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ON APPLICATION OF THE MONOTONE ITERATION SCHEME TO NONCOERCIVE ELLIPTIC AND HYPERBOLIC PROBLEMS

PHILIP KORMAN
The Ohio State University, Mathematics Department Columbus, Ohio 43210, U.S.A.

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1. INTRODUCTION

IN THIS paper we show that the monotone iteration scheme can be successfully applied to equations other than coercive elliptic and parabolic. For this we first state the monotone scheme in a more general situation than usual (see e.g. [2, 9])—theorem 1, and then apply it to noncoercive elliptic and hyperbolic problems. Our generalization consists essentially in allowing subelliptic estimates instead of elliptic ones and not requiring Schauder's estimates. Also, boundary conditions are allowed to be nonlinear. The price we pay is higher differentiability requirements.

Recently, D. Dunninger has extended the monotone scheme in another direction, namely to treat singular elliptic equations, see [3]. It appears that his results can be used in combination with ours.

Our conclusion is that the monotone scheme seems to be applicable in any problem with a weak maximum principle, provided there is some gain of derivatives for the corresponding linear problem. In addition to the applications considered in this paper, such a situation arises, for example, for Tricomi's equation, where there is a maximum principle due to Agmon, Nirenberg and Protter, see [1]. Telegraph equation is another example.

2. NOTATION AND THE PRELIMINARY LEMMAS

Let D be a bounded domain in R^n, and let $\partial D = \Gamma_1 \cup \Gamma_2 \cup \ldots \cup \Gamma_k$ denote a part (or the whole) of its boundary. By $\|\,.\,\|_m$ we denote the mth Sobolev norm and $|u|_{L^\infty} = \operatorname{ess\,sup}_D |u|$. We shall write c for all irrelevant constants.

We shall use the following standard lemmas, see e.g. [4] for proofs.

LEMMA 1. Suppose that $w(x) \in C^m$ and $\varphi = \varphi(x, w)$ has continuous derivatives up to order $m \geq 1$ bounded by c on $|w| \leq c_0$, $x \in D$. Then

$$\|\varphi(x, w)\|_m \leq c(\|w\|_m + 1) \quad \text{for } |w|_{L^\infty} \leq c_0.$$

LEMMA 2. Suppose $f_1, f_2 \in C^m(D)$ such that all norms appearing below are bounded; $m \geq [n/2] + 1$. Then

$$\|f_1 f_2\|_m \leq c\|f_1\|_m \|f_2\|_m.$$

LEMMA 3. (Interpolation inequality.) If $f \in C^m(D)$ and $0 \leqslant j < m$, then

$$\|f\|_j \leqslant c\|f\|_m^{j/m}\|f\|_0^{1-j/m}.$$

3. THE GENERAL THEOREMS

THEOREM 1. We consider the following nonlinear boundary value problem

$$\begin{aligned} Lu &= f(x,u) \quad \text{in } D \\ B_j u &= g_j(x,u) \quad \text{on } \Gamma_j, j = 1, 2, \ldots, k. \end{aligned} \tag{1}$$

Here L and B_j are linear partial differential operators of orders m_0 and m_j respectively. We make no explicit assumptions on their order, type and smoothness of coefficients (and on the domain D). Instead, we require problem (1) to satisfy the following conditions.

(i) Consider the linear problem

$$\begin{aligned} Lu - \Omega u &= F(x) \quad \text{in } D \\ B_j u + \Omega_j u &= G_j(x) \quad \text{on } \Gamma_j. \end{aligned} \tag{1$'$}$$

We assume that for any $F \in C^{m_0}(D)$ and $G_j \in C^{m_j}(\Gamma_j)$ the problem (1)′ is (uniquely) solvable for $u \in C^{m_0}(D) \cap C^{m_j}(\Gamma_j)$ and the following estimate holds (m – positive integer)

$$\|u\|_{m+1} \leqslant c\left(\|F\|_m + \sum_{j=1}^{k}\|G_j\|_m\right). \tag{2}$$

(ii) We assume that if

$$\begin{aligned} Lu - \Omega u &\geqslant 0 \quad \text{in } D \\ B_j u + \Omega_j u &\leqslant 0 \quad \text{on } \Gamma_j, j = 1, \ldots, k, \end{aligned} \tag{3}$$

for any constants Ω, $\Omega_j \geqslant 0$, then $u \leqslant 0$ in D. This "inverse positivity condition" usually follows from a weak maximum principle.

(iii) There exists a function $\varphi(x)$, called supersolution, such that

$$\begin{aligned} L\varphi - f(x,\varphi) &\leqslant 0 \quad \text{in } D \\ B_j\varphi &\geqslant g_j(x,\varphi) \quad \text{on each } \Gamma_j. \end{aligned} \tag{4}$$

(iv) There exists a subsolution $\psi(x)$, defined by reversing the inequalities in (4).

(v) $\psi(x) \leqslant \varphi(x)$ everywhere in D.

We denote

$$m = \max_{0 \leqslant j \leqslant k} m_j + \left[\frac{n}{2}\right] + 1; a = \min_{\bar{D}} \psi(x), b = \max_{\bar{D}} \varphi(x)$$

and assume finally that φ, $\psi \in C^{m_0}(D) \cap C^{m_j}(\Gamma)$; f, $g \in C^m$ in $V_0 \equiv D \times \{a \leqslant u \leqslant b\}$ and $V_j \equiv \Gamma_j \times \{a \leqslant u \leqslant b\}$ correspondingly.

Then the problem (1) has a solution $u(x) \in C^{m_0}(D) \cap C^{m_j}(\Gamma_j)$.

Proof. Without loss of generality we may assume that

$$f_u \leqslant 0 \text{ in } V_0 \text{ and } \frac{\partial g_j}{\partial u} \geqslant 0 \text{ in } V_j. \tag{5}$$

For if otherwise, we may set $\Omega = \max_{\bar{V}_0} |f(x, u)|$, $\Omega_j = \max_{\bar{V}_j} |g_j(x, u)|$, and consider instead of (1) an equivalent problem

$$\begin{aligned} Lu - \Omega u &= f(x, u) - \Omega u \\ B_j u + \Omega_j u &= g_j(x, u) + \Omega_j u, \end{aligned} \tag{6}$$

for which the condition (5) is satisfied.

Next, as is standard, we define a nonlinear transformation $v = Tu$ by solving

$$\begin{aligned} Lv &= f(x, u) && \text{in } D \\ B_j v &= g_j(x, u) && \text{on each } \Gamma_j. \end{aligned} \tag{7}$$

T is monotone, i.e. $u_1 \leqslant u_2$ implies $Tu_1 \leqslant Tu_2$. Indeed, setting $w = Tu_1 - Tu_2$, we have by (5)

$$\begin{aligned} Lw &= f(x, u_1) - f(x, u_2) \geqslant 0 \\ B_j w &= g(x, u_1) - g(x, u_2) \leqslant 0, \end{aligned}$$

So that by condition (ii) $w = Tu_1 - Tu_2 \leqslant 0$.

Next, we let $u_1 = T\varphi$ and show that $u_1 \leqslant \varphi$. Indeed, by (4)

$$\begin{aligned} L(u_1 - \varphi) &= f(x, \varphi) - L\varphi \geqslant 0 \\ B_j(u_1 - \varphi) &= g(x, \varphi) - B_j\varphi \leqslant 0 \end{aligned}$$

and hence $u_1 - \varphi \leqslant 0$ using condition (ii).

By induction we get a nonincreasing sequence of iterates $u_{i+1} = Tu_i$, $i = 1, 2, \ldots$, $u_{i+1} \leqslant u_i$. Similarly, we get a nondecreasing sequence of iterates $v_1 = T\psi$, $v_{i+1} = Tv_i$, $i = 1, 2, \ldots$, $v_{i+1} \geqslant v_i$. By monotonicity of T and the condition (v) we have $v_i \leqslant u_i$ for all i, and hence both the sequences $\{u_i\}$ and $\{v_i\}$ converge pointwise. Call $\lim_{i\to\infty} u_i(x) = \bar{u}(x)$, $\lim_{i\to\infty} v_i(x) = \bar{v}(x)$. We show next that $\bar{u}$ has the desired smoothness and $\bar{u} = T\bar{u}$.

Indeed, since $\bar{u} - u_n \to 0$ pointwise and is bounded (by $\varphi - \psi$) it follows that $\bar{u} - u_n \to 0$ in L^2, i.e. $\|\bar{u} - u_n\|_0 \to 0$. Also, $\|u_n\|_0 \leqslant c$ uniformly in n. Then by (2) and lemma 1

$$\|u_{n+1}\|_1 \leqslant c\left(\|f(x, u_n)\|_0 + \sum_{j=1}^{k} \|g_j(x, u_n)\|_0\right) \leqslant c,$$

uniformly in n. By induction

$$\|u_{n+1}\|_{m+1} \leqslant c\left(\|f(x, u_n)\|_m + \sum_{j=1}^{k} \|g_j(x, u_n)\|_m\right) \leqslant c$$

uniformly in n $(n \geqslant m)$. By lemma 3 we have for any $0 < \varepsilon < \frac{1}{2}$

$$\begin{aligned} \|u_p - u_n\|_{m+1-\varepsilon} &\leqslant c\|u_p - u_n\|_{m+1}^{(m+1-\varepsilon)/(m+1)} \|u_p - u_n\|_0^{\varepsilon/(m+1)} \\ &\leqslant c\|u_p - u_n\|_0^{(\varepsilon/m+1)} \to 0 \quad \text{as} \quad n, p \to \infty, \end{aligned}$$

so that $\{u_n\}$ is a Cauchy sequence in $H^{m+1-\varepsilon}$. Then by imbedding and trace theorems for Sobolev spaces and the choice of m we conclude that $\{u_n\}$ is a Cauchy sequence also in

$C^{m_0}(D)$ and $C^{m_j}(\Gamma_j)$. Hence $\bar{u} \in C^{m_0} \cap C^{m_j}(\Gamma_j)$, and we can pass to the limit in

$$\begin{aligned} Lu_{n+1} &= f(x, u_n) && \text{in } D \\ B_j u_{n+1} &= g_j(x, u_n) && \text{on each } \Gamma_j, \end{aligned}$$

to get the desired solution for the problem (1).

Remark 1. If conditions (i) and (ii) are simultaneously satisfied only for $\Omega = 0$, then the result holds provided $f_u(x, u) \leqslant 0$ in V_0, as it is clear from the proof. Similarly, if the same conditions can only be satisfied if some $\Omega_j = 0$, then the theorem holds provided $\partial g_j(x, u)/\partial u \geqslant 0$ in V_j, for that j.

Remark 2. In the case $g_j = g_j(x)$ the estimate (2) can be relaxed to consider

$$\|u\|_{m+1} \leqslant c\left(\|F\|_m + \sum \|G_j\|_{m-\sigma}\right), \; \sigma > 0, \tag{2)'}$$

provided $g_j(x) \in C^{m+\sigma}$ for that j.

Remark 3. Let P be a subspace of C^{m_0} which is preserved under action of the solution operator T. If in addition to the conditions of theorem 1 we have ψ, $\varphi \in P$ then the conclusion of the theorem holds and moreover $\bar{u} \in P$. As an example of P we may consider functions which are $\tau_1, \ldots, \tau_k$ periodic in variables $x_1, \ldots, x_k$. Another common example is spherically symmetric functions.

Remark 4. Boundary conditions need not be prescribed on all parts of ∂D. We do not study uniqueness questions at present.

Remark 5. If instead of (2) we have a stronger estimate

$$\|u\|_{m+2} \leqslant c\left(\|F\|_m + \sum_{j=1}^{k} \|G_j\|_m\right), \tag{2)''}$$

then it suffices to assume f, $g \in C^{m-1}$, with m as defined above.

THEOREM 2. Let $D_1 \subset D_2 \subset \ldots \subset D_n \subset \ldots$ be bounded domains in R^n, $D_\infty = \bigcup_1^\infty D_n$ may be unbounded, and let $\partial D = \Gamma_1 \cup \Gamma_2 \cup \ldots \cup \Gamma_k$ be a part (or the whole) of their common boundary. Assume that for each n the problem

$$\begin{aligned} Lu &= f(x, u) && \text{in } D_\infty \\ B_j u &= g_j(x, u) && \text{on each } \Gamma_j \end{aligned} \tag{8}$$

satisfies all conditions of the theorem 1. Assume that the problem

$$\begin{aligned} Lu &= f(x, u) && \text{in } D_\infty \\ B_j u &= g_j(x, u) && \text{on each } \Gamma_j \end{aligned} \tag{9}$$

has super- and subsolutions ψ, φ (as defined above) with $\psi \leqslant \varphi$ in D_∞. Then the problem (9) has a solution $u \in C^{m_0}(D_\infty) \cap C^{m_j}(\Gamma_j)$.

Proof. It is clear that for any n, φ and ψ are super- and subsolutions of (8). Call solution of (8) by u^n, $\psi \leq u^n \leq \varphi$. Consider the sequence $\{u^n\}$ on D_1. Since $\|u^n\|_0 \leq c$ independent of n, we have by repeated application of estimates (2): $\|u^n\|_{m-1} \leq c$, so that by Sobolev imbedding and definition of m

$$\|u^n\|_{C^{m_0-\alpha}(D_\infty)} + \sum_{j=1}^{k} \|u^n\|_{C^{m_j-\alpha}(\Gamma_j)} \leq c,$$

for some $\alpha > 0$. Since the imbedding $C^{p+\alpha} \to C^p$ is compact we can (in $k+1$ steps) select a subsequence $\{u^{n_i}\}$ converging in $C^{m_0}(D_1) \cap C^{m_j}(\Gamma_j)$ to a solution of (8). Next, we consider $\{u^{n_i}\}$ on D_2 $(n_i \geq 2)$. In the same way we extract a subsequence which converges in D_2 to a solution of (8). By repeating this process for $D_3, D_4, \ldots$ and then taking the usual diagonal subsequence, we establish the theorem.

Remark. The functions φ and ψ are allowed to be unbounded in D_∞.

4. A NONLINEAR NONCOERCIVE ELLIPTIC PROBLEM

We shall apply theorem 1 to the following boundary value problem:

$$u_y - u_{xx} = g(x, z, u), \qquad y = 1 \tag{10a}$$

$$\Delta u = f(x, y, z, u), \qquad 0 < y < 1 \tag{10b}$$

$$u = 0, \qquad y = 0. \tag{10c}$$

Here f and g are assumed to be 2π periodic in x and z, and we are looking for a 2π periodic in x and z solution $u(x, y, z)$. In [5, 6] we discussed the relevance of (10) as a model noncoercive problem and its connection with the theory of water waves. In order to prove our existence result we need the following lemmas.

LEMMA 4. Consider the problem

$$\begin{aligned} &u_y - u_{xx} + \Omega_1 u = g(x, z), && y = 1 \\ &\Delta u - \Omega u = f(x, y, z), && 0 < y < 1 \\ &u = 0, && y = 0. \end{aligned} \tag{11}$$

Let the functions $f, g \in H^m$, $m \geq 0$, be 2π periodic in x, z. Then for any Ω, $\Omega_1 \geq 0$ problem (11) has a unique 2π periodic in x, z solution $u(x, y, z)$ and

$$\|u\|_{m+2} \leq c(\|f\|_m + \|g\|_m + \|g_z\|_m). \tag{12}$$

Proof. Let $v(x, y, z)$ be any 2π periodic in x, z function, satisfying $v(x, 0, z) \equiv 0$. Multiply (10b) by v and integrate by parts. Periodicity and (10c) imply that the integral $\int_{\partial D} v\,(\partial u/\partial n)\,\mathrm{d}S$ will have contributions only from the top $(y \equiv 1)$ part of the boundary (where $\partial u/\partial n = u_y$), i.e. we have

$$-\int \nabla u \,.\, \nabla v - \Omega \int uv + \int_t vu_y = \int fv,$$

where we denote $\int w = \int_0^{2\pi}\int_0^1\int_0^{2\pi} w(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ and $\int_t w = \int_0^{2\pi}\int_0^{2\pi} w(x, 1, z)\,\mathrm{d}x\,\mathrm{d}z$.

Using (10a) and periodicity we get:

$$\int_t vu_y = \int_t v(u_{xx} - \Omega_1 u + g) = -\int_t u_x v_x - \Omega_1 \int_t uv + \int_t gv,$$

so that

$$-\int \nabla u \,.\, \nabla v - \Omega \int uv - \int_t u_x v_x - \Omega_1 \int_t uv + \int_t gv = \int fv. \tag{13}$$

Next we let v to be successively equal to u, u_{xx}, u_{zz} in (13), obtaining the following formulas

$$-\int |\nabla u|^2 - \Omega \int u^2 - \int_t u_x^2 - \Omega_1 \int_t u^2 + \int_t gu = \int fu, \tag{14}$$

$$-\int \nabla u \,.\, \nabla u_{xx} - \Omega \int uu_{xx} - \int_t u_x u_{xxx} - \Omega_1 \int_t uu_{xx} + \int_t gu_{xx} = \int fu_{xx}, \tag{15}$$

$$-\int \nabla u \,.\, \nabla u_{zz} - \Omega \int uu_{zz} - \int_t u_x u_{xzz} - \Omega_1 \int_t uu_{zz} + \int_t gu_{zz} = \int fu_{zz}. \tag{16}$$

Notice that by (10c) $\int_t u^2 \leqslant \int |\nabla u|^2$ and $\int u^2 \leqslant \int |\nabla u|^2$. Then we estimate [writing LHS (14) for the left-hand side of formula (14)]:

$$|\mathrm{LHS}(14)| \geqslant \int |\nabla u|^2 - \tfrac{1}{2}\int_t u^2 - \tfrac{1}{2}\int_t g^2 \geqslant \tfrac{1}{2}\int |\nabla u|^2 - \tfrac{1}{2}\int_t g^2.$$

Also

$$|\mathrm{RHS}(14)| \leqslant 2\int f^2 + \tfrac{1}{8}\int u^2 \leqslant 2\int f^2 + \tfrac{1}{8}\int |\nabla u|^2.$$

Combining, we get:

$$\int |\nabla u|^2 \leqslant c\left(\int f^2 + \int_t g^2\right). \tag{17}$$

In formula (15) we integrate by parts, obtaining

$$\int |\nabla u_x|^2 + \Omega \int u_x^2 + \int_t u_{xx}^2 + \Omega_1 \int_t u_x^2 + \int_t gu_{xx} = \int fu_{xx}.$$

Estimating exactly as in the case of (14) we get:

$$\int |\nabla u_x|^2 \leqslant c\left(\int f^2 + \int_t g^2\right). \tag{18}$$

Similarly starting with (16), we get:

$$\int |\nabla u_z|^2 \leqslant c\left(\int f^2 + \int_t g_z^2\right). \tag{19}$$

Then by (10b), (18) and (19) we estimate

$$\int u_{yy}^2 \leqslant c\left(\int f^2 + \int u_{xx}^2 + \int u_{zz}^2\right) \leqslant c\left(\int f^2 + \int_t g^2\right). \tag{20}$$

Adding the estimates (17)–(20) we obtain

$$\|u\|_2 \leqslant c(\|f\|_0 + \|g\|_0 + \|g_z\|_0),$$

which proves the lemma for $m = 0$.

The higher estimates are easily obtained by differentiation of equations (11), see [5–7].

Existence of solution for the linear problem (11) follows by an elementary Fourier series analysis.

LEMMA 5. Let $u(x, y, z)$ be 2π periodic in x, z and

$$\begin{aligned} u_y - u_{xx} + \Omega_1 u &\leqslant 0, & y &= 1 \\ \Delta u - \Omega u &\geqslant 0, & 0 &< y < 1 \\ u &\leqslant 0, & y &= 0 \end{aligned}$$

Then $u \leqslant 0$ in the entire strip S: $0 \leqslant y \leqslant 1$, $-\infty < x, z < \infty$.

Proof. By the maximum principle $u(x, y, z)$ assumes its maximum on the boundary of the strip S. We argue next that a positive maximum cannot be assumed on the top ($y = 1$) part of the boundary. Indeed, in such case at the point of maximum we would have $u_y > 0$ by Hopf's lemma, and then

$$u_{xx} \geqslant u_y + \Omega_1 u > 0,$$

a contradiction. Hence $u \leqslant 0$ in S.

We can now state our existence result.

THEOREM 3. Assume that problem (10) has a subsolution $\psi(x, y, z)$ and a supersolution $\varphi(x, y, z)$ with $\psi \leqslant \varphi$. Suppose that 2π periodic in x and z functions $\psi(x, y, z)$, $\varphi(x, y, z)$, $f(x, y, z, u)$, $g_z(x, z)$ belong to C^3 for $0 \leqslant y \leqslant 1$, $-\infty < x, z < \infty$; $\psi \leqslant u \leqslant \varphi$. Then problem (10) has a 2π periodic in x, z solution $u(x, y, z) \in C^2(\bar{S})$.

Proof. We apply theorem 1. Conditions (i) and (ii) are satisfied in view of lemmas 4 and 5 (see remark 5 for the smoothness requirement). The remaining conditions of theorem 1 are assumed here. As an example for theorem 3 we have the following.

PROPOSITION 1. Consider problem (10). Assume that the functions f and g are 2π periodic in x, z, belong to C^3 and are sublinear in u, i.e.

$$\begin{aligned} |f(x, y, z, u)| &\leqslant c_0(1 + |u|^\alpha), \\ |g(x, z, u)| &\leqslant c_0(1 + |u|^\alpha), \quad 0 < \alpha < 1 \end{aligned} \tag{21}$$

for all real u and (x, y, z) in the strip $0 \leqslant y \leqslant 1$. Then problem (10) has a C^2 solution.

Proof. According to theorem 3 we have only to exhibit super- and subsolutions. Let $\varphi = b(1 - e^{-y})$, $b = \text{const} > 0$. In order for φ to be a supersolution for (10), we need according to (4) and (21)

$$\begin{aligned} be^{-1} &\geqslant c_0\{1 + [b(1 - e^{-1})]^\alpha\} \geqslant g(x, z, \varphi) \\ -be^{-y} &\leqslant -c_0\{1 + [b(1 - e^{-y})]^\alpha\} \leqslant f(x, y, z, \varphi), \end{aligned}$$

which is easily achieved by taking b large enough. Similarly, one sees that $\psi = -b(1 - e^{-y})$ is a subsolution, completing the proof.

Remark. Proposition 1 can also be proved via Schauder's fixed point theorem.

5. NONLINEAR WAVE EQUATION

For simplicity we consider a one-dimensional wave equation for an infinite string (with prescribed initial conditions)

$$\begin{aligned} &u_{xx} - u_{tt} = f(x, t, u), \ -\infty < x < \infty, t > 0 \\ &u(x, 0) = g(x) \\ &u_t(x, 0) = h(x) \ (g, h \in C_0^\infty), \end{aligned} \tag{22}$$

although our results generalize to general hyperbolic equations in two and three dimensions.

THEOREM 4. Assume that for $0 \leqslant t \leqslant T$, $-\infty < x < \infty$ the following conditions hold:

(i) There exists a supersolution $\varphi(x, t)$, i.e.

$$\begin{aligned} &\varphi_{xx} - \varphi_{tt} \leqslant f(x, t, \varphi) \\ &\varphi(x, 0) \geqslant g(x) \\ &\varphi_t(x, 0) \geqslant h(x). \end{aligned}$$

(ii) There exists a subsolution $\psi(x, t)$, defined by reversing the inequalities in (i).

(iii) $\psi \leqslant \varphi$.

(iv) $f_u \leqslant 0$ for $\inf_{x,t} \psi \leqslant u \leqslant \sup_{x,t} \varphi$.

(v) $f \in C^4$ in all arguments for $\inf_{x,t} \psi \leqslant u \leqslant \sup_{x,t}$, and $g \in C_0^5$, $h \in C_0^4$, $\psi, \varphi \in C^2$.

Then problem (22) has a C^2 solution $u(x, t)$, with $\psi \leqslant u \leqslant \varphi$ for $0 \leqslant t \leqslant T$, $-\infty < x < \infty$.

Proof. We proceed to verify conditions of theorem 1. As is well-known, if

$$\begin{aligned} &u_{xx} - u_{tt} \geqslant 0, \qquad 0 \leqslant t \leqslant T \\ &u(x, 0) \leqslant 0 \\ &u_t(x, 0) \leqslant 0 \end{aligned}$$

then $u(x, t) \leqslant 0$ for $0 \leqslant t \leqslant T$, see e.g. [8, p. 196]. This verifies the condition (i). Condition (ii) follows from the following

LEMMA 6. Consider the problem

$$\begin{aligned} &u_{xx} - u_{tt} = f(x, t), \ -\infty < x < \infty, 0 \leqslant t \leqslant T \\ &u(x, 0) = g(x) \\ &u_t(x, 0) = h(x), \qquad g, h \in C_0^\infty. \end{aligned} \tag{22'}$$

Denote by $\|.\|_m$ (integer $m \geq 0$) the mth Sobolev norm in x, t space. Then

$$\|u\|_{m+1} \leq c(\|f\|_m + \|g\|_{m-1} + \|h\|_m), c = c(T). \tag{23}$$

We postpone the proof of this lemma, which supplies the desired estimate, in view of remark 2 to theorem 1.

Next we recall a standard fact, that for f, g, $h \in C^2$ problem (22)' has a unique solution $u(x, t) \in C^2$.

Now theorem 4 follows by applying theorem 1 to any finite domain D, containing the domain of influence for problem (22) for $t \leq T$.

Proof of lemma 6. We start with the standard energy inequality, see [4, p. 32].

$$\int_{-\infty}^{\infty} (u_x^2 + u_t^2)\,dx \leq c\left(\int_{-\infty}^{\infty} f^2\,dx + \int_{-\infty}^{\infty} (g'^2 + h^2)\,dx\right), c = c(T). \tag{24}$$

Notice that for each t, $u(x, t)$ is of compact support, so that

$$\int_{-\infty}^{\infty} u^2(x, t)\,dx \leq c\int_{-\infty}^{\infty} (u_x^2 + u_t^2)\,dx. \tag{25}$$

Then integrating (24) in t from 0 to T, and using (25), we get the estimate (23) for $m = 0$. The higher estimates are obtained by differentiation of (22)'.

An example for theorem 4 is given by proposition 2.

PROPOSITION 2. Assume that for $-\infty < x < \infty$, $0 \leq t \leq T$, $-\infty < u < \infty$, we have:

(i) $|f(x, t, u)| \leq c(1 + |u|^\alpha)$, $0 < \alpha < 1$

(ii) $f_u \leq 0$; $f, g \in C^3$.

Then problem (22) has a C^2 solution $u(x, t)$ for $0 \leq t \leq T$, $-\infty < x < \infty$.

Proof. Similarly to proposition 1, we show that $\varphi = C(2t - e^{-t})$ and $\psi = -\varphi$ are super- and subsolutions for (22), provided the constant b is chosen large enough.

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