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$B(H)$-Commutators: A Historical Survey

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Abstract. This is a historical survey that includes a progress report on the 1971 seminal paper of Pearcy and Topping and 32 years of subsequent investigations by a number of researchers culminating in a completely general characterization, for arbitrary ideal pairs, of their commutator ideal in terms of arithmetic means.

This characterization has applications to the study of generalized traces, linear functionals vanishing on a certain commutator ideal, and to the study of the $B(H)$-ideal lattice and certain special sublattices. The structure of commutator ideals is essential for investigating traces which in turn is relevant for the calculation of the cyclic homology and the algebraic K-theory of operator ideals.

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Commutators, linear operators of the form $AB - BA$, first appeared in physics, for instance in a mathematical formulation of Heisenberg's Uncertainty Principle ([17]). A simple concrete example is the product rule in calculus applied to xf expressed in terms of operators: $I = \frac{d}{dx}M_x - M_x\frac{d}{dx}$ where the operators act on the class of differentiable functions. The situation changes in $B(H)$, that is, when the operators act boundedly on a Hilbert space. Wintner ([32]) and Wielandt ([31]) in 1947 and 1949, respectively, gave two elegant distinct proofs that the identity is not a commutator of two bounded linear operators on a Hilbert. Both apply also to arbitrary complex normed algebras with unit, except Wintner's proof requires that the norm be complete. For the period preceding 1967, the Hilbert space problem book ([15], Chapter 24) provides a brief history of $B(H)$-commutators including their proofs. This survey starts with an elementary description of the subject similar to the viewpoint held by the author in the 1970's and continues with a report on the main contributions including references and some open problems spanning 1971–2003 from which our deeper understanding of the subject evolved.

$AB-BA$ (also denoted by $[A, B]$) represents in a sense the "degree" to which A and B do not commute, either via norm (the operator norm or some other norm, i.e., quantitative measures) or via the commutator's containment in a two-sided ideal (qualitative measures). (All ideals herein are two-sided $B(H)$-ideals where H is a separable, infinite-dimensional, complex Hilbert space.) For instance, one can study conditions for containment in the class $F(H)$ of finite rank operators, the smallest nonzero ideal in the lattice of all $B(H)$-ideals, or containment in the class $K(H)$ of compact operators, the largest proper ideal. The latter is equivalent to commuting when projected canonically into the Calkin algebra, $B(H)/K(H)$, commonly called "commuting modulo the compact operators." The Calkin algebra being a unital normed algebra, Wintner and Wielandt reveals that its identity is not a commutator. This translates in $B(H)$ immediately to the fact that the "thin" operators, $\lambda I + K$ where $\lambda \neq 0$ and $K \in K(H)$, are noncommutators.

Brown and Pearcy ([5]) in 1965 determined the structure of all commutators in $B(H)$ by proving that the thin operators are the only noncommutators. This then determines as well the commutators in the Calkin algebra – all except the nonzero scalars. (See also [15].)

Theorem 0.1 (Brown and Pearcy, 1965). *An operator is a commutator if and only if it is not thin.*

An approximation consequence of Theorem 0.1 is that thin operators are norm-close to commutators since thin operators are easily seen to be norm-close to non-thin operators. But trying to get norm-close to the identity with a sequence of commutators can only be achieved if the norms of the operators in the commutators are not all uniformly bounded. (Cf., [15].) In fact, $A_nB_n - B_nA_n \to I$ in the $B(H)$-norm implies $\max(\|A_nB_n\|, \|B_nA_n\|) \to \infty$. This provides a hint of what was to come. In a sense, it tells us that to obtain the identity as a commutator it is necessary in one's matrix design to "spread out" in order to exploit a lot of cancellation. An example of this phenomenon is described in the following paragraph. In later developments on solving commutator equations, various kinds of spreading out matrix designs were discovered that provided quantitative control (i.e., of various norms) and qualitative control (i.e., forcing membership in ideals-the smaller the ideal the better). (Cf., [27], [1], [29], [2], [3], [19] and [8].)

In finite-dimensional Hilbert space, the trace distinguishes between commutators and noncommutators. There the trace of a product of two operators is independent of their order (unlike three or more) and so, there, commutators have trace zero. Moreover having a nonzero trace is the only obstruction to being a commutator. Likewise for infinite-dimensions for the analogous class, $F(H)$. That is, *on infinite dimensional Hilbert space, a finite rank operator is a commutator if and only if it has trace zero.* That the infinite dimensional case is different is evident from Theorem 0.1, but more simply from the fact that the self-commutator of the unilateral shift, $U^*U - UU^* = \mathrm{diag}(1, 0, \dots)$, the diagonal matrix with diagonal sequence $(1, 0, \dots)$, is a rank one projection operator with trace one. This commutator provided a beginning framework from which to approach many matrix design problems in the subject and is therefore one of the underlying themes of this survey. For instance, since the unilateral shift is not compact, in order to express the rank one projection $\mathrm{diag}(1, 0, \dots)$ as a commutator or sum of commutators of compact operators, a key step in [25], another approach needed to be found. An illuminating and more transparent construction leading to its representation as such a sum follows from the identity illustrating the matricial spreading out phenomenon mentioned above:

$$\operatorname{diag}(1, 0, \dots) = 1 \oplus \Sigma^{\oplus}(-x_n D) + 0 \oplus \Sigma^{\oplus} x_n D$$

where $D := \operatorname{diag}(1, -1/2, -1/2)$ and $< x_n >$ is the sequence $\frac{1}{2^n}$-repeated 2^n-times for $n \geq 0$ (cf. [27], Chapter 1.3, pp. 34–40.) Noting that D is simply a self-commutator of a 3×3 weighted shift with weights $1, \frac{1}{\sqrt{2}}$ and that $\langle x_n \rangle \asymp \langle \frac{1}{n} \rangle$ (the harmonic sequence), each summand can be represented as a commutator of compact operators. Moreover, the summands are (I, J)-commutators when the diagonal operator with eigenvalues the harmonic sequence is contained in the ideal IJ (for notations see: the third paragraph below and the explanation following Theorem 2.3). That containment of $\operatorname{diag}(1, 1/2, 1/3, \dots)$ is necessary followed subsequently as one application of Theorem 2.3 (see the section below on Traces and Arithmetic Means, fourth paragraph).

When the trace of a trace class commutator must be zero is a deep question (cf. [13], Section III.8 for background). If AB and BA for instance are in the trace class (the trace ideal C_1), then their commutator has trace zero (cf., [21], Lemma 2.1). I know of no weaker general condition on AB and BA that would insure that $Tr\ (AB - BA) = 0$ whenever the commutator is trace class. At one extreme, it is easy to produce operators, even compact operators, with zero product but non-trace class commutator, e.g., $[\begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}]$. But when the operators are both in the Hilbert-Schmidt class (the Hilbert-Schmidt ideal C_2) or one is a trace class operator and the other a $B(H)$-operator, then their products, in either order, are trace class operators so their commutator has trace zero. Hence the questions: *Is every trace class operator with trace zero representable as a commutator of Hilbert-Schmidt operators or as a commutator of a trace class operator with a $B(H)$-operator?* I.e., is the trace the only obstruction for a trace class operator to be either type of commutator?

In 1971 Pearcy and Topping ([25]) posed four seminal problems described below that have dominated the subject of $B(H)$-commutators ever since and that led to breakthroughs in the study of generalized traces with applications to algebraic K-theory for operator ideals, e.g., the computation of certain K-groups. (For further details see [8], [2] and [3].) Indeed, the study of generalized traces depends fundamentally on understanding particularly the structure of the commutator ideals $[I, B(H)]$. This is because ideals are the natural domains of generalized traces (algebraic linear functionals that are unitarily invariant) and for algebraic linear functionals, unitary invariance is equivalent to their vanishing on the commutator ideal $[I, B(H)]$. This equivalence follows from the commutator identity $[TU^*, U] = T - UTU^*$ for T arbitrary and U unitary and from the fact that every operator is the linear combination of four unitary operators.

Commutator ideals also play a role in the Fong, Miers, Sourour characterization of the Lie ideals of $B(H)$ ([12]) where they prove: if $\mathcal{L}$ is a Lie ideal of $B(H)$ (i.e., $\mathcal{L}$ is a linear subspace of $B(H)$ containing $[\mathcal{L}, B(H)]$), then $[I, B(H)] \subseteq \mathcal{L} \subseteq I + \mathbb{C}1$ for some $B(H)$-ideal I.

For each pair of not necessarily proper $B(H)$-ideals I, J, denote by $[I, J]$ their commutator ideal, that is, the algebraic linear span of the class $[I, J]_1$ of single (I, J)-commutators (the commutators $AB - BA$ with $A \in I$ and $B \in J$). Let $[I, J]_n$ denote the class of all n-sums of (I, J)-commutators.

In 1954 ([14]) Halmos proved $[B(H), B(H)]_2 = B(H)$ and thereafter investigators on the structure of commutator ideals, once a particular commutator ideal representation was proven, would try to reduce the number of commutators needed in the sum. To date, the strongest result known on this is: $[I, J] = [I, J]_4$ (cf., [8], Corollary 6.5). Under various additional conditions the number can be further reduced. For instance, $[I, B(H)] = [I, B(H)]_3$ (cf., [8], Theorem 6.1). For more results of this type see [8], Chapter 6.

The Pearcy-Topping problems and progress to date on them are as follows.

Problem 1. (Pearcy-Topping [25], 1971) Is $K(H) = [K(H), K(H)]_1$?

In particular, is $\operatorname{diag}(1, 0, \dots)$ and hence any rank one projection operator a commutator of compact operators?

Pearcy and Topping considered this a key test question presumably because representing a rank one projection as sums of appropriate types of commutators was at the core of their proofs in [25]: that $K(H) = [K(H), K(H)]$ and that $C_p = [C_{2p}, C_{2p}]$ when $p > 1$. Until then, matrix forms designed to solve commutator equations were bounded but not compact operators, even when the target operator was compact. Indeed, representing this simplest nonzero compact operator without trace zero, $\operatorname{diag}(1, 0, \dots)$, provided the first inroad into controlling the matrix forms' membership in prescribed ideals.

Problem 2. Is $C_p = [C_{2p}, C_{2p}]_1$ when $p > 1$?

When $I \subset C_1$ let I^o denote the class of all operators in I with trace zero.

Problem 3. Is $C_1^o = [C_2, C_2]_1$? I.e., is every trace class trace zero operator a commutator of Hilbert-Schmidt operators?

Problem 3′. Is $C_1^o = [C_2, C_2]$? I.e., is every trace class trace zero operator a finite sum of commutators of Hilbert-Schmidt operators?

Problem 1 remains open and the structure of $[C_2, C_2]_1$ and $[C_1, B(H)]_1$ remain unknown. Indeed, little progress has been made on characterizing any of the single commutator classes $[I, J]_1$ or even on understanding their structure with four notable exceptions: the fundamental work of Anderson in 1977 ([1]) related contrasting work of L.G. Brown in 1994 ([4]), work related to both in [8] (Chapter 7), and the negative solution to Problem 3 in 1980 ([28]; respectively, Theorems 1.1–2.1 below).

Commutator ideals, in contrast, have seen significant progress. Of Problems 3–3′ Pearcy and Topping wrote "The techniques involved in giving an affirmative answer to this question would likely enable us to solve some stubborn problems in the theory of commutators in finite von Neumann algebras (see [24]). Problem 3 is so intractable that we cannot even answer the weaker Problem 3′".

1. Single commutators

Theorem 1.1 (Anderson, [1], 1977). *Rank one projections and more generally operators whose kernels contain infinite-dimensional reducing subspaces are commutators of compact operators.*

Consequently, every compact operator is a $(K(H), B(H))$-commutator, that is, $K(H) = [K(H), B(H)]_1$. Moreover, $C_p \subset [C_{2p}, B(H)]_1$ when $p > 1$.

Theorem 1.2 (L.G. Brown, [4], 1994). *If $A \in C_p$, $B \in C_q$, $p^{-1} + q^{-1} \geq \frac{1}{2}$ and the commutator $[A, B]$ has finite rank, then $Tr\ [A, B] = 0$.*

Theorem 1.3 (ii) below is a paired down version of a theorem in [8] (the main theorem in Chapter 7). Chapter 7 is an outgrowth of a prior question of Wodzicki on when $[I, B(H)]_1$ can contain a finite rank operator with nonzero trace. In particular, is containment in I of the diagonal operator $\operatorname{diag}\langle 1/\sqrt{n}\rangle$ necessary and sufficient. Sufficiency follows from [1] (cf., [8] (Chapter 7)). Chapter 7 provides conditions on necessity for general single commutator classes $[I, J]_1$. *Necessity remains an open question.* (The necessity of the weaker containment conditions: $\operatorname{diag}\langle 1/n\rangle \in I$, respectively IJ, follows by blending methods in [4] and [8].)

Problem 4. If $[I, J]_1$ contains a finite rank operator with nonzero trace, must IJ contain the diagonal operator $\operatorname{diag}\langle 1/\sqrt{n}\rangle$? If not, what about the cases $[I, B(H)]_1$?

Theorem 1.3 (Dykema, Figiel, Weiss and Wodzicki, [8], 2001, Theorems 7.1–7.3).

- (i) *If the diagonal operator* $\operatorname{diag}\langle 1/\sqrt{n}\rangle \in I$*, then* $[I, B(H)]_1$ *contains a finite rank operator with nonzero trace.*
- (ii) *If* $[I, B(H)]_1$ *contains a finite rank operator with nonzero trace, then* $\operatorname{diag}\langle 1/\sqrt{n}\rangle$ *is contained in the arithmetic mean closure of I (see below) and, in particular, in I itself when it is arithmetically mean closed.*
- (iii) *For every compact operator T, $T \in [(T \otimes \operatorname{diag}\langle 1/\sqrt{n}\rangle), B(H)]_1$, the first ideal being the principal ideal generated by $T \otimes \operatorname{diag}\langle 1/\sqrt{n}\rangle$.*

The arithmetic mean (am) closure of an ideal I is the smallest enveloping ideal that is solid under domination by the induced arithmetic mean sequences of the s-numbers of its operators (i.e., $T \in I$ whenever $s(T)_a \leq s(A)_a$ for some $A \in I$). This condition is equivalent to I being solid under domination in the sense of the Hardy-Littlewood-Polya-Schur majorization ($\prec$) of an operators' s-number sequences, that is, $T \in I$ whenever $s(T) \prec s(A)$ for some $A \in I$. ($s(T) \prec s(A)$ means $\sum_1^n s(T)_j \leq \sum_1^n s(A)_j\ \forall n$). This ordering for finite sequences with equality at n provided their characterization of when two nonincreasing sequences x, y of length n satisfy $\sum_1^n \phi(x_j) \leq \sum_1^n \phi(x_j)$ for every convex functions ϕ (cf., [16], Section 3.17, Theorem 108, and also for general reference, [23] beginning pp. 3–4).

An arithmetic mean closed ideal is one that is equal to its closure. (See the section below on arithmetic mean ideals for more details. Cf., [8] Theorem 7.3 and Section 2.8.) Theorem 1.3 strengthens some of the results in [1] blending some of Anderson's methods there with methods in [8].

Problems 3–3′ were settled in 1980 in the negative via Theorem 2.1 below [28]. A negative answer to Problem 3′ provides automatically the same for Problem 3. And since $C_1^o \supset [C_2, C_2]_1$ is automatic as discussed earlier, the problems reduce to whether or not $C_1^o \subset [C_2, C_2]_1$ (resp., $C_1^o \subset [C_2, C_2]$), that is, whether or not all trace class trace zero operators are commutators of Hilbert-Schmidt operators (respectively, sums of such commutators). An even more important question as it turned out is: if not, which ones are? Some of the important single commutator open problems are summarized below in Problem 5.

The sufficiency of the condition for this class of diagonal operators was originally proved in 1973 ([27]) with a shortened proof in 1986 ([29]) with the help of E. Azoff. The necessity was originally proved in 1976 ([28]).

2. Commutator ideals

Theorem 2.1 (Weiss [28], 1980). *Setting $d := \sum_1^\infty d_n$ for an arbitrary sequence $d_n \downarrow 0$, the following are equivalent.*

(i) $\operatorname{diag}(-d, d_1, d_2, \dots) \in [C_2, C_2]$

(ii) $\operatorname{diag}(-d, d_1, d_2, \dots) \in [C_1, B(H)]$

(iii) $\sum_1^\infty d_n \ \log n < \infty$.

In particular, if $\langle d_n \rangle = \langle \frac{1}{n \log^2 n} \rangle$, *then* $\operatorname{diag}(-d, d_1, d_2, \dots) \in C_1^o \setminus [C_2, C_2]$.

Theorem 2.1 (ii), although not included in [28], with minor modifications can be obtained from the same construction and computations.

Diagonal operators of the form $\operatorname{diag}(-d, d_1, d_2, \dots)$ became the test cases inside C_1^o in [27]–[29] for early approaches to commutator problems where having trace zero was a necessary condition, and they continue to play an essential role in the C_1 study, for instance in the proof of the main summability theorem in [8], Theorem 5.11 (iii) (Theorem 2.4 below). Indeed, when $I \subset C_1$, i.e., inside C_1 they are the building blocks of I^o in that every I^o-operator has its real and imaginary parts easily decomposable as the sum of two compact selfadjoint operators each unitarily equivalent to diagonals of this form. Another important point about these diagonal forms is their resemblance to $\operatorname{diag}(1, 0, 0, \dots)$ so that matricially they provide reasonable analogs to the rank one projection and its commutator problems mentioned earlier (for instance, Problem 1 and Problem 4 for the case $\operatorname{diag}(1, 0, 0, \dots)$).

The study of this class began in 1973 (cf., [27] and [28]) by considering low dimension finite-dimensional cases (especially important were dimensions $2, 3$ and

4), and this survey concludes with a discussion of some of the remaining open problems there that might lead to better insights into the structure of single commutator classes.

One negative weight and the rest positive in practice were the hardest operators to deal with in this investigation. In contrast, operators of the form $\operatorname{diag}\langle \pm d_n \rangle$ are easy to represent as single commutators using tensors of diagonals and 2×2 scalar matrices with the best possible control on their norms or on their membership in prescribed ideals. Indeed, for every product representation $d_n = a_n b_n$:

$$\operatorname{diag}\langle \pm d_n \rangle = \left[\operatorname{diag}\langle a_n \rangle \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \operatorname{diag}\langle b_n \rangle \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right].$$

Indeed, from this it follows easily that IJ is the ideal generated by $[I, J]$. Since 1971, much of the work on commutators depended significantly on increasingly complicated methods of organizing cancellation in operator equations. Another elementary example of this was discussed above in the third paragraph succeeding Theorem 0.1.

Regarding the monotone ordering on the sequence $\langle d_n \rangle$, since the summability condition in Theorem 2.1 along with other weighted summability conditions with nondecreasing weights appearing in [27] are minimal when the sequence $\langle d_n \rangle$ is arranged in nonincreasing order and since all the commutator classes under discussion are unitarily invariant, there is no loss of generality in assuming that the sequence is nonincreasing.

Anderson [3] in 1986 proved $C_p^o = [C_p, B(H)]$ when $p < 1$ and applied this to the computation of the K-groups, $K_1(C_p, B(H))$ (cf., [2]).

Kalton [19] in 1989 linked arithmetic means to $[C_1, B(H)]$ and to $[C_2, C_2]$ achieving the remarkable characterizations of the spaces:

Theorem 2.2 (Kalton, [19]**, 1989).** *An operator T is in $[C_1, B(H)]$ (or in $[C_2, C_2]$) if and only if $\lambda(T)_a$ is absolutely summable.*

Here $\lambda(T) := \langle \lambda_n(T) \rangle$ denotes the sequence of eigenvalues counting algebraic multiplicity and arranged in order of nonincreasing moduli and $\lambda(T)_a := \langle \frac{\lambda_1(T)+\cdots+\lambda_n(T)}{n} \rangle$ denotes its averaging sequence.

Surprisingly and only indirectly via their same characterization did Kalton obtain: $[C_1, B(H)] = [C_2, C_2]$. No direct proof of this is known.

Theorem 2.2 provided some of the inspiration and methodology for the general characterization of commutator ideals in the following theorem (in some respects the central result of [8]). For more of Kalton's work on this subject, some joint with Dykema and some related to the following theorem; see [20] and [9].

Theorem 2.3 (Dykema, Figiel, Weiss and Wodzicki, [8]**, 2001).** *If I, J are two arbitrary $B(H)$-ideals, at least one of which is proper, and $T = T^* \in IJ$, then*

$$T \in [I, J] \text{ if and only if } \operatorname{diag}\ \lambda(T)_a \in IJ.$$

Consequently, $[I, J] = [IJ, B(H)]$.

For normal compact operators, $\lambda(T)$ is simply the sequence of eigenvalues counting ordinary multiplicity and arranged in order of decreasing moduli, as described above. (A stronger version of Theorem 2.3 requires only a more general condition than monotoniztion: that $|\lambda(T)| \leq \nu$ for some $\nu \in \Sigma(I)$.) Moreover, IJ traditionally defined in ring theory as the ideal generated by all (I, J)-products, in $B(H)$ it is precisely the class of all single (I, J)-products (cf., [8], Lemma 6.3). Alternatively, IJ is the class of compact operators T dominated by (I, J)-products in the sense of s-numbers, i.e., all compact operators T for which $s(T) \leq s(A)s(B)$ for some $A \in I, B \in J$, and this reveals that the ideal product is a commutative operation. These facts follow in part from Calkin's inclusion preserving lattice isomorphism, $I \to \Sigma(I)$, from the class of $B(H)$-ideals onto the class of characteristic subsets of c_o^* (i.e., solid subsets of the class of nonincreasing nonnegative sequences tending to zero that are invariant under ampliation, $D_2(s(T)) := (s(T)_1, s(T)_1, s(T)_2, s(T)_2, \dots)$, and closed under addition) [6]. (See also [8], Chapter 2.)

Part of the value of Theorem 2.3 lies in the fact that, because IJ is a commutative ideal operation, the theorem reduces many noncommutative problems in this study to commutative problems involving c_o^*-sequences, often achieving greater accessibility. This is evidenced in [8] and [18].

Like Kalton's result, that $[I, J] = [IJ, B(H)]$ follows likewise only indirectly from their same characterization and no direct proof is known. (That $[I, J]_2 \supset [IJ, B(H)]_1$ is straightforward.) Hence the primary general single commutator questions (cf., [8], Chapter 7):

Problem 5. (i) Characterize $[I, J]_1$.

(ii) Absent this, characterize $[I, J]_1$ for any pair $I, J \neq F(H)$ distinct from those covered in [8], Corollaries 7.2 and 7.6.
(iii) Is $[I, J]_1 = [IJ, B(H)]_1$?
(iv) In particular, is $[C_1, B(H)]_1 = [C_2, C_2]_1$?

In C_1, Theorem 2.3 also reveals the summability theory.

Theorem 2.4 (Dykema, Figiel, Weiss and Wodzicki [8]**, 2001, Theorem 5.11-(iii)).** *If I, J, L are $B(H)$-ideals where $IJ \subset K(H)$, then*

(i) $\operatorname{diag}(-d, d_1, d_2, \dots) \in [I, J]$ *if and only if* $\operatorname{diag}\langle d_n \rangle_{a_\infty} \in IJ$
(ii) *If $L \subset C_1$, then $L^o \subset [I, J]$ if and only if $L_{a_\infty} \subset IJ$.*
(iii) *If $I \subset C_1$, then $I^o = [I, B(H)]$ if and only if $I_{a_\infty} = I$*

(equivalently $I_{a_\infty} \subset I$ since the reverse inclusion is automatic).

Here the arithmetic mean at infinity is defined for summable sequences λ by: $\lambda_{a_\infty} := \frac{\lambda_{n+1} + \lambda_{n+2} + \cdots}{n}$. See the third paragraph below for the definition of the arithmetic mean at infinity I_{a_∞}.

3. Traces and arithmetic mean ideals

Consequences of Theorem 2.3 can be found in [8] and in [18]. It provides a new point of view on traces and arithmetic mean ideals and a brief introduction including a few of its consequences are presented below. Applications of Theorem 2.3 to cyclic homology and the algebraic K-theory of operator ideals are described in [8], Introduction.

Theorem 2.3 characterizes when traces and nonsingular traces exist. Traces (also called generalized traces), as mentioned earlier, are unitarily invariant linear functionals on an ideal I and unitary invariance is equivalent to the linear functional vanishing on the commutator ideal $[I, B(H)]$. Alternatively, traces are simply natural liftings to I of linear functionals on the quotient $\frac{I}{[I,B(H)]}$, so they exist precisely when $I \neq [I, B(H)]$, or equivalently, by Theorem 2.3, when $I \neq I_a$ (i.e., when $\Sigma(I)_a \not\subset \Sigma(I)$).

I_a and ${}_aI$, the basic arithmetic mean ideals called respectively the arithmetic mean ideal and the pre-arithmetic mean ideal of I, are defined as:

$$I_a := \{T \in K(H) \mid s(T) \leq s(A)_a \text{ for some } A \in I\} \text{ and}$$
$${}_aI := \{T \in K(H) \mid s(T)_a \leq s(A) \text{ for some } A \in I\}.$$

Likewise I_{a_∞} and ${}_{a_\infty}I$, the basic arithmetic mean ideals at infinity called respectively the arithmetic mean ideal at infinity and the pre-arithmetic mean ideal at infinity of I, are defined as:

$$I_{a_\infty} := \{T \in K(H) \mid s(T) \leq s(A)_{a_\infty} \text{ for some } A \in I\} \text{ and}$$
$${}_{a_\infty}I := \{T \in K(H) \mid s(T)_{a_\infty} \leq s(A) \text{ for some } A \in I\}.$$

Nonsingular traces exist precisely when $\operatorname{diag}(1, 1/2, 1/3, \dots) \notin I$. Nonsingular traces are those generalized traces nonvanishing on $F(H)$, or equivalently, whose restrictions to $F(H)$ up to scalar multiplication are simply the classical trace. The necessity of $\operatorname{diag}(1, 1/2, 1/3, \dots) \notin I$ follows by applying Theorem 2.3 to obtain the key idea that

$$\operatorname{diag}(1, 0, 0, \dots) \in [I, B(H)] \text{ if and only if } \operatorname{diag}\langle 1/n\rangle \in I.$$

Examples of nonsingular traces are: the classical trace on $F(H)$ or C_1; and an example of a singular trace is the Dixmier trace on the Köthe dual of the Macaev ideal (cf., [7] and [22]). In the language of arithmetic mean ideals (see below), the Macaev ideal is the Lorentz ideal $\mathcal{L}(\log\ n)$ and its Kothe dual is the pre-arithmetic mean ideal of a principal ideal, ${}_a(\operatorname{diag}\langle\frac{\log\ n}{n}\rangle)$, also known as the Marcinkiewicz ideal $\mathcal{M}(\langle\frac{n}{\log\ n}\rangle)$, and which is also the arithmetic mean closure of a principal ideal, ${}_a((\operatorname{diag}\langle 1/n\rangle)_a)$. (Cf., [8], Sections 4.7 and 4.10.) The arithmetic mean (am) closure was discussed earlier in the paragraph following Theorem 1.3. Indeed many classical ideals arising from classical spaces in the literature (e.g., Lorentz, Marcinkiewicz and Orlicz sequence spaces) fit this new context identified in [8] and the additional notions of "softness" and "soft enveloping ideals" in [18] arose from Theorem 2.3 applied to the work of Dixmier ([7]) and Varga ([26]).

One consequence of Theorem 2.3 for nonsingular traces is that the classical trace extends beyond the trace class to the strictly larger ideal of operators T whose s-numbers $s(T) = o(1/n)$ (see also [10]). This is the soft interior of the principal ideal generated by $\mathrm{diag}\langle 1/n\rangle$.

By Theorem 2.3, $({}_aI)^+ = [I, B(H)]^+$ and the ideal ${}_aI \subset [I, B(H)]$ consequently. As mentioned earlier, the ideal generated by $[I, B(H)]$ is I. So ${}_aI \subset [I, B(H)] \subset I$ where ${}_aI, I$ form the optimal upper and lower ideal envelopes for $[I, B(H)]$ and the inclusions become equalities if and only if $[I, B(H)]$ is itself an ideal.

In c_o^*, the new double inequality linking tensors with arithmetic means (see [8], Proposition 3.14): $(x \otimes \omega)^* \leq x_a \leq 2\ (x \otimes \omega)^*$ provides an alternate characterization of the arithmetic mean ideal of I: $I_a = I \otimes (\mathrm{diag}\ \omega)$. Here $*$ means monotonization and $\omega := \langle 1/n\rangle$.

Am-stability, $I = I_a$, is equivalent to ${}_aI = I$, and from the previous comment, am-stability is equivalent to the condition $I = I \otimes (\mathrm{diag}\ \omega)$ (i.e., $I \supset I \otimes (\mathrm{diag}\ \omega)$ since the reverse inclusion is automatic). This is related to the tensor product closure property (TPCP), $I = I \otimes I$, investigated in [27]. (See [8], Sections 2.8 and 4.3 for more details on the tensor operation for ideals.)

The following 5-chain with some arithmetic mean relations links the basic arithmetic mean ideals: ${}_aI \subset ({}_aI)_a \subset I \subset {}_a(I_a) \subset I_a$. Denoting $A(I) := I_a$ and $A^{-1}(I) := {}_aI$ for convenience of notation, $\forall\ n \in \mathbb{Z}$ am-ideals satisfy the relations $A^nA^{-n}A^n = A^n$ which imply the 5-chain and that A^nA^{-n} and $A^nA^{-2n}A^n$ are idempotents. The interior $I^\circ := ({}_aI)_a$ and the closure $I^- := {}_a(I_a)$ reveal a topological like structure (except of course the complement of an ideal is not an ideal) where the am-closed ideals are those equal to their closures (equivalently as mentioned earlier, those solid under Hardy-Littlewood-Polya-Schur majorization), and the am-open ideals are those equal to their interiors. More generally, am-interiors/am-closures form the optimal am-open/am-closed inner and outer envelopes, respectively. For am-stability, all the inclusions become equality. An open question here is *whether or not there exists a clopen ideal that is not am-stable.*

Three notable features of this structure are that every ideal contains a largest am-closed ideal and is contained in a smallest am-open ideal; that they can be expressed in terms of certain inner and outer convex envelopes; and that the am-closure operation distributes over finite sums of ideals. (Cf., [18] for details on this topological like structure.)

There is also an emerging summability theory with analogous phenomena related to the slightly subtler 5-chain for arithmetic mean ideals at infinity: ${}_{a_\infty}I \subset ({}_{a_\infty}I)_{a_\infty} \subset I \subset {}_{a_\infty}(I_{a_\infty}) \subset I_{a_\infty}$.

4. Single commutator problems on $\mathrm{diag}(-d, d_1, d_2, \dots)$

The development leading to Theorem 2.1 left some interesting open questions. This is a brief description of the related work and problems from [27], Section 1.8, pp. 122–136, [28], Section I, pp. 576–580, and [29], p. 885.

An early construct used by the author in this study was that, using the standard notation $[A,B] := AB - BA$ and a parameter $0 \le t \le 1$:

$$\begin{pmatrix} -\sum_1^\infty d_n & 0 & 0 & \cdots \\ 0 & d_1 & 0 & \cdots \\ 0 & 0 & d_2 & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

$$= \left[\begin{pmatrix} 0 & 0 & 0 & \cdots \\ (\sum_1^\infty d_n)^{1-t} & 0 & 0 & \cdots \\ 0 & (\sum_2^\infty d_n)^{1-t} & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}, \begin{pmatrix} 0 & (\sum_1^\infty d_n)^t & 0 & \cdots \\ 0 & 0 & (\sum_2^\infty d_n)^t & \cdots \\ 0 & 0 & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \right].$$

Here the IJ-norm in both the $[C_1, B(H)]_1$ and $[C_2, C_2]_1$ contexts is

$$\sum_1^\infty \sum_k^\infty d_n = \sum_1^\infty n d_n.$$

That is, a sufficient condition that

$$\operatorname{diag}(-d, d_1, d_2, \dots) \in [C_2, C_2]_1 \text{ or } [C_1, B(H)]_1 \text{ is that } \sum_1^\infty n d_n < \infty.$$

Weighted shifts such as these play a central role throughout the theory in building solution operators for commutator equations. For instance, they were essential for the sufficiency part of Theorem 2.1 (iii). The strategy back then was to look for suitable matrix forms with smaller trace norms in the form of weighted sums where the weights increase slower than n, say like $o(n)$. The point of Theorem 2.1 (iii) was that the sufficiency of the weights n were strengthened to the sufficiency of the weights $\log\, n$ thereby weakening the condition on $\langle d_n \rangle$ and hence expanding the class of achievable $\operatorname{diag}(-d, d_1, d_2, \dots)$. Attempts to reduce further the summability condition qualitatively failed, but quantitatively, the following construct succeeded.

First note that

$$\operatorname{diag}(-d, d_1, d_2, \dots) \cong \operatorname{diag}(-d, d_1, d_3, \dots) \oplus \operatorname{diag}(d_2, d_4, \dots) = [A, B]$$

where

$$A = \begin{pmatrix} U_t & -\big(\sum_{n=1}^\infty d_{2n}\big)^t P \\ 0 & -V_t^* \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} U_{1-t}^* & 0 \\ \big(\sum_{n=1}^\infty d_{2n}\big)^{1-t} P & V_{1-t} \end{pmatrix}$$

with $P := \operatorname{diag}(1, 0, \dots)$, the parameter $0 \le t \le 1$, the weighted shift

$$U_t := U\Big(\Big(\sum_1^\infty d_{2n-1}\Big)^t, \sum_2^\infty d_{2n-1}\Big)^t, \dots$$

and the weighted shift

$$V_t := U\Big(\Big(\sum_2^\infty d_{2n}\Big)^t, \sum_3^\infty d_{2n}\Big)^t, \dots .$$

When $t = 1$, $\|A\|_1 = \sum_1^\infty [\frac{n+1}{2}]\ d_n$ where $[x]$ denotes the greatest integer function, and $\|B\| = 1$. When $t = \frac{1}{2}$, $\|A\|_2^2 = \|B\|_2^2 = \sum_1^\infty [\frac{n+1}{2}]\ d_n$. And for any $0 \le t \le 1$, $\|AB\|_1 = \|BA\|_1 = \sum_1^\infty [\frac{n+1}{2}]\ d_n$. These are all quantitative improvements over $\sum_1^\infty nd_n$ for the respective norms, but not qualitative improvements. Hence the following test questions for investigating the structure of $[C_1, B(H)]_1$ and $[C_2, C_2]_1$:

Problem 6. Is $\sum_1^\infty [\frac{n+1}{2}]\ d_n$ minimal for each of these three contexts $(\max(\|AB\|_1, \|BA\|_1),\ \|A\|_2\|B\|_2$ and $\|A\|_1\|B\|)$?

For which of these three contexts is the condition $\sum_1^\infty nd_n < \infty$ necessary?

Finally, the problems on $\operatorname{diag}(-d,\ d_1,\ d_2,\ \dots)$ all have finite-dimensional analogs by simply taking the sequence $\langle d_n \rangle$ finite. Indeed Theorem 2.1 owes its discovery as described in [28] to the test cases $d_1 = d_2 = \cdots = d_N = \frac{1}{N}$. The formula in Problem 6 for the cases $\operatorname{diag}(-1, \frac{1}{N}, \frac{1}{N}, \dots, \frac{1}{N})$ is $\frac{N+2}{4}$ for N even and $\frac{(N+1)^2}{4N}$ for N odd. Since $\|\operatorname{diag}(-1, d_1, d_2, \dots)\|_1 = 2$ (by normalizing, $d = 1$), we see that $\operatorname{diag}(-1, d_1, d_2, \dots) = AB - BA$ implies

$$2 = \|AB - BA\|_1 \le \|AB\|_1 + \|BA\|_1 \text{ hence by Hölder's inequality,}$$

$\|A\|_2\|B\|_2,\ \|A\|_1\|B\|,\ \max(\|AB\|_1, \|BA\|_1) \ge 1$.

For $N = 1, 2$, the inequality

$$2 = \|AB - BA\|_1 \le \|AB\|_1 + \|BA\|_1 \le 2\ \|A\|_2\|B\|_2$$

in the "minimal" case is actually equality. But for $N = 3$, the $\|A\|_2\|B\|_2$-minimum turned out strictly larger than 1, namely $4/3$, and for $N = 4$, the minimum is $3/2$. So for $1 \le N \le 4$, the formula in Problem 6 is sharp for minimizing $\|A\|_2\|B\|_2$ and perhaps something interesting is occurring at $N = 5$ like new solution operators providing smaller Hilbert-Schmidt norms or a new stronger proof of sharpness. The proof for $1 \le N \le 4$ provided the keys in [28] to proving Theorem 2.1.

Problem 7. Minimize $\|A\|_2\|B\|_2$ for $N \ge 5$ for the target operators $\operatorname{diag}(-1, \frac{1}{N}, \frac{1}{N}, \dots, \frac{1}{N})$ and $\operatorname{diag}(-1, \frac{1}{N}, \frac{1}{N}, \dots, \frac{1}{N}) \oplus 0$ where 0 is finite- and infinite-dimensional?

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