Calculating Global Solution Curves for Boundary Value Problems

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This program calculates curves of solutions for two-point Dirichlet problems

(1) $u''(x)+g(u(x))=\mu \sin x+e(x), \quad 0<x<\pi, \quad u(0)=u(\pi)=0.$

Here the functions g(u) and e(x) are given, μ is a parameter. Observe that a given function on the right hand side of (1) has been decomposed into the sum of the first harmonic $\mu \sin x$, and the "the rest" i.e., the function e(x) with $\int_0^\pi e(x) \sin x \, dx = 0$. Likewise, let us decompose the solution of (1) as u(x)=$\xi \sin x$+U(x), where ξ is a number and $\int_0^\pi U(x) \sin x \, dx=0$. For the eigenvalue problem

$u''(x)+\lambda u=0, \quad 0<x<\pi, \quad u(0)=u(\pi)=0$

we have $\lambda_1=1$ with the corresponding eigenfunction sin x, and $\lambda_2=4$ with the corresponding eigenfunction sin 2x. It was shown in P. Korman, Curves of equiharmonic solutions, and problems at resonance, Discrete Contin. Dyn. Syst. 34, no. 7, 2847-2860 (2014), that the value of ξ is a global parameter, $0<\xi<\infty$, uniquely identifying the solution pair (u(x), μ) of (1), provided that $g'(u)<4$ for all u>0. For a direct access to the paper click here:

https://homepages.uc.edu/~kormanp/George2.pdf

Note that other conditions on g(u) are described there. Thus we can have a solution curve (u(x), μ)(ξ) with different domains. A section of this curve $\mu(\xi)$ governs the multiplicity of solutions of (1). We now describe the program for computing $\mu=\mu(\xi)$, which is presented below.

The program begins by implementing the "linear solver", i.e., the numerical solution of the following problem: given any ξ, and any continuous functions a(x) and f(x), find u(x) and μ solving

(3) $u''(x)+a(x)u=\mu \sin x+f(x), \quad 0<x<\pi,$

$u(0)=u(\pi)=0$

$\frac{2}{\pi} \int_0^\pi u(x) \sin x \, dx = \xi.$

The general solution of the differential equation in (3) is

$u(x)=Y(x)+c_1 u_1(x)+c_2 u_2(x),$

where Y(x) is any particular solution, $u_1(x)$ and $u_2(x)$ are two linearly independent solutions of the corresponding homogeneous equation $u''+a(x)u=0$, $u_1(0)=u_2(\pi)=0$.

The particular solution is calculated in the form $Y(x)=\mu Y_1(x)+Y_2(x)$, where $Y_1(x)$ solves

$u''(x)+a(x)u=\sin x, \; u(0)=0, \; u'(0)=1,$

and Y_2(x) solves

u''(x) +a(x)u= f(x), u(0)=0, u'(0)=1.

The solution of the equation in (3) and satisfying the first boundary condition is then

Y(x)=$\mu\, Y_1$(x)+Y_2(x)+$c_2\, u_2$(x).

Then we choose μ and c_2 to satisfy the other boundary condition and the third line in (3).

We now compute the solution curve (u(x), μ)(ξ_i) by using the above algorithm combined with Newton's method at each ξ_i.

The program below computes the solution curve μ=μ(ξ) for a problem at resonance. The functions g(u) and e(x) are defined at the beginning of the program.

The program computes st=100 mesh points, then joins them to plot the solution curve. The length of the interval in ξ can be controlled by changing xi0 and delxi. (Please execute the program.) The solution curve illustrates the fact that at μ = 0 there are infinitely many solutions. If you change to p =4 in line 2, the program will confirm that a drastic change occurs.

This program solves the problem (1) and draws the solution curve μ=μ(ξ) (the bifurcation diagram)

In[1]:=
```
Clear["`*"]
p = 1;
st = 100;
xi0 = 0;
delxi = .4;
T = π  // N;
nu = π / T;
e[x_] =  Sin[3 * nu * x];
g[u_] = nu ^2 u + Sin[u] / Sqrt[u^p + 4];

linear := Module[{den, aa, bb, cc, dd, p, q},
  u1 = NDSolveValue[{ y''[x] + a[x] * y[x] == 0, y[0] == 0, y′[0] == 1}, y, {x, 0, T}];
  Y1 = NDSolveValue[{ y''[x] + a[x] * y[x] ==  Sin[x], y[0] == 0, y′[0] == 0}, y, {x, 0, T}];
  Y2 = NDSolveValue[{ y''[x] + a[x] * y[x] == f[x], y[0] == 0, y′[0] == 0}, y, {x, 0, T}];

  aa = u1[T];
  bb = Y1[T];
  cc = NIntegrate[u1[x] * Sin[x], {x, 0, T}];
  dd = NIntegrate[Y1[x] * Sin[x], {x, 0, T}];
  den = aa * dd - bb * cc;
  If[Abs[den] < 10^-7, Print["approaching singularity of linear system den=", den]];
  p = -Y2[T];
  q = ξ * T / 2 - NIntegrate[Y2[x]  * Sin[x], {x, 0, T}];

  c1 = (dd * p - bb * q) / den;
  μ  = (aa * q - cc * p) / den ;
```

```
  u[x_] = c1 * u1[x] + μ * Y1[x] + Y2[x];
 ]

dg[u_] = D[g[u], u];
uold[x_] = 12 * Sin[nu * x];

For[i = 1, i ≤ st, i++,
 xi[i] = xi0 + i delxi;
 ξ = xi[i] * 2 / T;
 mu[i] = 0;
 jc = 1;
 While[True,
  a[t_] = dg[uold[t]];
  f[t_] = e[t] - g[uold[t]] + dg[uold[t]] * uold[t];
  linear;
  If[Abs[(mu[i] - μ)] < 10^-7, mu[i] = μ;
   Print["At xi[", i, "]=", xi[i], "   number of iterations=", jc]; Break[]];

  If[jc > 10,
   Print["At xi[", i, "]=", xi[i], " no convergence for μ=", μ, "  mu[", i, "]=", mu[i]];
   Break[]];
  mu[i] = μ;
  uold[t_] = u[t];
  jc++;
 ]
]
```

```
At xi[1]=0.4   number of iterations=4
At xi[2]=0.8   number of iterations=3
At xi[3]=1.2   number of iterations=3
At xi[4]=1.6   number of iterations=3
At xi[5]=2.   number of iterations=3
At xi[6]=2.4   number of iterations=3
At xi[7]=2.8   number of iterations=3
At xi[8]=3.2   number of iterations=3
At xi[9]=3.6   number of iterations=3
At xi[10]=4.   number of iterations=3
At xi[11]=4.4   number of iterations=3
At xi[12]=4.8   number of iterations=3
At xi[13]=5.2   number of iterations=3
At xi[14]=5.6   number of iterations=3
At xi[15]=6.   number of iterations=5
At xi[16]=6.4   number of iterations=4
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At xi[17]=6.8   number of iterations=5
At xi[18]=7.2   number of iterations=4
At xi[19]=7.6   number of iterations=3
At xi[20]=8.   number of iterations=3
At xi[21]=8.4   number of iterations=3
At xi[22]=8.8   number of iterations=4
At xi[23]=9.2   number of iterations=3
At xi[24]=9.6   number of iterations=3
At xi[25]=10.   number of iterations=3
At xi[26]=10.4   number of iterations=3
At xi[27]=10.8   number of iterations=4
At xi[28]=11.2   number of iterations=6
At xi[29]=11.6   number of iterations=3
At xi[30]=12.   number of iterations=4
At xi[31]=12.4   number of iterations=3
At xi[32]=12.8   number of iterations=3
At xi[33]=13.2   number of iterations=3
At xi[34]=13.6   number of iterations=4
At xi[35]=14.   number of iterations=3
At xi[36]=14.4   number of iterations=3
At xi[37]=14.8   number of iterations=3
At xi[38]=15.2   number of iterations=5
At xi[39]=15.6   number of iterations=3
At xi[40]=16.   number of iterations=3
At xi[41]=16.4   number of iterations=5
At xi[42]=16.8   number of iterations=4
At xi[43]=17.2   number of iterations=3
At xi[44]=17.6   number of iterations=5
At xi[45]=18.   number of iterations=3
At xi[46]=18.4   number of iterations=4
At xi[47]=18.8   number of iterations=3
At xi[48]=19.2   number of iterations=4
At xi[49]=19.6   number of iterations=3
At xi[50]=20.   number of iterations=3
At xi[51]=20.4   number of iterations=4
At xi[52]=20.8   number of iterations=4
```

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At xi[53]=21.2   number of iterations=3
At xi[54]=21.6   number of iterations=3
At xi[55]=22.   number of iterations=4
At xi[56]=22.4   number of iterations=4
At xi[57]=22.8   number of iterations=3
At xi[58]=23.2   number of iterations=4
At xi[59]=23.6   number of iterations=3
At xi[60]=24.   number of iterations=4
At xi[61]=24.4   number of iterations=3
At xi[62]=24.8   number of iterations=3
At xi[63]=25.2   number of iterations=5
At xi[64]=25.6   number of iterations=3
At xi[65]=26.   number of iterations=5
At xi[66]=26.4   number of iterations=5
At xi[67]=26.8   number of iterations=4
At xi[68]=27.2   number of iterations=3
At xi[69]=27.6   number of iterations=5
At xi[70]=28.   number of iterations=5
At xi[71]=28.4   number of iterations=7
At xi[72]=28.8   number of iterations=5
At xi[73]=29.2   number of iterations=3
At xi[74]=29.6   number of iterations=4
At xi[75]=30.   number of iterations=3
At xi[76]=30.4   number of iterations=5
At xi[77]=30.8   number of iterations=3
At xi[78]=31.2   number of iterations=4
At xi[79]=31.6   number of iterations=4
At xi[80]=32.   number of iterations=6
At xi[81]=32.4   number of iterations=4
At xi[82]=32.8   number of iterations=6
At xi[83]=33.2   number of iterations=4
At xi[84]=33.6   number of iterations=4
At xi[85]=34.   number of iterations=4
At xi[86]=34.4   number of iterations=4
At xi[87]=34.8   number of iterations=4
At xi[88]=35.2   number of iterations=5
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At xi[89]=35.6   number of iterations=3
At xi[90]=36.   number of iterations=6
At xi[91]=36.4   number of iterations=4
At xi[92]=36.8   number of iterations=7
At xi[93]=37.2   number of iterations=3
At xi[94]=37.6   number of iterations=5
At xi[95]=38.   number of iterations=3
At xi[96]=38.4   number of iterations=6
At xi[97]=38.8   number of iterations=4
At xi[98]=39.2   number of iterations=3
At xi[99]=39.6   number of iterations=4
At xi[100]=40.   number of iterations=3
```

In[14]:=
```
table = Table[{xi[i], mu[i]}, {i, 1, st}];
```

In[15]:=
```
g1 = ListPlot[table, Joined → True, AxesOrigin → {0, 0},
  AxesLabel → {"\!\(\*StyleBox[\"ξ\",FontSize->18]\)",
    "\!\(\*StyleBox[\"μ\",FontSize->18]\)"}, PlotStyle → Thickness[0.01]]
```

Out[ ]=