Nonlinear oscillators at resonance with periodic forcing

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Abstract

In this note we unify the results of A.C. Lazer and P.O. Frederickson [3], A.C. Lazer [6], A.C. Lazer and D.E. Leach [7], J.M. Alonso and R. Ortega [1], and P. Korman and Y. Li [4] on periodic oscillations and unbounded solutions of nonlinear equations with linear part at resonance and periodic forcing. We give conditions for the existence and non-existence of periodic solutions, and obtain a rather detailed description of the dynamics for nonlinear oscillations at resonance, in case periodic solutions do not exist.

Key words: Resonance, periodic oscillations, unbounded solutions.

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1 Introduction

We are interested in the existence of 2π periodic solutions to the problem (here $x = x(t)$)

(1.1) $$x'' + f(x)x' + g(x) + n^2 x = e(t)\,.$$

The linear part

$$x'' + n^2 x = e(t)$$

is at resonance, with the null space spanned by $\cos nt$ and $\sin nt$. Define $F(x) = \int_0^x f(z)\,dz$. We assume throughout this paper that $e(t) \in C(R)$ satisfies $e(t+2\pi) = e(t)$ for all t, $f(x), g(x) \in C(R)$, $n \geq 1$ is an integer; moreover, we assume that the finite limits at infinity $F(\infty)$, $F(-\infty)$, $g(\infty)$, $g(-\infty)$ exist, and

$$F(-\infty) < F(x) < F(\infty), \quad \text{for all } x \in R\,, \tag{1.2}$$

$$g(-\infty) < g(x) < g(\infty), \quad \text{for all } x \in R\,. \tag{1.3}$$

Define

$$A_n = \int_0^{2\pi} e(t)\cos nt\,dt, \quad B_n = \int_0^{2\pi} e(t)\sin nt\,dt\,.$$

In the case when $f = 0$, the equation

$$x'' + g(x) + n^2 x = e(t) \tag{1.4}$$

was considered in the paper of A.C. Lazer and D.E. Leach [7] who proved the following classical theorem.

Theorem 1.1 *([7]) The condition*

$$\sqrt{A_n^2 + B_n^2} < 2\,(g(\infty) - g(-\infty))$$

is necessary and sufficient for the existence of 2π *periodic solutions of (1.4).*

In case $g = 0$, for the corresponding equation

$$x'' + f(x)x' + n^2 x = e(t) \tag{1.5}$$

one has the following result.

Theorem 1.2 *The condition*

$$\sqrt{A_n^2 + B_n^2} < 2n\,(F(\infty) - F(-\infty))$$

is necessary and sufficient for the existence of 2π *periodic solutions of (1.5).*

This theorem was proved by A.C. Lazer [6] for $n = 1$ (with an earlier result by P.O. Frederickson and A.C. Lazer [3]), and for all $n \geq 1$ by P. Korman and Y. Li [4], who used a small modification of A.C. Lazer's proof.

Question: is it possible to combine these theorems for the equation (1.1)? It turns out that the necessary conditions can be combined, while sufficient conditions cannot be combined.

Proposition 1 *The condition*

$$\sqrt{A_n^2 + B_n^2} < 2n\,(F(\infty) - F(-\infty)) + 2\,(g(\infty) - g(-\infty)) \tag{1.6}$$

is necessary for the existence of 2π *periodic solution of (1.1).*

Proposition 2 *The condition (1.6) is not sufficient for the existence of* 2π *periodic solution of (1.1).*

In case (1.4) has no 2π periodic solutions, all solutions of (1.4) are unbounded as $t \to \pm\infty$, as follows by the second Massera's theorem, as was observed first by G. Seifert [8]. Later J.M. Alonso and R. Ortega [1] gave an elementary approach to this result (with a more refined statement, asserting that solutions tend to infinity in C^1 norm). We observe next that the approach of [1] works for the equation (1.1) as well.

Proposition 3 *Assume in addition to the assumptions above that* $f(x)$ *is uniformly bounded from below (the assumption (2.10) below). Then in case*

$$\sqrt{A_n^2 + B_n^2} \geq 2n\,(F(\infty) - F(-\infty)) + 2\,(g(\infty) - g(-\infty))\,, \tag{1.7}$$

all solutions of (1.1) satisfy $\lim_{t\to\pm\infty}\left(x^2(t) + x'^2(t)\right) = \infty$.

Clearly, there are no 2π periodic solutions in this case, in view of Proposition 1. Proposition 3 shows that the absence of 2π periodic solutions turns out to be more decisive in determining the overall dynamics of (1.1) than the existence of 2π periodic solutions.

2 The proofs

The following elementary lemmas are easy to prove.

Lemma 2.1 *Consider a function* $\cos(nt - \varphi)$*, with an integer* n *and any real* φ*. Denote* $P_c = \{t \in (0, 2\pi) \,|\, \cos(nt - \varphi) > 0\}$ *and* $N_c = \{t \in (0, 2\pi) \,|\, \cos(nt - \varphi) < 0\}$*. Then*

$$\int_{P_c} \cos(nt - \varphi)\, dt = 2, \quad \int_{N_c} \cos(nt - \varphi)\, dt = -2\,.$$

Lemma 2.2 *Consider a function* $\sin(nt-\varphi)$*, with an integer* n *and any real* φ*. Denote* $P_s = \{t \in (0,2\pi) \,|\, \sin(nt-\varphi) > 0\}$ *and* $N_s = \{t \in (0,2\pi) \,|\, \sin(nt-\varphi) < 0\}$*. Then*

$$\int_{P_s} \sin(nt-\varphi)\, dt = 2, \quad \int_{N_s} \sin(nt-\varphi)\, dt = -2\,.$$

Proof of Proposition 1. Given arbitrary numbers a and b, one can find a $\delta \in [0,2\pi)$, so that

$$a\cos nt + b\sin nt = \sqrt{a^2+b^2}\cos(nt-\delta)\,.$$

(with $\cos\delta = \frac{a}{\sqrt{a^2+b^2}}$, $\sin\delta = \frac{b}{\sqrt{a^2+b^2}}$.) It follows that

$$\text{(2.1)} \qquad \frac{A_n}{\sqrt{A_n^2+B_n^2}}\cos nt + \frac{B_n}{\sqrt{A_n^2+B_n^2}}\sin nt = \cos(nt-\delta)\,,$$

for some $\delta \in [0,2\pi)$. Multiply (1.1) by $\frac{A_n}{\sqrt{A_n^2+B_n^2}}\cos nt$ and integrate, then multiply (1.1) by $\frac{B_n}{\sqrt{A_n^2+B_n^2}}\sin nt$ and integrate, and add the results:

$$\text{(2.2)} \quad \sqrt{A_n^2+B_n^2} = \int_0^{2\pi} F(x(t))'\cos(nt-\delta)\,dt + \int_0^{2\pi} g(x(t))\cos(nt-\delta)\,dt,$$

in view of (2.1). Using that $x(t)$ is a 2π periodic solution, and Lemma 2.2, obtain

$$\begin{aligned} &\textstyle\int_0^{2\pi} F(x(t))'\cos(nt-\delta)\,dt = n\int_0^{2\pi} F(x(t))\sin(nt-\delta)\,dt \\ &= n\textstyle\int_{P_s} F(x(t))\sin(nt-\delta)\,dt + n\int_{N_s} F(x(t))\sin(nt-\delta)\,dt \\ &< 2n\left(F(\infty)-F(-\infty)\right)\,. \end{aligned}$$

Similarly, using Lemma 2.1

$$\begin{aligned} &\textstyle\int_0^{2\pi} g(x(t))\cos(nt-\delta)\,dt < g(\infty)\int_{P_c}\cos(nt-\delta)\,dt + g(-\infty)\int_{N_c}\cos(nt-\delta)\,dt \\ &= 2\left(g(\infty)-g(-\infty)\right)\,. \end{aligned}$$

The condition (1.6) follows. ♢

Proof of Proposition 2. Consider the equation

$$\text{(2.3)} \qquad x'' + f(x)x' + g(x) + n^2x = E\cos nt,$$

with a parameter E. Calculate $A_n = \int_0^{2\pi} E\cos^2 nt\,dt = E\pi$, $B_n = \int_0^{2\pi} E\cos nt\sin nt\,dt = 0$, and $\sqrt{A_n^2+B_n^2} = E\pi$. Choose E so that

$$\text{(2.4)} \quad E\pi = \sqrt{A_n^2+B_n^2} = 2n\left(F(\infty)-F(-\infty)\right) + 2\left(g(\infty)-g(-\infty)\right) - \epsilon,$$

with $\epsilon > 0$ small, to be specified. The condition (1.6) holds for the equation (2.3). If this condition were sufficient, we would have a 2π periodic solution of (2.3), and hence

$$x'' + F(x)' + n^2 x = E \cos nt - g(x) \equiv \bar{e}(t) \,. \tag{2.5}$$

Calculate the coefficients A_n, B_n for (2.5):

$$\begin{aligned} \bar{A}_n &= \textstyle\int_0^{2\pi} \bar{e}(t) \cos nt \, dt = E\pi - \int_0^{2\pi} g(x) \cos nt \, dt, \\ \bar{B}_n &= \textstyle\int_0^{2\pi} \bar{e}(t) \sin nt \, dt = -\int_0^{2\pi} g(x) \sin nt \, dt. \end{aligned} \tag{2.6}$$

Since (2.5) is solvable, by Theorem 1.2 we have

$$\sqrt{\bar{A}_n^2 + \bar{B}_n^2} < 2n \left(F(\infty) - F(-\infty) \right) . \tag{2.7}$$

For any $\epsilon > 0$ we can choose an index n_0, so that for $n \geq n_0$

$$| \int_0^{2\pi} g(x) \sin nt \, dt | < \epsilon, \tag{2.8}$$

as follows by well known results on oscillatory integrals, see e.g., O. Costin et al [2]. Using (2.6), (2.8), followed by (2.4), obtain

$$\begin{aligned} &\sqrt{\bar{A}_n^2 + \bar{B}_n^2} > |\bar{A}_n| > E\pi - \epsilon \\ = \; &2n \left(F(\infty) - F(-\infty) \right) + 2 \left(g(\infty) - g(-\infty) \right) - 2\epsilon \\ &> 2n \left(F(\infty) - F(-\infty) \right) , \end{aligned}$$

contradicting (2.7), provided we fix $\epsilon < g(\infty) - g(-\infty)$. ♢

We shall prove a generalization of Proposition 3 after several preliminary results. By an obvious modification of its proof, one obtains the following generalization of Proposition 1.

Proposition 4 *Assume that the functions $F(x)$ and $g(x)$ have finite infimums and supremums on $(-\infty, \infty)$. Then the condition*

$$\sqrt{A_n^2 + B_n^2} < 2n \left(\sup F - \inf F \right) + 2 \left(\sup g - \inf g \right)$$

is necessary for the existence of 2π periodic solution of (1.1).

We shall use the following result that is included in J.M. Alonso and R. Ortega [1].

Proposition 5 *([1]) Let $G(\zeta,\eta): R^2 \to R^2$ be a continuous vector function, and let $V(\zeta,\eta): R^2 \to R$ be a continuous function. With $\xi \equiv (\zeta,\eta)$ consider a sequence*

$$\xi_{n+1} = G(\xi_n), \quad n \geq 0,$$

beginning with an arbitrary vector ξ_0. Assume that

$$V(G(\xi)) > V(\xi), \quad \forall \xi \in R^2. \tag{2.9}$$

Then $\lim_{n\to\infty} ||\xi_n|| = \lim_{n\to\infty} \sqrt{\zeta_n^2 + \eta_n^2} = \infty$.

Proof: If the sequence $\{||\xi_n||\}$ fails to tend to infinity, then $\{\xi_n\}$ has a finite accumulation point $\xi^* \in R^2$. Let $\{\xi_{n_k}\}$ be a subsequence tending to ξ^*, with $n_1 < n_2 < \cdots$. Since $V(\xi_{n+1}) = V(G(\xi_n)) > V(\xi_n)$ by (2.9), the sequence $V(\xi_n)$ is increasing. Then one has

$$V(G(\xi^*)) = \lim_{k\to\infty} V(G(\xi_{n_k})) = \lim_{k\to\infty} V(\xi_{n_k+1})$$
$$\leq \lim_{k\to\infty} V(\xi_{n_{k+1}}) = V(\xi^*),$$

contradicting (2.9). (Observing that $n_k + 1 \leq n_{k+1}$.) $\diamondsuit$

The next lemma says that for solution of (1.1), $x^2(t) + x'^2(t)$ cannot increase too much over an interval of length 2π.

Lemma 2.3 *Assume assume that $e(t) \in C(R)$ is 2π periodic, the condition (1.3) holds, and moreover assume that*

$$f(x) \geq \alpha, \quad \text{for some } \alpha \in R, \text{ and all } x \in R. \tag{2.10}$$

Then for any initial data $(x(0), x'(0))$, with $c_0 = x^2(0) + n^2 x'^2(0)$, there is a number $c = c(c_0)$ so that the corresponding solution of (1.1) satisfies

$$x^2(t) + n^2 x'^2(t) \leq c, \quad \text{for all } t \in [0, 2\pi].$$

Proof: Consider the "energy" $E(t) = \frac{1}{2}x'^2(t) + \frac{1}{2}n^2x^2(t)$. Since

$$E'(t) = -f(x)x'^2 - g(x)x' + e(t)x'.$$

By our conditions obtain $E'(t) \leq c_1 E(t) + c_2$, and the proof follows. $\diamondsuit$

We now prove the following generalization of Proposition 3.

Proposition 6 *Assume that the functions $F(x)$ and $g(x)$ have finite infimums and supremums on $(-\infty,\infty)$ Assume also that (2.10) holds. In case*

$$\sqrt{A_n^2+B_n^2} \geq 2n\,(\sup F - \inf F) + 2\,(\sup g - \inf g)\,,$$

all solutions of (1.1) satisfy $\lim_{t\to\pm\infty}\left(x^2(t)+{x'}^2(t)\right)=\infty$.

Proof: Following J.M. Alonso and R. Ortega [1], we shall use Proposition 5. Given $\xi=(\zeta,\eta)\in R^2$, denote by $x(t,\xi)$ the solution of (1.1) satisfying $x(0)=\zeta$, $x'(0)=\eta$. Define the map $R^2\to R^2$ by $G(\xi)=(x(2\pi,\xi),x'(2\pi,\xi))$, and we shall show that the sequence of iterates

$$\xi_{n+1}=G(\xi_n)\,,\quad n=0,1,2,\ldots$$

is unbounded for any ξ_0. With δ as defined by (2.1), define the function

$$V(\xi)=\eta\cos\delta - n\zeta\sin\delta + F(\zeta)\cos\delta\,.$$

Multiply (1.1) by $\frac{A_n}{\sqrt{A_n^2+B_n^2}}\cos nt$ and integrate, then multiply (1.1) by $\frac{B_n}{\sqrt{A_n^2+B_n^2}}\sin nt$ and integrate, and add the results. In view of (2.1) obtain as above

$$\begin{aligned}(2.11)\qquad &\textstyle\int_0^{2\pi}(x''+n^2x)\cos(nt-\delta)\,dt+\int_0^{2\pi}F(x(t))'\cos(nt-\delta)\,dt\\ &\textstyle+\int_0^{2\pi}g(x(t))\cos(nt-\delta)\,dt=\sqrt{A_n^2+B_n^2}\,.\end{aligned}$$

Integrating by parts, we express the first term on the left as

$$\begin{aligned}&[x'(t)\cos(nt-\delta)]\,|_0^{2\pi}+n\,[x(t)\sin(nt-\delta)]\,|_0^{2\pi}\\ &=[x'(2\pi)-\eta]\cos\delta-n\,[x(2\pi)-x(\zeta)]\sin\delta\,,\end{aligned}$$

and the second term as

$$[F(x(2\pi))-F(x(0))]\cos\delta+n\int_0^{2\pi}F(x)\sin(nt-\delta)\,dt\,.$$

We combine the non-integral terms in (2.11) as $V(G(\xi))-V(\xi)$. Then (2.11) gives

$$\begin{aligned}V(G(\xi))-V(\xi)&=\textstyle\sqrt{A_n^2+B_n^2}-n\int_0^{2\pi}F(x(t))\sin(nt-\delta)\,dt\\ &\quad\textstyle-\int_0^{2\pi}g(x(t))\cos(nt-\delta)\,dt\\ &>\sqrt{A_n^2+B_n^2}-2n\,(\sup F-\inf F)-2\,(\sup g-\inf g)\geq 0\,.\end{aligned}$$

(The first inequality is strict because the functions g and F are non-constant by (1.2) and (1.3).) Hence, the condition (2.9) holds, and Proposition 5 applies, proving the unboundness of the sequence $(x(2n\pi,\xi),x'(2n\pi,\xi))$. If there was a sequence $\{t_k\}\to\infty$ with bounded ${x'}^2(t_k)+x^2(t_k)$, we would obtain a contradiction with Lemma 2.3. ♢

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