The Global Solution Set for a Class of Semilinear Problems

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Abstract

For a class of semilinear Dirichlet problems we present an exact multiplicity result. Our proof simplifies the previous one in T. Ouyang and J. Shi [11]. By an indirect argument we sidestep the necessity of proving positivity for linearized equation, which was the most difficult step in [11], as well as in the earlier paper of P. Korman, Y. Li and T. Ouyang [6].

1 Introduction

We consider a class of semilinear Dirichlet problems

$$\Delta u + \lambda f(u) = 0 \quad \text{for } |x| < R,\ u = 0 \quad \text{for } |x| = R, \tag{1.1}$$

on a ball of radius R in R^n. Here λ is a positive parameter, and the nonlinearity $f(u)$ is assumed to generalize a model case $f(u) = u(u-b)(c-u)$, with positive constants b and c, and $c > 2b$ (in case $c \leq 2b$ the problem (1.1) has no nontrivial solutions, see e.g., [6]).

We now list our assumptions on the nonlinearity $f(u)$. We assume that $f(u) \in C^2(\bar{R}_+)$, and it has the following properties

$$f(0) = f(b) = f(c) = 0 \text{ for some constants } 0 < b < c, \tag{1.2}$$

$$(1.3)\qquad \begin{aligned} f(u) < 0 \ \text{ for } \ u \in (0,b) \cup (c,\infty),\\ f(u) > 0 \ \text{ for } \ u \in (-\infty,0) \cup (b,c), \end{aligned}$$

$$(1.4)\qquad \int_0^c f(u)du > 0,$$

(1.5) There exists an $\alpha \in (0,c)$, such that

$$f''(u) > 0 \ \text{ for } u \in (0,\alpha) \text{ and } f''(u) < 0 \text{ for } u \in (\alpha,c).$$

We define θ to be the smallest positive number, such that $\int_0^\theta f(s)\,ds = 0$. Clearly, $\theta \in (b,c)$. After T. Ouyang and J. Shi [11], we set $\rho = \alpha - \frac{f(\alpha)}{f'(\alpha)}$ (i.e. ρ is the first Newton iterate when solving $f(u) = 0$ with the initial guess α). We define $K(u) = \dfrac{uf'(u)}{f(u)}$. Our final assumption is the following. If $\theta < \rho$ we assume that

$$(1.6)\qquad \begin{aligned} K(u) > K(\theta) \quad &\text{on } (b,\theta)\\ K(u) \text{ is nonincreasing on } &(\theta,\rho)\\ K(u) < K(\rho) \quad &\text{on } (\rho,\alpha). \end{aligned}$$

(If $\theta \geq \rho$ this assumption is empty.)

We are now ready to state the main result.

Theorem 1.1 *Assume that $f(u)$ satisfies the conditions listed above. For the problem (1.1) there is a critical $\lambda_0 > 0$ such that the problem (1.1) has exactly* 0, 1 *or* 2 *nontrivial solutions, depending on whether $\lambda < \lambda_0$, $\lambda = \lambda_0$ or $\lambda > \lambda_0$. Moreover, all solutions lie on a single smooth solution curve, which for $\lambda > \lambda_0$ has two branches denoted by $0 < u^-(r,\lambda) < u^+(r,\lambda)$, with $u^+(r,\lambda)$ strictly monotone increasing in λ and* $\lim_{\lambda\to\infty} u^+(r,\lambda) = c$ *for $r \in [0,1)$. For the lower branch* $\lim_{\lambda\to\infty} u^-(r,\lambda) = 0$ *for $r \neq 0$, while $u^-(0,\lambda) > b$ for all $\lambda > \lambda_0$.*

In present generality this theorem was proved first by T. Ouyang and J. Shi [11]. In two dimensions (with some extra assumptions on $f(u)$) this theorem was proved in P. Korman, Y. Li and T. Ouyang [6], where the general scheme for proving such results was developed. One of the crucial things in that approach was proving positivity of any non-trivial solution of the linearized problem

$$(1.7)\qquad \Delta w + \lambda f'(u)w = 0 \quad \text{for } |x| < R,\ w = 0 \quad \text{for } |x| = R.$$

This turned out to be a difficult task, and it was the only reason the paper [6] was restricted to two dimensions. Later, T. Ouyang and J. Shi [11] were able to prove that $w(r) > 0$ by using Pohozhaev type identity. Their proof is rather involved. We also mention that one-dimensional version of this result was proved in P. Korman, Y. Li and T. Ouyang [7], where more general nonlinearities of the type $f(u) = (u-a)(u-b)(c-u)$ were considered, and where references to earlier work in case $n = 1$ by J. Smoller and A. Wasserman, and S.-H. Wang can be found.

In this work by using an indirect argument, we are able to avoid having to prove that $w(r) > 0$, which considerably simplifies the proof, and makes it more transparent. We show that it suffices to prove that $w(r)$ cannot vanish exactly once. We show that our assumptions on $f(u)$ make the function $\frac{uf'(u)}{f(u)}$ behave almost the same way as in the important paper of M.K. Kwong and L. Zhang [8], and then our proof that $w(r)$ cannot vanish exactly once is similar to Lemma 8 of [8].

We outline our arguments next. It is known that for large λ our problem (1.1) has a positive solution. When continued for increasing λ this solution, after possibly some turns, has to tend to c as $\lambda \to \infty$. When continued for decreasing λ this solution has to turn, since no positive solutions exist for $\lambda > 0$ small. The lower end of our solution curve, after possibly some turns, has to tend to 0 as $\lambda \to \infty$. If one assumes that $w(r) > 0$ at any one of the turns, we show that the result follows. It is important on this step that $w(r)$ cannot vanish exactly once. It then remains to consider the case when condition $w(r) > 0$ is violated at all turning points. Assume for simplicity there is only one turning point on the solution curve. Since condition $w(r) > 0$ is violated, it follows by the Crandall-Rabinowitz bifurcation theorem (which is recalled below) that the lower and upper solution branches intersect near the turning point. By uniqueness for initial-value problem these branches would have to intersect for all λ. But the upper branch tends to c, while the lower one tends to zero, and hence they have to separate eventually, a contradiction. In case of more than one turning point, the argument is more involved, although the idea is similar.

Next we state a bifurcation theorem of Crandall-Rabinowitz [1].

Theorem 1.2 *[1] Let X and Y be Banach spaces. Let $(\overline{\lambda}, \overline{x}) \in \mathbf{R} \times X$ and let F be a continuously differentiable mapping of an open neighborhood of $(\overline{\lambda}, \overline{x})$ into Y. Let the null-space $N(F_x(\overline{\lambda}, \overline{x})) = span\,\{x_0\}$ be one-dimensional and codim $R(F_x(\overline{\lambda}, \overline{x})) = 1$. Let $F_\lambda(\overline{\lambda}, \overline{x}) \notin R(F_x(\overline{\lambda}, \overline{x}))$. If Z is a complement of* *span* $\{x_0\}$ *in* X, *then the solutions of* $F(\lambda, x) = F(\overline{\lambda}, \overline{x})$ *near* $(\overline{\lambda}, \overline{x})$ *form a curve* $(\lambda(s), x(s)) = (\overline{\lambda}+\tau(s), \overline{x}+sx_0+z(s))$, *where* $s \to (\tau(s), z(s)) \in \mathbf{R}\times Z$ *is a continuously differentiable function near* $s = 0$ *and* $\tau(0) = \tau'(0) = 0$, $z(0) = z'(0) = 0$.

Throughout the paper we consider only the classical solutions of (1.1). Without loss of generality we set $R = 1$. Also notice that by the maximum principle all non-trivial solutions of (1.1) are positive.

2 Preliminary results

We list some consequences of our conditions on $f(u)$. We define $\beta > 0$ to be the unique number where $f'(\beta) = \dfrac{f(\beta)}{\beta}$. Clearly, $\beta \in (\alpha, \gamma)$, where γ is the larger root of $f'(u) = 0$. The following lemma was proved in [6].

Lemma 2.1 *We have*

$$uf'(u) - f(u) \begin{cases} > 0, & \text{for } u \in (0, \beta) \\ < 0 & \text{for } u \in (\beta, c). \end{cases} \tag{2.1}$$

Lemma 2.2 $K(0) = 1$, $K(u) < 1$ *on* $(0, b)$.

Proof: The first statement follows by L'Hospital rule. Notice next that for $u < b$ we have $f(u) < 0$, and also by the previous lemma $f'(u)u > f(u)$. It follows that $K(u) < 1$.

Lemma 2.3 $K'(u) < 0$ *on* (α, β).

Proof: Compute

$$K'(u) = \frac{uf''f + f'(f - uf')}{f^2}.$$

The first term in the numerator is negative for $u > \alpha$, and the second one is negative by Lemma 2.1 (notice that $f'(u) > 0$ on (α, β)).

Lemma 2.4 $K(u) < 1$ *on* (β, c).

Proof: On (β, c) we have $f(u) > 0$, and $f'(u)u < f(u)$ by Lemma 2.1, and the proof follows.

Lemma 2.5 *If $\theta < \rho$ then $K(\rho) > 1$.*

Proof: By the definition of ρ

$$f'(\alpha)(\alpha - \rho) - f(\alpha) = 0.$$

Using this and our last condition in (1.6),

$$K(\rho) > K(\alpha) = 1 + \frac{\rho f'(\alpha)}{f(\alpha)} > 1,$$

proving the lemma.

The lemmas above imply the following result.

Theorem 2.1 *Assume $\theta < \rho$. For any $u_0 \in (\theta, \rho)$ we define $\gamma = K(u_0)$. Then $\gamma > 1$, and*

$$uf'(u) - \gamma f(u) \begin{cases} > 0, & \text{for } u \in (0, u_0) \\ < 0 & \text{for } u \in (u_0, c). \end{cases} \tag{2.2}$$

Proof: The above lemmas imply that the horizontal line $y = \gamma$ intersects the graph of $y = K(u)$ exactly once, and the graph of $K(u)$ lies above the line $y = \gamma$ in the region where $f(u) > 0$. This proves the first inequality in (2.2), and second one follows similarly.

We study multiplicity of positive solutions of the Dirichlet problem, depending on a positive parameter λ

$$\Delta u + \lambda f(u) = 0 \text{ for } |x| < 1, \quad u = 0 \text{ on } |x| = 1, \tag{2.3}$$

with nonlinearity $f(u)$ satisfying all of our assumptions. By the classical theorem of B. Gidas, W.-M. Ni and L. Nirenberg [3] positive solutions of (2.3) are radially symmetric, which reduces (2.3) to

$$u'' + \frac{n-1}{r}u' + \lambda f(u) = 0 \text{ for } 0 < r < 1, \quad u'(0) = u(1) = 0. \tag{2.4}$$

We shall also need the corresponding linearized equation

$$w'' + \frac{n-1}{r}w' + \lambda f'(u)w = 0 \text{ for } 0 < r < 1, \quad w'(0) = w(1) = 0. \tag{2.5}$$

The following lemma was proved in [4].

Lemma 2.6 *Assume that the function $f(u) \in C^2(\bar{R}_+)$, and the problem (2.5) has a nontrivial solution w at some λ. Then*

$$\int_0^1 f(u) w r^{n-1}\, dr = \frac{1}{2\lambda} u'(1) w'(1). \tag{2.6}$$

We recall that solution of (2.4) is called singular provided the corresponding linearized problem (2.5) has a nontrivial solution. The following lemma follows immediately from the equations (2.4) and (2.5).

Lemma 2.7 *Let (λ, u) be a singular solution of (2.4). Then*

$$\int_0^1 \left(f(u) - f'(u)u\right) w r^{n-1}\, dr = 0. \tag{2.7}$$

The following lemma is a consequence of the previous two.

Lemma 2.8 *Let (λ, u) be a singular solution of (2.4). Then for any real γ*

$$\int_0^1 \left(\gamma f(u) - f'(u)u\right) w r^{n-1}\, dr = \frac{\gamma - 1}{2\lambda} u'(1) w'(1). \tag{2.8}$$

Proof. Multiplying (2.6) by $\gamma - 1$, and adding (2.7), we obtain (2.8).

The following lemma is known, see e.g. E.N. Dancer [2]. We present its proof for completeness.

Lemma 2.9 *Positive solutions of the problem (2.4) are globally parameterized by their maximum values $u(0, \lambda)$. I.e., for every $p > 0$ there is at most one $\lambda > 0$, for which $u(0, \lambda) = p$.*

Proof. If $u(r, \lambda)$ is a solution of (2.4) with $u(0, \lambda) = p$, then $v \equiv u(\frac{1}{\sqrt{\lambda}} r)$ solves

$$v'' + \frac{n-1}{r} v' + f(v) = 0, \quad v(0) = p,\ v'(0) = 0. \tag{2.9}$$

If $u(0, \mu) = p$ for some $\mu \neq \lambda$, then $u(\frac{1}{\sqrt{\mu}} r)$ is another solution of the same problem. This is a contradiction, in view of the uniqueness of solutions for initial value problems of the type (2.21), see [12].

The following lemma restricts the region where $w(r)$, solution of the linearized problem (2.5), may vanish. Its first part is due to T. Ouyang and J. Shi [11], see also J. Wei [13], and its second part is due to M.K. Kwong and L. Zhang [8].

Lemma 2.10 *Any nontrivial solution of (2.5) cannot vanish in either interval where* $0 < u < \theta$, *and where* $\rho < u < 1$.

In case $\theta \geq \rho$ it follows that any nontrivial solution of (2.5) is positive, and the main result of the present paper then follows similarly to [6].

The following lemma follows the idea of Lemma 8 of M.K. Kwong and L. Zhang [8].

Lemma 2.11 *Under our conditions on* $f(u)$ *any non-trivial solution of (2.5)* $w(r)$ *cannot have exactly one zero on* $(0, 1)$.

Proof. Since $w(0) \neq 0$, see [12] for the appropriate uniqueness result (if $w(0) = w'(0) = 0$ then $w \equiv 0$), we may assume that $w(0) > 0$. Assume that on the contrary $w(r)$ has exactly one root at some $r = r_0$, i.e.

$$w(r) > 0 \text{ on } (0, r_0), \quad w(r) < 0 \text{ on } (r_0, 1). \tag{2.10}$$

By Lemma 2.10 $u(r_0) \in (\theta, \rho)$. Setting $\gamma = K(u(r_0))$, we see by Theorem 2.1 that

$$\gamma f(u) - uf'(u) \begin{cases} < 0 & \text{for all } \; u < u(r_0) \\ > 0 & \text{for all } \; u > u(r_0). \end{cases} \tag{2.11}$$

Since $\gamma > 1$, we obtain by Lemma 2.8 (notice that $w'(1) > 0$ by (2.10))

$$\int_0^1 [\gamma f(u) - uf'(u)]\, w(r) r^{n-1}\, dr < 0. \tag{2.12}$$

In view of (2.10) and (2.11) the quantity on the left is positive, and we have a contradiction in (2.12).

Lemma 2.12 *Let* $u(r, \lambda)$ *and* $v(r, \lambda)$ *be two solution curves of (2.4), which are continuous in* λ, *when the parameter* λ *varies in some interval* I. *Assume that for some* $\lambda_0 \in I$ *solutions* $u(r, \lambda_0)$ *and* $v(r, \lambda_0)$ *intersect. Then* $u(r, \lambda)$ *and* $v(r, \lambda)$ *intersect for all* $\lambda \in I$.

Proof: In order for the solution curves to separate, there must exist λ_1 (the last λ at which they intersect) and a point $r_1 \in [0, 1]$ at which $u(r_1, \lambda_1) = v(r_1, \lambda_1)$ and $u_r(r_1, \lambda_1) = v_r(r_1, \lambda_1)$. But this contradicts uniqueness for initial value problems.

Next we study the linearized eigenvalue problem corresponding to any solution of (2.4):

$$(2.13)\varphi'' + \frac{n-1}{r}\varphi' + \lambda f'(u)\varphi + \mu\varphi = 0 \text{ on } (0,1), \ \ \varphi'(0) = \varphi(1) = 0.$$

Comparing this to (2.5), we see that at any singular solution of (2.4) $\mu = 0$ is an eigenvalue, corresponding to an eigenfunction $\varphi = w$.

We shall need the following generalization of Lemma 2.6.

Lemma 2.13 *Let $\varphi > 0$ be a solution of (2.13) with $\mu \le 0$. (I.e. φ is a principal eigenfunction of (2.13).) Then*

$$(2.14) \qquad \int_0^1 f(u)\varphi r^{n-1} dr \ge \frac{1}{2\lambda} u'(1)\varphi'(1).$$

Proof. The function $v = ru_r - u_r(1)$ satisfies

$$(2.15)\ \Delta v + \lambda f'(u)v + \mu v = \mu v - 2\lambda f(u) - \lambda f'(u)u'(1) \text{ for } |x| < 1,$$
$$v = 0 \text{ on } |x| = 1.$$

Comparing (2.15) with (2.13) we conclude by the Fredholm alternative

$$(2.16) \qquad \mu \int_0^1 v\varphi r^{n-1} dr - 2\lambda \int_0^1 f(u)\varphi r^{n-1} dr - \lambda u'(1) \int_0^1 f'(u)\varphi r^{n-1} dr = 0.$$

Integrating (2.13)

$$-\lambda \int_0^1 f'(u)\varphi r^{n-1} dr = \varphi'(1) + \mu \int_0^1 \varphi r^{n-1} dr.$$

Using this in (2.16), we have

$$2\lambda \int_0^1 f(u)\varphi r^{n-1} dr = \mu \int_0^1 ru_r\varphi r^{n-1} dr + u'(1)\varphi'(1) + \mu u'(1) \int_0^1 \varphi r^{n-1} dr,$$

and the proof follows.

We now define Morse index of any solution of (2.4) to be the number of negative eigenvalues of (2.13). The following lemma is based on K. Nagasaki and T. Suzuki [10].

Lemma 2.14 *Assume that* (λ, u) *is a singular solution of (2.4) such that* $w'(1) < 0$ *and*

$$\int_0^1 f''(u)w^3 r^{n-1} dr < 0. \tag{2.17}$$

Then at (λ, u) *a turn to "the right" in* (λ, u) *"plane" occurs, and as we follow the solution curve in the direction of decreasing* $u(0, \lambda)$*, the Morse index is increased by one.*

Proof. To see that the turn is to the right, we observe that the function $\tau(s)$, defined in Crandall-Rabinowitz theorem, satisfies $\tau(0) = \tau'(0) = 0$ and

$$\tau''(0) = -\frac{\lambda \int_0^1 f''(u)w^3 r^{n-1} dr}{\int_0^1 f(u)wr^{n-1} dr}, \tag{2.18}$$

see [6] for more details. By our assumption the numerator in (2.18) is negative, while by Lemma 2.2 the denominator is positive. It follows that $\tau''(0) > 0$, and hence $\tau(s)$ is positive for s close to 0, which means that the turn is to the right.

At a turning point one of the eigenvalues of (2.13) is zero. Assume it is the ℓ-th one, and denote $\mu = \mu_e$. Here $\mu = \mu(s)$, and $\mu(0) = 0$. We now write (2.13) in the corresponding PDE form and differentiate this equation in s

$$\Delta\varphi_s + \lambda f'(u)\varphi_s + \lambda' f'(u)\varphi + \lambda f''(u)u_s\varphi + \mu'\varphi + \mu\varphi_s = 0 \quad \text{for } |x| < 1, \quad \varphi_s = 0 \text{ on } |x| = 1. \tag{2.19}$$

At (λ, u) the Crandall-Rabinowitz theorem applies, and hence we have: $\mu(0) = 0$, $\varphi(0) = w$, $\lambda'(0) = 0$, and $u_s(0) = -w$ (considering the chosen parameterization). Here w is a solution of the linearized equation (2.5). The equation (2.19) becomes

$$\Delta\varphi_s - \lambda f''(u)w^2 + \lambda f'(u)\varphi_s + \mu'(0)w = 0. \tag{2.20}$$

Multiplying (2.5) by φ_s, (2.20) by w, subtracting and integrating, we have

$$\mu'(0) = \frac{\lambda \int_0^1 f''(u)w^3 r^{n-1} dr}{\int_0^1 w^2 r^{n-1} dr} < 0.$$

It follows that across the turning point one of the positive eigenvalues crosses into the negative region, increasing the Morse index by one.

Lemma 2.15 *Assume that (λ_0, u_0) is a singular solution of (2.4), i.e. the problem (2.5) has a nontrivial solution $w(r)$. Then*

$$w(r) > 0 \quad \text{for all } r \in [0,1) \tag{2.21}$$

if and only if for λ close to λ_0 any two solutions on the solution curve passing through (λ_0, u_0) do not intersect.

Proof: In view of Lemma 2.6 the Crandall-Rabinowitz theorem applies at (λ_0, u_0) (see [6] for more details). According to that theorem near the point (λ_0, u_0) solutions differ asymptotically by a factor of $w(r)$, which implies the lemma.

The following lemma was proved in [6], see also [11] and [13].

Lemma 2.16 *Assume that (λ_0, u_0) is a singular solution of (2.4), such that (2.21) holds. Then the inequality (2.17) holds, and the conclusions of the Lemma 2.14 apply.*

Next we study eigenvalues and eigenfunctions of radial solutions of Laplace equation on a ball. Since singularity at $r = 0$ is introduced by the polar coordinates, and is not present in the original equation, it is natural to expect spectral properties similar to that of regular Sturm-Liouville problems. Surprisingly, we were not able to find any references.

Lemma 2.17 *Consider an eigenvalue problem*

$$y'' + \frac{a}{r}y' + b(r)y + \lambda y = 0, \quad \text{for } 0 < r < 1, \; y'(0) = y(1) = 0, \tag{2.22}$$

with a constant $a > 1$, and $b(r) \in C^2[0,1]$. Assume that $\lambda = 0$ is an eigenvalue of (2.22), and let $y_0(r)$ be the corresponding eigenfunction. Then the problem (2.22) has an infinite sequence of eigenvalues $\lambda_1 < \lambda_2 < \ldots$, with $\lambda_n \to \infty$ as $n \to \infty$, and the n-th eigenfunction has precisely $n-1$ roots on $(0,1)$ for all $n \geq 1$. (One of λ_k's is equal to zero.)

Proof: We convert the problem (2.22) into an integral equation, using the modified Green's function. We claim that any solution of the equation

$$y'' + \frac{a}{r}y' + b(r)y = 0, \tag{2.23}$$

that is bounded at $r = 0$ must be a multiple of the first eigenfunction $y_0(r)$. Indeed, writing the first two terms of the Taylor's series of the solution with a remainder term, we easily conclude that $y'(0) = 0$ for any bounded solution. If we now fix a constant α so that $y(0) = \alpha y_0(0)$, then we shall have $y(r) = \alpha y_0(r)$ for all $r > 0$, in view of uniqueness for initial value problems of the type (2.23), see [12]. Let $y_2(r)$ be a solution of (2.23) with $y_2(1) = 1$. Since y_2 is not a multiple of y_0, it follows that $y_2(r) \to \infty$ as $r \to +0$. A formal use of Frobenius method at $r = 0$ shows that $y_2 \sim \beta r^{-a+1}$ as $r \to 0$, with some constant β. Setting $y(r) = r^{-a+1}z(r)$, we see that the resulting equation for $z(r)$ has all solutions bounded near $r = 0$, which justifies the asymptotic formula for $y_2(r)$ near $r = 0$.

Notice that the problem (2.23) can be put into an equivalent self-adjoint form

$$(r^a y')' + r^a b(r) y = 0. \tag{2.24}$$

The modified Green's function for (2.24) subject to the boundary conditions $y'(0) = y(1) = 0$ has the form

$$G(r,\xi) = \begin{cases} \frac{y_0(r)y_2(\xi)}{K} & \text{for} \quad r < \xi \\ \frac{y_0(\xi)y_2(r)}{K} & \text{for} \quad r > \xi, \end{cases} \tag{2.25}$$

where K is a constant. By the above remarks we have, with some constant $c > 0$,

$$\begin{aligned} |G(r,\xi)| &\le c\xi^{1-a} \ \text{ for } r < \xi \\ |G(r,\xi)| &\le cr^{1-a} \ \text{ for } r > \xi. \end{aligned} \tag{2.26}$$

We now multiply the equation (2.22) by r^a, and convert it into an integral equation for the function $z(r) = r^{\frac{a}{2}} y(r)$

$$z(r) = \lambda \int_0^1 \bar{G}(r,\xi) z(\xi)\, d\xi, \tag{2.27}$$

with the kernel $\bar{G}(r,\xi) = G(r,\xi) r^{\frac{a}{2}} \xi^{\frac{a}{2}}$. Using (2.26) it is a standard exercise to show that $\int_0^1 \int_0^1 \bar{G}^2 \, dr d\xi < \infty$, see pages 178 and 421 in [14]. This means that (2.27) is an integral equation with a compact and symmetric kernel. It follows that its spectrum is discrete, and eigenvalues tend to infinity. Moreover, we conclude that the minimum characterization of eigenvalues applies, from which it follows that the k-th eigenfunction cannot have more than $k-1$ interior roots, see p. 173 in [14]. On the other hand, the same minimum characterization implies that y_1 is of one sign, and y_2 must vanish at least once. Also, by Sturm's comparison theorem y_{k+1} must have at least one more interior root than y_k. We then conclude that y_2, then y_3, and so on have the desired number of interior roots.

3 Proof of the main result

We are now ready to prove the Theorem 1.1. We begin by noticing that existence of positive solutions under our conditions follows by the Theorem 1.5 in P.L. Lions [9], see also [11]. Indeed the result in [9] implies existence of a critical $\bar{\lambda}$, so that for $\lambda \geq \bar{\lambda}$ there exists a maximal positive solution of (2.3), while for $\lambda > \bar{\lambda}$ there exists at least two positive solutions. Since positive solutions of our problem (2.3) are radial, we consider its ODE version (2.4). We now continue the curve of maximal solutions for decreasing λ. It was shown in [6] that this curve cannot be continued for all $\lambda > 0$, and hence a critical point (λ_0, u_0) must be reached, at which the curve will turn. By the definition of a critical point, the linearized equation (2.5) has a nontrivial solution $w(r)$. We claim that the theorem follows provided that

$$w(r) > 0 \quad \text{for all } r \in [0,1). \tag{3.1}$$

By the Crandall-Rabinowitz Theorem near the turning point (λ_0, u_0) the solution set has two branches $u^-(r,\lambda) < u^+(r,\lambda)$, for $r \in [0,1)$, $\lambda > \lambda_0$. By the Crandall-Rabinowitz Theorem we also conclude

$$u_\lambda^+(r,\lambda) > 0 \ \text{ for } \lambda \text{ close to } \lambda_0 \ (\text{for all } r \in [0,1)). \tag{3.2}$$

Arguing like in P. Korman, Y. Li and T. Ouyang [6], we show that the same inequality holds for all $\lambda > \lambda_0$ (until a possible turn), see also T. Ouyang and J. Shi [11] and J. Wei [13]. We claim next that solutions $u^+(r,\lambda)$ are stable, i.e. all eigenvalues of (2.13) are positive. Indeed, let on the contrary $\mu \leq 0$ be the principal eigenvalue of (2.13), and $\varphi > 0$ the corresponding eigenvector. The equation for u_λ is

$$\begin{aligned} u_\lambda'' + \tfrac{n-1}{r} u_\lambda' + \lambda f'(u) u_\lambda + f(u) = 0 \ \text{ for } r \in (0,1], \\ u_\lambda'(0) = u_\lambda(1) = 0. \end{aligned} \tag{3.3}$$

From the equations (2.13) and (3.3) we obtain

$$r^{n-1}(\varphi' u_\lambda - u_\lambda' \varphi)|_0^1 = -\mu \int_0^1 \varphi u_\lambda r^{n-1} dr + \int_0^1 f(u) \varphi r^{n-1} dr. \tag{3.4}$$

The right hand side in (3.4) is positive by our assumptions, inequality (3.2), and Lemma 2.13, while the quantity on the left is zero, a contradiction.

We show next that for $\lambda > \lambda_0$ both branches $u^+(r,\lambda)$ and $u^-(r,\lambda)$ have no critical points. Indeed, if we had a critical point on the upper branch $u^+(r,\lambda)$ at some $\overline{\lambda} > \lambda_0$, then by the Crandall-Rabinowitz Theorem solution of the linearized equation would be positive at $\lambda = \overline{\lambda}$ (since $u_\lambda > 0$ as we enter the critical point). But then by Lemma 2.14 we know precisely the structure of solution set near $(\overline{\lambda}, u^+(r,\overline{\lambda}))$, namely it is a parabola-like curve with a turn to the right. This is impossible, since the solution curve has arrived at this point from the left. Turning to the lower branch $u^-(r,\lambda)$, we know by Lemma 2.14 that each solution on this branch has Morse index of one, until a possible critical point. At the next possible turning point one of the eigenvalues becomes zero, which means that the Morse index of the turning point is either zero or one. If Morse index is zero, it means that zero is a principal eigenvalue, and so solutions of the corresponding linearized equation are of one sign, and then we obtain a contradiction the same way as on the upper branch. If Morse index $= 1$, it means that zero is a second eigenvalue, i.e. by Lemma 2.17 $w(r)$ changes sign exactly once, but that is impossible by Lemma 2.11. It follows that if condition (3.1) is satisfied at the first turning point, then our Theorem 1.1. follows. But exactly the same arguments show that having $w(r) > 0$ at any turning point will imply Theorem 1.1.

It remains to rule out the possibility that condition (3.1) fails at all turning points. By Lemma 2.15 this means that the branches $u^-(r,\lambda)$ and $u^+(r,\lambda)$ intersect near any turning point (λ_0, u_0). When we continue the upper branch $u^+(r,\lambda)$ for increasing λ, then, after possibly some more turns, $u^+(r,\lambda) \to c$ as $\lambda \to \infty$ for all $r \in [0,1)$, see [6] for more details. Similarly, for the lower branch we have $u^-(r,\lambda) \to 0$ as $\lambda \to \infty$ for all $r \in (0,1)$, after possibly some additional turns, see [6]. (Notice that $u^-(0,\lambda) > \theta$.) It follows that for λ sufficiently large

$$(3.5) \qquad u^-(r,\lambda) < u^+(r,\lambda) \quad \text{for all } r \in [0,1).$$

We now pick the leftmost turning point on our curve (i.e. the turning point with smallest λ; if there is more than one such point, take any one of them). In Figure 1 this is the point A. By above, condition (3.1) is violated at this point, and hence solution branches contain intersecting solutions near A. As we increase λ solutions on both branches continue to intersect by Lemma 2.12, until a possible turning point. If both branches have no more

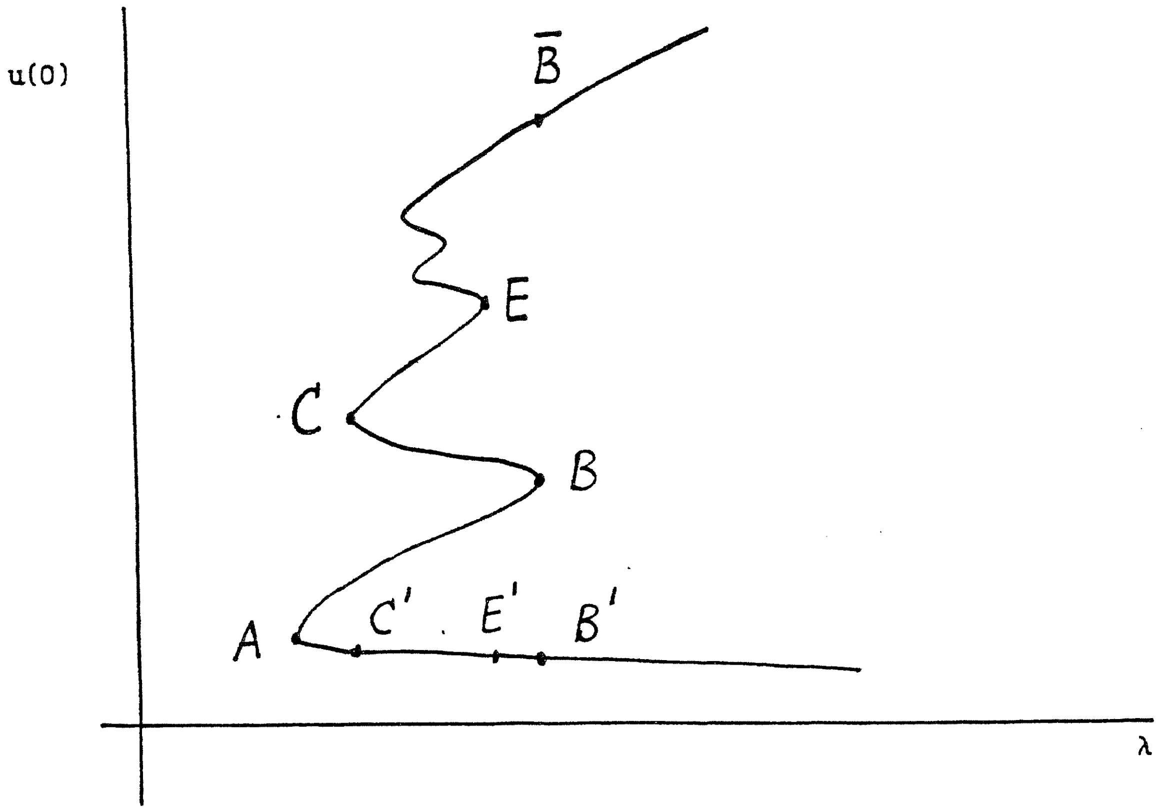

Figure. 1. Solution curve with several turning points.

turning points, solutions will intersect for all λ, contradicting (3.5). Assume that as we continue both branches $u^-(r,\lambda)$ and $u^-(r,\lambda)$ for increasing λ the first turning point happens, say, at the upper branch at a point B. By B' we denote the point on the lower branch, which has the same λ coordinate as B. By Lemma 2.12 solutions at B and B' intersect. We now continue the upper branch for decreasing λ until the next turning point, which we call C. By C' we denote the point on the lower branch, which has the same λ coordinate as C. Moving leftwards on both branches, we conclude by Lemma 2.12 that solutions at C and C' intersect. We denote by E the next turning point on the upper branch (if it exists), and by E' the corresponding point under it on the lower branch. By moving to the right on both branches and using Lemma 2.12 , we conclude that solutions at E and E' intersect. We continue the process until the upper branch passes over B for the last time at a point $\bar{B}$. We conclude that solutions at B' and $\bar{B}$ intersect. We now resume moving forward in λ on both lower and upper branches. If another turning point is encountered, we repeat the above procedure. We conclude that solutions on upper and lower branches corresponding to the same λ intersect for all λ. This contradicts (3.5). We conclude that $w(r) > 0$ at any turning point, and the theorem follows.

Remark. After completing the proof, we conclude that the solution curve has exactly one turn, and $w(r) > 0$ there. This is simpler than the previous strategy (of [6], [11] and [13]) of directly proving that $w(r) > 0$.

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