Paving Small Matrices and
The Kadison-Singer Extension Problem
AIM Workshop Notes

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Part 1

Pavings
CHAPTER 1

Notation

\[ \mathbb{M}_n = n \times n \text{ complex matrices} \]
\[ \mathbb{M}_n^0 = n \times n \text{ complex matrices with zero diagonal} \]
\[ \mathbb{M}_{n,sa} = n \times n \text{ selfadjoint complex matrices} \]
\[ \mathbb{M}_n^0,sa = n \times n \text{ selfadjoint complex matrices with zero diagonal} \]
\[ \mathbb{M}_{n,sym} = n \times n \text{ real symmetric matrices} \]
\[ \mathbb{M}_n^0,sym = n \times n \text{ real symmetric matrices with zero diagonal} \]
\[ \mathbb{M}_{n,++} = n \times n \text{ non-negative matrices} \]
\[ \mathbb{M}_n^0,++ = n \times n \text{ non-negative matrices with zero diagonal} \]
\[ \mathbb{D}_n = n \times n \text{ diagonal matrices} \]

If \( A \in \mathbb{M}_n \), define

\[ \alpha_k(A) = \min_{\text{diagonal projections } P_1 + \cdots + P_k = I_n} \max_{1 \leq j \leq k} ||P_j AP_j|| \]

If \( 0 \neq A \in \mathbb{M}_n \), define

\[ \tilde{\alpha}_k(A) = \frac{\alpha_k(A)}{||A||} \]

If \( S \subset \mathbb{M}_n \), define

\[ \tilde{\alpha}_k(S) = \sup_{0 \neq A \in S} \tilde{\alpha}_k(A) \]
CHAPTER 2

2-Pavings

**Theorem 2.1 (2-pavings).**

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\tilde{\alpha}_2(M_n^{0,0})$</th>
<th>$\tilde{\alpha}<em>2(M</em>{n,sa}^{0})$</th>
<th>$\tilde{\alpha}<em>2(M</em>{n,sym}^{0})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>$1$</td>
<td>$\frac{1}{\sqrt{3}}$</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>4</td>
<td>$\tilde{\alpha}<em>2(M</em>{n,sa}^{0})$</td>
<td>$[0.5493, 0.5773]$</td>
<td>$[0.5000, 0.5773]$</td>
</tr>
<tr>
<td>5</td>
<td>$\frac{\sqrt{3}}{2}$</td>
<td>$\frac{\sqrt{3}}{2}$</td>
<td>$\frac{\sqrt{3}}{2}$</td>
</tr>
</tbody>
</table>

1. Selfadjoint

**Proposition 2.2 (3 × 3 selfadjoint).** $\tilde{\alpha}_2(M_{3,sa}^{0}) = \frac{1}{\sqrt{3}} \approx 0.5773$.

**Proof.** Suppose $A = \begin{bmatrix} 0 & a & b \\ \pi & 0 & c \\ \pi & \pi & 0 \end{bmatrix} \in M_{3,sa}^{0}$ with $\alpha_2(A) = 1$.

Then $|a|, |b|, |c| \geq 1$. By the Universal Selfadjoint 3-Identity (Lemma 7.1),

$$1 = \frac{|a|^2 + |b|^2 + |c|^2}{\|A\|^2} + \frac{2|\text{Re}(abc)|}{\|A\|^3} \geq \frac{3}{\|A\|^2}.$$ 

Thus, $\|A\| \geq \sqrt{3} \Rightarrow \tilde{\alpha}_2(A) \leq \frac{1}{\sqrt{3}}$. This bound is attained by

$$A = \begin{bmatrix} 0 & 1 & i \\ 1 & 0 & 1 \\ -i & 1 & 0 \end{bmatrix}$$

because $\alpha_2(A) = 1$ and $\|A\| = \sqrt{3}$ by Corollary 7.2. $$\square$$

**Proposition 2.3 (4 × 4 selfadjoint).** $\tilde{\alpha}_2(M_{4,sa}^{0}) = \frac{1}{\sqrt{3}}$.

**Proof.** Suppose $A \in M_{4,sa}^{0}$, with $\alpha_2(A) = 1$. Create a graph $G = (V, E)$ as follows: $V = \{1, 2, 3, 4\}$ and $(i, j) \in E$ if $|a_{ij}| < 1$. We have the following axioms:

1. $G_{11}$ is not a subgraph of $G$. Otherwise, $A$ admits a 2-2 paving of norm $< 1$, violating the assumption $\alpha_2(A) = 1$.
2. For all $i$, the degree of $i$ is greater than 0. Otherwise, row $i$ of $A$ has three entries of absolute value $\geq 1 \Rightarrow \|A\| \geq \sqrt{3} \Rightarrow \tilde{\alpha}_3(A) \leq \frac{1}{\sqrt{3}}$.
3. By removing a vertex from $G$, one cannot arrive at $G_4$. Otherwise, $A$ has a 3-compression of norm $\geq \sqrt{3} \Rightarrow \|A\| \geq \sqrt{3} \Rightarrow \tilde{\alpha}_2(A) \leq \frac{1}{\sqrt{3}}$. 

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This exhausts all possible 4-graphs and hence proves the inequality.

**Proposition 2.4** (5 × 5 selfadjoint). Let \( \tilde{\alpha}_2(M_{5,sa}^0) = \frac{2}{\sqrt{5}} \approx 0.8944. \)

**Proof.** Suppose \( A \in M_{5,sa}^0 \) with \( \alpha_2(A) = 1 \). Create a graph \( G = (V, E) \) as follows: \( V = \{1, 2, 3, 4, 5\} \) and \( (i, j) \in E \) if \( |a_{ij}| < 1 \). We may assume the following axiom:

1. For all \( i, \ deg(i) \geq 3 \). Otherwise, row \( i \) of \( A \) has at least two entries of absolute value \( \geq 1 \) \( \Rightarrow \|A\| \geq \sqrt{2} \Rightarrow \tilde{\alpha}_2(A) \leq \frac{\sqrt{2}}{\sqrt{5}} \approx 0.7071 \).

This leaves graphs \( G50, G51, \) and \( G52. \)

**Case G50:** Only two 2-compressions have norm \( \geq 1 \), and they are disjoint. Without loss of generality, \( \|A_{12}\|, \|A_{34}\| \geq 1 \). We claim that every 3-compression has norm \( \geq 1 \). Indeed, \( \|A_{125}\| \geq \|A_{12}\| \geq 1, \|A_{345}\| \geq \|A_{34}\| \geq 1 \), and the remaining 3-compressions have norm \( \geq 1 \) because their complementary 2-compressions have norm \( < 1 \). It follows that \( \|A\| \geq \frac{\sqrt{2}}{\sqrt{5}} \Rightarrow \tilde{\alpha}_2(A) \leq \frac{2}{\sqrt{5}}. \)

**Case G51:** Only one 2-compression has norm \( \geq 1 \). Without loss of generality, \( \|A_{12}\| \geq 1 \). It follows that

\[
\|A\|^2 \geq \frac{1}{4} \|A\|^2_{HS} = \frac{1}{4} \left( \|A_{12}\|^2_{HS} + \frac{1}{2} \sum_{1 \in B, 2 \notin B} \|B\|^2_{HS} + \frac{1}{2} \sum_{2 \in B, 1 \notin B} \|B\|^2_{HS} \right) \\
\geq \frac{1}{4} \left( 2 + \frac{1}{2} \cdot 3 \cdot \frac{3}{2} + \frac{1}{2} \cdot 3 \cdot \frac{3}{2} \right) = \frac{13}{8}
\]

Thus, \( \|A\| \geq \sqrt{\frac{13}{8}} \Rightarrow \tilde{\alpha}_2(A) \leq \sqrt{\frac{13}{13}} \approx 0.7845. \)

**Case G52:** Every 2-compression has norm \( < 1 \) \( \Rightarrow \) every 3-compression has norm \( \geq 1 \) \( \Rightarrow \|A\| \geq \frac{\sqrt{2}}{\sqrt{5}} \Rightarrow \tilde{\alpha}_2(A) \leq \frac{2}{\sqrt{5}}. \)

The matrix

\[
A = \begin{bmatrix}
0 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & -1 \\
1 & 1 & 0 & -1 & 1 \\
1 & 1 & -1 & 0 & -1 \\
1 & -1 & 1 & -1 & 0
\end{bmatrix}
\]

shows that the inequality is sharp. The unimodular circulant

\[
B = \begin{bmatrix}
0 & e^{2\pi i/5} & e^{-\pi i/5} & e^{\pi i/5} & e^{-2\pi i/5} \\
e^{-2\pi i/5} & 0 & e^{\pi i/5} & e^{-\pi i/5} & e^{2\pi i/5} \\
e^{\pi i/5} & e^{-2\pi i/5} & 0 & e^{\pi i/5} & e^{-\pi i/5} \\
e^{-\pi i/5} & e^{\pi i/5} & e^{-2\pi i/5} & 0 & e^{2\pi i/5} \\
e^{2\pi i/5} & e^{-\pi i/5} & e^{\pi i/5} & e^{-2\pi i/5} & 0
\end{bmatrix}
\]

also works. Note: \( A \) and \( B \) are unitarily equivalent. \( \Box \)
Alternate Proof. Suppose $A \in M_{5,sa}^0$, with $\alpha_2(A) = 1$.

1. Assume that all 3-compressions of $A$ have norm $\geq 1$. Then $\tilde{\alpha}_2(A) \leq \frac{2}{\sqrt{5}}$ (see the previous proof).

2. Assume that exactly one 3-compression, say $A_{345}$, has norm $< 1$, then $\|A_{12}\| \geq 1 \Rightarrow \tilde{\alpha}_2(A) \leq \sqrt{\frac{8}{13}}$ (see the previous proof).

3. Assume that exactly two 3-compressions have norm $< 1$. We may assume that the complementary 2-compressions are disjoint. Otherwise, $\|A\| \geq \sqrt{2} \Rightarrow \tilde{\alpha}_2(A) \leq \frac{1}{\sqrt{2}}$. Without loss of generality, $\|A_{12}\|, \|A_{34}\| \geq 1$ and $\|A_{345}\|, \|A_{123}\| < 1$. This is a contradiction.

4. Assume that more than two 3-compressions have norm $< 1$. Then their complementary 2-compressions cannot be disjoint. Thus, $\|A\| \geq \sqrt{2} \Rightarrow \tilde{\alpha}_2(A) \leq \frac{1}{\sqrt{2}}$. 

$\square$
2. Real Symmetric

**Proposition 2.5** ($3 \times 3$ real symmetric). $\tilde{\alpha}_2(M_{3,\text{sym}}^0) = \frac{1}{2}$.

**Proof.** Suppose

$$A = \begin{bmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{bmatrix} \in M_{3,\text{sym}}^0$$

with $\alpha_2(A) = 1$.

Then $|a|, |b|, |c| \geq 1$. By the Universal Selfadjoint 3-Identity (Lemma 7.1),

$$1 = \frac{a^2 + b^2 + c^2}{\|A\|^2} + \frac{2|abc|}{\|A\|^3} \geq \frac{3}{\|A\|^2} + \frac{2}{\|A\|^3}$$

which implies $\|A\| \geq 2$, hence $\tilde{\alpha}_2(A) \leq \frac{1}{2}$. This bound is attained by

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \in M_{3,\text{sym}}^0$$

since $\alpha_2(A) = 1$ and $\|A\| = 2$ by Corollary 7.2.

**Lemma 2.6.** Let

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & d & e \\ 1 & d & 0 & f \\ 1 & e & f & 0 \end{bmatrix} \in M_{4,\text{sym}}^0.$$ 

If

$$\begin{bmatrix} 0 & d & e \\ d & 0 & f \\ e & f & 0 \end{bmatrix} \geq 1,$$

then $\|A\| \geq (9.75)^{1/4} \approx 1.767$.

**Proof.** Let $x = [1 \ 1 \ 1]$ and

$$B = \begin{bmatrix} 0 & d & e \\ d & 0 & f \\ e & f & 0 \end{bmatrix}.$$ 

Then

$$A = \begin{bmatrix} 0 \ x^* \\ x^* \ B \end{bmatrix} \Rightarrow A^*A = \begin{bmatrix} xx^* & xB \\ B^*x^* & x^*x + B^*B \end{bmatrix}.$$ 

Thus

$$\|A\|^4 = \|A^*A\|^2 \geq \|xx^* \ xB\|^2$$

$$= 9 + (d + e)^2 + (d + f)^2 + (e + f)^2.$$ 

We claim that

$$(d + e)^2 + (d + f)^2 + (e + f)^2 \geq d^2 + e^2 + f^2.$$ 

Indeed, let $F(d, e, f) = (d + e)^2 + (d + f)^2 + (e + f)^2$ and $G(d, e, f) = d^2 + e^2 + f^2$. Using the Method of Lagrange Multipliers, we minimize $F(d, e, f)$ subject to the constraint $G(d, e, f) = r^2$. 


\[2(d + e) + 2(d + f) = 2\lambda d\]
\[2(d + e) + 2(e + f) = 2\lambda e\]
\[2(d + f) + 2(e + f) = 2\lambda f\]
\[
\begin{bmatrix}
2 & 1 & 1 \\
1 & 2 & 1 \\
1 & 1 & 2 \\
\end{bmatrix}
\begin{bmatrix}
d \\
e \\
f \\
\end{bmatrix}
= \lambda
\begin{bmatrix}
d \\
e \\
f \\
\end{bmatrix}
\]
\[
\begin{bmatrix}
d \\
e \\
f \\
\end{bmatrix}
\begin{bmatrix}
dx \\
e \\
fx \\
\end{bmatrix}
\]
or
\[
\begin{bmatrix}
dx \\
e \\
fx \\
\end{bmatrix}
\begin{bmatrix}
x + y \\
x - y \\
-2x \\
\end{bmatrix}
\]

In the former case,
\[3x^2 = d^2 + e^2 + f^2 = r^2 \Rightarrow (d + e)^2 + (d + f)^2 + (e + f)^2 = 12x^2 = 4r^2.\]

In the later case,
\[(x + y)^2 + (x - y)^2 + (-2x)^2 = d^2 + e^2 + f^2 = r^2\]
\[\Rightarrow (d + e)^2 + (d + f)^2 + (e + f)^2 = (2x)^2 + (-x + y)^2 + (-x - y)^2 = r^2.\]
Thus, \(r^2 \leq (d + e)^2 + (d + f)^2 + (e + f)^2 \leq 4r^2\), which proves the claim. Now
\[\|B\| \geq 1 \Rightarrow \|B\|_{2H} \geq 1.5 \Rightarrow d^2 + e^2 + f^2 \geq 0.75.\]

Hence, \(\|A\|^4 \geq 9.75\), which proves the lemma. \(\square\)

**PROPOSITION 2.7 (4 \times 4 real symmetric).** \(\tilde{\alpha}_2(M_0^4, \text{sym}) \in [0.5493, 0.5773].\)

**Proof.** Suppose \(A \in M_0^4, \text{sym}\), with \(\alpha_2(A) = 1\). Create a graph \(G = (V, E)\) as follows: \(V = \{1, 2, 3, 4\}\) and \((i, j) \in E\) if \(|a_{ij}| < 1\). We have the following axioms:

1. \(G11\) is not a subgraph of \(G\). Otherwise, \(A\) admits a 2-2 paving of norm \(< 1\), violating the assumption \(\alpha_2(A) = 1\).

2. By removing a vertex from \(G\), one cannot arrive at \(G4\). Otherwise, \(A\) has a 3-compression of norm \(\geq 2 \Rightarrow \|A\| \geq 2 \Rightarrow \tilde{\alpha}_2(A) \leq \frac{1}{2}\).

This leaves only graph \(G12\). Thus,
\[A = \begin{bmatrix}
0 & a & b & c \\
a & 0 & d & e \\
b & d & 0 & f \\
c & e & f & 0 \\
\end{bmatrix},\]
where \(|a|, |b|, |c| \geq 1, |d|, |e|, |f| < 1\), and
\[
\begin{bmatrix}
0 & d & e \\
d & 0 & f \\
e & f & 0 \\
\end{bmatrix}
\geq 1.
\]

**Lower bound:**
\[A = \begin{bmatrix}
0 & 1 & -0.3946 & 0.6854 \\
1 & 0 & -0.3946 & 0 & -0.3986 \\
1 & -0.3946 & 0 & -0.3986 & 0 \\
\end{bmatrix}.
\]
\(\square\)
CHAPTER 3

3-Pavings

In 1987 the 3-paving problem was posed: whether or not 3-pavings suffice for Anderson’s Paving Conjecture and hence for Kadison-Singer. To date we have heard of no refutation to this. Recall also the $\frac{2}{3}$-challenge from then: whether or not $\hat{\alpha}_3(M_n^0) \leq \frac{2}{3}$, which the following table refutes.

**Theorem 3.1 (3-pavings).**

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\hat{\alpha}_3(M_n^0)$</th>
<th>$\hat{\alpha}<em>3(M</em>{n,sa}^0)$</th>
<th>$\hat{\alpha}<em>3(M</em>{n,++}^0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>$\frac{2}{1+\sqrt{5}}$</td>
<td>$\frac{1}{\sqrt{3}}$</td>
<td>$\kappa$</td>
</tr>
<tr>
<td></td>
<td>0.6180</td>
<td>0.5773</td>
<td>0.5550</td>
</tr>
<tr>
<td>5</td>
<td>&quot;</td>
<td>&quot;</td>
<td>$[\kappa, \frac{2}{1+\sqrt{5}}]$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$[0.5550, 0.6180]$</td>
</tr>
<tr>
<td>6</td>
<td>$\frac{1}{\sqrt{2}}$</td>
<td>&quot;</td>
<td>&quot;</td>
</tr>
<tr>
<td></td>
<td>0.7071</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>$[?, 1]$</td>
<td>$[\frac{2}{3}, \frac{2}{\sqrt{3}}]$</td>
<td>$[\kappa, \frac{4}{3}]$</td>
</tr>
<tr>
<td></td>
<td>$[0.8231, 1]$</td>
<td>$[0.6667, 0.7559]$</td>
<td>$[0.5550, 0.6667]$</td>
</tr>
<tr>
<td>8</td>
<td>$[?, 1]$</td>
<td>$[\frac{2}{3}, \frac{2}{\sqrt{3}}]$</td>
<td>&quot;</td>
</tr>
<tr>
<td></td>
<td>$[0.8231, 1]$</td>
<td>$[0.6667, 0.8944]$</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>&quot;</td>
<td>$[\frac{2}{3}, 1]$</td>
<td>&quot;</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$[0.7454, 1]$</td>
<td></td>
</tr>
</tbody>
</table>

where

$$\kappa = \sqrt{\frac{3}{5 + 2\sqrt{7}\cos(\tan^{-1}(3\sqrt{3}/3))}}$$

**boldface** signifies what we feel are the most interesting facts, "?" signifies a lack of a closed form, and "" signifies “ditto from above”.
1. General

**Lemma 3.2.** Let
\[ A = \begin{bmatrix} r_1 e^{i \theta_1} & r_2 e^{i \theta_2} \\ 0 & r_3 e^{i \theta_3} \end{bmatrix} \in M_2. \]

Then there exist unitaries \( U, V \in \mathbb{D}_2 \) such that
\[ UAV = \begin{bmatrix} r_1 & r_2 \\ 0 & r_3 \end{bmatrix}. \]

**Proof.** Let
\[ U = \begin{bmatrix} e^{-i \theta_2} & 0 \\ 0 & e^{-i \theta_3} \end{bmatrix}, \quad V = \begin{bmatrix} e^{i(\theta_2 - \theta_1)} & 0 \\ 0 & 1 \end{bmatrix}. \]

\[ \square \]

**Corollary 3.3.** Let
\[ A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in M_2. \]

If \(|a|, |b|, |c| \geq 1\), then \( \|A\| \geq \frac{1 + \sqrt{5}}{2} \).

**Proof.** By the previous lemma,
\[ \|A\| = \left\| \begin{bmatrix} |a| & |b| \\ 0 & |c| \end{bmatrix} \right\| \geq \left\| \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\| = \frac{1 + \sqrt{5}}{2}. \]

\[ \square \]

**Proposition 3.4 (4 \times 4 general).** \( \tilde{\alpha}_3(M_{4 \times 4}^{\text{d}}) = \frac{2}{1 + \sqrt{5}} \approx 0.6180. \)

**Proof.** Let
\[ A = \begin{bmatrix} 0 & 1 & 1 & -\frac{2}{1 + \sqrt{5}} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \in M_{4 \times 4}^{\text{d}}. \]

Then \( \tilde{\alpha}_3(A) = \frac{2}{1 + \sqrt{5}} \) (\( \alpha_3(A) = 1 \) and \( \|A\| = \frac{1 + \sqrt{5}}{2} \) by applying to the upper-right \( 3 \times 3 \) corner either Parrott’s Completion Lemma with Formula, or factoring the characteristic polynomial of the square of its absolute value, or Matlab).

Now suppose \( A \in M_{4 \times 4}^{\text{d}} \), with \( \alpha_3(A) = 1 \). Create a digraph \( D = (V, E) \) as follows:
\( V = \{1, 2, 3, 4\} \) and \( (i, j) \in E \) if \( |a_{ij}| \geq 1 \). We may assume the following axioms:

1. For all \( i \neq j \), either \( (i, j) \in E \) or \( (j, i) \in E \). Otherwise \( A \) admits a 1-1-2 paving of norm \( < 1 \), violating the assumption \( \alpha_3(A) = 1 \).

2. For all \( i \), the in-degree of \( i \) and the out-degree of \( i \) are less than 3. Otherwise, \( \|A\| \geq \sqrt{3} \Rightarrow \tilde{\alpha}_3(A) \leq \frac{1}{\sqrt{3}} \approx 0.5774. \)

This leaves only digraphs \( D_{149}, D_{185}, D_{186}, \) and \( D_{218} \) as labeled in [1]. Now each of these digraphs has \( D_{12} \) as a subgraph [ibid.]. Thus, \( \|A\| \geq \frac{1 + \sqrt{5}}{2} \) (Corollary 3.3) \( \Rightarrow \tilde{\alpha}_3(A) \leq \frac{2}{1 + \sqrt{5}}. \)

\[ \square \]
Proposition 3.5 (5 × 5 general). \( \tilde{\alpha}_3(M^0_{5,4}) = \frac{2}{1 + \sqrt{3}} \approx 0.6180. \)

Proof. Clearly,
\[
\tilde{\alpha}_3(M^0_{5,4}) \geq \tilde{\alpha}_3(M^0_{4,4}) = \frac{2}{1 + \sqrt{3}}.
\]
Now let \( A \in M^0_{5,5} \), with \( \alpha_3(A) = 1 \). Construct a graph \( G = (V, E) \) as follows: \( V = \{1, 2, 3, 4, 5\} \) and \((i, j) \in E \) if \( |a_{ij}|, |a_{ji}| < 1 \). We may assume the following axioms:

(1) \( G11 \) is not a subgraph of \( G \). Otherwise, \( G \) has a 1-2-2 paving of norm \(< 1 \), violating the fact that \( \alpha_3(A) = 1 \).

(2) By removing a vertex from \( G \) one cannot arrive at \( G8 \). Otherwise, there exists a 4-compression \( B \) of \( A \) such that \( \alpha_3(B) \geq 1 \). Since \( \tilde{\alpha}_3(M^0_{4,4}) = \frac{2}{1 + \sqrt{3}} \), this would imply \( \|B\| \geq \frac{1 + \sqrt{5}}{2} \Rightarrow \|A\| \geq \frac{1 + \sqrt{5}}{2} \Rightarrow \tilde{\alpha}_3(A) \leq \frac{2}{1 + \sqrt{3}} \).

This leaves \( G23 \). After permuting indices, we may assume that
\[
A = \begin{bmatrix}
0 & s_{12} & s_{13} & b_{14} & b_{15} \\
s_{21} & 0 & s_{23} & b_{24} & b_{25} \\
s_{31} & s_{32} & 0 & b_{34} & b_{35} \\
b_{41} & b_{42} & b_{43} & 0 & b_{45} \\
b_{51} & b_{52} & b_{53} & b_{54} & 0
\end{bmatrix},
\]
where \( |s_{ij}| < 1 \) and \( \max\{|b_{ij}|, |b_{ji}|\} \geq 1 \) for all \( i \neq j \). Permuting the indices 4 and 5, if necessary, we may assume \( |b_{15}| \geq 1 \). If \( b_{51}, b_{52}, \) and \( b_{53} \) all have magnitude \( \geq 1 \), then \( \|A\| \geq \sqrt{3} \Rightarrow \tilde{\alpha}_3(A) \leq \frac{1}{\sqrt{3}} < \frac{2}{1 + \sqrt{3}} \). Thus, we may assume that one of them has magnitude \( < 1 \) \( \Rightarrow \) either \( b_{15}, b_{25}, \) or \( b_{35} \) has magnitude \( \geq 1 \). Permuting the indices 1, 2, and 3, if necessary, we may assume \( |b_{35}| \geq 1 \). If \( |b_{34}| \geq 1 \), then
\[
\|A\| \geq \left\|\begin{bmatrix}
b_{34} & b_{35} \\
0 & b_{45}
\end{bmatrix}\right\| \geq \frac{1 + \sqrt{5}}{2}.
\]
Likewise, if \( |b_{43}| \geq 1 \), then
\[
\|A\| \geq \left\|\begin{bmatrix}
b_{43} & b_{35} \\
b_{45} & 0
\end{bmatrix}\right\| \geq \frac{1 + \sqrt{5}}{2}.
\]
It follows that \( \tilde{\alpha}_3(A) \leq \frac{2}{1 + \sqrt{3}}. \) \hfill \( \square \)

Proposition 3.6 (6 × 6 general). \( \tilde{\alpha}_3(M^0_{6,6}) = \frac{1}{\sqrt{2}} \approx 0.7071. \)

Proof. Construct a graph \( G = (V, E) \) as follows: \( V = \{1, 2, 3, 4, 5, 6\} \) and \((i, j) \in E \) if \( |a_{ij}|, |a_{ji}| < 1 \). We may assume the following axioms:

(1) \( G61 \) is not a subgraph of \( G \). Otherwise \( A \) would have a 2-2-2 paving of norm \(< 1 \), violating the fact that \( \alpha_3(A) = 1 \).

(2) By removing vertices from \( G \), one cannot arrive at \( G8 \). Otherwise \( A \) would have a 4-compression \( B \) such that \( \alpha_3(B) \geq 1 \). Since \( \tilde{\alpha}_3(M^0_{6,6}) = \frac{2}{1 + \sqrt{3}} \), this would imply \( \|B\| \geq \frac{1 + \sqrt{5}}{2} \Rightarrow \|A\| \geq \frac{1 + \sqrt{5}}{2} \Rightarrow \tilde{\alpha}_3(A) \leq \frac{2}{1 + \sqrt{3}} < \frac{1}{\sqrt{2}}. \)

(3) For all vertices \( i \), \( \deg(i) \geq 3 \). Otherwise, if \( \deg(i) \leq 2 \), then either row \( i \) or column \( i \) of \( A \) would have at least two entries of magnitude \( \geq 1 \) \( \Rightarrow \|A\| \geq \sqrt{2} \Rightarrow \tilde{\alpha}_3(A) \leq \frac{1}{\sqrt{2}}. \)
This eliminates all graphs. Now let

\[
A = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 1 \\
\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 & 1 & 0 \\
0 & 0 & 0 & -1 & 0 & 1 \\
-\frac{1}{2} & 1 & \frac{1}{2} & 0 & \frac{1}{\sqrt{2}} & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
\frac{1}{2} & 1 & -\frac{1}{2} & 0 & -\frac{1}{\sqrt{2}} & 0 \\
\end{bmatrix} \in M_6^0.
\]

Then \( \alpha_3(A) = 1 \) and \( A^*A = 2I \). \( \square \)

**Proposition 3.7 (7 \times 7 general).** \( \tilde{\alpha}_3(\mathcal{M}_7^0) \in [0.8231, 1) \).

**Proof.** The following matrix was discovered by searching among \( 7 \times 7 \) unitary circulants for bad pavers. The starting point for the search was a \( 7 \times 7 \) unitary circulant with the eigenvalue distribution \((1, e^{\pi i/3}, e^{-\pi i/3}, i, -i, -1, -1, -1)\).

\[
A = \begin{bmatrix}
a & b & c & d & e & f \\
f & a & b & c & d \\
e & f & 0 & a & b & c \\
d & e & f & 0 & a & b \\
c & d & e & f & 0 & a \\
b & c & d & e & f & 0 \\
a & b & c & d & e & f \\
\end{bmatrix},
\]

where

\[
a = -0.19104830537481 - 0.1857143276728i \\
b = 0.03404378754044 + 0.00110165928527i \\
c = -0.13926357252448 + 0.42165365488402i \\
d = 0.21474405201775 - 0.42217403069332i \\
e = -0.28337369310887 - 0.48101315713848i \\
f = 0.29151538363540 - 0.33115367910212i.
\]

Then \( \alpha_3(A) = 0.82305627367962 \) and \( A^*A = I \), i.e. \( \tilde{\alpha}_3(A) = 0.82305627367962 \).

It remains to show that \( \tilde{\alpha}_3(\mathcal{M}_7^0) \neq 1 \). To that end, let \( A \in \mathcal{M}_7^0 \), with \( \alpha_3(A) = 1 \). If every 3-compression of \( A \) has norm \( \geq 1 \), then \( \| A \| > 1 \) (Corollary 7.10). If, on the other hand, some 3-compression of \( A \) has norm \( < 1 \), then the complementary 4-compression \( B \) satisfies \( \alpha_2(B) \geq 1 \). In particular, every 2-2 paving of \( B \) has norm \( \geq 1 \). By Lemma 7.11, we may assume that

\[
A = \begin{bmatrix}
0 & 0 & 0 & 0 & * & * & * \\
0 & a & 0 & 0 & 0 & 0 \\
0 & 0 & b & 0 & 0 & 0 \\
0 & c & 0 & 0 & 0 & 0 \\
* & 0 & 0 & 0 & * & * & * \\
* & 0 & 0 & 0 & * & 0 & * \\
* & 0 & 0 & 0 & * & 0 & 0
\end{bmatrix},
\]

where \( |a| = |b| = |c| = 1 \) and \( \| A_{567} \| < 1 \). Since \( \| A_{12} \| = \| A_{35} \| = 0, \| A_{467} \| = 1 \Rightarrow \| A_{567} \| = 1 \Rightarrow \| A_{567} \| = 1 \), a contradiction. \( \square \)
2. Selfadjoint

Proposition 3.8 (4 x 4 selfadjoint). \( \tilde{\alpha}_3(M_{4,sa}^0) = \frac{1}{\sqrt{3}} \approx 0.5773 \).

Proof. Suppose \( A \in M_{4,sa}^0 \), with \( \alpha_3(A) = 1 \). Then \( |a_{ij}| \geq 1 \) for all \( i \neq j \).

Thus, \( \|A\| \geq \sqrt{3} \Rightarrow \tilde{\alpha}_3(A) \leq \frac{1}{\sqrt{3}} \). Now let

\[
A = \begin{bmatrix}
0 & i & 1 & 1 \\
-i & 0 & 1 & -1 \\
1 & 1 & 0 & i \\
1 & -1 & -i & 0
\end{bmatrix} \in M_{4,sa}^0.
\]

Then \( \tilde{\alpha}_3(A) = \frac{1}{\sqrt{3}} \) (\( \alpha_3(A) = 1 \) and \( A^*A = 3I \)). \( \square \)

Proposition 3.9 (5 x 5 selfadjoint). \( \tilde{\alpha}_3(M_{5,sa}^0) = \frac{1}{\sqrt{3}} \).

Proof. Clearly,

\[
\tilde{\alpha}_3(M_{5,sa}^0) \geq \tilde{\alpha}_3(M_{4,sa}^0) = \frac{1}{\sqrt{3}}
\]

Now let \( A \in M_{5,sa}^0 \), with \( \alpha_3(A) = 1 \). Construct a graph \( G = (V, E) \) as follows:

\( V = \{1, 2, 3, 4, 5\} \) and \( (i, j) \in E \) if \( |a_{ij}| < 1 \) \( \Rightarrow |a_{ji}| < 1 \). We may assume the following axioms:

1. G11 is not a subgraph of \( G \). Otherwise, \( A \) would have a 1-2-2 paving of norm \( < 1 \), violating the assumption \( \alpha_3(A) = 1 \).
2. By removing a vertex from \( G \), one cannot arrive at G8. Otherwise, \( A \) would have a 4-compression \( B \) such that \( \alpha_3(B) \geq 1 \).

Since \( \tilde{\alpha}_3(M_{4,sa}^0) = \frac{1}{\sqrt{3}} \), this would imply \( \|B\| \geq \sqrt{3} \Rightarrow \|A\| \geq \sqrt{3} \Rightarrow \tilde{\alpha}_3(A) \leq \frac{1}{\sqrt{3}} \).
3. For every vertex \( i \), \( \deg(i) \geq 2 \). Otherwise, if \( \deg(i) \leq 1 \), then row \( i \) of \( A \) has at least three entries of magnitude \( \geq 1 \) \( \Rightarrow \|A\| \geq \sqrt{3} \Rightarrow \tilde{\alpha}_3(A) \leq \frac{1}{\sqrt{3}} \).

This eliminates all graphs. \( \square \)

Proposition 3.10 (6 x 6 selfadjoint). \( \tilde{\alpha}_3(M_{6,sa}^0) = \frac{1}{\sqrt{3}} \).

Proof. Clearly,

\[
\tilde{\alpha}_3(M_{6,sa}^0) \geq \tilde{\alpha}_3(M_{5,sa}^0) = \frac{1}{\sqrt{3}}
\]

Now let \( A \in M_{6,sa}^0 \), with \( \alpha_3(A) = 1 \). Construct a graph \( G = (V, E) \) as follows:

\( V = \{1, 2, 3, 4, 5, 6\} \) and \( (i, j) \in E \) if \( |a_{ij}| < 1 \) \( \Rightarrow |a_{ji}| < 1 \). We may assume the following axioms:

1. G61 is not a subgraph of \( G \). Otherwise, \( A \) would have a 2-2-2 paving of norm \( < 1 \), violating the assumption \( \alpha_3(A) = 1 \).
2. By removing vertices from \( G \), one cannot arrive at G8. Otherwise, \( A \) would have a 4-compression \( B \) such that \( \alpha_3(B) \geq 1 \).

Since \( \tilde{\alpha}_3(M_{4,sa}^0) = \frac{1}{\sqrt{3}} \), this would imply \( \|B\| \geq \sqrt{3} \Rightarrow \|A\| \geq \sqrt{3} \Rightarrow \tilde{\alpha}_3(A) \leq \frac{1}{\sqrt{3}} \).
3. For every vertex \( i \), \( \deg(i) \geq 3 \). Otherwise, if \( \deg(i) \leq 2 \), then row \( i \) of \( A \) has at least three entries of magnitude \( \geq 1 \) \( \Rightarrow \|A\| \geq \sqrt{3} \Rightarrow \tilde{\alpha}_3(A) \leq \frac{1}{\sqrt{3}} \).

This eliminates all graphs. \( \square \)
Preliminaries for $7 \times 7$ Selfadjoints

Notation: $F = [1 - \delta_{ij}] \in M^0_{n,sa}$ (the “fat” operator)

**Lemma 3.11.** Let $0 \neq A \in M^0_{n,sa}$. Then the following are equivalent:

i. $\|A\|_2^2 = \frac{n-1}{n} \|A\|_{HS}^2$.

ii. There exists a nonzero $\alpha \in \mathbb{R}$ such that

$$\sigma(\alpha^{-1}A) = \left(1, -\frac{1}{n-1}, -\frac{1}{n-1}, \ldots, -\frac{1}{n-1}\right).$$

iii. There exists a diagonal unitary $U \in D_n$ and a nonzero $\beta \in \mathbb{R}$ such that

$$U^* AU = \beta F.$$

**Proof.** (i $\Leftrightarrow$ ii): We have seen that $\|A\|_2^2 = \frac{n-1}{n} \|A\|_{HS}^2$ if and only if

$$\sigma(A) = \pm \|A\| \left(1, -\frac{1}{n-1}, -\frac{1}{n-1}, \ldots, -\frac{1}{n-1}\right).$$

(ii $\Leftrightarrow$ iii): Set $\tilde{A} = \alpha^{-1}A$. If $\sigma(\tilde{A}) = \left(1, -\frac{1}{n-1}, -\frac{1}{n-1}, \ldots, -\frac{1}{n-1}\right)$, then there exists a unitary $U \in M_n$ such that

$$\tilde{A} = V \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -\frac{1}{n-1} & 0 & \cdots & 0 \\ 0 & 0 & -\frac{1}{n-1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -\frac{1}{n-1} \end{bmatrix} V^*.$$

Letting $v$ stand for the first column of $V$, we have that

$$A = \frac{n}{n-1} vv^* - \frac{1}{n-1}I = \left[\frac{n}{n-1}v_i v_j - \frac{1}{n-1}\delta_{ij}\right].$$

Since $\tilde{A} \in M^0_{n,sa}$,

$$\frac{n}{n-1}|v_i|^2 - \frac{1}{n-1} = 0 \Rightarrow v_i = \frac{1}{\sqrt{n}} e^{i\theta_i}$$

for some $\theta_i \in \mathbb{R}$. It follows that

$$\tilde{A} = \frac{1}{n-1} \left[e^{i(\theta_i - \theta_j)} - \delta_{ij}\right] = \frac{1}{n-1} UFU^*,$$

where

$$U = \text{diag}(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n}) \in D_n.$$  

Thus, $U^* AU = \beta F$, where $\beta = \frac{\alpha}{n-1}$.

(iii $\Rightarrow$ ii): Clearly

$$F = nE - I,$$

where all the off-diagonal entries of $E \in M_n$ equal $\frac{1}{n}$. Since $E$ is a rank-one projection,

$$\sigma(F) = \{n-1, -1, -1, \ldots, -1\}.$$  

The result follows. \(\square\)
LEMMA 3.12. Let $0 \neq A \in M_{n,sa}^0$. Fix $k \geq 3$ and assume $\|B\|^2 = \frac{k-1}{k} \|B\|_{HS}^2$ for all $k$-compressions $B$ of $A$. Then there exists a diagonal unitary $U \in \mathbb{D}_n$ and an $\alpha > 0$ such that

$$U^* A U = \alpha S,$$

where all the off-diagonal entries of $S \in M_{n,sa}^0$ equal $\pm 1$.

PROOF. Let $B$ be a $k$-compression of $A$. By Lemma 3.11, all the off-diagonal entries of $B$ have the same modulus. It follows that all the off-diagonal entries of $A$ have the same modulus, say $\alpha$ (here we use $k \geq 3$). Set $C = \alpha^{-1} A$. Then all the off-diagonal entries of $C$ have modulus 1, and $\|B\|^2 = \frac{k-1}{k} \|B\|_{HS}^2$ for all $k$-compressions $B$ of $C$. We claim that $c_{rs,ct} = \pm c_{rt}$ for all $r < s < t$. Indeed, this follows from Lemma 3.11 applied to any $k$-compression of $C$. In particular, this follows from Lemma 3.11 applied to any $k$-compression of $B$ containing $r, s,$ and $t$ (again we use $k \geq 3$). Now let $\phi_1, \phi_2, ..., \phi_{n-1} \in \mathbb{R}$ be such that $c_{i,i+1} = e^{i\phi_i}$, $i = 1, 2, ..., n - 1$. For $j = 1, 2, ..., n$, define $\theta_j = -\sum_{i=1}^{j-1} \phi_i$. We claim that

$$c_{rs} = \pm e^{i(\theta_r - \theta_s)}, \ r < s.$$

Indeed,

$$c_{rs} = \pm c_{r,r+1} c_{r+1,r+2} \cdots \cdot_{s-1,s} = \pm e^{i\phi_r} e^{i\phi_{r+1}} \cdots e^{i\phi_{s-1}}$$

$$= \pm e^{i \sum_{i=r}^{s-1} \phi_i} = \pm e^{i(\sum_{i=1}^{s-1} \phi_i - \sum_{i=1}^{r-1} \phi_i)} = \pm e^{i(\theta_r - \theta_s)}.$$ Setting

$$U = \text{diag}(e^{i\theta_1}, e^{i\theta_2}, ..., e^{i\theta_n}) \in \mathbb{D}_n,$$

we have that $U^* C U = S \in M_{n,sa}^0$, where all the off-diagonal entries of $S$ are $\pm 1$. \qed

PROPOSITION 3.13 ($7 \times 7$ selfadjoint). $\tilde{\alpha}_3(M_{7,sa}^0) \in \left[\frac{2}{3}, \frac{2}{\sqrt{7}}\right] \approx [0.6667, 0.7559]$.

PROOF. Let

$$A = \begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 & -1 & -1 \\
1 & 1 & 0 & -1 & 1 & -1 & -1 \\
1 & 1 & -1 & 0 & -1 & -1 & 1 \\
1 & 1 & 1 & -1 & 0 & 1 & 1 \\
1 & -1 & -1 & -1 & 1 & 0 & 1 \\
1 & -1 & -1 & 1 & 1 & 1 & 0
\end{bmatrix} \in M_{7,sa}^0.$$

Then $\tilde{\alpha}_3(A) = \frac{2}{3}$ ($\alpha_3(A) = 2$ and $\|A\| = 3$). Thus, $\tilde{\alpha}_3(M_{7,sa}^0) \geq \frac{2}{3}$. Now let $A \in M_{7,sa}^0$, with $\alpha_3(A) = 1$.

If every 3-compression of $B$ has norm $\geq 1$, then $\|B\|_{HS}^2 \geq \frac{3}{4} \|B\|^2$ by selfadjointness using Proposition 7.5 ($p = 2$, $n = 3$). General identity: $\sum_B \|B\|_{HS}^2 = 5 \|A\|_{HS}^2$ by a counting argument. From general selfadjoint trace zero inequality for odd rank: $\|A\|_{HS}^2 \leq 6 \|A\|^2$ by Corollary 7.4 ($n = 7$). Thus

$$35 \leq \sum_B \|B\|^2 \leq \frac{2}{3} \sum_B \|B\|_{HS}^2 = \frac{10}{3} \|A\|_{HS}^2 \leq 20 \|A\|^2$$

and hence $\|A\| \geq \frac{\sqrt{7}}{2} \Rightarrow \tilde{\alpha}_3(A) \leq \frac{2}{\sqrt{7}}$.

That $\|A\| \geq \frac{\sqrt{7}}{2}$ is a special case of Corollary 7.6 ($n = 7$, $k = 3$), so the above internal proof of this can alternatively be referenced.
If, on the other hand, some 3-compression of A has norm < 1, then the complementary 4-compression B satisfies \( \alpha_2(B) \geq 1 \). Since \( \tilde{\alpha}_2(M^0_{4,sa}) = \frac{2}{\sqrt{3}} \), \( \|B\| \geq \sqrt{3} \Rightarrow \tilde{\alpha}_3(A) \leq \frac{2}{\sqrt{3}} < \frac{2}{\sqrt{7}} \).

Now assume \( \alpha_3(A) = 1 \) and \( \|A\| = \frac{\sqrt{7}}{2} \). By the previous discussion, every 3-compression \( B \) of \( A \) has norm \( \geq 1 \). Thus

\[
35 \leq \sum_B \|B\|^2 \leq \frac{2}{3} \sum_B \|B\|^2_{HS} = \frac{10}{3} \|A\|^2_{HS} \leq 20 \|A\|^2 = 35.
\]

It follows that \( \|B\|^2 = \frac{2}{3} \|B\|^2_{HS} \) for all 3-compressions \( B \) of \( A \). By Lemma 3.13, there exists a diagonal unitary \( U \in \mathbb{D}_n \) and an \( \alpha > 0 \) such that \( U^*AU = \alpha S \), where all the off-diagonal entries of \( S \in \mathbb{M}^0_{n,sa} \) are \( \pm 1 \). Searching exhaustively among all such \( S \), we see that \( \tilde{\alpha}_3(A) \leq \frac{2}{\sqrt{7}} < \frac{2}{\sqrt{5}}, \) a contradiction.

**PROPOSITION 3.14 (8 \times 8 selfadjoint).** \( \tilde{\alpha}_3(M^0_{8,sa}) \in \left[\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{5}}\right] \approx [0.6667, 0.8944] \).

**PROOF.** Clearly,

\[
\tilde{\alpha}_3(M^0_{8,sa}) \geq \tilde{\alpha}_3(M^0_{7,sa}) \geq \frac{2}{3}.
\]

Now let \( A \in \mathbb{M}^0_{8,sa} \), with \( \alpha_3(A) = 1 \). If every 3-compression of \( A \) has norm \( \geq 1 \), then \( \|A\| \geq \frac{\sqrt{7}}{2} \) (by proof of 3.13 every 7-compression has norm \( \geq \frac{\sqrt{7}}{2} \)) \( \Rightarrow \tilde{\alpha}_3(A) \leq \frac{2}{\sqrt{7}} < \frac{2}{\sqrt{5}} \). If, on the other hand, some 3-compression of \( A \) has norm < 1, then the complementary 5-compression \( B \) satisfies \( \alpha_2(B) \geq 1 \). Since \( \tilde{\alpha}_2(M^0_{5,sa}) = \frac{2}{\sqrt{5}} \), \( \|B\| \geq \frac{\sqrt{5}}{2} \Rightarrow \|A\| \geq \frac{\sqrt{5}}{2} \Rightarrow \tilde{\alpha}_3(A) \leq \frac{2}{\sqrt{5}} \).

**PROPOSITION 3.15 (10 \times 10 selfadjoint).** \( \tilde{\alpha}_3(M^0_{10,sa}) \in \left[\frac{\sqrt{5}}{3}, 1\right] \approx [0.7454, 1] \).

**PROOF.** Let

\[
A = \begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 \\
1 & 1 & 0 & 1 & -1 & 1 & 1 & 1 & -1 & -1 \\
1 & 1 & 1 & 0 & -1 & -1 & -1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 & 0 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 & 1 & 0 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 & -1 & 0 & 1 & 1 & 1 & -1 \\
1 & -1 & 1 & -1 & 1 & 0 & 1 & 1 & 1 & -1 \\
1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & 0 \\
1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 0
\end{bmatrix} \in \mathbb{M}^0_{10,sa}.
\]

Then \( \tilde{\alpha}_3(A) = \frac{\sqrt{5}}{3} \) (\( \alpha_3(A) = \sqrt{5} \) and \( A^*A = 9I \)).

**Remark:** \( A \) is a conference matrix.
3. Nonnegative

**Lemma 3.16.** Let \( A \in \mathbb{M}_{4,++}^0 \). If \( \alpha_3(A) = 1 \) and a row or column of \( A \) has three entries \( \geq 1 \), then \( \| A \| \geq 2 \). This inequality is sharp.

**Proof.** We may assume the first row of \( A \) has three entries \( \geq 1 \). Then

\[
\| A \| \geq \begin{pmatrix}
0 & 1 & 1 & 1 \\
0 & 0 & b_{23} & b_{24} \\
0 & b_{32} & 0 & b_{34} \\
0 & b_{42} & b_{43} & 0
\end{pmatrix},
\]

where \( \max\{b_{ij}, b_{ji}\} \geq 1 \) for all \( i \neq j \). Since

\[
\min\left\{ \begin{pmatrix}
0 & 1 & 1 & 1 \\
0 & 0 & \delta_{23} & \delta_{24} \\
0 & 1 - \delta_{23} & 0 & \delta_{34} \\
0 & 1 - \delta_{24} & 1 - \delta_{34} & 0
\end{pmatrix} : \delta_{23}, \delta_{24}, \delta_{34} \in \{0, 1\} \right\} = 2,
\]

we have that \( \| A \| \geq 2 \). A sharp example is furnished by the matrix

\[
A = \begin{pmatrix}
0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0
\end{pmatrix}.
\]

\[\square\]

**Proposition 3.17** \((4 \times 4 \) nonnegative\). \( \tilde{\alpha}_3(\mathbb{M}_{4,++}^0) = 0.5550 \).

**Proof.** Suppose \( A \in \mathbb{M}_{4,++}^0 \) with \( \alpha_3(A) = 1 \). Create a digraph \( D = (V, E) \) as follows: \( V = \{1, 2, 3, 4\} \) and \( (i, j) \in E \) if \( a_{ij} \geq 1 \). We may assume the following axioms:

1. For all \( i \neq j \), either \( (i, j) \in E \) or \( (j, i) \in E \). Otherwise, \( A \) admits a 1-1-2 paving of norm \(< 1\), violating the assumption \( \alpha_3(A) = 1 \).
2. For all vertices \( i \), the in-degree of \( i \) and the out-degree of \( i \) are less than 3. Otherwise, row \( i \) or column \( i \) of \( A \) has three entries \( \geq 1 \Rightarrow \| A \| \geq 2 \) (Lemma 3.16) \( \Rightarrow \tilde{\alpha}_3(A) \leq \frac{2}{\kappa} < \kappa \).

This leaves digraphs \( D149, D185, D186, \) and \( D218, \) which all have \( D149 \) as a subgraph. Thus,

\[
\| A \| \geq \begin{pmatrix}
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{pmatrix} = \frac{1}{\kappa} \Rightarrow \tilde{\alpha}_3(A) \leq \kappa.
\]

Now let

\[
A = \begin{pmatrix}
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{pmatrix}.
\]

Then \( \tilde{\alpha}_3(A) = \kappa \Rightarrow \tilde{\alpha}_3(\mathbb{M}_{4,++}^0) \geq \kappa. \)

\[\square\]
Proposition 3.18 $(6 \times 6$ nonnegative). $\tilde{\alpha}_3(M_{6,++}^0) \in \left[ \kappa, \frac{2}{1+\sqrt{5}} \right] \approx [0.5550, 0.6180].$

Proof. Suppose $A \in M_{6,++}^0$, with $\alpha_3(A) = 1$. Create a graph $G = (V,E)$ as follows: $V = \{1, 2, 3, 4, 5, 6\}$ and $(i, j) \in E$ if $a_{ij}, a_{ji} < 1$. We may assume the following axioms:

1. $G_61$ is not a subgraph of $G$. Otherwise, $A$ has a 2-2-2 paving of norm $< 1$, violating the assumption $\alpha_3(A) = 1$.
2. By removing vertices, one cannot arrive at $G_8$. Otherwise, $A$ has a 4-compression $B$ with $\alpha_3(B) \geq 1 \Rightarrow \|B\| \geq \frac{1}{\kappa} \Rightarrow \|A\| \geq \frac{1}{\kappa} \Rightarrow \tilde{\alpha}_3(A) \leq \kappa$.
3. $G$ has no isolated vertices. Otherwise, if vertex $i$ is isolated, then either row $i$ or column $i$ of $A$ has at least three entries $\geq 1 \Rightarrow \|A\| \geq \sqrt{3} \Rightarrow \tilde{\alpha}_3(A) \leq \frac{1}{\sqrt{3}}$.
4. There does not exist a partition $V = \{i, j, k\} \bigcup \{i', j', k'\}$ such that $(r, s) \notin E, r, s \in \{i, j, k\}$. Otherwise, some $3 \times 3$ submatrix of $A$ has at least five entries $\geq 1 \Rightarrow \|A\| \geq \frac{1}{\kappa} \Rightarrow \tilde{\alpha}_3(A) \leq \kappa$ (by exhaustive search of 0-1 $3 \times 3$ matrices with five 1’s).

This leaves $G_{114}$ and $G_{133}$, both of which have a 5-compression of the form

$$
\begin{bmatrix}
0 & * & * & * \\
* & 0 & * & * \\
* & * & 0 & * \\
* & * & * & 0
\end{bmatrix},
$$

where a “*” in the $(i, j)$ position indicates that $a_{ij} \geq 1$ or $a_{ji} \geq 1$, and a “.” in the $(i, j)$ position indicates that $a_{ij} < 1$. Searching exhaustively over all 0-1 $5 \times 5$ matrices satisfying this pattern yields $\|A\| \geq \frac{1+\sqrt{5}}{2} \Rightarrow \tilde{\alpha}_3(A) \leq \frac{2}{1+\sqrt{5}}$. $\square$
## 2,3-Pavings Summary Table

<table>
<thead>
<tr>
<th>n</th>
<th>( \alpha_2(M_n^0) )</th>
<th>( \alpha_2(M_{n,sq}^0) )</th>
<th>( \alpha_2(M_{n,sym}^0) )</th>
<th>( \alpha_3(M_n^0) )</th>
<th>( \alpha_3(M_{n,sq}^0) )</th>
<th>( \alpha_3(M_{n,++}^0) )</th>
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<td>( \frac{1}{2} )</td>
<td>.5000</td>
<td>0</td>
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<td>( \sqrt{3} )</td>
<td>[?, \frac{1}{\sqrt{3}}]</td>
<td>[.5493, .5773]</td>
<td>( \frac{2}{1+\sqrt{5}} )</td>
<td>.6180</td>
<td>.5773</td>
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Part 2

Supplementary Material and Tools
CHAPTER 5

Supplementary Material: 2-Pavings
CHAPTER 6

Supplementary Material: 3-Pavings

1. 4 × 4 General

Lemma 6.1. Let $A \in M^0_4$. If $\alpha_3(A) = 1$ and $\|A\| < \sqrt{3}$, then there exists a permutation matrix $U \in M_4$ such that

$$U^*AU = \begin{bmatrix} 0 & \hat{a} & \hat{b} & \hat{c} \\ \hat{a} & 0 & \hat{d} & \hat{e} \\ \hat{b} & \hat{d} & 0 & \hat{f} \\ \hat{c} & \hat{e} & \hat{f} & 0 \end{bmatrix},$$

where $|\hat{x}| \leq |\hat{x}|$ for all $x \in \{a, b, c, d, e, f\}$. The result remains true if $A \gg 0$ and $\|A\| < 2$.

Proof. Let

$$A = \begin{bmatrix} 0 & a_{12} & a_{13} & a_{14} \\ a_{21} & 0 & a_{23} & a_{24} \\ a_{31} & a_{32} & 0 & a_{34} \\ a_{41} & a_{42} & a_{43} & 0 \end{bmatrix}.$$

The condition $\alpha_3(A) = 1$ implies that $\max\{|a_{ij}|, |a_{ji}|\} \geq 1$ for all $i < j$. The condition $\|A\| < \sqrt{3}$ (resp. $A \gg 0$ and $\|A\| < 2$) ensures that each row and each column has at most two entries of magnitude greater than or equal to 1 (see Lemma 6.1). Conjugating by $U_{(12)}$, if necessary, we may assume that $|a_{12}| \geq |a_{21}|$, which we indicate as follows:

$$A = \begin{bmatrix} 0 & \hat{a}_{12} & \hat{a}_{13} & \hat{a}_{14} \\ \hat{a}_{21} & 0 & a_{23} & a_{24} \\ \hat{a}_{31} & a_{32} & 0 & a_{34} \\ \hat{a}_{41} & a_{42} & a_{43} & 0 \end{bmatrix}.$$

Case 1: Suppose $|a_{13}| \geq |a_{31}|$. Then

$$A = \begin{bmatrix} 0 & \hat{a}_{12} & \hat{a}_{13} & \hat{a}_{14} \\ \hat{a}_{21} & 0 & a_{23} & a_{24} \\ \hat{a}_{31} & a_{32} & 0 & a_{34} \\ \hat{a}_{41} & a_{42} & a_{43} & 0 \end{bmatrix}.$$

Conjugating by $U_{(23)}$, if necessary, we may assume that $|a_{23}| \geq |a_{32}|$. Then

$$A = \begin{bmatrix} 0 & \hat{a}_{12} & \hat{a}_{13} & \hat{a}_{14} \\ \hat{a}_{21} & 0 & a_{23} & a_{24} \\ \hat{a}_{31} & a_{32} & 0 & \hat{a}_{34} \\ \hat{a}_{41} & a_{42} & \hat{a}_{43} & 0 \end{bmatrix}.$$
If $|a_{24}| \geq |a_{42}|$, then we are done. Thus, we may assume the opposite. That is,

$$ A = \begin{bmatrix} 0 & \hat{a}_{12} & \hat{a}_{13} & \hat{a}_{14} \\ \hat{a}_{21} & 0 & \hat{a}_{23} & \hat{a}_{24} \\ \hat{a}_{31} & \hat{a}_{32} & 0 & \hat{a}_{34} \\ \hat{a}_{41} & \hat{a}_{42} & \hat{a}_{43} & 0 \end{bmatrix}. $$

Conjugating by $U = U_{(1432)}$ yields

$$ U^*AU = \begin{bmatrix} 0 & \hat{a}_{14} & \hat{a}_{12} & \hat{a}_{13} \\ \hat{a}_{21} & 0 & \hat{a}_{23} & \hat{a}_{24} \\ \hat{a}_{34} & \hat{a}_{31} & \hat{a}_{32} & 0 \\ \hat{a}_{34} & \hat{a}_{31} & \hat{a}_{32} & 0 \end{bmatrix}. $$

Case 2: Suppose $|a_{13}| < |a_{31}|$. Then

$$ A = \begin{bmatrix} 0 & \hat{a}_{12} & \hat{a}_{13} & \hat{a}_{14} \\ \hat{a}_{21} & 0 & a_{23} & a_{24} \\ \hat{a}_{31} & a_{32} & 0 & a_{34} \\ \hat{a}_{41} & a_{42} & a_{43} & 0 \end{bmatrix}. $$

Case 2.1: If $|a_{14}| \geq |a_{41}|$, then

$$ A = \begin{bmatrix} 0 & \hat{a}_{12} & \hat{a}_{13} & \hat{a}_{14} \\ \hat{a}_{21} & 0 & a_{23} & a_{24} \\ \hat{a}_{31} & a_{32} & 0 & a_{34} \\ \hat{a}_{41} & a_{42} & a_{43} & 0 \end{bmatrix}. $$

Conjugating by $U_{(34)}$ yields

$$ U_{(34)}^*AU_{(34)} = \begin{bmatrix} 0 & \hat{a}_{12} & \hat{a}_{14} & \hat{a}_{13} \\ \hat{a}_{21} & 0 & a_{24} & a_{23} \\ \hat{a}_{31} & a_{42} & 0 & a_{43} \\ \hat{a}_{31} & a_{32} & a_{34} & 0 \end{bmatrix}, $$

and we may proceed as in Case 1.

Case 2.2: If $|a_{14}| < |a_{41}|$, then

$$ A = \begin{bmatrix} 0 & \hat{a}_{12} & \hat{a}_{13} & \hat{a}_{14} \\ \hat{a}_{21} & 0 & a_{23} & a_{24} \\ \hat{a}_{31} & a_{32} & 0 & a_{34} \\ \hat{a}_{41} & a_{42} & a_{43} & 0 \end{bmatrix}. $$

Conjugating by $U_{(34)}$ if necessary, we may assume that $|a_{34}| \geq |a_{43}|$. Then

$$ A = \begin{bmatrix} 0 & \hat{a}_{12} & \hat{a}_{13} & \hat{a}_{14} \\ \hat{a}_{21} & 0 & a_{23} & a_{24} \\ \hat{a}_{31} & \hat{a}_{32} & 0 & \hat{a}_{34} \\ \hat{a}_{41} & \hat{a}_{42} & \hat{a}_{43} & 0 \end{bmatrix}. $$

Case 2.2.1: If $|a_{24}| \geq |a_{42}|$, then

$$ A = \begin{bmatrix} 0 & \hat{a}_{12} & \hat{a}_{13} & \hat{a}_{14} \\ \hat{a}_{21} & 0 & a_{23} & a_{24} \\ \hat{a}_{31} & \hat{a}_{32} & 0 & \hat{a}_{34} \\ \hat{a}_{41} & \hat{a}_{42} & \hat{a}_{43} & 0 \end{bmatrix}. $$
Conjugating by $U = U_{(1234)}$ yields

$$U^*AU = \begin{bmatrix}
\hat{a}_{23} & \hat{a}_{24} & \hat{a}_{21} \\
\hat{a}_{32} & 0 & \hat{a}_{34} & \hat{a}_{31} \\
\hat{a}_{42} & \hat{a}_{43} & 0 & \hat{a}_{41} \\
\hat{a}_{12} & \hat{a}_{13} & \hat{a}_{14} & 0
\end{bmatrix}.$$

**Case 2.2.2:** If $|a_{24}| < |a_{42}|$, then

$$A = \begin{bmatrix}
\hat{a}_{12} & \hat{a}_{13} & \hat{a}_{14} \\
\hat{a}_{21} & 0 & \hat{a}_{23} & \hat{a}_{24} \\
\hat{a}_{31} & \hat{a}_{32} & 0 & \hat{a}_{34} \\
\hat{a}_{41} & \hat{a}_{42} & \hat{a}_{43} & 0
\end{bmatrix}.$$

Conjugating by $U = U_{(13)(24)}$ yields

$$U^*AU = \begin{bmatrix}
\hat{a}_{34} & \hat{a}_{31} & \hat{a}_{32} \\
\hat{a}_{43} & 0 & \hat{a}_{41} & \hat{a}_{42} \\
\hat{a}_{13} & \hat{a}_{14} & 0 & \hat{a}_{12} \\
\hat{a}_{23} & \hat{a}_{24} & \hat{a}_{21} & 0
\end{bmatrix}.$$

\[\square\]

**Remark 6.2.** Although this example doesn’t satisfy the hypotheses of Lemma 6.1, it satisfies the conclusion. Also, the extreme example doesn’t satisfy the graph theory, since $|\cdot| < 1$.

**D149:** breadth-first labeling 2134

$$\begin{bmatrix}
0 & * & * & . \\
* & 0 & * & * \\
* & * & 0 & * \\
. & . & . & 0
\end{bmatrix}$$

$$\inf \left\{ \begin{bmatrix}
0 & 1 & 1 & . \\
. & 0 & 1 & 1 \\
. & . & 0 & 1 \\
1 & . & . & 0
\end{bmatrix} \right\} = \begin{bmatrix}
0 & 1 & 1 & -\frac{2}{1+\sqrt{5}} \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{bmatrix} \approx \frac{1 + \sqrt{5}}{2} \approx 1.6180$$

**D185:** breadth-first labeling 2341

$$\begin{bmatrix}
0 & * & * & . \\
* & 0 & * & * \\
* & * & 0 & * \\
. & . & . & 0
\end{bmatrix}$$

$$\inf \left\{ \begin{bmatrix}
0 & 1 & 1 & . \\
. & 0 & 1 & 1 \\
. & 1 & 0 & 1 \\
1 & . & . & 0
\end{bmatrix} \right\} = \begin{bmatrix}
0 & 1 & 1 & -1 \\
0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0
\end{bmatrix} = \sqrt{3} \approx 1.7321$$

**Remark 6.2.** Although this example doesn’t satisfy the hypotheses of Lemma 6.1, it satisfies the conclusion. Also, the extreme example doesn’t satisfy the graph theory, since $|\cdot| < 1$.

**D186:** breadth-first labeling 3124

$$\begin{bmatrix}
0 & * & * & . \\
* & 0 & * & * \\
* & . & 0 & * \\
. & . & . & 0
\end{bmatrix}$$
D218: breadth-first labeling 3124

$$\inf \left\{ \left\| \begin{bmatrix} 0 & 1 & 1 \\
0 & 0 & 1 \\
1 & 0 & 0 \\
1 & \cdot & 0 
\end{bmatrix} \right\| \right\} = \left\| \begin{bmatrix} 0 & 1 & 1 & -1/2 \\
-1/2 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 \\
1 & 1/2 & 0 & 0 
\end{bmatrix} \right\| = \frac{\sqrt{11}}{2} \approx 1.6583$$

Remark 6.3. Notice that this is a circulant. Best among circulants?
CHAPTER 7

Tools

1. Universal Selfadjoint 3-Identity and consequences

LEMMA 7.1 (Universal Selfadjoint 3-Identity). Arbitrary $3 \times 3$ selfadjoint trace zero matrices $S$ satisfy:

$$\frac{|S|^2}{2|S|^2} + \left|\text{Det } S\right| = 1$$

PROOF. Since all trace zero finite (or trace class) matrices have a basis in which their representation has zero diagonal, without loss of generality we can assume $S$ has the form:

$$S = \begin{pmatrix}
0 & a & b \\
\pi & 0 & c \\
b & c & 0
\end{pmatrix}$$

and by computation, the characteristic polynomial:

$$c_\lambda(S) = \det (\lambda - S) = \lambda^3 - 2 \text{Re } abc - \lambda(|a|^2 + |b|^2 + |c|^2)$$

$$= \lambda^3 - (|a|^2 + |b|^2 + |c|^2)\lambda - 2 \text{Re } abc$$

$$= \frac{||S||^2}{2} \lambda - \text{Det } S.$$ 

An alternative way to see this is that the characteristic polynomial has the form $\lambda^3 + p\lambda^2 + q\lambda + r$, with $p = 0$ because the sum of the roots is the trace of $S$, the latter also implying

$$q = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = \frac{1}{2} ((\lambda_1 + \lambda_2 + \lambda_3)^2 - (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)) = \frac{-||S||^2}{2}$$

where $\lambda_j$, $j = 1, 2, 3$ denotes its roots, and $r = -\lambda_1\lambda_2\lambda_3 = -\text{Det } S$.

Since $S$ is selfadjoint, $\lambda = \pm||S||$ is an eigenvalue of $S$. Also, because this is the largest eigenvalue in modulus and $S$ has trace zero, the other two real eigenvalues are opposite this in sign making their product, $\text{Det } S$, the same sign as $\lambda$. Hence $(\pm||S||)^3 = \frac{||S||^2}{2}(\pm||S||) + (\pm|\text{Det } S|)$, whence the Universal Selfadjoint 3-Identity in either case. □

COROLLARY 7.2 (Universal Selfadjoint 3-Identity consequences). For arbitrary $3 \times 3$ selfadjoint trace zero matrices $S$,

$$||S|| = 1 \iff \frac{|S|^2}{2} + |\text{Det } S| = 1.$$ 

For greater or less than 1, the respective conditions are equivalent. A necessary condition for equality is $3/2 \leq ||S||^2 \leq 2$. 35
7. TOOLS

PROOF. The Universal Selfadjoint 3-Identity, \( \frac{||S||^2}{2||S||^2} + |\text{Det } S| = 1 \), implies that if \( ||S|| > 1 \) then \( \frac{||S||^2}{2||S||^2} + |\text{Det } S| > 1 \), and likewise, if \( ||S|| < 1 \) then \( \frac{||S||^2}{2||S||^2} + |\text{Det } S| < 1 \). Therefore \( ||S|| = 1 \) if and only if \( \frac{||S||^2}{2||S||^2} + |\text{Det } S| = 1 \).

Moreover, if \( \frac{||S||^2}{2||S||^2} + |\text{Det } S| = 1 \), then \( ||S||^2 \leq 2 \). Also in this case when \( ||S|| = 1 \), \( ||S||^2 \geq \frac{1}{2} ||S||^2 = \frac{1}{2} \) is the \( n = 3 \), \( p = 2 \) case of Proposition 7.5. \( \square \)

2. Universal Selfadjoint 4-Identity and consequences

Universal Selfadjoint 4-Identity (for \( 4 \times 4 \) selfadjoint zero-trace):

\[
\frac{||S||^2}{2||S||^2} + \frac{|\text{Tr } S^3|}{3||S||^3} - \frac{|\text{Det } S|}{||S||^2} = 1
\]

Unpolished and unverified work (for proofs see file UniversalIdentities.Tex):

Consequence: Since \( \frac{|\text{Det } S|}{||S||^2} \leq 1 \)

\[
\frac{||S||^2}{2||S||^2} + \frac{|\text{Tr } S^3|}{3||S||^3} \leq 2
\]

Separate Fact (\( ||S||^2 \geq \frac{n}{n^2} ||S||^2 \)): \( ||S||^2 \geq \frac{4}{3} ||S||^2 \) so \( \frac{||S||^2}{2||S||^2} \geq \frac{2}{3} \)

Implying: \( \frac{|\text{Tr } S^3|}{3||S||^3} \leq \frac{4}{3} \)

(Trivially also follows generally from Hölder: \( |\text{Tr } S^3|^{1/3} \leq ||S||_3 \leq 4^{1/3}||S|| \))

Development of Universal Selfadjoint 4-Identity:

Let \( S \) denote a \( 4 \times 4 \) selfadjoint zero-trace matrix with eigenvalues

\( 1 = \lambda_1 \geq |\lambda_2| \geq |\lambda_3| \geq |\lambda_4| \).

\( c_\lambda(S) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)(\lambda - \lambda_4) = \lambda^4 - (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)\lambda^3 + (\sum_{i<j} \lambda_i\lambda_j)\lambda^2 - (\sum_{i<j<k} \lambda_i\lambda_j\lambda_k)\lambda + \lambda_1\lambda_2\lambda_3\lambda_4 = \lambda^4 + p\lambda^2 - q\lambda + r \)

SUMMARY: NASC for \( ||S|| = 1 \) (unverified)

1. \( p \geq \frac{q}{3} \)
2. \( p + |q| + r = 1 \)
3. \( 0 \leq p + |q| \leq 2 \) (equivalent to \( |\text{product of roots}| \leq 1 \))
4. When \( p < 1 \), \( \frac{20}{27} - \frac{2}{3}p - \frac{2}{27}(3p - 2)^{3/2} \leq q \leq \frac{20}{27} - \frac{2}{3}p + \frac{2}{27}(3p - 2)^{3/2} \).
5. When \( p \geq 1 \), \( 0 \leq q \leq \frac{20}{27} - \frac{2}{3}p + \frac{2}{27}(3p - 2)^{3/2} \).

(4-5: \( \max (0, \frac{20}{27} - \frac{2}{3}p - \frac{2}{27}(3p - 2)^{3/2}) \leq q \leq \frac{20}{27} - \frac{2}{3}p + \frac{2}{27}(3p - 2)^{3/2} \))
3. Operator Norm/p-Norm Comparisons

**Proposition 7.3 (Operator Norm/p-Norm).** If \( A \) is a finite rank selfadjoint trace 0 matrix and

\[
|\lambda| = \left|\frac{\text{rank } A - k}{2} \right| \text{ pairs of them}, 1, -\frac{k}{k+1}, -\frac{1}{k(k+1)}, \ldots, -\frac{1}{k(k+1)};
\]

(Sharp example: \diag (-1, 1))

(Sharp asymptotically: \diag (±1, \ldots, ±1, (\frac{\text{rank } A - k}{2} - 1) pairs of them), 1, -\frac{k}{k+1}, -\frac{1}{k(k+1)}, \ldots, -\frac{1}{k(k+1)});

(note: rank \( A - k \) must be even)

**Proof.** Easy proof for \( p \geq 2 \) based on the \( p = 2 \) case:

If \( \lambda_j > \lambda_1 \) are the (real) eigenvalues of \( A \), then

\[
\sum_{k=1}^{n} |\lambda_j|^p = \sum_{k=1}^{n} |\lambda_j|^{p-2} |\lambda_j|^2 \leq |\lambda_1|^{p-2} \sum_{k=1}^{n} |\lambda_j|^2 \leq |\lambda_1|^{p-2} (n-k)|\lambda_1|^2 = (n-k)|\lambda_1|^p.
\]

For all \( p \geq 1 \), we describe informally the following variational approach:

Maximize \( \sum |\lambda_j|^p \) subject to \( \lambda_1 + \cdots + \lambda_n = 0 \).

Without loss of generality, \( A \neq 0 \), \( ||A|| \leq 1 \) and \( \text{tr } A \neq 0 \) implies that for some \( n > m \geq 1 \) the eigenvalues of \( A \) have the \([-1, 1]\) distribution:

\[-1 \leq \lambda_n \leq \cdots \leq \lambda_{m+1} < 0 < \lambda_m \leq \cdots \leq \lambda_1 \leq 1,\]

We induct on \( n-k \). Since \( A \neq 0 \), \( n-k > 0 \) and is even and so \( n-k \geq 2 \).

Increase \( \lambda_1 \) and decrease \( \lambda_n \) equally so to preserve the trace, until one of them reaches 1 or \(-1\), respectively. (Increasing both moduli increases the sum \( \sum |\lambda_j|^p \) and so permits reduction of the proof to this case.) If they both reach 1 or \(-1\), then dropping them leaves \( k \) invariant and reduces to the \( n-k-2 \) case.

If now \( \lambda_1 = 1 \) and \( \lambda_n = -1 \) (handle the reverse case the same), decrease \( \lambda_n \) and increase \( \lambda_{n-1} \) equally to preserve their sum. Elementary calculus shows that this will increase \( |\lambda_n|^p + |\lambda_{n-1}|^p \). Continue this until either \( \lambda_n \) reaches \(-1\) or \( \lambda_{n-1} \) reaches \( \lambda_{n-2} \). If the former, then drop \( \lambda_n \) and \( \lambda_1 \), and again apply the induction hypothesis. If the latter, then decrease both until \( \lambda_n \) reaches \(-1\) or both \( \lambda_{n-1} \) and \( \lambda_{n-2} \) reaches \( \lambda_{n-3} \), and so on. This process will increase \( \sum |\lambda_j|^p \) and unless \( m = 1 \), one has \( m > 1 \) or equivalently, \( \lambda_n + \cdots + \lambda_{m+1} < -1 \) implying that eventually in this process \( \lambda_n \) will reach \(-1\) so we can apply again the induction hypothesis while preserving \( k \). If \( m = 1 \), then this process ends in one pair of \( \pm 1 \) with sum 2 so \( \sum_{j=1}^{n} |\lambda_j|^p \leq 2 \leq n-k \).

**Corollary 7.4.** If \( A \) is an \( n \times n \) selfadjoint trace 0 matrix with \( n \) odd, then \( ||A||_2 \leq \sqrt{n-1} ||A|| \).
Proposition 7.5. If $A$ is an $n \times n$ selfadjoint trace 0 matrix and $p \geq 1$ (or more generally rank $A = n$), then
\[ ||A||_p \geq \left[ 1 + \frac{1}{(n-1)^{p-1}} \right]^{1/p} ||A|| \]
with equality iff $A = c \text{diag}(-1, \frac{1}{n-1}, \ldots, \frac{1}{n-1})$.

Proof. Suffices to show the sequence analog for $\lambda_1 + \cdots + \lambda_n = 0$, all $\lambda_j$ real. Since the inequality is obvious for $p = 1$, needing selfadjoint with trace 0 to see it, we can assume without loss of generality that $p > 1$. Then
\[ |\lambda_1| = \left| - \sum_{2}^{n} \lambda_j \right| \leq ||1||_p^p ||\lambda||_p \]
where $\lambda := < \lambda_j >_{2 \leq j \leq n}$, $1 := < 1 >_{2 \leq j \leq n}$, and $\frac{1}{p} + \frac{1}{p'} = 1$, i.e., $\frac{p}{p'} = p - 1$.
Equality holds if and only if $\lambda$ is a constant multiple of $1$. (This is the $p$-case for Cauchy-Schwartz equality which I presume holds true for $p \neq 2$ like it does for $p = 2$—except I don’t know a reference.) So
\[ |\lambda_1|^p \leq (n-1)^{p/p'} \sum_{2}^{n} |\lambda_j|^p = (n-1)^{p-1} \sum_{2}^{n} |\lambda_j|^p. \]
Adding $(n-1)^{p-1}|\lambda_1|^p$ to both sides yields: $[1 + (n-1)^{p-1}]||A||_p^p \leq (n-1)^{p-1}||A||_p^p$, from which (iii) follows. The case for equality also follows from the previous comment about equality. \qed
COROLLARY 7.6. If every k-compression of $A \in \mathbb{M}_{n,sa}^0$ has norm $\geq 1$, then

$$\|A\| \geq \begin{cases} \sqrt{\frac{n-1}{k-1}} & \text{n even} \\ \sqrt{\frac{n}{k}} & \text{n odd} \end{cases}.$$  

PROOF. Denote by $\Pi_k$ the set of all k-compressions of $A$. Then $\|B\|^2 \leq \frac{k-1}{k} \|B\|_{HS}^2$ for all $B \in \Pi_k$ by Proposition 7.5 ($p = 2$ & take n to be k). Thus,

$$\|A\|^2 \geq \frac{\binom{n}{k}}{(n-1)} \frac{k-1}{k} \frac{n-2}{k-2} \geq \frac{\sqrt{n-1}}{k-1} \text{ or } \frac{\sqrt{n}}{k-1}.$$  

□

COROLLARY 7.7. If $\tilde{\alpha}_2(\mathbb{M}_{n-k,sa}^0) < \tilde{\alpha}_3(\mathbb{M}_{n,sa}^0)$ and

$$\tilde{\alpha}_3(\mathbb{M}_{n,sa}^0) \cap \{\text{all zero-diagonals with } \pm 1 \text{ off diagonal entries}\} < \begin{cases} \frac{k-1}{\sqrt{n}} & \text{n even} \\ \sqrt{\frac{n}{k}} & \text{n odd} \end{cases},$$  

then

$$\tilde{\alpha}_3(\mathbb{M}_{n,sa}^0) < \begin{cases} \frac{k-1}{\sqrt{n}} & \text{n even} \\ \sqrt{\frac{n}{k}} & \text{n odd} \end{cases}.$$  

PROOF. Fix an extremal $A = A_n$, that is, $\tilde{\alpha}_3(\mathbb{M}_{n,sa}^0) = \frac{\alpha_3(A)}{\|A\|}$ and without loss of generality assume $\alpha_3(A) = 1$ and $\|A\| = \frac{1}{\alpha_3(\mathbb{M}_{n,sa}^0)}$.

Either $\|B\| < 1$ for some k-compression or every k-compression $B$ of $A$ has norm $\geq 1$.

Assume first $\|B\| < 1$ for some k-compression $B = PAP$. Because $\alpha_3(A) = 1$, every 3-paving has norm $\geq 1$ and by definition, $\tilde{\alpha}_2(\mathbb{M}_{n-k,sa}^0) \geq \frac{\alpha_2((I-P)A(I-P))}{\alpha_3(\mathbb{M}_{n-k,sa}^0)}$, so $\|(I-P)A(I-P)\| \geq \alpha_3((I-P)A(I-P))$. So if additionally $\|B\| < 1$ and $\alpha_3(A) = 1$, then $\alpha_2((I-P)A(I-P)) = 1$ so all 2-pavings of $(I-P)A(I-P)$ have norm $\geq 1$, in which case

$$\|A\| \geq \|(I-P)A(I-P)\| \geq \frac{1}{\alpha_2(\mathbb{M}_{n-k,sa}^0)} > \frac{1}{\alpha_3(\mathbb{M}_{n,sa}^0)}$$  

(the last $>$ by hypothesis), contradicting $\tilde{\alpha}_3(\mathbb{M}_{n,sa}^0) = \frac{\alpha_3(A)}{\|A\|} = \frac{1}{\alpha_3}$.

On the other hand, if every k-compression $B$ of $A$ has norm $\geq 1$, then the displayed inequality in Corollary 7.6 becomes equality throughout:

$$\binom{n}{k} \sum_{B \in \Pi_k} \|B\|^2 \leq \frac{k-1}{k} \sum_{B \in \Pi_k} \|B\|_{HS}^2 = \frac{k-1}{k} \binom{n-2}{k-2} \|A\|_{HS}^2 = (n - 1) \frac{k-1}{k} \binom{n-2}{k-2} \|A\|^2.$$

So each $\|B\|^2 = \frac{k-1}{k} \|B\|_{HS}^2$. Now apply Lemma 3.12 so that

$$A \cong S \in \mathbb{M}_{n,sa}^0 \cap \{\text{all zero-diagonals with } \pm 1 \text{ off diagonal entries}\}$$

and apply the hypothesis to $S$ to contradict the extremality of $A$. □
For what $n$, $k = \frac{n}{3}$ is this good for? Solve for $n \equiv 3 \mod 0$, $\frac{n^2 - 1}{\sqrt{n - 1}} < 1$ ($n$ odd) and $\frac{n^2 - 1}{\sqrt{n}} < 1$ ($n$ even). Answer: $n = 12, 9, 6$. 
4. Operator Norm/Hilbert-Schmidt Norm Comparisons

**Lemma 7.8.** Let $A \in \mathbb{M}_n$. Then
\[
\|A\| \leq \|A\|_{HS} \leq \sqrt{n}\|A\|.
\]
Furthermore,
\begin{enumerate}
  \item $\|A\| = \|A\|_{HS}$ if and only if $\text{rank}(A) \leq 1$.
  \item $\|A\|_{HS} = \sqrt{n}\|A\|$ if and only if $A$ is a scalar multiple of a unitary.
\end{enumerate}

**Proof.** The inequalities are well-known and easy to prove. Now let
\[A = U\Sigma V^*\]
be a singular value decomposition of $A$ (i.e. $U, V$ are unitary and $\Sigma = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n)$, where $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n \geq 0$). Assume $\|A\| = \|A\|_{HS}$. Then
\[
\sigma_1^2 = \|A\|^2 = \|A\|_{HS}^2 = \sum_{i=1}^{n} \sigma_i^2 \Rightarrow \sigma_2 = \sigma_3 = \ldots = \sigma_n = 0.
\]
Thus, $A = \sigma_1 u_1 v_1^*$, where $u_1$ and $v_1$ are the first columns of $U$ and $V$, respectively. Hence, $\text{rank}(A) \leq 1$. Conversely, if $\text{rank}(A) \leq 1$, then
\[
\sigma_2 = \sigma_3 = \ldots = \sigma_n = 0 \Rightarrow \|A\| = \|A\|_{HS}.
\]

Now assume $\|A\|_{HS} = \sqrt{n}\|A\|$. Then
\[
\sum_{i=1}^{n} \sigma_i^2 = \|A\|_{HS}^2 = n\|A\|^2 = n\sigma_1^2 \Rightarrow \sigma_1 = \sigma_2 = \ldots = \sigma_n.
\]
Thus, $A = \sigma_1 U V^*$, which is a scalar multiple of a unitary. Conversely, if $A = \alpha W$, where $\alpha \in \mathbb{C}$ and $W$ is a unitary, then
\[
\|A\|_{HS}^2 = \text{Tr}(A^* A) = |\alpha|^2 \text{Tr}(W^* W) = |\alpha|^2 \text{Tr}(I) = n|\alpha|^2 = n\|A\|^2.
\]

**Corollary 7.9.** If every 3-compression of $A \in \mathbb{M}_7^0$ has norm $\geq 1$, then
\[
\|A\| \geq \sqrt{\frac{n-1}{k(k-1)}}.
\]
Equality occurs if and only if $A$ is a multiple of a unitary and every $k$-compression of $A$ has rank one.

**Proof.** Denote by $\Pi_k$ the set of all $k$-compressions of $A$. Then
\[
\binom{n}{k} \leq \sum_{B \in \Pi_k} \|B\|^2 \leq \sum_{B \in \Pi_k} \|B\|_{HS}^2 = \binom{n-2}{k-2} \|A\|_{HS}^2 \leq n \binom{n-2}{k-2} \|A\|^2.
\]
Thus,
\[
\|A\|^2 \geq \frac{\binom{n}{k}}{n \binom{n-2}{k-2}} = \frac{n-1}{k(k-1)}.
\]
The stated equality condition follows immediately from Lemma 7.8.

**Corollary 7.10.** If every 3-compression of $A \in \mathbb{M}_7^0$ has norm $\geq 1$, then
\[
\|A\| > 1.
\]
PROOF. By Lemma 7.9,
\[ \|A\|^2 \geq \frac{7 - 1}{3(3 - 1)} = 1. \]
Suppose \( \|A\| = 1 \). Again by Lemma 7.9, \( A \) is unitary and every 3-compression of \( A \) has rank one. It follows that every 3-compression of \( A \) has exactly two zero columns or exactly two zero rows. Consider \( A_{123} \), the \( \{1, 2, 3\} \)-compression of \( A \). Without loss of generality, we may assume that the second and third columns of \( A_{123} \) are zero. It follows that every 3-compression of \( A \) has exactly two zero columns or exactly two zero rows. Consider \( A_{123}^{12} \), the \( \{1, 2, 3\} \)-compression of \( A_{123} \).

Without loss of generality, we may assume that the second and third columns of \( A_{123}^{12} \) are zero. It follows that the first column of \( A_{123}^{12} \) has norm 1. Thus,
\[
A = \begin{bmatrix}
0 & 0 & 0 & * & * & * \\
0 & 0 & * & * & * & * \\
0 & * & * & 0 & * & * \\
0 & * & * & 0 & * & * \\
0 & * & * & 0 & * & * \\
0 & * & * & 0 & * & * \\
\end{bmatrix},
\]
where \( |a_{21}|^2 + |a_{31}|^2 = 1 \). Conjugating by \( U_{(23)} \), if necessary, we may assume that \( a_{21} \neq 0 \). Case 1: Suppose \( |a_{21}| = 1 \). By considering, in order, \( A_{123}, A_{124}, A_{125}, A_{126}, \) and \( A_{127} \), we have that
\[
A = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
a_{21} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & * & * & * & * \\
0 & 0 & 0 & * & * & * \\
0 & 0 & * & * & 0 & * \\
0 & 0 & * & * & 0 & * \\
\end{bmatrix}.
\]
Considering \( A_{234} \), we have that either \( |a_{34}| = 1 \) or \( |a_{43}| = 1 \). Conjugating by \( U_{(34)} \), if necessary, we may assume the former. Considering, in order, \( A_{234}, A_{345}, A_{346}, \) and \( A_{347} \), we have that
\[
A = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
a_{21} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & * \\
0 & 0 & 0 & 0 & 0 & * \\
\end{bmatrix}.
\]
But then \( \|A_{235}\| = 0 \), a contradiction.

Case 2: Suppose \( |a_{21}| < 1 \). By considering, in order, \( A_{124}, A_{234}, \) and \( A_{345} \), we have that
\[
A = \begin{bmatrix}
0 & 0 & 0 & 0 & * & * \\
a_{21} & 0 & 0 & a_{24} & 0 & 0 \\
a_{31} & 0 & a_{34} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & * \\
0 & * & 0 & 0 & 0 & * \\
0 & * & 0 & 0 & 0 & * \\
\end{bmatrix}.
\]
where $|a_{21}|^2 + |a_{24}|^2 = 1$ and $|a_{24}|^2 + |a_{44}|^2 = 1$. But then $\|A_{345}\| < 1$, a contradiction.

□

Lemma 7.11. Let $A \in \mathbb{M}_4^0$. If every 2-2 paving of $A$ has norm $\geq 1$, then either $\|A\| > 1$ or, up to permutation,

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \\ 0 & c & 0 & 0 \end{bmatrix},$$

where $|a| = |b| = |c| = 1$.

Proof. Assume $\|A\| = 1$. Create a graph $G = (V, E)$ as follows: $V = \{1, 2, 3, 4\}$ and $(i, j) \in E$ if $|a_{ij}|, |a_{ji}| < 1$. We may assume the following axioms:

1. $G_{11}$ is not a subgraph of $G$. Otherwise, $A$ has a 2-2 paving of norm $< 1$.
2. For all $i$, $\deg(i) > 0$. Otherwise, either row $i$ or column $i$ of $A$ has at least two entries of modulus $\geq 1 \Rightarrow \|A\| \geq \sqrt{2}$. This leaves $G_{13}$, which proves the result. □
5. Averaging and Constrained Averaging

Let $A^* = A = (a_{ij})$, $E(A) = 0$, with the reduction assumption for $M_{7,8}^0$ that the $B$’s range over all the $3 \times 3$ zero-diagonal matrices with norm at least 1 (in which case each Hilbert-Schmidt norm is at least $\frac{2}{7}$) or in the case of constrained averaging, all the $B$’s with diagonal projection not containing prescribed $i, j$ pairs.

The following weighted formulas for the Hilbert-Schmidt norm of a $7 \times 7$ zero-diagonal selfadjoint matrix in terms of the Hilbert-Schmidt norms of some or all of its 3-diagonal compressions $P A P$ for averaging and constrained averaging are obtained by careful groupings of triplet integer subsets of $[1,7]$ to compensate for overcounting due to multiple occurrences, analogous to the elementary counting formula for finite sets: $|A \cup B| = |A| + |B| - |A \cap B|.$

\begin{align*}
6||A||^2 &\geq ||A||_{HS}^2 = \frac{1}{5} \sum_{all}^{|B|^2_{HS}} \quad (\text{Averaging}) \\
(12) \quad 6||A||^2 &\geq ||A||_{HS}^2 = 2|a_{12}|^2 + \left( \frac{1}{4} \sum_{134-267}^{|B|^2_{HS}} + \frac{1}{6} \sum_{345-567}^{|B|^2_{HS}} \right) ||B||_{HS}^2 \quad (\text{Constrained Averaging here and below}) \\
(\text{row}) \quad 6||A||^2 &\geq ||A||_{HS}^2 = 2||Ae_1||^2 + \frac{1}{4} \sum_{1 \in B}^{|B|^2_{HS}} \\
(12,23) \quad 6||A||^2 &\geq ||A||_{HS}^2 = 2|a_{12}|^2 + 2|a_{23}|^2 + \left( \frac{1}{4} \sum_{1 \in B, 2 \notin B}^{|B|^2_{HS}} + \frac{1}{3} \sum_{1 \notin B, 2 \in B}^{|B|^2_{HS}} + \frac{1}{4} \sum_{1,2 \notin B, 3 \notin B}^{|B|^2_{HS}} + \frac{1}{12} \sum_{1,2,3 \notin B}^{|B|^2_{HS}} \right) \\
(12,13) \quad 6||A||^2 &\geq ||A||_{HS}^2 = 2|a_{12}|^2 + 2|a_{13}|^2 + \left( \frac{1}{3} \sum_{1 \in B, 2 \notin B}^{|B|^2_{HS}} + \frac{1}{4} \sum_{1 \notin B, 2 \in B}^{|B|^2_{HS}} + \frac{1}{6} \sum_{1,2 \notin B}^{|B|^2_{HS}} \right) \\
(12,23,34) \quad 6||A||^2 &\geq ||A||_{HS}^2 = 2|a_{12}|^2 + 2|a_{23}|^2 + 2|a_{34}|^2 + \left( \frac{1}{3} \sum_{135-147, all \notin B'}^{|B|^2_{HS}} + \frac{1}{6} \sum_{156-167, 456-467}^{|B|^2_{HS}} + (0) \sum_{567}^{|B|^2_{HS}} \right)
\end{align*}
5. AVERAGING AND CONSTRAINED AVERAGING

Application of constrained averaging:

If $|a_{ij}| \geq 1$ (wlog $i, j = 1, 2$) and $A$ satisfies the 3-compression reduction given above, then by (12),

$$6||A||^2 \geq ||A||_{HS}^2 = 2|a_{12}|^2 + \left(\frac{1}{4} \sum_{134-267} + \frac{1}{6} \sum_{345-567} \right) ||B||_{HS}^2$$

$$\geq 2 + \left(\frac{1}{4} \sum_{134-267} + \frac{1}{6} \sum_{345-567} \right) \frac{3}{2} ||B||$$

$$\geq 2 + \left(\frac{20}{4} + \frac{10}{6} \right) \frac{3}{2} = 2 + \left(5 + \frac{5}{3} \right) \frac{3}{2} = 12$$

So $6||A||^2 \geq 12$, $||A|| \geq \sqrt{2}$, $\bar{\alpha}_3(A) \leq \frac{1}{\sqrt{2}} \approx .7071$, smaller than the $\bar{\alpha}_3(M_{0,7}^{0,0})$-table upper range in $[\frac{3}{4}, \frac{2}{\sqrt{7}}] = [.6667, .7559]$. This then rules out entries with larger than 1 modulus for an extremal bad paver in case one succeeds in proving $\bar{\alpha}_3(M_{0,7}^{0,0}, sa) \in (\frac{3}{4}, \frac{2}{\sqrt{7}})$.

Moreover, since $\frac{1}{||A||} = \bar{\alpha}_3(M_{0,7}^{0,0}, sa)$, if $A$ were extremal, and wlog $|a_{12}| = \max_{i,j} |a_{ij}|$, then $||A||^2 = \frac{1}{\alpha_3(M_{0,7}^{0,0}, sa)^2} \in (\frac{7}{4}, \frac{9}{4}]$ and

$$||A||^2 \geq \frac{1}{6} ||A||_{HS}^2 = \frac{1}{3} |a_{12}|^2 + \frac{1}{6} \left(\frac{1}{4} \sum_{134-267} + \frac{1}{6} \sum_{345-567} \right) ||B||_{HS}^2$$

$$\geq \frac{|a_{12}|^2}{3} + \left(\frac{1}{4} \sum_{134-267} + \frac{1}{6} \sum_{345-567} \right) \frac{3}{2} ||B||$$

$$\geq \frac{|a_{12}|^2}{3} + \frac{1}{6} \left(\frac{20}{4} + \frac{10}{6} \right) \frac{3}{2} = \frac{|a_{12}|^2}{3} + \frac{5}{3} > \frac{9}{4}$$

leads to the contradiction: $\bar{\alpha}_3(M_{0,7}^{0,0}, sa) = \frac{1}{||A||} < \frac{7}{3}$. Hence

$$|a_{12}|^2 \leq \frac{27}{4} - 5 = \frac{7}{4}$$

i.e., $\max_{i,j} |a_{ij}| \leq \frac{\sqrt{7}}{2} < ||A||$. 

Bibliography