lambda}, \bar{x}) \in \mathbf{R} \times X$ and let F be a continuously differentiable mapping of an open neighborhood of $(\bar{\lambda}, \bar{x})$ into Y. Let the null-space $N(F_x(\bar{\lambda}, \bar{x})) = \operatorname{span}\{x_0\}$ be one-dimensional and $\operatorname{codim} R(F_x(\bar{\lambda}, \bar{x})) = 1$. Let $F_\lambda(\bar{\lambda}, \bar{x}) \notin R(F_x(\bar{\lambda}, \bar{x}))$. If Z is a complement of $\operatorname{span}\{x_0\}$ in X, then the solutions of $F(\lambda, x) = F(\bar{\lambda}, \bar{x})$ near $(\bar{\lambda}, \bar{x})$ form a curve $(\lambda(s), x(s)) = (\bar{\lambda} + \tau(s), \bar{x} + sx_0 + z(s))$, where $s \to (\tau(s), z(s)) \in \mathbf{R} \times Z$ is a continuously differentiable function near $s = 0$ and $\tau(0) = \tau'(0) = 0$, $z(0) = z'(0) = 0$.

A word on notation. We shall denote derivatives of $u(r)$ by either $u'(r)$ or u_r, and mix both notations to make our proofs more transparent. Throughout the paper we consider only the classical solutions.

2. RADIAL SOLUTIONS FOR A CLASS OF DIRICHLET PROBLEMS ON AN ANNULUS

On an annulus $\Omega = \{x \mid A < |x| < B\}$ in R^n, $n \geq 2$, we consider the problem

$$\Delta U + \lambda f(U) = 0 \quad \text{in } \Omega, \qquad U = 0 \quad \text{on } \partial\Omega. \tag{2.1}$$

We study positive radially symmetric solutions of (2.1), depending on a positive parameter λ. Writing $U = U(r)$ with $r = |x|$, we are then led to consider the problem

$$U'' + \frac{n-1}{r} U' + \lambda f(U) = 0, \quad \text{for } A < r < B, \qquad U(A) = U(B) = 0. \tag{2.2}$$

We make a standard change of variables. In case $n \geq 3$ we let $s = r^{2-n}$ and $u(s) = U(r)$, transforming (2.2) into the problem

$$u'' + \lambda\alpha(s)f(u) = 0, \quad \text{for } a < s < b, \qquad u(a) = u(b) = 0, \tag{2.3}$$

where $\alpha(s) = (n-2)^{-2}s^{-2k}$ with $k = 1 + 1/(n-2)$, $a = B^{2-n}$ and $b = A^{2-n}$. In case $n = 2$ we set $s = -\log r$, and $u(s) = U(r)$, obtaining again the problem (2.3), this time with $\alpha(s) = \mathrm{e}^{-2s}$ and $a = -\log B$, $b = -\log A$. Finally, in case $n \geq 3$, without loss of generality we may take $b = a + 1$, and consider the positive solutions of

$$u'' + \lambda\alpha(s)f(u) = 0 \quad \text{for } a < s < a+1, \qquad u(a) = u(a+1) = 0. \tag{2.4}$$

Indeed, letting $s = (b-a)\xi$, we change (2.3) into the same family of equations on the interval $(a/(b-a), b/(b-a))$ of length one. Notice that considering (2.3) with $b - a$ small is equivalent to equation (2.4) with $a > 0$ large. (In case $n \geq 3$ we always have $a > 0$.) In particular, condition $a > \alpha$ for (2.4) is equivalent to requiring that $b/a < (\alpha+1)/\alpha$ for (2.3) ($\alpha > 0$ is a constant).

We shall also need the corresponding linearized equation

$$w'' + \lambda\alpha(s)f'(u)w = 0, \quad \text{for } a < s < a+1, \qquad w(a) = w(a+1) = 0. \tag{2.5}$$

The following theorem provides a key to obtaining various exact multiplicity results.

THEOREM 2.1. Assume that $f(u) \in C^2[0, \infty)$ satisfies $f(u) > 0$ for almost all $u > 0$. If $n = 2$ then any nontrivial solution of (2.5) can be chosen to be positive. If $n \geq 3$ then any nontrivial solution of (2.5) can be chosen to be positive, provided $a \geq 1/(n-2)$.

Proof. Case I. Dimensions $n \geq 3$.

Since $u(s)$ is a concave function, it has a unique point of maximum, say at $s_0 \in (a, a+1)$. Assume that $w(s)$ vanishes somewhere on $(a, a+1)$. We consider two possibilities.

Case 1. $w(s)$ has a root on (a, s_0).

Assume that $w(\gamma) = 0$ for some $\gamma \in (a, s_0)$ which is the smallest root of $w(s)$. We may assume that $w > 0$ on (a, γ). Differentiate (2.4)

$$u_s'' + \lambda\alpha(s)f'(u)u_s + \lambda\alpha'(s)f(u) = 0. \tag{2.6}$$

Similarly to Korman and Ouyang [7], with the function $g(s) > 0$ on $(a, a+1)$, which will be specified later, we now multiply equation (2.6) by $g(s)w$ and substract from it equation (2.5) multiplied by $g(s)u_s$. Then we integrate over (a, γ) to obtain

$$-gu_s w'|_a^\gamma - \int_a^\gamma g'wu_s' \,\mathrm{d}s + \int_a^\gamma g'w'u_s \,\mathrm{d}s + \lambda \int_a^\gamma \alpha' gfw \,\mathrm{d}s = 0. \tag{2.7}$$

The nonintegral terms on the left in (2.7) are nonnegative. Integrating by parts in the second integral on the left, and using equation (2.4), we combine all the integral terms in (2.7) as

$$-\int_a^\gamma g''u_s w \,\mathrm{d}s + \lambda \int_a^\gamma (2g'\alpha + \alpha' g)fw \,\mathrm{d}s.$$

Hence, we shall obtain a contradiction in (2.7), provided we can find a function $g(s) > 0$ on (a, b), such that

$$g''(s) < 0 \tag{2.8}$$

$$2g'\alpha + \alpha' g > 0, \tag{2.9}$$

for all $s \in (a, a + 1)$. We rewrite (2.9) as

$$g' s^{-2k} - k s^{-2k-1} g > 0. \tag{2.10}$$

We search for an appropriate solution $g(s)$ of (2.10) by solving the equation

$$g' s^{-2k} - k s^{-2k-1} g = s^{-2k} h(s), \tag{2.11}$$

with $h(s) > 0$ to be specified. A solution of (2.11) is

$$g(s) = s^k \int_a^s \tau^{-k} h(\tau)\, d\tau. \tag{2.12}$$

Compute

$$g'' = k(k-1)s^{k-2} \int_a^s \tau^{-k} h(\tau)\, d\tau + \frac{k}{s} h + h'. \tag{2.13}$$

Set $z(s) = \int_a^s \tau^{-k} h(\tau)\, d\tau$, $z' = s^{-k} h$, $h = s^k z'$ and $h' = ks^{k-1} z' + s^k z''$. We then rewrite (2.13)

$$g'' = s^{k-2}[s^2 z'' + 2ksz' + k(k-1)z]. \tag{2.14}$$

To obtain $g'' < 0$, we choose z satisying

$$s^2 z'' + 2ksz' + k(k-1)z = -1, \; z(a) = 0, \; z > 0, \; z' > 0. \tag{2.15}$$

That we also need to require that $z > 0$ and $z' > 0$ can be seen from the formulas relating z to h. The equation in (2.15) is the nonhomogeneous Euler's equation. Its solution satisfying the initial conditions $z(a) = 0$, and $z'(a) = \beta > 0$ is

$$z(s) = -\frac{1}{k(k-1)} + c_1 s^{-k} + c_2 s^{-k+1}, \tag{2.16}$$

where, denoting $\bar{\alpha} = 1/k(k-1)$,

$$c_1 = a^{2k}[\bar{\alpha}(-k+1)a^{-k} - \beta a^{-k+1}], \tag{2.17}$$

$$c_2 = a^{2k}[\bar{\alpha} k a^{-k-1} + \beta a^{-k}]. \tag{2.18}$$

Near $s = a$ the function $z(s)$ is a positive and increasing solution of (2.15). Everything in (2.15) is satisfied until $s = \bar{s}$, such that $z'(\bar{s}) = 0$. Compute

$$\bar{s} = \frac{c_1 k}{c_2(-k+1)} = -\frac{c_1}{c_2}(n-1). \tag{2.19}$$

We now let $\beta \to \infty$. From (2.17) and (2.18) we see that $c_1/c_2 \to -a$. From (2.19) it follows that $\bar{s} \to a(n-1)$. Hence, we obtain a contradiction in (2.7) provided that a satisfies

$$a(n-1) \geq a + 1,$$

which holds by our assumptions.

Case 2. $w(s)$ has a root γ on $(s_0, a+1)$. We may assume that $w > 0$ on $(\gamma, a+1)$. We proceed as in the derivation of (2.7), only now we integrate over $(\gamma, a+1)$,

$$-gu_s w'|_\gamma^{a+1} - \int_\gamma^{a+1} g' w u_s' \, ds + \int_\gamma^{a+1} g' w' u_s \, ds + \int_\gamma^{a+1} \alpha' g f w \, ds = 0. \tag{2.20}$$

The nonintegral terms in (2.20) are now negative. Setting $g(s) \equiv 1$, we obtain a contradiction in (2.20), completing the proof of the theorem for $n \geq 3$.

Case II. Dimension $n = 2$. We proceed as before. In case $\gamma \in (a, s_0)$ we obtain a contradiction if we find a function $g(s)$ satisfying (2.8) and (2.9), and we require also that $g(a) = 1$. Condition (2.9) now takes the form

$$g' - g > 0.$$

We search for $g(s)$ as solution of

$$g'(s) - g(s) = h(s) \quad \text{for } s > a \qquad g(a) = 1, \tag{2.21}$$

with the function $h(s) > 0$ to be determined. Solution of (2.21) is

$$g(s) = e^{s-a} + e^s \int_a^s e^{-t} h(t) \, dt > 0.$$

We need to select $h(s) > 0$ so that $g'' < 0$. Compute

$$g'' = e^{s-a} + e^s \int_a^s e^{-t} h(t) \, dt + h + h'. \tag{2.22}$$

Set $z(s) = \int_a^s e^{-t} h(t) \, dt$, $z' = e^{-s} h(s)$, $h(s) = e^s z'$ and $h' = e^s z' + e^s z''$. We then rewrite (2.22)

$$g'' = e^{s-a} + e^s (z'' + 2z' + z). \tag{2.23}$$

We obtain $g'' < 0$, if we choose z satisfying

$$z'' + 2z' + z = -2\, e^{-a}, \qquad z(a) = 0, \qquad z > 0,\ z' > 0. \tag{2.24}$$

That we also need to require that $z > 0$ and $z' > 0$, can be seen from the formulas relating z to h. The solution of the equation in (2.24) satisfying the initial conditions $z(a) = 0$, $z'(a) = \beta > 0$ is

$$z = -2\, e^{-a} + 2\, e^{-a} e^{-(s-a)} + (\beta + 2\, e^{-a})(s-a)\, e^{-(s-a)}. \tag{2.25}$$

Everything in (2.24) is satisfied until $\bar{s}$, such that $z'(\bar{s}) = 0$. Compute

$$\bar{s} = a + \frac{\beta}{\beta + 2\, e^{-a}}.$$

Hence by choosing β sufficiently large, we obtain $\bar{s} - a > \gamma$, so that $g'' < 0$ holds on (a, γ), and then we proceed to get the same contradiction as before.

In the case $\gamma \in (s_0, a+1)$, we again integrate over $(\gamma, a+1)$ and select $g \equiv 1$, obtaining a contradiction.

Remark 1. As mentioned above, in case $n \geq 3$ our condition $a \geq 1/(n-2)$ implies that $b/a \leq n-1$ in (2.3). Returning to the original variables, this implies $B/A \leq (n-1)^{1/(n-2)}$. In case $n = 2$, we notice that the shift $s \to s - p$ changes (2.4) into the same family of equations for any constant p. It follows that $w > 0$ on any interval of length one, i.e. for $b \leq a + 1$. In the original variables this implies $B \leq eA$.

Remark 2. Notice that the functions $g(s)$, constructed in the proof, satisfy $g'(s) > 0$. Examining the proof, one verifies that if α_0, $\mu > 0$ are any constants, any nontrivial solution of

$$w'' + \lambda(\mu\alpha(s) + \alpha_0)f'(u)w = 0, \quad \text{for } a < s < a + 1, \qquad w(a) = w(a+1) = 0$$

is of one sign.

Theorem 2.1 can be used to give a number of uniqueness and multiplicity results. For the multiplicity results we assume that $f(u)$ is positive:

$$f \in C^2(\bar{R}_+) \qquad \text{and} \qquad f(u) > 0 \qquad \text{for } u \geq 0, \tag{2.26}$$

and it is superlinear, i.e.

$$\lim_{u \to \infty} \frac{f(u)}{u} = \infty. \tag{2.27}$$

The following lemma is known, see e.g. Lin [2]. For completeness we present a different proof.

LEMMA 2.1. Under the conditions (2.26) and (2.27) solutions of (2.4) are uniformly bounded for $\lambda > 0$, i.e. for each $\lambda > 0$ there is a constant $c = c(\lambda)$, such that

$$|u|_{C^2[a,a+1]} \leq c.$$

Proof. We follow the argument of [10]. Define for $c > 0$

$$I(c) = \{s \in (a, a+1) : u(s) > c\} \equiv (a(c), b(c)).$$

Assume that for every $c > 0$ the set $I(c)$ is nonempty for λ close to some $\lambda_0 > 0$ (otherwise, there is nothing to prove). Rewrite equation (2.4) in the form $u'' + \lambda\rho_c(s)u = 0$, where $\rho_c(s) = \alpha(s)(f(u)/u)$. By (2.27) $\rho_c(s) \geq \phi(c)$ on I_c, where $\phi(c) \to \infty$ as $c \to \infty$. Since $u(s)$ has no zeros on I_c, it follows by Sturm's comparison theorem that (for λ close to λ_0)

$$b(c) - a(c) < \frac{\pi}{\sqrt{\lambda\phi(c)}} < \frac{\pi}{\sqrt{(\lambda_0/2)\phi(c)}}.$$

Chose c_0 so large that $b(c) - a(c) < 1/4$ for all $c \geq c_0$. It follows that as $\lambda \to \lambda_0$ the function $u(s)$ is bounded by c_0 on a set of measure $> 3/4$. Since $u(s)$ is a concave function, it follows that it cannot become unbounded, a contradiction. It follows that $u(s)$ is bounded in $C_0^2(a, a+1)$ as $\lambda \to \lambda_0$.

THEOREM 2.2. Assume that $f \in C^2[0, \infty)$ satisfies (2.26) and (2.27), $B \le c_n A$. Then all positive solutions of (2.1) lie on a unique solution curve in (λ, u) "plane". This curve starts at $\lambda = 0$, $u = 0$, it projects on a bounded interval $(0, \lambda_0)$ along the λ axis, with some $\lambda_0 > 0$. Near each turning point we have two branches, one strictly increasing in λ, and another strictly decreasing. Solutions of (2.1) corresponding to any fixed λ are finite in number, and strictly ordered. As $\lambda \downarrow 0$ the open end of the solution curve approaches infinity (i.e. $\max_s u(s, \lambda) \to \infty$).

Proof. When $\lambda = 0$ there is a trivial solution $u = 0$. It follows by the implicit function theorem that for $\lambda > 0$ small there is a smooth curve of solutions passing through (0, 0). By the maximum principle all solutions of (2.1) are positive. This curve cannot be continued indefinitely for increasing λ, since it is well known that under conditions (2.26) and (2.27), the problem (2.1) has no solutions for λ sufficiently large, see, e.g. Amann [11]. Let λ_0 denote the supremum of λ's for which the solution curve can be continued to the right. By Lemma 2.1 it follows that as $\lambda \to \lambda_0$, the solution $u(s, \lambda)$ of (2.4) remains uniformly bounded (recall that the problems (2.1) and (2.4) are equivalent). Passing to the limit in the integral version of (2.4), we see that at $\lambda = \lambda_0$ problem (2.4) has a solution $u(s, \lambda_0) \in C_0^2(a, a + 1)$.

As mentioned above problem (2.1) is equivalent to (2.4). Rewrite (2.4) as

$$F(\lambda, u) \equiv u'' + \lambda\alpha(s) f(u) = 0,$$

where $F: R_+ \times C_0^2(a, a + 1) \to C(a, a + 1)$. Notice that $F_u(\lambda, u)w$ is given by the left-hand side of (2.5). By the definition of λ_0, $F_u(\lambda_0, u_0)$ has to be singular, i.e. (2.5) has a nontrivial solution $w(s)$, which is positive by Theorem 2.1. Clearly the null-space $N(F_u(\lambda_0, u_0)) = \operatorname{span}\{w(s)\}$ is one-dimensional (it can be parameterized by $w'(a)$). It follows that $\operatorname{codim} R(F_u(\lambda_0, u_0)) = 1$, since $F_u(\lambda_0, u_0)$ is a Fredholm operator of index zero. To apply the Crandall–Rabinowitz Theorem 1.1 it remains to check that $F_\lambda(\lambda_0, u_0) \notin R(F_u(\lambda_0, u_0))$. Assuming otherwise would imply existence of $v(s) \not\equiv 0$, such that

$$\begin{gathered} v'' + \lambda_0 \alpha(s) f'(u_0) v = \alpha(s) f(u_0), \qquad a < s < a + 1, \\ v(a) = v(a + 1) = 0. \end{gathered} \tag{2.28}$$

Multiplying (2.28) by w, (2.5) by v, subtracting and integrating over $(a, a + 1)$, we obtain

$$\int_a^{a+1} \alpha(s) f(u_0) w \, ds = 0,$$

which is impossible, since both $f(u_0)$ and w are positive.

Applying the Crandall–Rabinowitz Theorem 1.1, we conclude that (λ_0, u_0) is a turning point, near which the solutions of (2.1) form a curve $(\lambda_0 + \tau(\xi), u_0 + \xi w + z(\xi))$ with ξ near $\xi = 0$, and $\tau(0) = \tau'(0) = 0$, $z(0) = z'(0) = 0$. It follows that for λ close to λ_0 and $\lambda < \lambda_0$ there are two solutions with

$$0 < u^-(s, \lambda) < u^+(s, \lambda) \qquad \text{for all } s \in (a, a + 1),$$

and that $u^-(s, \lambda)$ is strictly increasing in λ, while $u^+(s, \lambda)$ is strictly decreasing. By the strong maximum principle the above inequality is true for all λ.

We now continue the solution curve for decreasing λ until a possible next turn, and so on. By Lemma 2.1 solutions cannot become unbounded unless $\lambda \to 0$. If there were infinitely many turns at some $\bar{\lambda}$, then we would have infinitely many singular solutions at $\bar{\lambda}$, converging to a limit solution, which is singular, but the solution set near it is not a simple curve, contradicting the Crandall-Rabinowitz Theorem, see [8] for more details. Finally, as $\lambda \downarrow 0$ the solution must become unbounded, as can be seen from equation (2.4).

Finally, we rule out the possibility of another curve of solutions. Since by the implicit function theorem we have uniqueness near $\lambda = 0$, $u = 0$, we cannot have another curve, similar to the one just described. By the Crandall-Rabinowitz Theorem, near each turning point there are two branches, one which is increasing in λ, and one decreasing in λ (close to the turning point). We show next that the increasing branch, call it $u(s, \lambda)$ continues to increase for all λ (until the next turning point). Indeed, assuming otherwise, let λ_1 be such that $u_\lambda(s, \lambda_1) \geq 0$ for all $s \in (a, a+1)$, but $u_\lambda(s_1, \lambda_1) = 0$ for some $s_1 \in (a, a+1)$. Differentiating (2.4) in λ,

$$u_\lambda'' + \lambda a(s) f'(u) u_\lambda = -\alpha(s) f(u) < 0, \qquad u_\lambda(a) = u_\lambda(a+1) = 0.$$

By the strong maximum principle we conclude that $u_\lambda(s, \lambda_1) > 0$ for all $s \in (a, a+1)$, a contradiction. This argument implies that if a curve "turns to the right" then the upper branch is increasing in λ until the next turning point (or for all λ if there are no more turning points). Similarly, at each "turn to the left" the lower branch will be increasing.

Next we rule out a singleton, i.e. a closed bounded curve of solutions. Assuming existence of such a curve, denote by λ_0 the smallest λ on the curve, and by u_0 any solution at λ_0. Clearly (λ_0, u_0) is a singular point (since the solution curve cannot be continued to the left), and in fact it is a turning point, as was explained previously. At (λ_0, u_0) we have an increasing upper branch, call it $u(s, \lambda)$. Since the solution curve is bounded, $u(s, \lambda)$ will reach another turning point, call it (λ_1, u_1), at which a turn to the left will occur. At (λ_1, u_1) $u(s, \lambda)$ must be the lower branch, since otherwise we will have two increasing branches there, which is impossible by Theorem 1.2. Denote by $v(s, \lambda)$ the upper branch at (λ_1, u_1). By Theorems 1.2 and 2.1 we see that for λ near λ_1

$$u(s, \lambda) < v(s, \lambda) \qquad \text{for all } s \in (a, a+1),$$

and by the strong maximum principle the same inequality is true for all λ until the next turning point (λ_2, u_2). Hence $u_2 > u_0$. Also $v(s, \lambda)$ is the lower branch at (λ_2, u_2), since otherwise we would again get two increasing branches at (λ_1, u_1). We can now repeat the argument, obtaining $u_4 > u_2$, and so on. This leads to a contradiction, since the solution curve can never come back to (λ_0, u_0).

Since any unbounded curve must approach infinity as $\lambda \downarrow 0$, only one other possibility remains: a curve whose both ends approach infinity at $\lambda = 0$. Denote by λ_0 the largest λ on this curve. If there is more than one solution at λ_0 let u_0 be the largest one (arguing as in the previous paragraph, we see that all solutions at λ_0 are strictly ordered). There are two branches at (λ_0, u_0). We refer to the upper one as $u(s, \lambda)$ as we follow it for decreasing λ through all the possible turns, and similarly we refer to the lower branch as $v(s, \lambda)$. Choose any $\lambda_1 < \lambda_0$. Let $\bar{v}$ be the largest of all solutions on the branch $v(s, \lambda)$ at λ_1. Clearly this is the last time this branch visits $\lambda = \lambda_1$ as we trace it from (λ_0, u_0). Also notice that $\bar{v} < u(s, \lambda_1)$, where $u(s, \lambda_1)$ is any solution at the upper branch at λ_1. Next we consider a family of functions corresponding to all solutions on the branch $v(s, \lambda)$ after $\bar{v}$.

All these functions are subsolutions at $\lambda = \lambda_1$. Hence, $u(s, \lambda_1)$ must be greater than all of these, which is impossible since $v(s, \lambda) \to \infty$. (This consequence of strong maximum principle is sometimes referred to as "Serrin's sweeping principle".)

THEOREM 2.3. In addition to the conditions of Theorem 2.2, assume that

$$f''(u) > 0 \text{ for almost all } u > 0. \tag{2.29}$$

Then there is a critical $\lambda_0 > 0$, such that problem (2.1) has exactly two positive solutions for $0 < \lambda < \lambda_0$, it has exactly one positive solution at $\lambda = \lambda_0$, and no solutions for $\lambda > \lambda_0$. Moreover, all positive solutions lie on a single smooth solution curve, which for $\lambda \in (0, \lambda_0)$ has two branches $u^-(r, \lambda)$ and $u^+(r, \lambda)$, with $u^-(r, \lambda) < u^-(r, \lambda)$ for all $r = |x| \in (A, B)$. The lower branch $u^-(r, \lambda)$ is strictly monotone increasing in λ, and $\lim_{\lambda \to 0^+} u(r, \lambda) = 0$ for all $r \in (A, B)$. For the upper branch $\lim_{\lambda \to 0^+} \max_r u(r, \lambda) = \infty$.

Proof. The proof is similar to that of the previous theorem, only this time we can compute the direction of the bifurcation. We recall that the solution curve started at $\lambda = 0$, $u = 0$, and we continued it for increasing λ until (λ_0, u_0), the first turning point, near which the solutions of (2.1) form a curve $(\lambda_0 + \tau(\xi), u_0 + \xi w + z(\xi))$ with ξ near $\xi = 0$, and $\tau(0) = \tau'(0) = 0$, $z(0) = z'(0) = 0$. We claim that

$$\tau''(0) > 0, \tag{2.30}$$

which implies that only "turns to the left" in the (λ, u) "plane" are possible. We use the formula

$$\tau''(0) = -\lambda_0 \frac{\int_a^{a+1} \alpha f''(u_0) w^3 \, ds}{\int_a^{a+1} \alpha f(u_0) w \, ds}. \tag{2.31}$$

To derive (2.31), we differentiate (2.4) twice in ξ,

$$u''_{\xi\xi} + \lambda \alpha f'(u) u_{\xi\xi} + 2\lambda' \alpha f'(u) u_\xi + \lambda \alpha f''(u) u_\xi^2 + \lambda'' \alpha f(u) = 0. \tag{2.32}$$

Setting here $\xi = 0$, and using that $\tau'(0) = 0$ and $u_\xi|_{\xi=0} = w(x)$, we obtain

$$u''_{\xi\xi} + \lambda_0 \alpha f'(u_0) u_{\xi\xi} + \lambda_0 \alpha f''(u_0) w^2 + \lambda''(0) \alpha f(u_0) = 0. \tag{2.33}$$

Multiplying (2.33) by w, and equation (2.5) by $u_{\xi\xi}$, subtracting and integrating, we obtain (2.31), and hence (2.30). Formula (2.30) implies that the solution curve has exactly one turn, giving us two branches of solutions. Similar to the previous theorem, we prove that the upper branch tends to infinity as $\lambda \downarrow 0$. Monotinicity of the lower branch is easily proved as in [7] or [8], completing the proof. ■

We now turn to proving uniqueness of solutions under the additional condition:

$$uf'(u) > f(u) \text{ for almost all } u \geq 0. \tag{2.34}$$

We begin with the following lemma, which appears to be of independent interest.

LEMMA 2.2. In addition to the conditions of Theorem 2.1 assume that (2.34) holds. Then any positive solution of problem (2.1) is nondegenerate, i.e. the corresponding linearized problem admits only the trivial solution.

Proof. As above we work with the equivalent equations (2.4) and (2.5). By Theorem 2.1 if a nontrivial solution w of (2.5) exists, it is positive. Rewriting equation (2.4) in the form

$$u'' + \lambda\alpha(s)\frac{f(u)}{u}u = 0,$$

and comparing it with (2.5), we obtain a contradiction, in view of Sturm comparison theorem. ■

LEMMA 2.3. Assume that the function $f \in C^1(\bar{R}_+)$ satisfies (2.34). Then the problem

$$u'' + f(u) = 0, \quad \text{for } a < s < a + 1, \qquad u(a) = u(a + 1) = 0 \tag{2.35}$$

has at most one positive solution.

Proof. It is well-known that any positive solution of (2.35) is symmetric about its point of maximum $x = a + \frac{1}{2}$, and any two positive solutions are strictly ordered. Writing $f(u) = ug(u)$, we see using (2.34) that the function $g(u)$ satisfies $g'(u) \geq 0$ for $u \geq 0$. If $v \geq u$ is another positive solution of (2.35), then multiplying (2.35) by v, and the corresponding equation for v by u, integrating and subtracting, we obtain

$$\int_a^{a+1} uv(g(u) - g(v))\,ds = 0,$$

which implies a contradiction, unless $u = v$. ■

THEOREM 2.4. Assume that $B \leq c_n A$ and conditions (2.26), (2.27) and (2.34) are satisfied. Then problem (2.1) has at most one positive solution. Moreover, all positive solutions lie on a unique solution curve, bifurcating from infinity at $\lambda = 0$.

Proof. Letting $\alpha_0 > 0$ be any constant, we imbed (2.4) into a family of problems $(0 \leq \mu \leq 1)$

$$\begin{gathered} u'' + \lambda[\mu(\alpha(s) - \alpha_0) + \alpha_0]f(u) = 0, \qquad \text{for } a < s < a + 1, \\ u(a) = u(a + 1) = 0. \end{gathered} \tag{2.36}$$

When $\mu = 0$ problem (2.36) has at most one positive solution, as follows by Lemma 2.3. The linearized problem for (2.36) is given by

$$\begin{gathered} w'' + \lambda\mu\left[\alpha(s) + \frac{1 - \mu}{\mu}\alpha_0\right]f'(u)w = 0, \qquad \text{for } a < s < a + 1, \\ w(a) = w(a + 1) = 0. \end{gathered} \tag{2.37}$$

By Remark 2 any solution of (2.37) is of one sign, and then as in Lemma 2.2 it follows that all solution of (2.36) are nondegenerate, and hence they can be continued using the implicit function theorem. When $\mu = 1$ we have the original problem (which is equivalent to (2.1)). If problem (2.36) had two positive solutions for some fixed λ, we could continue both of them for decreasing μ on smooth solution curves, without any turns, obtaining two positive solutions at $\mu = 0$, giving a contradiction. Two things need to be checked: that solutions on these curves stay bounded and nontrivial. By Lemma 2.1 solution curves cannot escape to infinity.

If at some $\bar{\mu} \in [0, 1]$ the solution becomes zero, then since by (2.34) $f(0) = 0$, we would have at $\mu = \bar{\mu}$ bifurcation of the positive solution from the trivial one, which can only happen if problem (2.37) at $u = 0$ and $\mu = \bar{\mu}$ has a positive solution. Define μ_1 to be the principal eigenvalue of the problem

$$\begin{gathered} w'' + \lambda\alpha(s)f'(u)w = \mu_1 w, \qquad \text{for } a < s < a+1, \\ w(a) = w(a+1) = 0. \end{gathered} \tag{2.38}$$

Assume first $\mu_1 < 0$. Then we choose a constant α_0 in (2.36), such that $\alpha_0 < \min_{[a,a+1]} \alpha(s)$. Then $\bar{\mu}(\alpha(s) - \alpha_0) + \alpha_0 < \alpha(s)$ for all $s \in [a, a+1]$, and $\bar{\mu} \in [0, 1)$. It follows that problem (2.37) at $u = 0$ and $\mu = \bar{\mu}$ cannot have a positive solution, since the principal eigenvalue for the operator on the left would have to be negative, rather than zero. In case $\mu_1 \geq 0$ we select α_0 such that $\alpha_0 > \max_{[a,a+1]} \alpha(s)$, and obtain a similar contradiction.

The last assertion of the theorem is easily proved, using arguments similar to those of Theorem 2.2.

3. UNIQUENESS FOR A CLASS OF BOUNDARY VALUE PROBLEM

In this section we consider a class of symmetric boundary value problems of the type

$$u'' + \lambda f(x, u) = 0 \quad \text{for } -1 < x < 1, \qquad u(-1) = u(1) = 0. \tag{3.1}$$

We assume that the nonlinearity $f(x, u) \in C^2([-1, 1] \times \bar{R}_+)$ is even in x (i.e. $f(-x, u) = f(x, u)$ for all $x \in (-1, 1)$ and $u > 0$) and it satisfies

$$xf_x(x, u) \leq 0 \quad \text{for all } x \neq 0 \text{ and } u > 0; \tag{3.2}$$

$$\begin{gathered} uf_u(x, u) > \mu f(x, u) \\ \text{for some } \mu \geq 1, \text{ and almost all } x \in (-1, 1),\ u > 0; \end{gathered} \tag{3.3}$$

$$f(x, 0) \leq 0 \quad \text{for all } x \in (-1, 1); \qquad f(\pm 1, 0) < 0. \tag{3.4}$$

Finally we assume existence of a constant $p > 1$ and a continuous and strictly positive on $[-1, 1]$ function $h(x)$, such that

$$\lim_{u \to \infty} \frac{f(x, u)}{u^p} = h(x) \qquad \text{uniformly in } x \in [-1, 1]. \tag{3.5}$$

Similarly to Kwong and Zhang [5], we notice that our conditions include nonlinearities of the type $f(x, u) = a(x)u^p - \sum_{i=0}^{q} b_i(x)u^i$, with $0 \leq q < p$, and even functions $a(x) > 0$ and $b_i(x) > 0$ satisfying $xa'(x) \leq 0$, $xb_i'(x) \geq 0$ for all $x \neq 0$, and $b_0(\pm 1) > 0$.

In view of the well-known results of Gidas *et al.* [4], it follows that any positive solution of (3.1) is an even function with $u'(x) < 0$ for all $x \in (0, 1)$. Moreover, it follows from Korman and Ouyang [7] that any two positive solutions of (3.1) do not intersect.

We shall need the linearization of (3.1)

$$w'' + \lambda f_u(x, u)w = 0 \quad \text{on } (-1, 1), \qquad w(-1) = w(1) = 0. \tag{3.6}$$

In Korman and Ouyang [8] the following lemma was proved.

LEMMA 3.1. Let $u(x)$ be a positive solution of (3.1), and assume that condition (3.2) is satisfied. If problem (3.6) admits a nontrivial solution, this solution does not change sign, i.e. we can assume that $w(x) > 0$ on $(-1, 1)$. Moreover, $w(x)$ is an even function.

The following lemma will be central for our uniqueness result. The test function $v(x)$, which appears in the proof, was used previously in [5].

LEMMA 3.2. Let $u(x)$ be a positive solution of (3.1), and assume that conditions (3.2) and (3.3) are satisfied. Then problem (3.6) admits no nontrivial solutions.

Proof. The function $v(x) = xu'(x) + \beta u(x)$, with a constant β satisfies

$$v'' + \lambda f_u v = \lambda\beta\left(f_u u - \frac{\beta + 2}{\beta} f\right) - \lambda x f_x. \tag{3.7}$$

If we now choose β so that $1 + (2/\beta) = \mu$, then by our conditions v will satisfy

$$v'' + \lambda f_u v > 0, \qquad \text{on } (-1, 1). \tag{3.8}$$

Recall that by Lemma 3.1 we may assume that

$$w(x) < 0 \qquad \text{on } (-1, 1). \tag{3.9}$$

We consider first the case when

$$u'(1) < 0. \tag{3.10}$$

Since $v(0) > 0$, while by (3.10) $v(1) < 0$, it follows that $v(x)$ changes sign on $(-1, 1)$. Let $\xi \in (0, 1)$ be the smallest root of $v(x)$. We now multiply equation (3.7) by $w(x)$, and subtract from this equation (3.6) multiplied by $v(x)$. Integrating over $(0, \xi)$, we obtain

$$w(\xi)v'(\xi) < 0. \tag{3.11}$$

Since the left-hand side of (3.11) is nonnegative by (3.9), we obtain a contradiction.

We now consider the second possibility that

$$u'(1) = 0. \tag{3.12}$$

If one can still find a $\xi \in (0, 1)$ where $v(\xi) = 0$, then we obtain a contradiction as above. Otherwise, $v(x) > 0$ on $(0, 1)$ and, in view of (3.12), $v(1) = 0$. We now multiply equation (3.7) by $w(x)$, and subtract from it equation (3.6) multiplied by $v(x)$. Integrating over $(0, 1)$, we obtain a contradiction, similar to the above. ■

Examining the proof of this lemma, we verify the following lemma.

LEMMA 3.3. Let $u(x)$ be a positive solution of (3.1), and assume that conditions (3.2) and (3.3) are satisfied. Then the problem

$$w'' + \lambda f_u(x, u)w \geq 0 \quad \text{on } (-1, 1), \qquad w(-1) = w(1) = 0$$

admits no negative solution (i.e. $w(x) < 0$ on $(-1, 1)$).

When $f(x, 0) < 0$ it is possible for a solution branch of (3.1) to lose its positivity. The next lemma shows that this can occur only for increasing λ.

LEMMA 3.4. Let $u(x, \lambda) > 0$ be a positive solution of (3.1), and assume that conditions (3.2)–(3.4) are satisfied. Then $u(x, \lambda)$ can lose its positivity only for increasing λ (i.e. the solution stays positive, when continued for decreasing λ).

Proof. Assume on the contrary that $u(x, \lambda)$ loses its positivity for decreasing λ. Since any positive solution $u(x, \lambda)$ of (3.1) is a decreasing function of x, the only way it may lose its positivity is that there is a λ_1, such that $u'(1, \lambda_1) = 0$, and then for $\lambda < \lambda_1$ the solution $u(x, \lambda)$ becomes negative near $x = 1$.

We claim that $u_\lambda(x, \lambda_1)$ is positive near $x = 1$. By the definition of λ_1 it is clear that $u_\lambda(x, \lambda_1)$ cannot be negative on an interval containing $x = 1$. If $u_\lambda(x, \lambda_1)$ failed to be positive in some interval containing $x = 1$, we could find a sequence $x_n \to 1$, such that $u_\lambda(x_n, \lambda_1) = 0$. Let $\mu_n \to 1$ be points of nonnegative local maximums of $u_\lambda(x, \lambda_1)$, i.e. $u_\lambda(\mu_n, \lambda_1) \geq 0$ and $u_\lambda'(\mu_n, \lambda_1) = 0$. Differentiate (3.1) in λ, denoting $u_\lambda = u_\lambda(x, \lambda_1)$,

$$u_\lambda'' + \lambda f_u u_\lambda = -f(x, u) \quad \text{for } -1 < x < 1, \qquad u_\lambda(-1) = u_\lambda(1) = 0. \tag{3.13}$$

We now evalute (3.13) at $x = \mu_n$. The first term on the left is negative, the second one is tending to zero, and the right-hand side tends to $-f(1, 0) > 0$, a contradiction.

We claim next that

$$u_\lambda(x, \lambda_1) > 0 \qquad \text{for all } x \in [0, 1). \tag{3.14}$$

Assuming otherwise, let $\xi \in [0, 1)$ be the largest root of $u_\lambda(x, \lambda_1)$. Multiplying equation (3.1) by u' and integrating over $(\xi, 1)$, we conclude that at $\lambda = \lambda_1$

$$\int_\xi^1 f(x, u)u' \,\mathrm{d}x = \frac{1}{2\lambda} u'^2(\xi) > 0. \tag{3.15}$$

Differentiate equation (3.1) in x,

$$u_x'' + \lambda f_u u_x + \lambda f_x(x, u) = 0 \qquad \text{for } -1 < x < 1. \tag{3.16}$$

We now multiply equation (3.13) by u' and subtract from it equation (3.16) multiplied by u_λ. Then integrate over $(0, 1)$. In view of (3.15), we obtain

$$(u'u_\lambda' - u_\lambda u'')|_\xi^1 - \lambda \int_\xi^1 f_x u_\lambda \,\mathrm{d}x + \int_\xi^1 fu' \,\mathrm{d}x = 0. \tag{3.17}$$

The integral terms in (3.17) are positive. The other terms are equal to $-u'(\xi)u_\lambda'(\xi) > 0$, a contradiction.

The function $w(x) \equiv xu_x(x, \lambda_1) - 2\lambda_1 u_\lambda(x, \lambda_1)$ satisfies

$$\begin{aligned} w'' + \lambda_1 f_u w &= -\lambda_1 x f_x(x, u) > 0 \qquad \text{for } -1 < x < 1, \\ w(-1) &= w(1) = 0. \end{aligned} \tag{3.18}$$

We know by the above that $w(x)$ is negative near $x = 1$, and $w(0) < 0$. By Lemma 3.3 $w(x)$ cannot be negative everywhere on $(-1,1)$. Since $w(x)$ is an even function, it has to vanish on $(0, 1)$. Hence we can find $0 < \xi < \eta < 1$, such that $w(\xi) = w(\eta) = 0$ and $w > 0$ on (ξ, η). (The possibility of $w < 0$ around one of its roots is ruled out from (3.18).) We now multiply equation (3.18) by $u' < 0$, and subtract from this equation (3.16) multiplied by w, and then integrate over (ξ, η), obtaining

$$u'(\eta)w'(\eta) - u'(\xi)w'(\xi) < 0.$$

Since both terms on the left are positive, we obtain a contradiction.

We now state our uniqueness result.

THEOREM 3.1. Assume that for problem (3.1) conditions (3.2)–(3.5) are satisfied. Then for any $\lambda > 0$ problem (3.1) has at most one positive solution.

Proof. Let $u(x, \lambda)$ be a solution of (3.1) at a certain λ. We now continue this solution for decreasing λ. Since by Lemma 3.1 (3.6) has only the trivial solution, this can be done using the implicit function theorem. By the well known estimates of Gidas and Spruck [12] the solution branch remains bounded for any $\lambda > 0$. (I.e. $u(0, \lambda)$ remains bounded.) We show next that as $\lambda \to 0$, the solution $u(x, \lambda)$ cannot approach zero. In view of condition (3.4) we can find a constant $c_1 \geq 0$, such that $f(x, u) \leq c_1 u$ for small $u > 0$ and all $x \in (-1, 1)$. Multiplying (3.1) by u, integrating by parts and using the Poincare's inequality, we obtain

$$\lambda c_1 \int_{-1}^{1} u^2\, dx \geq \lambda \int_{-1}^{1} f(x, u)u\, dx = \int_{-1}^{1} u'^2\, dx \geq \frac{\pi^2}{4} \int_{-1}^{1} u^2\, dx,$$

which implies a contradiction for λ small. It follows that as $\lambda \to 0$, solution $u(x, \lambda) \to \infty$. Since any other solution branch can have no turns, and has the same behavior near $\lambda = 0$, we conclude uniqueness of solutions, if we show that there is only one solution branch going to infinity as $\lambda \to 0$. I.e. we show next that the curve "bifurcating from infinity" is unique. (This is more or less known, but we give details for completeness.)

We rescale equation (3.1), setting $u = (1/\sqrt[p-1]{\lambda})z$. Setting $f(x, u) = h(x)u^p + g(x, u)$, we obtain

$$z'' + h(x)z^p + G(x, z, \lambda) = 0 \quad \text{on } (-1, 1), \qquad z(-1) = z(1) = 0, \tag{3.19}$$

where $G(x, z, \lambda) = \lambda^{p/(p-1)} g(x, (1/\sqrt[p-1]{\lambda})z)$ for $\lambda > 0$, and in view of (3.5) we may define by continuity $G(x, z, 0) = 0$.

Equation (3.19), when λ is small, is a perturbation of

$$z'' + h(x)z^p = 0 \quad \text{on } (-1, 1), \qquad z(-1) = z(1) = 0. \tag{3.20}$$

The problem (3.20) has two solutions: $z \equiv 0$, and a unique positive solution. To see existence and uniqueness of the positive solution, we imbed (3.20) into a family of problems

$$z'' + (\nu h(x) + (1 - \nu)h_0)z^p = 0 \quad \text{on } (-1, 1), \qquad z(-1) = z(1) = 0, \tag{3.21}$$

where $0 \leq \nu \leq 1$ is a parameter, and h_0 is any positive constant. If the linearized equation for (3.21)

$$\begin{gathered} w'' + p(\nu h(x) + (1 - \nu)h_0)w^{p-1} = 0 \quad \text{on } (-1, 1), \\ w(-1) = w(1) = 0, \end{gathered} \tag{3.22}$$

has a nontrivial solution, it has to be of one sign by Lemma 3.1. Using the Sturm comparison theorem, we easily see from (3.21) and (3.22) that this is impossible, i.e. (3.21) admits only the trivial solution. By the implicit function theorem equation (3.21) has a unique positive solution, for any $0 \leq \nu \leq 1$, since for $\nu = 0$ this follows by phase plane analysis.

The trivial solution of equation (3.20) does not continue for $\lambda > 0$. Indeed, assuming that for small $\lambda > 0$ equation (3.19) has a small and positive solution z, we multiply (3.19) by z and integrate over $(-1, 1)$, easily obtaining a contradiction. (The first term after integration by parts, $I \equiv -\int_{-1}^{1} z'^2 \, dx$, dominates all other terms as $(z, \lambda) \to (0, 0)$, except the zero order term in G, which is negative, and so it "pulls" the same way as I.) The positive solution of (3.20) continues by the implicit function theorem to a unique positive solution for small $\lambda > 0$. It follows that the curve of solutions of (3.1), which bifurcates from infinity at $\lambda = 0$ is unique, concluding the proof of the theorem. ∎

Remark. Our conditions allow for a positive solution of (3.1) to exist. For example the problem

$$u'' + \lambda(h(x)u^p - 1) = 0 \quad \text{for } -1 < x < 1, \qquad u(-1) = u(1) = 0, \tag{3.23}$$

with positive $h(x)$ satisfying (3.2), and $p > 1$, will have a (unique) positive solution for $\lambda > 0$ small. Indeed, rescaling $u = \lambda^{-1/(p-1)}z$, we obtain a small perturbation of equation (3.20), and so existence follows by the implicit function theorem. For increasing λ the solution of (3.23) loses its positivity.

Note that one of the crucial things in the proof of the above result was to show that positivity of a solution branch can be lost only for increasing λ. To prove this fact in Lemma 3.4 we used our conditions (3.1) and (3.2). It is interesting that if $f = f(u)$, the corresponding result holds in much greater generality.

Proposition. Consider the problem

$$u'' + \lambda f(u) = 0 \quad \text{for } -1 < x < 1, \qquad u(-1) = u(1) = 0. \tag{3.24}$$

Assume that $f(u) \in C^2(\bar{R}_+)$ and $f(0) < 0$. A positive branch $u(x, \lambda) > 0$ of solutions of (3.24) can lose it positivity only for increasing λ.

Proof. Assuming the contrary, we can as before find $\lambda_1 > 0$, such that $u_x(1, \lambda_1) = 0$ and $u_\lambda(x, \lambda_1) > 0$ near $x = 1$. We notice that

$$u_{xx}(0, \lambda_1) < 0. \tag{3.25}$$

Indeed, assuming $u_{xx}(0, \lambda_1) = 0$, we differentiate (3.1) in x, and discover that $v = u_x$ solves a linear homogeneous equation with zero initial conditions, a contradiction. We claim next that

$$u_\lambda(0, \lambda_1) = 0. \tag{3.26}$$

Indeed, from equations (3.13) and (3.16) we obtain as before

$$(u'u'_\lambda - u_\lambda u'')' + f(u)u' = 0.$$

Integrating over (0, 1), and using (3.15), we conclude

$$u_\lambda(0)u''(0) = 0.$$

In view of (3.25), the claim (3.26) follows.

The function $w(x) \equiv 2\lambda_1 u_\lambda - xu_x$ is positive near $x = 1$, and it is a nontrivial solution of

$$w'' + \lambda_1 f'(u)w = 0 \quad \text{on } (0, 1), \qquad w(0) = w'(0) = 0,$$

which is a contradiction, completing the proof. ∎

Finally, to complete the discussion we consider the remaining case when

$$f(x, 0) = 0 \qquad \text{for all } x \in [-1, 1]. \tag{3.27}$$

Notice that the possibility of $f(x, 0) > 0$ for some x, is inconsistent with the condition (3.3).

THEOREM 3.2. Assume that for problem (3.1) conditions (3.2), (3.3), (3.5) and (3.27) are satisfied. Then for any $\lambda > 0$ problem (3.1) has at most one positive solution.

Proof. Similar to the previous theorem, and actually easier, since now we know by Hopf's lemma that positive solution branches can never lose positivity. ∎

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REFERENCES

1. Ni, W.-M. and Nussbaum, R., Uniqueness and non-uniqueness for positive radial solution of $\Delta u + f(u, r) = 0$. *Comm. Pure Appl. Math.*, 1985, **38**, 67-108.
2. Lin, S. S., Positive radial solutions and non-radial bifurcation for semi-linear elliptic equations in annular domains. *J. Differential Equations*, 1990, **86**, 367-391.
3. Nagasaki, K. and Suzuki, T., Spectral and related properties abou' the Emden-Fowler equation $-\Delta u = \lambda\, e^u$ on circular domains. *Math. Ann.*, 1994, **299**, 1-15.
4. Gidas, B., Ni, W.-M. and Nirenberg, L., Symmetry and related properties via the maximum principle. *Commun. Math. Phys.*, 1979, **68**, 209-243.
5. Kwong, M. K. and Zhang, L., Uniqueness of the positive solutions of $\Delta u + f(u) = 0$ in an annulus. *Differential and Integral Equations*, 1991, **4**, 583-599.

6. Korman, P., Li, Y. and Ouyang, T., Exact multiplicity results for boundary-value problems with nonlinearities generalizing cubic. *Proc. Royal Soc. Edinburgh, Ser. A*, 1996, **126A**, 599-616.
7. Korman, P. and Ouyang. T., Solution curves for two classes of boundary-value problems. *Nonlinear Analysis*, 1996, **27**(9), 1031-1047.
8. Korman, P. and Ouyang, T., Exact multiplicity results for two classes of boundary-value problems. *Differential and Integral Equations*, 1993, **6**, 1507-1517.
9. Korman, P. and Ouyang, T., Exact multiplicity results for a class of boundary-value problems with cubic nonlinearities. *J. Math. Anal. Applic.*, 1995, **194**, 328-341.
10. Crandall, M. G. and Rabinowitz, P. H., Bifurcation, perturbation of simple eigenvalues and linearized stability. *Arch. Rational Mech. Anal.*, 1973, **52**, 161-180.
11. Amann, H., Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces. *SIAM Rev.*, 1976, **18**, 620-709.
12. Gidas, B. and Spruck, J., A priori bounds for positive solutions of nonlinear elliptic equations. *Comm. in PDE*, 1986, **6**(8), 883-901.