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THE FUGLEDE COMMUTATIVITY THEOREM MODULO OPERATOR IDEALS

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ABSTRACT. Let H denote a separable, infinite-dimensional complex Hilbert space. A two-sided ideal I of operators on H possesses the generalized Fuglede property (GFP) if, for every normal operator N and every $X \in L(H)$, $NX - XN \in I$ implies $N^*X - XN^* \in I$. Fuglede's Theorem says that $I = \{0\}$ has the GFP. It is known that the class of compact operators and the class of Hilbert-Schmidt operators have the GFP.

We prove that the class of finite rank operators and the Schatten p-classes for $0 < p < 1$ fail to have the GFP. The operator we use as an example in the case of the Schatten p-classes is multiplication by $z + w$ on L^2 of the torus.

Introduction. Let H denote a separable, infinite-dimensional complex Hilbert space. Let $L(H) \supset K(H) \supset C_p \supset F(H)$ $(0 < p < \infty)$ denote, respectively, the class of all bounded linear operators, the class of compact operators, the Schatten p-class, and the class of finite rank operators on H. All operators herein are assumed to be linear and bounded. Let $\|\cdot\|_p$ denote the C_p-norm.

Let I be any two-sided ideal in $L(H)$ (every ideal herein is assumed to be two-sided). It is well known that if $I \neq \{0\}$ or $L(H)$, then $K(H) \supset I \supset F(H)$.

DEFINITION. The ideal I is said to possess the *generalized Fuglede property* (GFP) if, for every normal operator N and every bounded operator X, we have $NX - XN \in I$ implies $N^*X - XN^* \in I$ (i.e., $NX = XN$ modulo I implies $N^*X = XN^*$ modulo I).

Fuglede's Theorem essentially states that $I = \{0\}$ has the GFP [**1**].

There is a connection between the GFP for C_1 (which is not known to hold true) and a possible generalization of the trace result [**5**, Question 3]. Namely, if N is a normal operator and X is a compact operator such that $NX - XN \in C_1$, must $\operatorname{trace}(NX - XN) = 0$?

If C_1 possessed the GFP, then the answer to this question would be yes. The proof is the same as the proof for the well-known fact that if S is a selfadjoint operator and X is a compact operator then $SX - XS \in C_1$ implies $\operatorname{trace}(SX - XS) = 0$ (cf. [**8**, p. 279, Lemma 1.3]). Say $T = NX - XN \in C_1$. If C_1 possessed the GFP, then $S = N^*X - XN^* \in C_1$. Hence $-S^* = NX^* - X^*N \in C_1$ and $T - S^* = N(X + X^*) - (X + X^*)N$ and $T + S^* = N(X - X^*) - (X - X^*)N$. But $X + X^*$ and $X - X^*$ are scalar multiples of compact, selfadjoint

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operators. By the spectral theorem they are diagonalizable. By a computation the above commutators must have trace 0. Therefore

$$\operatorname{trace}(T - S^*) = \operatorname{trace}(T + S^*) = 0.$$

Hence trace $T = 0$.

With these remarks, we pose the following questions:

1. Does C_1 possess the GFP?

2. If N is normal and X is compact, does $NX - XN \in C_1$ imply $\operatorname{trace}(NX - XN) = 0$?

3. Are there any ideals in $L(H)$ other than $\{0\}$, C_2 and $K(H)$ that possess the GFP?

Some positive results. Let $|T|$ denote $\sqrt{T^*T}$.

The first interesting observation is that if I is any ideal and N is normal and $X \in L(H)$, then $|NX| = |N^*X|$. This is a trivial computation. In fact, if N is hyponormal, then $|NX| \geq |N^*X|$. From this we get N hyponormal implies $NX \in I \Rightarrow N^*X \in I$. Furthermore if N is normal and I is a normed ideal where $\|T\|_I = \||T|\|_I$ for all $T \in I$, then $\|NX\|_I = \|N^*X\|_I$. The same holds for XN and XN^*.

The compacts $K(H)$ have the GFP. This is clear by applying Fuglede's Theorem to the Calkin algebra.

The Class C_2 has the GFP [**6**, Theorem 1]. (For another proof use Voiculescu's Theorem [3].)

The Putnam generalization of Fuglede's Theorem [**2**] applies to the GFP. That is any ideal I has the GFP if and only if for every two normal operators N_1, N_2 and $X \in L(H)$, $N_1X - XN_2 \in I$ implies $N_1X - XN_2 \in I$. This is proved in an analogous way to the well-known proof of Putnam's generalization [**2**] using a 2×2 matrix trick due to S. K. Berberian [**7**].

The counterexamples.

THEOREM 1. *The class of finite rank operators $F(H)$ does not possess the GFP.*

PROOF. By the previous remark, it suffices to produce two trace class diagonal operators D_1, D_2 and a Hilbert-Schmidt operator X such that the rank of $D_1X - XD_2$ is 1, but the rank of $D_1^*X - XD_2^*$ is infinite.

Let D_1 denote the diagonal matrix with diagonal entries $1/2, 1/4, 1/8, \ldots$. Clearly, $D_1 \in C_1$. Formally let D_2 denote the diagonal matrix with diagonal entries $z_1, z_2, \ldots$ and $X_1 = (x_{ij})$ where $x_{ij} = 4^{-(i+j)}(2^{-i} - z_j)^{-1}$. We shall choose the sequence (z_n) inductively to satisfy several conditions. First of all, we want to be sure that $|2^{-i} - z_j|$ is large enough to insure that $X_1 \in C_2$, that is, x_{ij} is square summable. Secondly, we need to be sure that $|z_n|$ is small enough to insure that $D_2 \in C_1$.

In the first place, if we can choose (z_n) to be in the closed, left half-plane, then $|2^{-i} - z_j| \geq 2^{-i}$, and so $|x_{ij}| \leq 4^{-(i+j)}2^i = 2^{-i}4^{-j}$ which is square summable.

In the second place, we can insure that $D_2 \in C_1$ if we can choose (z_n) such that $|z_n| \leq 2^{-n}$.

In regard to the rank requirements on $D_1X_1 - X_1D_2$ and $D_1^*X_1 - X_1D_2^*$, note that

$$(D_1X_1 - X_1D_2)(i,j) = (2^{-i} - z_j)4^{-(i+j)}(2^{-i} - z_j)^{-1} = 4^{-(i+j)},$$

and so the range of $D_1X_1 - X_1D_2$ is the 1-dimensional subspace spanned by the vector $(1, 1/4, 1/16, \ldots)$. Therefore, since

$$(D_1^*X_1 - X_1D_2^*)(i,j) = 4^{-(i+j)}((2^{-i} - \bar{z}_j)/(2^{-i} - z_j)),$$

it is clear that $D_1^*X_1 - X_1D_2^*$ has an infinite-dimensional range provided that we choose the sequence (z_n) with one additional property. That is, for each positive integer N, the N vectors given by $4^{-(i+j)}((2^{-i} - \bar{z}_j)/(2^{-i} - z_j))_{i=1}^{\infty}$ for $1 \leqslant j \leqslant N$ form a linearly independent set. Clearly, for this to hold, it is sufficient that these N vectors be linearly independent in the first N coordinates. Equivalently, it is sufficient to show that the determinant of the corresponding $N \times N$ matrix be nonzero; that is, it suffices to prove that there exists a sequence (z_n) in the closed left half-plane for which $|z_n| < 2^{-n}$ for every n and for which

$$\det\left(4^{-(i+j)}((2^{-i} - \bar{z}_j)/(2^{-i} - z_j))\right)_{i,j=1}^{N} \neq 0$$

for every N. We prove this by induction.

Let $z_1 = 0$. Then the case $N = 1$ is trivial. (Observe that $N = 2$ means $\dim H = 4$; it is curious to note and easy to prove that if $\dim H = 3$ then $\operatorname{rank}(DX - XD) = \operatorname{rank}(D^*X - XD^*)$.)

Assume $\langle z_n\rangle_{n=1}^{N}$ has been chosen to satisfy the induction hypothesis. Let z_{N+1} denote the free complex variable which ranges over the intersection of the open, left half-plane and the open disc $[|z| < 2^{-(N+1)}]$. Then

$$\begin{aligned} f(z_{N+1}, \bar{z}_{N+1}) &= \det\left(4^{-(i+j)}((2^{-i} - \bar{z}_j)/(2^{-i} - z_j))\right)_{i,j=1}^{N+1} \\ &= \sum_{i=1}^{N+1} (-1)^{N+1-i} D_i 4^{-(i+N+1)}((2^{-i} - \bar{z}_{N+1})/(2^{-i} - z_{N+1})), \end{aligned}$$

where D_i is the subdeterminant of the $(i, N+1)$ entry which is

$$4^{-(i+N+1)}((2^{-i} - \bar{z}_{N+1})/(2^{-i} - z_{N+1})).$$

Clearly, by inspection and the induction hypothesis,

$$D_{N+1} = \det\left(4^{-(i+j)}((2^{-i} - \bar{z}_j)/(2^{-i} - z_j))\right)_{i,j=1}^{N} \neq 0,$$

and we will use this fact presently. If we let $a_i = (-1)^{N+1-i}D_i4^{-(i+N+1)}$ and $z = z_{N+1}$, then it suffices to show that there exists a z contained in the intersection of the open, left half-plane and the open disc $[|z| < 2^{-(N+1)}]$ such that

$$f(z, \bar{z}) = \sum_{i=1}^{N+1} a_i((2^{-i} - \bar{z})/(2^{-i} - z)) \neq 0.$$

To see that such a complex number z exists, suppose to the contrary that $f(z, \bar{z}) = 0$ in this region. Taking the $\bar{z}$ derivative of both sides of this equation, we obtain $0 = f_{\bar{z}}(z, \bar{z}) = \sum_{i=1}^{N+1} -a_i/(2^{-i} - z)$, for every z in this region. However,

$a_{N+1} = D_{N+1}4^{-2(N+1)} \neq 0$ and so $f_{\bar z}(z, \bar z) = \Sigma_{i=1}^{N+1} -a_i/(2^{-i} - z)$ becomes unbounded in the open disc $[|z| < 2^{-(N+1)}]$ near the point $z = 2^{-(N+1)}$. Therefore, $f_{\bar z}(z, \bar z)$ is not identically 0 in this open disc. But since $f_{\bar z}(z, \bar z) = 0$ throughout the intersection of the open, left half-plane and the open disc, we must have that $f_{\bar z}(z, \bar z)$ is identically 0 in this open disc, which is a contradiction. Q.E.D.

We now present the result of the Schatten p-classes. Let T^2 denote the torus.

THEOREM 2. *The classes C_p, for $0 < p < 1$, do not possess the GFP. In fact, if $N = M_{z+w}$ acting on $L^2(T^2)$, then for each $0 < p < 1$, there exists $X \in C_p$ such that $NX - XN \in C_p$ and yet $N^*X - XN^* \notin C_p$.*

PROOF. It is well known that $\{z^i w^j\}_{i,j=-\infty}^{\infty}$ is an orthonormal basis for $L^2(T^2)$. In this basis, the matrix of M_{z+w} is $N = U \otimes I + I \otimes U$, where U denotes the bilateral shift. Therefore the matrix of M_{z+w} has nonzero blocks on the diagonal and lower diagonal, with all other blocks 0. The diagonal blocks are bilateral shifts and the lower diagonal blocks are the identity I.

We choose X to look exactly the same as M_{z+w} except that the diagonal blocks will be weighted shifts and the lower diagonal blocks will be diagonal matrices. Let y_{ij} denote the entries of the jth diagonal block weighted shift, where y_{ij} is the ith weight on its lower diagonal (i.e. in the $(i, i-1)$ position). In the jth lower diagonal block (in the jth block row) let x_{ij} denote the ith diagonal entry (i.e. in the (i, i) position).

By computing the matrix entries of the commutators $NX - XN$ and $N^*X - XN^*$ we obtain

$$\|NX - XN\|_p^p \leqslant \sum_{i,j} |x_{ij} - x_{ij-1}|^p + |y_{ij} - y_{i-1j}|^p + |(x_{ij} - x_{i-1j}) - (y_{ij} - y_{ij-1})|^p$$

and

$$\|N^*X - XN^*\|_p^p \geqslant \sum_{i,j} |x_{ij} - x_{i-1j}|^p.$$

To accomplish that $\|NX - XN\|_p < \infty$ and $\|N^*X - XN^*\|_p = \infty$, it clearly suffices to obtain x_{ij}, y_{ij} uniformly bounded, doubly indexed sequences (which insures that $X \in L(H \otimes H)$) such that

(1) $\Sigma_{i,j}|x_{ij} - x_{ij-1}|^p < \infty$,
(2) $\Sigma_{i,j}|y_{ij} - y_{i-1j}|^p < \infty$,
(3) $\Sigma_{i,j}|(x_{ij} - x_{i-1j}) - (y_{ij} - y_{ij-1})|^p < \infty$,
(4) $\Sigma_{i,j}|x_{ij} - x_{i-1j}|^p = \infty$.

We shall construct matrix arrays for (x_{ij}) and (y_{ij}) by taking the direct sums of finite arrays. Let $a_n A_n$ and $b_n B_n$ denote the square finite matrix arrays to be constructed. The scalars a_n, b_n represent positive numbers we shall choose in order to guarantee that the entires of $\Sigma \oplus a_n A_n$ and $\Sigma \oplus b_n B_n$ remain uniformly bounded (to insure that X be a bounded operator) and also to guarantee that they satisfy conditions (1)–(4). Choose A_n and B_n to be the matrix arrays

$$
\begin{array}{ccccccc}
 & & & A_n & & & \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 1 & 1 & \cdots & 1 & 0 \\
0 & 2 & 2 & 2 & \cdots & 2 & 0 \\
0 & 3 & 3 & 3 & \cdots & 3 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
0 & n & n & n & \cdots & n & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0
\end{array}
\qquad\qquad
\begin{array}{ccccccc}
 & & & & B_n & & \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 2 & . & \cdots & n & 0 \\
0 & 1 & 2 & \cdots & n & 0 & \\
0 & 1 & 2 & \cdots & n & 0 & \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
0 & 1 & 2 & \cdot & \cdots & n & 0 \\
0 & 0 & 0 & \cdot & \cdots & 0 & 0
\end{array}
$$

For convenience, let (1)–(4) also denote the corresponding sums in conditions (1)–(4). Let us investigate how (1)–(4) act on A_n, B_n where (x_{ij}) is A_n and (y_{ij}) is B_n (considering them as two-way infinite arrays by extending them to have all other entries 0). By computing quantities (1)–(4) we see that

$$
\begin{aligned}
&(1) = (2) = 2\sum_{k=1}^{n} k^p < 2n \cdot n^p = 2n^{p+1}, \\
&(3) = 2 \cdot n \cdot n^p = 2n^{p+1}, \\
&(4) = \sum_{n^2 \text{ times}} 1 + n \cdot n^p > n^2.
\end{aligned}
$$

In other words, (1), (2) and (3) applied to A_n, B_n have orders of magnitude of n^{p+1}, whereas (3) grows with an order of magnitude of n^2. The clincher is that $p + 1 < 2$ (since $p < 1$). Set $a_n = 1/n^{3/2} = b_n$. Clearly $a_n, b_n < 1/n$ and so the entries of a_nA_n and b_nB_n are all less than or equal to 1. Furthermore, applying (1)–(4) to a_nA_n and b_nB_n we obtain

$$
\begin{aligned}
&(1) < (1/n^{3/2})^2 \cdot 2n^{p+1} = 2/n^{2-p} \quad \text{and similarly,} \\
&(2) < 2/n^{2-p}, \\
&(3) < 2/n^{2-p}, \\
&(4) \geqslant n^2/n^3 \geqslant 1/n.
\end{aligned}
$$

Now take (x_{ij}) to be the two-way infinite matrix array $\Sigma \oplus a_nA_n$ and (y_{ij}) to be $\Sigma \oplus b_nB_n$. However here we must just consider a_nA_n and b_nB_n as finite blocks surrounded by enough zeros so that when "pasted" together in the matrix they do not overlap and so (1)–(4) remain as before. The quantities (1)–(4) applied to $\Sigma \oplus a_nA_n$ and $\Sigma \oplus b_nB_n$ are simply the sum of the corresponding quantities (1)–(4) respectively, applied to a_nA_n, b_nB_n. This is because the sums (1)–(4) act independently on each finite block array in $\Sigma \oplus a_nA_n$ and $\Sigma \oplus b_nB_n$. Hence since

$2 - p > 1$, we have

$$(1) \leqslant \Sigma \frac{2}{n^{2-p}} < \infty, \qquad (2) \leqslant \Sigma \frac{2}{n^{2-p}} < \infty,$$

$$(3) \leqslant \Sigma \frac{2}{n^{2-p}} < \infty, \qquad (4) \geqslant \Sigma \frac{1}{n} = \infty. \quad \text{Q.E.D.}$$

REMARK. These investigations have led to another kind of possible generalization of Fuglede's Theorem. Namely, let $\langle M_k \rangle$ and $\langle N_k \rangle$ be commuting sequences of normal operators (i.e., $M_k M_j = M_j M_k$ and $N_k N_j = N_j N_k$). Let $X \in L(H)$. The question is: Does $\Sigma_{k=1}^N M_k X N_k = 0$ imply $\Sigma_{k=1}^N M_k^* X N_k^* = 0$? Also if $\Sigma \|M_k\| \, \|N_k\| < \infty$, does $\Sigma_1^\infty M_k X N_k = 0$ imply $\Sigma_1^\infty M_k^* X N_k^* = 0$? Also are the corresponding statements true modulo the Hilbert-Schmidt class?

Thus far we know easily that all answers are yes when all normal operators are diagonalizable, and C. Apostol settled the questions in the affirmative in the case $N = 2$ using [**6**].

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