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THE COMMUTATOR STRUCTURE OF OPERATOR IDEALS

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Abstract. Additive commutators of operators belonging to two–sided ideals of $B(\mathcal{H})$ are studied. For ideals $\mathcal{I}$ and $\mathcal{J}$, the space, $[\mathcal{I},\mathcal{J}]$, of all finite sums of $(\mathcal{I},\mathcal{J})$–commutators is characterized and found to equal $[\mathcal{I}\mathcal{J},B(\mathcal{H})]$. It is shown that three commutators are enough in $[\mathcal{I},B(\mathcal{H})]$, and that four are enough in $[\mathcal{I},\mathcal{J}]$. Finally, some results about single commutators are proved, for example that every element of the Schatten class $\mathcal{C}_p$ is a single $(\mathcal{C}_p,B(\mathcal{H}))$–commutator if and only if $p>2$.

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Introduction.

The study of the two–sided ideals of the algebra, $B(\mathcal{H})$, of bounded operators on separable, infinite dimensional, complex Hilbert space, $\mathcal{H}$, was initiated by von Neumann and Calkin [**6**]. Calkin showed that the smallest nonzero ideal is the set of finite rank operators, denoted $\mathcal{F}$, that the largest proper ideal is the set of compact operators, denoted $\mathcal{K}$, and that every ideal, $\mathcal{I}$, of $B(\mathcal{H})$ is characterized by its *characteristic set*, which is

$$s(\mathcal{I}) \overset{\text{def}}{=} \{(\lambda_n(T))_{n=1}^{\infty} \mid T \in \mathcal{I},\, T \geq 0\},$$

where $\lambda_1(T) \geq \lambda_2(T) \geq \cdots$ is the sequence of nonzero eigenvalues of T, listed according to multiplicity. (If there are only finitely many nonzero eigenvalues, then $(\lambda_n(T))_{n=1}^{\infty}$ is allowed to have a tail of zeros.) Moreover, by [**6**], every suitable set of nonincreasing, non–negative sequences is the characteristic set of some ideal of $B(\mathcal{H})$; (this characterization is repeated as Theorem 1.1 below). Note that the characteristic set of $\mathcal{F}$ is the set of all eventually zero, nonincreasing, non–negative sequences and the characteristic set of $\mathcal{K}$ is the set of all nonincreasing, non–negative sequences tending to zero. Thus, loosely speaking, ideals are classified in terms of the rates of fall–off of the spectra of their positive elements.

Proper, nonzero, two–sided ideals of $B(\mathcal{H})$ are also called *operator ideals*, and have been widely studied. We seek to understand properties of additive commutators, $[A, B] = AB - BA$, of elements from ideals of $B(\mathcal{H})$. More particularly, this paper is about commutators and commutator spaces. If $\mathcal{I}$ and $\mathcal{J}$ are ideals of $B(\mathcal{H})$, then denote

$$C(\mathcal{I},\mathcal{J}) = \{[A,B] \mid A \in \mathcal{I},\, B \in \mathcal{J}\}$$

$$C_n(\mathcal{I},\mathcal{J}) = \underbrace{C(\mathcal{I},\mathcal{J}) + \cdots + C(\mathcal{I},\mathcal{J})}_{n \text{ times}}$$

$$[\mathcal{I},\mathcal{J}] = \bigcup_{n\geq 1} C_n(\mathcal{I},\mathcal{J}).$$

We call $[\mathcal{I},\mathcal{J}]$ the $(\mathcal{I},\mathcal{J})$*–commutator space.* There are some known results about these classes, especially for the Schatten ideals, $\mathcal{C}_p$, (whose characteristic set consists of the p–summable, non–negative, nonincreasing sequences). A proof in Pearcy and Topping [**11**] (cf. [**4**]) shows that, for $1 < p < \infty$,

$$[\mathcal{C}_p, B(\mathcal{H})] = [\mathcal{C}_{2p}, \mathcal{C}_{2p}] = \mathcal{C}_p.$$

As is well known, there is a trace on $\mathcal{C}_1$, denoted $\mathrm{Tr} : \mathcal{C}_1 \to \mathbf{C}$ and given by $\mathrm{Tr}(A) = \sum_{j=1}^{\infty}\langle A\xi_j, \xi_j\rangle$, where $(\xi_j)_{j=1}^{\infty}$ is any orthonormal basis for $\mathcal{H}$. J.H. Anderson showed [**2**] that for $0 < p < 1$,

$$[\mathcal{C}_p, B(\mathcal{H})] = [\mathcal{C}_{2p}, \mathcal{C}_{2p}] = \mathcal{C}_p^0,$$

where $\mathcal{C}_p^0$ is the kernel of the trace on $\mathcal{C}_p$. The case of the trace–class operators, i.e. when $p = 1$, exhibits remarkable behavior. The first sign of this was [**16**], (see also [**17**]): while it is clear that $[\mathcal{C}_1, B(\mathcal{H})] \subseteq [\mathcal{C}_2, \mathcal{C}_2] \subseteq \mathcal{C}_1^0$, Weiss proved a result implying that

$$\frac{\mathcal{C}_1^0}{[\mathcal{C}_2, \mathcal{C}_2]}$$

has uncountable dimension. Kalton [**10**] later showed that $[\mathcal{C}_1, B(\mathcal{H})] = [\mathcal{C}_2, \mathcal{C}_2]$ and found a characterization of this set in terms of eigenvalues. More precisely, if $T \in \mathcal{C}_1$ has $(\lambda_j)_{j=1}^{\infty}$ for nonzero eigenvalues, listed according to algebraic multiplicity and arranged in order of nonincreasing absolute value, then $T \in [\mathcal{C}_1, B(\mathcal{H})]$ if and only if the arithmetic mean sequence

$$\left(\frac{|\lambda_1 + \ldots + \lambda_n|}{n}\right)_{n=1}^{\infty}$$

is summable. (See also 5.1.)

In §2, we show for every pair, $\mathcal{I}$, $\mathcal{J}$, of proper ideals of $B(\mathcal{H})$, that

$$[\mathcal{I}, \mathcal{J}] = [\mathcal{I}\mathcal{J}, B(\mathcal{H})],$$

where $\mathcal{I}\mathcal{J}$ is the ideal $\mathcal{I}\mathcal{J} = \operatorname{span}\{AB \mid A \in \mathcal{I}, B \in \mathcal{J}\}$; (note that $\mathcal{I}\mathcal{J} = \mathcal{J}\mathcal{I}$). We also completely characterize $[\mathcal{I}, B(\mathcal{H})]$ with a criterion analogous to Kalton's as follows: if $T = T^* \in \mathcal{I}$ then $T \in [\mathcal{I}, B(\mathcal{H})]$ if and only if

$$\left(\frac{|\lambda_1 + \cdots + \lambda_n|}{n}\right)_{n=1}^{\infty}$$

belongs to the characteristic set of $\mathcal{I}$, where $\lambda_1, \lambda_2, \ldots$ are the eigenvalues of T listed according to multiplicity and such that $|\lambda_1| \geq |\lambda_2| \geq \cdots$. To see that this characterizes $[\mathcal{I}, B(\mathcal{H})]$, note that $A \in [\mathcal{I}, B(\mathcal{H})]$ if and only if $A^* \in [\mathcal{I}, B(\mathcal{H})]$, so that

$$A \in [\mathcal{I}, B(\mathcal{H})] \qquad \Leftrightarrow \qquad \frac{A + A^*}{2}, \frac{A - A^*}{2i} \in [\mathcal{I}, B(\mathcal{H})];$$

thus A is a finite sum of $(\mathcal{I}, B(\mathcal{H}))$–commutators if and only if the arithmetic mean sequence of the eigenvalues (listed according to multiplicity and in order of decreasing absolute value) of each of its real and imaginary parts is in the characteristic set of I.

It is a natural question, for a particular operator ideal $\mathcal{I}$ or for general such: how many commutators are needed? More precisely, what is the least $n \in \mathbf{N}$, if any, such that $[\mathcal{I}, B(\mathcal{H})] =$

$C_n(\mathcal{I}, B(\mathcal{H}))$? Kalton [**10**] showed $[\mathcal{C}_1, B(\mathcal{H})] = C_6(\mathcal{C}_1, B(\mathcal{H}))$. In §3, in addition to several more specialized results, we show for a general operator ideal $\mathcal{I}$ that

$$[\mathcal{I}, B(\mathcal{H})] = C_3(\mathcal{I}, B(\mathcal{H}))$$

and in §4 we exhibit examples where $[\mathcal{I}, B(\mathcal{H})] \neq C_1(\mathcal{I}, B(\mathcal{H}))$.

It is also interesting to study the set, $C(\mathcal{I}, B(\mathcal{H}))$, of single commutators. Here, much less is known. Anderson [**1**] showed that

$$\forall p > 1 \qquad C(\mathcal{C}_{2p}, B(\mathcal{H})) \supseteq \mathcal{C}_p$$

and, letting $\mathcal{C}_p \oplus 0_\infty$ denote the set of elements of $\mathcal{C}_p$ having an infinite dimensional reducing subspace on which it is zero, that for $p > 0$, if $q, r \geq 2p$ and if $\frac{1}{q} + \frac{1}{r} < \frac{1}{2}$ then

$$C(\mathcal{C}_q, \mathcal{C}_r) \supseteq \mathcal{C}_p \oplus 0_\infty.$$

Brown [**5**] showed that this is sharp by proving if $q, r > 0$ and if $\frac{1}{q} + \frac{1}{r} \geq \frac{1}{2}$ then

$$C(\mathcal{C}_q, \mathcal{C}_r) \cap \mathcal{F} = \mathcal{F}^0,$$

where $\mathcal{F}^0$ denotes the finite rank operators having zero trace.

In §4, we obtain additional results about single commutators. For example, we show that

$$C(\mathcal{C}_q, B(\mathcal{H})) = \mathcal{C}_q$$

whenever $2 < q < \infty$, but that

$$C(\mathcal{C}_q, B(\mathcal{H})) \neq \mathcal{C}_q$$

whenever $0 < q \leq 2$.

§1. Sequences and characteristic sets.

In this section, we recall Calkin's classification of operator ideals in terms of characteristic sets, and we prove a few results about sequences. While the ideas of Calkin's classification underly the whole of this paper, the results about sequences will not be needed until §4.

By c_0^{++} we denote the set of all sequences, $\lambda = (\lambda_n)_{n=1}^\infty$, of non–negative, real numbers such that $\forall n\ \lambda_n \geq \lambda_{n+1}$ and $\lim_{n\to\infty} \lambda_n = 0$.

For an operator $A \in B(\mathcal{H})$, the *s–numbers* (cf. [**9**]) of A are denoted

$$s_1(A) \geq s_2(A) \geq \cdots,$$

and are defined by

$$s_n(A) = \min\{||AE^\perp|| \mid E \text{ is a projection in } \mathcal{H} \text{ of rank } n-1\},$$

where $E^\perp = 1-E$ is the projection onto the orthogonal complement of $E\mathcal{H}$. (Throughout this paper, "projection" always means a self–adjoint idempotent.) Thus $s_1(A) = ||A||$; moreover A is compact if and only if its s–numbers tend to zero. For a compact operator A, we write $s(A) = (s_n(A))_{n=1}^\infty \in c_0^{++}$. The s–numbers of a positive, compact operator, A, are just the nonzero eigenvalues of A listed in nonincreasing order; (but if $\mathrm{rank}(A) < \infty$ then $s_n(A) = 0$ for $n > \mathrm{rank}(A)$). Hence, using the polar decomposition, it is seen that the characteristic set of an operator ideal $\mathcal{I}$ is just

$$s(\mathcal{I}) = \{s(A) \mid A \in \mathcal{I}\} \subseteq c_0^{++}.$$

If λ, $\mu \in c_0^{++}$ where $\lambda = (\lambda_n)_{n=1}^\infty$ and $\mu = (\mu_n)_{n=1}^\infty$ then we define

$$\begin{aligned} \lambda^{\oplus n} &= (\underbrace{\lambda_1, \dots, \lambda_1}_{n \text{ times}}, \underbrace{\lambda_2, \dots, \lambda_2}_{n \text{ times}}, \dots) \\ \lambda + \mu &= (\lambda_1 + \mu_1, \lambda_2 + \mu_2, \dots) \\ \mu &\le \lambda \text{ if } \forall n\ \mu_n \le \lambda_n. \end{aligned}$$

The next theorem follows directly from Calkin's characterization [**6**] and was formulated in [**8**].

Theorem 1.1. *A subset $\mathfrak{S}$ of c_0^{++} is the characteristic set of an operator ideal if and only if the following conditions are satisfied:*

(i) $\lambda \in \mathfrak{S}$ *implies* $\lambda \oplus \lambda \in \mathfrak{S}$*;*

(ii) $\lambda, \mu \in \mathfrak{S}$ *implies* $\lambda + \mu \in \mathfrak{S}$*;*

(iii) $\lambda \in \mathfrak{S}$*,* $\mu \in c_0^{++}$ *and* $\mu \le \lambda$ *implies* $\mu \in \mathfrak{S}$*.*

For $\lambda = (\lambda_n)_{n=1}^\infty \in c_0^{++}$ and $c \ge 0$ we also define $c\lambda = (c\lambda_n)_{n=1}^\infty$. Hence, for a compact operator A, the principal ideal, $\langle A \rangle$, of $B(\mathcal{H})$ generated by A has characteristic set equal to

$$s(\langle A \rangle) = \{\mu \in c_0^{++} \mid \mu \le c\, s(A)^{\oplus n},\ n \in \mathbf{N},\ c > 0\}.$$

We now turn to results about sequences. For $\lambda, \mu \in c_0^{++}$,

$$\lambda \asymp \mu$$

is defined to mean that there is $c \geq 0$ such that $\lambda \leq c\mu$ and $\mu \leq c\lambda$. If $\lambda = (\lambda_n)_{n=1}^{\infty} \in c_0^{++}$ and $\mu = (\mu_n)_{n=1}^{\infty} \in c_0^{++}$ then $\lambda \otimes \mu$ denotes the element of c_0^{++} obtained by putting the terms

$$\begin{array}{l} \lambda_1\mu_1, \lambda_1\mu_2, \lambda_1\mu_3, \ldots \\ \lambda_2\mu_1, \lambda_2\mu_2, \lambda_2\mu_3, \ldots \\ \lambda_3\mu_1, \lambda_3\mu_2, \lambda_3\mu_3, \ldots \\ \quad\vdots \end{array}$$

in nonincreasing order and throwing away zeros if necessary. If $\mathfrak{S}$ and $\mathfrak{T}$ are characteristic sets of some ideal, then using Theorem 1.1 we easily see that

$$\{\lambda \otimes \mu \mid \lambda \in \mathfrak{S},\, \mu \in \mathfrak{T}\}$$

is the characteristic set of an ideal. This allows us to make the following definition.

Definition 1.2. Let $\mathcal{I}$ and $\mathcal{J}$ be proper ideals in $B(\mathcal{H})$, and denote their characterstic sets by $s(\mathcal{I})$ and $s(\mathcal{J})$, respectively. Then $\mathcal{I} \diamondsuit \mathcal{J}$ is the operator ideal whose characteristic set is

$$\{\lambda \otimes \mu \mid \lambda \in s(\mathcal{I}),\, \mu \in s(\mathcal{J})\}.$$

For $t > 0$ we define

$$\lambda^t = (\lambda_n^t)_{n=1}^{\infty} \in c_0^{++}.$$

We will consider the specific sequence

$$\omega \stackrel{\text{def}}{=} (1, 1/2, 1/3, 1/4, \ldots).$$

For $\lambda \in c_0^{++}$, the *arithmetic mean sequence* of λ is

$$\lambda^{(\mathrm{a})} \stackrel{\text{def}}{=} \left(\frac{\lambda_1 + \lambda_2 + \ldots \lambda_n}{n} \right)_{n=1}^{\infty} \in c_0^{++}.$$

Theorem 1.3. *Let $\lambda \in c_0^{++}$. Then*

$$\lambda \otimes \omega \leq \lambda^{(\mathrm{a})} \leq 2(\lambda \otimes \omega), \tag{1}$$

and consequently $\lambda \otimes \omega \asymp \lambda^{(\mathrm{a})}$.

Proof. Let $k \mapsto (i_k, j_k)$ denote the injective map from $\mathbf{N}$ into $\mathbf{N} \times \mathbf{N}$ such that $\lambda \otimes \omega = (\lambda_{i_n}/j_n)_{n=1}^{\infty}$, i.e. the nonincreasing ordering of $\{\lambda_i/j \mid i, j \in \mathbf{N}\}$. (If every $\lambda_i > 0$ then this map is onto.)

Let us show that $\lambda \otimes \omega \leq \lambda^{(\mathrm{a})}$. Suppose for contradiction that

$$\frac{\lambda_{i_n}}{j_n} > \frac{\lambda_1 + \cdots + \lambda_n}{n}$$

for some $n \geq 1$. Let

$$S_n = \{(i_k, j_k) \mid 1 \leq k \leq n\}.$$

If $(i,j) \in S_n$ then $(i',j') \in S_n$ for every $i' \leq i$ and $j' \leq j$. Let $m_n = \max(i_1, \ldots, i_n)$ and for $i \in \mathbf{N}$ let

$$j(i) = \max(\{j \mid (i,j) \in S_n\} \cup \{0\}).$$

We then have

$$\forall 1 \leq i \leq m_n \qquad \frac{\lambda_i}{j(i)} \geq \frac{\lambda_{i_n}}{j_n} > \frac{\lambda_1 + \cdots + \lambda_n}{n}.$$

Multiplying by $j(i)$ and summing we get

$$\sum_{i=1}^{m_n} \lambda_i > \left(\sum_{i=1}^{m_n} j(i)\right) \frac{\lambda_1 + \cdots + \lambda_n}{n}.$$

Clearly $\sum_{i=1}^{m_n} j(i) = |S_n| = n$. Thus we have $\sum_{i=1}^{m_n} \lambda_i > \lambda_1 + \cdots + \lambda_n$. Since $m_n \leq n$, this is a contradiction.

Now we show that $\lambda^{(\mathrm{a})} \leq 2(\lambda \otimes \omega)$. The bound with this constant, and this particularly nice proof, was first obtained by C. Vaqui [**14**]. Fix $n \in \mathbf{N}$ and let S_n and $j(i)$ be as above. Then for every $i \in \mathbf{N}$ we have

$$\frac{\lambda_{i_n}}{j_n} \geq \frac{\lambda_i}{j(i)+1}, \tag{2}$$

which implies that

$$\sum_{i=1}^{n} \lambda_i \leq \sum_{i=1}^{n} (j(i)+1) \frac{\lambda_{i_n}}{j_n} = \left(n + \sum_{i=1}^{n} j(i)\right) \frac{\lambda_{i_n}}{j_n}.$$

Since $\sum_{i=1}^{n} j(i) = n$, we get

$$\frac{\lambda_1 + \lambda_2 + \cdots + \lambda_n}{n} \leq 2 \frac{\lambda_{i_n}}{j_n}. \tag{3}$$

□

Remark 1.4. In the above proof, if $i = i_n$ then the inequality (2) can be improved to

$$\frac{\lambda_{i_n}}{j_n} = \frac{\lambda_{i_n}}{j(i_n)}.$$

This shows that for every n, the inequality (3) can be improved to give

$$\frac{\lambda_1 + \lambda_2 + \cdots + \lambda_n}{n} \leq (2 - \tfrac{1}{n}) \frac{\lambda_{i_n}}{j_n} = (2 - \tfrac{1}{n})(\lambda \otimes \omega)_n. \tag{4}$$

This improved result was also first obtained by C. Vaqui [**14**]. The inequality (4) is, in general, the best possible. Indeed, for arbitrary fixed $n \in \mathbf{N}$, a sequence beginning with

$$\lambda_j = \begin{cases} 1 & \text{if } j < n \\ \frac{1}{2} & \text{if } j = n \end{cases}$$

gives equality in (4).

Proposition 1.5. *Let $0 < t < s$. Then $\omega^t \otimes \omega^s \asymp \omega^t$.*

Proof. Clearly $\omega^t \leq \omega^t \otimes \omega^s$. We will find a constant, c, such that

$$\omega^t \otimes \omega^s \leq c\, \omega^t. \tag{5}$$

Taking the $1/s$ power of both sides of (5) and letting $r = t/s$, it will be enough to find a constant, c', such that

$$\omega^r \otimes \omega \leq c' \omega^r. \tag{6}$$

From Theorem 1.3 we know that $\omega^r \otimes \omega \leq (\omega^r)^{(\mathrm{a})}$. But

$$(\omega^r)^{(\mathrm{a})}_n = \frac{1}{n} \sum_{k=1}^{n} k^{-r} \leq \frac{1}{n} \left(1 + \textstyle\int_1^n t^{-r}\, dt\right) = \frac{1}{1-r} \left(n^{-r} - \frac{r}{n}\right).$$

Hence we get (6) with $c' = s/(s-t)$ and thus (5) with

$$c = \left(\frac{s}{s-t}\right)^s.$$

□

The remainder of this section, while not used later in the paper, may be of interest. Indeed, the following proposition shows that the constants in the bounds in (1) are the best possible, even asymptotically, and there is a single λ which shows both constants to be optimal. For $\lambda \in c_0^{++}$ and $p \in \mathbf{N}$ let $\lambda_p^{(\mathrm{a})}$ denote the pth term of the arithmetic mean sequence of λ, i.e. $\lambda_p^{(\mathrm{a})} = (\lambda_1 + \cdots + \lambda_p)/p$, and let $(\lambda \otimes \omega)_p$ be the pth term of $\lambda \otimes \omega$.

Proposition 1.6. *There is a sequence* $\lambda \in c_0^{++}$ *for which*

$$\limsup_{p\to\infty} \frac{\lambda_p^{(\mathrm{a})}}{(\lambda \otimes \omega)_p} = 2 \tag{7}$$

and

$$\liminf_{p\to\infty} \frac{\lambda_p^{(\mathrm{a})}}{(\lambda \otimes \omega)_p} = 1. \tag{8}$$

Proof. Let

$$\lambda = (\underbrace{1,\dots,1}_{n_0}, \underbrace{2^{-1},\dots,2^{-1}}_{n_1}, \underbrace{2^{-2},\dots,2^{-2}}_{n_2}, \dots, \underbrace{2^{-k},\dots,2^{-k}}_{n_k}, \dots), \tag{9}$$

with each $n_k > 0$. The arithmetic mean sequence of λ is thus

$$\begin{aligned}\lambda^{(\mathrm{a})} = (&\underbrace{1,\dots,1}_{n_0}, \underbrace{\frac{n_0 + 2^{-1}j}{n_0 + j}}_{1\le j\le n_1}, \underbrace{\frac{n_0 + 2^{-1}n_1 + 2^{-2}j}{n_0 + n_1 + j}}_{1\le j\le n_2}, \dots, \\ &\dots, \underbrace{\frac{n_0 + 2^{-1}n_1 + \cdots + 2^{-(k-1)}n_{k-1} + 2^{-k}j}{n_0 + n_1 + \cdots + n_{k-1} + j}}_{1\le j\le n_k}, \dots)\end{aligned} \tag{10}$$

and

$$\lambda \otimes \omega = (\underbrace{1,\dots,1}_{n_0}, \underbrace{2^{-1},\dots,2^{-1}}_{n_0+n_1}, \underbrace{3^{-1},\dots,3^{-1}}_{n_0}, \underbrace{4^{-1},\dots,4^{-1}}_{n_0+n_1+n_2}, \dots, \underbrace{m^{-1},\dots,m^{-1}}_{n_0+\cdots+n_{k(m)}}, \dots),$$

where for each $m \in \mathbf{N}$, m^{-1} appears $n_0 + \cdots + n_{k(m)}$ times, where $m = 2^{k(m)}j$ with j odd. Hence the first term of $\lambda \otimes \omega$ that is equal to 2^{-k} is the p_kth, where

$$\begin{aligned}p_k &= 1 + 2^{k-1}n_0 + 2^{k-2}(n_0 + n_1) + \cdots + 2(n_0 + \cdots + n_{k-1}) \\ &= 1 + (2^k - 1)n_0 + (2^{k-1} - 1)n_1 + \cdots + (2^2 - 1)n_{k-2} + (2 - 1)n_{k-1} \\ &= 1 + \sum_{j=0}^{k-1}(2^{k-j} - 1)n_j.\end{aligned}$$

Clearly $p_k > n_0 + \cdots + n_{k-1}$. If one chooses

$$n_k \ge 1 + \sum_{j=0}^{k-1}(2^{k-j} - 2)n_j \tag{11}$$

then $p_k \le n_0 + \cdots + n_k$. But then the p_kth term of $\lambda^{(a)}$ falls in the block in (10) associated with $1 \le j \le n_k$. Thus

$$\begin{aligned}
\frac{\lambda^{(a)}_{p_k}}{(\lambda\otimes\omega)_{p_k}} &= 2^k \lambda^{(a)}_{p_k} \\
&= 2^k \cdot \frac{n_0 + 2^{-1}n_1 + \cdots + 2^{-(k-1)}n_{k-1} + 2^{-k}(p_k - (n_0 + \cdots + n_{k-1}))}{p_k} \\
&= \frac{\sum_{j=0}^{k-1} 2^{k-j} n_j + 1 + \sum_{j=0}^{k-1}(2^{k-j}-2)n_j}{1 + \sum_{j=0}^{k-1}(2^{k-j}-1)n_j} = \frac{1 + 2\sum_{j=0}^{k-1}(2^{k-j}-1)n_j}{1 + \sum_{j=0}^{k-1}(2^{k-j}-1)n_j} \\
&= 2 - \frac{1}{p_k}.
\end{aligned}$$

Hence any choice of $n_0, n_1, \ldots$ growing fast enough to ensure that (11) holds gives λ such that (7) holds.

Now we show that $n_0, n_1, \ldots$ can be chosen so that λ simultaneously satisfies (8). The idea of the proof is as follows. If $n_0, \ldots, n_{k-1}$ have been chosen, then by choosing n_k arbitrarily large, $\lambda^{(a)}_{n_0+\cdots+n_k}$ can be made arbitarily close to 2^{-k}. Moreover, if n_k is large enough then $(\lambda\otimes\omega)_{n_0+\cdots+n_k} = 2^{-k}$. Hence $\lambda^{(a)}_p/(\lambda\otimes\omega)_p$ for $p = n_0 + \cdots + n_k$ can be made arbitarily close to 1. So for certain choices of $n_0, n_1, \ldots$, (8) will hold. Let us now make this discussion more quantitative. Since 2^{-k} is repeated $n_0 + n_1 + \cdots + n_k$ times in $\lambda\otimes\omega$, in order to ensure that $(\lambda\otimes\omega)_{n_0+\cdots+n_k} = 2^{-k}$ it is enough that $p_k \le n_0 + \cdots + n_k$, *i.e.* that (11) is satisfied. If we want $\lambda^{(a)}_{n_0+\cdots+n_k} \le 2^{-k}(1+\frac{1}{k})$, then using

$$\lambda^{(a)}_{n_0+\cdots+n_k} = \frac{n_0 + 2^{-1}n_1 + \cdots + 2^{-(k-1)}n_{k-1} + 2^{-k}n_k}{n_0 + \cdots + n_k}$$

we see it is enough that

$$n_k \ge \sum_{j=0}^{k-1} (k2^{k-j} - k - 1)n_j. \tag{12}$$

Now choosing, for example, $n_j = 4^j(j!)$ makes both (11) and (12) hold, hence both (7) and (8) hold.

□

Proposition 1.7. *Let λ be the sequence found in Proposition 1.6. Then for every $1 \le x \le 2$ there is a subsequence of $\lambda^{(a)}_n/(\lambda\otimes\omega)_n$ converging to x.*

Proof. Let λ be as in (9) with $n_0, n_1, \ldots$ as found in the proof of Proposition 1.6. Then for each k, as q increases from p_k to $n_0 + \cdots + n_k$, $\lambda^{(a)}_q$ decreases from $2^{-k}(2 - \frac{1}{p_k})$ to a value not greater than $2^{-k}(1+\frac{1}{k})$ in steps of size no greater than $2^{-k}/p_k$, while $(\lambda\otimes\omega)_q$ remains constant at 2^{-k}. Because p_k increases without bound, we can find $p_k \le q_k \le n_0 + \cdots + n_k$ such that

$$\lim_{k\to\infty} \frac{\lambda^{(\mathrm{a})}_{q_k}}{(\lambda \otimes \omega)_{q_k}} = x.$$

□

§2. Sums of Commutators.

In this section, we characterize, for $\mathcal{I}$ and $\mathcal{J}$ arbitrary two–sided ideals of $B(\mathcal{H})$ at least one of which is proper, the self–adjoint operators, T, that are sums of $(\mathcal{I},\mathcal{J})$–commutators. This characterization is in terms of the rate of fall–off of the arithmetic means of the eigenvalues of T.

Notation 2.1. For a sequence of complex numbers $(a_n)_{n=1}^\infty$ and an operator ideal $\mathcal{I}$, we will often write $(a_n)_{n=1}^\infty \in \mathcal{I}$ to mean that the diagonal operator $\mathrm{diag}(a_1, a_2, \ldots, a_n, \ldots)$ belongs to $\mathcal{I}$. Clearly, this happens if and only if the nonincreasing re–arrangement of $(|a_n|)_{n=1}^\infty$ belongs to the characteristic set of $\mathcal{I}$.

Lemma 2.2. *Let $\mathcal{I}$ and $\mathcal{J}$ be ideals of $B(\mathcal{H})$ and let $T \in [\mathcal{I},\mathcal{J}]$. Suppose P_n $(n \in \mathbf{N})$ are projections in $\mathcal{H}$ with* $\mathrm{rank}(P_n) = n$ *and* $(\|P_n^\perp T P_n^\perp\|)_{n=1}^\infty \in \mathcal{I}\mathcal{J}$. *Then*

$$\left(\frac{\mathrm{Tr}(P_n T P_n)}{n}\right)_{n=1}^\infty \in \mathcal{I}\mathcal{J}. \tag{13}$$

Proof. Write

$$T = \sum_{\iota=1}^N [A_\iota, B_\iota],$$

with $A_\iota \in \mathcal{I}$ and $B_\iota \in \mathcal{J}$. Using the definition of s–numbers, for each positive integer n there is a projection, E_n, with $\mathrm{rank}(E_n) \le (4N+1)n$, $E_n \ge P_n$ and such that $\forall 1 \le \iota \le N$,

$$\|E_n^\perp A_\iota\| \le s_n(A_\iota), \quad \|A_\iota E_n^\perp\| \le s_n(A_\iota)$$
$$\|E_n^\perp B_\iota\| \le s_n(B_\iota), \quad \|B_\iota E_n^\perp\| \le s_n(B_\iota).$$

Then, since $\mathrm{Tr}([E_n A_\iota E_n, E_n B_\iota E_n]) = 0$, we have

$$\mathrm{Tr}(E_n T E_n) = \sum_{\iota=1}^N \mathrm{Tr}(E_n A_\iota B_\iota E_n - E_n B_\iota A_\iota E_n) = \sum_{\iota=1}^N \mathrm{Tr}(E_n(A_\iota E_n^\perp B_\iota - B_\iota E_n^\perp A_\iota)E_n),$$

and since $|\mathrm{Tr}(E_nA)| \leq (\mathrm{rank}E_n)||A||$ for every $A \in B(\mathcal{H})$,

$$|\mathrm{Tr}(E_nTE_n)| \leq 2(4N+1)n\sum_{\iota=1}^{N} s_n(A_\iota)s_n(B_\iota).$$

But

$$\mathrm{Tr}(E_nTE_n) = \mathrm{Tr}(P_nTP_n) + \mathrm{Tr}((E_n - P_n)T(E_n - P_n))$$

and

$$|\mathrm{Tr}((E_n - P_n)T(E_n - P_n))| \leq 4Nn||P_n^\perp TP_n^\perp||,$$

so

$$\left|\frac{\mathrm{Tr}(P_nTP_n)}{n}\right| \leq 4N||P_n^\perp TP_n^\perp|| + (8N+2)\sum_{\iota=1}^{N} s_n(A_\iota)s_n(B_\iota).$$

Hence (13) holds.

□

Lemma 2.3. *Let $\mathcal{I}$ and $\mathcal{J}$ be ideals of $B(\mathcal{H})$, at least one of them proper, let $T = T^* \in [\mathcal{I}, \mathcal{J}]$ and let $\lambda_1, \lambda_2, \ldots$ be the nonzero eigenvalues of T listed according to multiplicity (with zeros added if T has only finitely many nonzero eigenvalues) and such that $(\max_{j\geq n}|\lambda_j|)_{n=1}^\infty \in \mathcal{IJ}$, (which holds, for example, if the eigenvalues are arranged in order of nonincreasing absolute value). Then*

$$\left(\frac{\lambda_1 + \cdots + \lambda_n}{n}\right)_{n=1}^\infty \in \mathcal{IJ}. \tag{14}$$

Proof. Let $\xi_1, \xi_2, \ldots$ be an orthonormal sequence in $\mathcal{H}$ consisting of eigenvectors of T such that $T\xi_n = \lambda_n\xi_n$. Let P_n be the projection in $B(\mathcal{H})$ with range space $\mathrm{span}\{\xi_1, \ldots, \xi_n\}$. Then $||P_n^\perp TP_n^\perp|| = \max_{j\geq n+1}|\lambda_j|$ and $\mathrm{Tr}(P_nTP_n) = \lambda_1 + \cdots + \lambda_n$. Hence the hypotheses of Lemma 2.2 hold and (14) follows from (13).

□

The proof of the following lemma is a straightforward generalization of Kalton's proof [**10**, Theorem 7.1] for the case $\mathcal{I} = \mathcal{C}_1$.

Lemma 2.4. *Let $\mathcal{I}$ be a proper ideal of $B(\mathcal{H})$, let $T = T^* \in \mathcal{I}$ and let $\lambda_1, \lambda_2, \ldots$ be the eigenvalues of T listed according to multiplicity (with zeros added if T has only finitely many nonzero eigenvalues) and such that $(\max_{j\geq n}|\lambda_j|)_{n=1}^\infty \in \mathcal{I}$, (which holds, for example, if the eigenvalues are arranged in order of nonincreasing absolute value). Suppose*

$$\left(\frac{\lambda_1 + \cdots + \lambda_n}{n}\right)_{n=1}^\infty \in \mathcal{I}.$$

Then $T \in [\mathcal{I}, B(\mathcal{H})]$ and T is the sum of three $(\mathcal{I}, B(\mathcal{H}))$–commutators.

Proof. We will show that A, the operator obtained from T by averaging the "2^n–blocks" (see below), is the sum of two commutators and that $T - A$ is a commutator. For $n \geq 1$ let $t_n = -2^{-n} \sum_{j=1}^{2^n-1} \lambda_j$, and consider the sequence

$$X = (t_1, t_2, t_2, t_3, t_3, t_3, t_3, \dots, \underbrace{t_n, \dots, t_n}_{2^{n-1} \text{ times}}, \dots).$$

We will show that $X \in \mathcal{I}$. If $2^{n-1} \leq k < 2^n$ then

$$\begin{aligned} |t_n| &= 2^{-n} \left| \sum_{j=1}^{2^n-1} \lambda_j \right| \\ &\leq 2^{-n} \left| \sum_{j=1}^{k} \lambda_j \right| + 2^{-n} \left| \sum_{j=k+1}^{2^n-1} \lambda_j \right| \\ &\leq \frac{1}{k} \left| \sum_{j=1}^{k} \lambda_j \right| + \max_{j \geq k} |\lambda_j|, \end{aligned}$$

so $X \in \mathcal{I}$. Let $e_1, e_2, \dots$ be an orthonormal sequence in $\mathcal{H}$ such that $Te_j = \lambda_j e_j$ $\forall j$, and let $\mathcal{H}_0 = \mathcal{H} \ominus \overline{\text{span}}\{e_1, e_2, \dots\}$. Let $U_1, U_2 \in \mathcal{I}$, $V_1, V_2 \in B(\mathcal{H})$ be such that

$$\begin{aligned} U_1 e_k &= t_n e_{2k} && 2^{n-1} \leq k < 2^n \text{ and } n \geq 1 \\ U_2 e_k &= t_n e_{2k+1} && 2^{n-1} \leq k < 2^n \text{ and } n \geq 1 \\ V_1 e_{2k+1} &= 0 && k \geq 0 \\ V_1 e_{2k} &= e_k && k \geq 1 \\ V_2 e_{2k+1} &= e_k && k \geq 1 \\ V_2 e_{2k} &= 0 && k \geq 1 \\ V_2 e_1 &= 0 \end{aligned}$$

and U_1, U_2, V_1, V_2 vanish on $\mathcal{H}_0$. We have $U_1, U_2 \in \mathcal{I}$ because $X \in \mathcal{I}$. For $2^{n-1} \leq k < 2^n$ we have

$$([U_1, V_1] + [U_2, V_2])e_k = \begin{cases} -2t_1 e_1 & k = 1 \\ (t_{n-1} - 2t_n)e_k & k > 1. \end{cases}$$

For $n \geq 1$ let $\alpha_n = 2^{-(n-1)} \sum_{j=2^{n-1}}^{2^n-1} \lambda_j$. Then $-2t_1 = \alpha_1$ and $t_{n-1} - 2t_n = \alpha_n$ for $n \geq 2$. Let $A \in B(\mathcal{H})$ be such that $Ae_k = \alpha_n e_k$ if $2^{n-1} \leq k < 2^n$ and $A|_{\mathcal{H}_0} = 0$. Then $[U_1, V_1] + [U_2, V_2] = A$.

Now we show that $T - A$ is a commutator. Let $\theta_n = |\lambda_{2^{n-1}}|$ for each $n \geq 1$. Since $|\lambda_k| \geq \theta_{n+1}$ whenever $k \leq 2^n$, we have that

$$(\theta_1, \theta_2, \theta_2, \theta_3, \theta_3, \theta_3, \theta_3, \dots, \underbrace{\theta_n, \dots, \theta_n}_{2^{n-1} \text{ times}}, \dots) \in \mathcal{I}.$$

Now since $\sum_{k=2^{n-1}}^{2^n-1}(\alpha_n - \lambda_k) = 0$ and $|\alpha_n - \lambda_k| \leq 2\theta_n$ for every $2^{n-1} \leq k < 2^n$, there is a re–arrangement $(\mu_{2^{n-1}}, \dots, \mu_{2^n-1})$ of $(\lambda_{2^{n-1}}, \dots, \lambda_{2^n-1})$ such that $|\sum_{k=2^{n-1}}^{r}(\alpha_n - \mu_k)| \leq 2\theta_n$ for every $2^{n-1} \leq r < 2^n$. Let $\gamma_r = \sum_{k=2^{n-1}}^{r}(\alpha_n - \mu_k)$ for $2^{n-1} \leq r < 2^n$. Then $(\gamma_r)_{r=1}^{\infty} \in \mathcal{I}$. Let σ be the permutation of $\mathbf{N}$:

$$\sigma(k) = \begin{cases} \frac{1}{2}(k+1) & \text{if } k+1 = 2^n, \text{ some } n \geq 1 \\ k+1 & \text{otherwise.} \end{cases}$$

Let $U_3 \in \mathcal{I}$, $V_3 \in B(\mathcal{H})$ be

$$U_3 e_k = \gamma_k e_{\sigma(k)} \qquad k \geq 1$$

$$V_3 e_k = e_{\sigma^{-1}(k)} \qquad k \geq 1$$

$U_3|_{\mathcal{H}_0} = 0 = V_3|_{\mathcal{H}_0}$. Then $[U_3, V_3]e_k = (\gamma_{\sigma^{-1}(k)} - \gamma_k)e_k$ and

$$\gamma_{\sigma^{-1}(k)} - \gamma_k = \begin{cases} \gamma_{k-1} - \gamma_k = \mu_k - \alpha_n & \text{if } 2^{n-1} < k < 2^n \\ \gamma_{2^n-1} - \gamma_k = -\gamma_k = \mu_k - \alpha_n & \text{if } k = 2^{n-1}, \end{cases}$$

so $[U_3, V_3] = T - A$ and thus

$$T = \sum_{i=1}^{3}[U_i, V_i] \in [\mathcal{I}, B(\mathcal{H})].$$

□

Theorem 2.5. *Let $\mathcal{I}$ be a proper ideal of $B(\mathcal{H})$.*

(i) *Suppose $T = T^* \in \mathcal{I}$. Let $\lambda_1, \lambda_2, \dots$ be the nonzero eigenvalues of T listed according to multiplicity (with zeros added if T has only finitely many nonzero eigenvalues) and such that $|\lambda_1| \geq |\lambda_2| \geq \cdots$. Then $T \in [\mathcal{I}, B(\mathcal{H})]$ if and only if*

$$\left(\frac{\lambda_1 + \cdots + \lambda_n}{n}\right)_{n=1}^{\infty} \in \mathcal{I}.$$

(ii) *The same as (i), but with the condition $|\lambda_1| \geq |\lambda_2| \geq \cdots$ replaced by the weaker condition $(\max_{j \geq n} |\lambda_j|)_{n=1}^{\infty} \in \mathcal{I}$.*

Proof. Clearly (ii) $\Longrightarrow$ (i). Part (ii) is true as a consequence of Lemma 2.3 (with $\mathcal{J} = B(\mathcal{H})$) and Lemma 2.4.

□

In the next Corollary, we consider the sequence $\omega = (1, 1/2, 1/3, \ldots)$ and the principal ideal, $\Omega = \langle \mathrm{diag}(\omega) \rangle$, where $\mathrm{diag}(\omega)$ is the diagonal operator with $1, 1/2, 1/3, \ldots$ for diagonal entries. See Definition 1.2 for the tensor products used below.

Corollary 2.6. *Let $\mathcal{I}$ be a proper ideal of $B(\mathcal{H})$.*

(i) *If R is a positive, compact operator whose sequence of s–numbers is $s(R)$, then $R \in [\mathcal{I}, B(\mathcal{H})]$ if and only if the sequence $s(R) \otimes \omega$ is in the characteristic set of $\mathcal{I}$.*
(ii) *If $\mathcal{L}$ is a proper ideal of $B(\mathcal{H})$ then $\mathcal{L} \subseteq [\mathcal{I}, B(\mathcal{H})]$ if and only if $\mathcal{L} \diamondsuit \Omega \subseteq \mathcal{I}$.*

Proof. Use Theorem 2.5 and Theorem 1.3.

□

Theorem 2.7. *If $\mathcal{I}$ and $\mathcal{J}$ are proper ideals of $B(\mathcal{H})$ then $[\mathcal{I}, \mathcal{J}] = [\mathcal{I}\mathcal{J}, B(\mathcal{H})]$.*

Proof. We easily see that $[\mathcal{I}, \mathcal{J}] \supseteq [\mathcal{I}\mathcal{J}, B(\mathcal{H})]$, because if $A \in \mathcal{I}$, $B \in \mathcal{J}$ and $X \in B(\mathcal{H})$, then

$$[X, AB] = XAB - ABX = [XA, B] + [BX, A]. \tag{15}$$

To see the reverse inclusion, let $T = T^* \in [\mathcal{I}, \mathcal{J}]$ and let $\lambda_1, \lambda_2, \cdots$ be the nonzero eigenvalues of T arranged so that $|\lambda_1| \geq |\lambda_2| \geq \cdots$. Then (14) holds by Lemma 2.3, hence $T \in [\mathcal{I}\mathcal{J}, B(\mathcal{H})]$ by Lemma 2.4.

□

Of course, commutators are intimately connected with traces. A *trace* on an operator ideal, $\mathcal{I}$, is a linear mapping $\tau : \mathcal{I} \to \mathbf{C}$ satisfying $\tau(XY) = \tau(YX)$ for every $X \in \mathcal{I}$ and $Y \in B(\mathcal{H})$. Now a linear functional is a trace if and only if it vanishes on the commutator space $[\mathcal{I}, B(\mathcal{H})]$. Thus, the set of traces is canonically the same as the linear space dual to the quotient $\mathcal{I}/[\mathcal{I}, B(\mathcal{H})]$. A trace is said to be *positive* if $A \in \mathcal{I}$ and $A \geq 0$ implies $\tau(A) \geq 0$.

We give three applications of Theorem 2.5 involving traces. The first relates to extending the trace from the finite rank operators to larger ideals.

Proposition 2.8. *Let $\mathcal{I}$ be a nonzero proper ideal of $B(\mathcal{H})$. There is a trace τ on $\mathcal{I}$ which is nonvanishing on the ideal of finite rank operators if and only if*

$$\mathrm{diag}(1, 1/2, 1/3, \ldots, 1/n, \ldots) \notin \mathcal{I},$$

i.e. if and only if the sequence $\omega = (1, 1/2, 1/3, \dots)$ *is not in the characteristic set of* $\mathcal{I}$. *Moreover, if such a trace* τ *is positive, then* $\mathcal{I} \subseteq \mathcal{C}_1$.

Proof. Theorem 2.5 and the fact that $[\mathcal{F}, B(\mathcal{H})] = \mathcal{F}^0$ give immediately that

$$[\mathcal{I}, B(\mathcal{H})] \cap \mathcal{F} = \mathcal{F}^0 \tag{16}$$

if and only if ω is not in the characteristic set of $\mathcal{I}$. Since traces on $\mathcal{F}$ are unique up to scalar multiples, if τ is a trace on $\mathcal{I}$ which is nonvanishing on $\mathcal{F}$, then clearly (16) must hold. But if (16) holds then, using a Hamel basis argument, one can construct a linear map $\tau : \mathcal{I} \to \mathbf{C}$ that vanishes on $[\mathcal{I}, B(\mathcal{H})]$ but not on a given rank one projection P. Thus, the condition involving ω is necessary and sufficient for the existence of τ nonvanishing on $\mathcal{F}$.

Now suppose there is a positive trace τ on $\mathcal{I}$ which is nonvanishing on the finite rank operators. Multiplying by a constant, we may suppose τ restricts to the usual trace on the finite rank operators. Let $T \in \mathcal{I}$, $T \geq 0$ and let $\lambda_1 \geq \lambda_2 \geq \cdots$ be the s–numbers of T. Then

$$\tau(T) \geq \tau(\operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n, 0, 0, \dots)) = \sum_{j=1}^{n} \lambda_j.$$

Thus, $\sum_1^\infty \lambda_j \leq \tau(T) < \infty$, so $T \in \mathcal{C}_1$. Hence $\mathcal{I} \subseteq \mathcal{C}_1$.

□

Our second application of Theorem 2.5, in combination with a result of Varga [**15**], is the following.

Theorem 2.9. *Let* $\mathcal{I} = \langle A \rangle$ *be a principal ideal of* $B(\mathcal{H})$ *generated by a compact operator* A. *Then* $\mathcal{I}$ *has a nonzero trace if and only if it has a nonzero positive trace.*

Proof. If $\mathcal{I}$ has a nonzero trace then necessarily $[\mathcal{I}, B(\mathcal{H})] \neq \mathcal{I}$. Since A and $|A|$ generate the same ideal, we may assume $A \geq 0$, hence the eigenvalues of A arranged in nonincreasing order are also its s–numbers. From Theorem 2.5 we must have that

$$\left(\frac{s_1(A) + \cdots + s_n(A)}{n} \right)_{n=1}^{\infty} \notin \mathcal{I},$$

hence $s_1 + \cdots s_n \neq O(ns_n)$, namely, the sequence $(s_n(A))_1^\infty$ is irregular. This is precisely the property that Varga [**15**] proved is necessary and sufficient for $\mathcal{I}$ to have a nonzero positive trace.

□

One can easily see, as in the following example, that this kind of result does not hold for general operator ideals.

Proposition 2.10. *The ideal $\mathcal{I}_{o(1/n)}$, consisting of all operators T such that $s_n(T) = o(1/n)$, has nonzero traces but no positive traces.*

Proof. By Proposition 2.8, $\mathcal{I}_{o(1/n)}$ has nonzero traces. But suppose to obtain a contradiction that

$$\tau : \mathcal{I}_{o(1/n)} \to \mathbf{C}$$

is a nonzero positive trace. Then since $\mathcal{I}_{o(1/n)} \not\subseteq \mathcal{C}_1$, by Proposition 2.8, τ must vanish on the finite rank operators. Then there is $T \in \mathcal{I}_{o(1/n)}$ such that $T \geq 0$ and $\tau(T) > 0$. Let $\lambda_1 \geq \lambda_2 \geq \cdots$ be the s–numbers of T. There are $1 = t_1 \leq t_2 \leq t_3 \leq \cdots$ such that $\lim_{n\to\infty} t_n = \infty$ and $\mu_n \stackrel{\text{def}}{=} t_n \lambda_n = o(1/n)$. Let $T_1 = \operatorname{diag}(\mu_1, \mu_2, \ldots)$. Since τ vanishes on finite rank operators, we easily see that for every n, $\tau(T_1) \geq t_n \tau(T)$. Hence $\tau(T_1) = \infty$, which is impossible.

□

Our third application of Theorem 2.5 is to the interpolation ideals $\mathcal{L}^{(p,q)}$ and related ideals considered by A. Connes in [**7**, IV.2.α]. For $1 < p < \infty$ and $1 \leq q < \infty$, these ideals are defined as follows:

$$\begin{aligned}
s(\mathcal{L}^{(p,q)}) &= \{\lambda \in c_0^{++} \mid \sum_{n=1}^{\infty} n^{-q-1+(q/p)}(\lambda_1 + \lambda_2 + \cdots + \lambda_n)^q < \infty\} \\
s(\mathcal{L}^{(p,\infty)}) &= \{\lambda \in c_0^{++} \mid \lambda_1 + \lambda_2 + \cdots + \lambda_n = O(n^{1-(1/p)})\} \\
s(\mathcal{L}_0^{(p,\infty)}) &= \{\lambda \in c_0^{++} \mid \lambda_1 + \lambda_2 + \cdots + \lambda_n = o(n^{1-(1/p)})\} \\
s(\mathcal{L}^{(1,\infty)}) &= \{\lambda \in c_0^{++} \mid \lambda_1 + \lambda_2 + \cdots + \lambda_n = O(\log n)\} \\
s(\mathcal{L}_0^{(1,\infty)}) &= \{\lambda \in c_0^{++} \mid \lambda_1 + \lambda_2 + \cdots + \lambda_n = o(\log n)\} \\
s(\mathcal{L}^{(\infty,1)}) &= \{\lambda \in c_0^{++} \mid \sum_{n=1}^{\infty} \frac{\lambda_n}{n} < \infty\}.
\end{aligned}$$

The ideal $\mathcal{L}^{(1,\infty)}$ has lots of nonzero, positive traces, namely the Dixmier traces (see [**7**, IV.2.β]).

Proposition 2.11. *Let $1 < p < \infty$ and $1 \leq q \leq \infty$. Then the ideals $\mathcal{L}^{(p,q)}$, $\mathcal{L}_0^{(p,\infty)}$ and $\mathcal{L}^{(\infty,1)}$ have no nonzero traces. The ideal $\mathcal{L}_0^{(1,\infty)}$ has nonzero traces, but no nonzero positive traces.*

Proof. By Theorem 2.5, for an ideal $\mathcal{I}$ to be without nonzero traces, it is necessary and sufficient that $\lambda \in s(\mathcal{I})$ imply $\lambda^{(\mathrm{a})} \in s(\mathcal{I})$, where $\lambda_n^{(\mathrm{a})} = (\lambda_1 + \cdots + \lambda_n)/n$.

We first examine $\mathcal{L}^{(p,q)}$ when $q < \infty$. We have

$$s(\mathcal{L}^{(p,q)}) = \{\lambda \in c_0^{++} \mid \sum_{n=1}^{\infty} \left(n^{-\frac{1}{q}+\frac{1}{p}} \left(\frac{\lambda_1 + \cdots + \lambda_n}{n} \right) \right)^q < \infty\}.$$

It is well–known that

$$\sum_{n=1}^{\infty}(n^{-\frac{1}{q}+\frac{1}{p}}\lambda_n)^q<\infty$$

implies

$$\sum_{n=1}^{\infty}\Big(n^{-\frac{1}{q}+\frac{1}{p}}\Big(\frac{\lambda_1+\cdots+\lambda_n}{n}\Big)\Big)^q<\infty,$$

(see [**12**, 2.1.7] for a short and direct argument) and the reverse implication is clear. Hence

$$s(\mathcal{L}^{(p,q)})=\{\lambda\in c_0^{++}\mid\sum_{n=1}^{\infty}(n^{-\frac{1}{q}+\frac{1}{p}}\lambda_n)^q<\infty\}$$

and $\lambda\in s(\mathcal{L}^{(p,q)})$ implies $\lambda^{(\mathrm{a})}\in s(\mathcal{L}^{(p,q)})$. Thus $\mathcal{L}^{(p,q)}$ has no nonzero traces.

The proof of lack of traces on $\mathcal{L}^{(p,\infty)}$ and $\mathcal{L}_0^{(p,\infty)}$ is similar, using that

$$\begin{aligned}s(\mathcal{L}^{(p,\infty)})&=\Big\{\lambda\in c_0^{++}\Big|\frac{\lambda_1+\cdots+\lambda_n}{n}=O(n^{-1/p})\Big\}\\&=\Big\{\lambda\in c_0^{++}\mid\lambda_n=O(n^{-1/p})\Big\}\end{aligned}$$

and the analogous statement for $\mathcal{L}_0^{(p,\infty)}$ and $o(n^{-1/p})$.

Suppose $\lambda\in s(\mathcal{L}^{(\infty,1)})$. Then

$$\sum_{n=1}^{\infty}\frac{\lambda_n^{(\mathrm{a})}}{n}=\sum_{n=1}^{\infty}\frac{1}{n^2}\sum_{k=1}^{n}\lambda_k=\sum_{k=1}^{\infty}\lambda_k\sum_{n=k}^{\infty}\frac{1}{n^2}.$$

But $\sum_{n=k}^{\infty}\frac{1}{n^2}\le\frac{c}{k}$ for some constant c, which shows that $\lambda^{(\mathrm{a})}\in s(\mathcal{L}^{(\infty,1)})$. Hence $\mathcal{L}^{(\infty,1)}$ has no nonzero traces.

We now show that $\mathcal{L}_0^{(1,\infty)}$ has traces by exhibiting $\lambda\in s(\mathcal{L}_0^{(1,\infty)})$ for which $\lambda^{(\mathrm{a})}\notin s(\mathcal{L}_0^{(1,\infty)})$. Let $\lambda\in c_0^{++}$ be such that $\lambda_n=o(1/n)$ and $\lambda_1\neq 0$. Then clearly $\lambda\in s(\mathcal{L}_0^{(1,\infty)})$. But $\lambda_n^{(\mathrm{a})}\ge\lambda_1/n$, so $\lambda^{(\mathrm{a})}\notin s(\mathcal{L}_0^{(1,\infty)})$. Thus, $\mathcal{L}_0^{(1,\infty)}$ has nonzero traces. (In fact, it is easy to see that if $\sum_1^{\infty}\lambda_n=\infty$ then even $\lambda^{(\mathrm{a})}\notin s(\mathcal{L}^{(1,\infty)})$.)

To see that $\mathcal{L}_0^{(1,\infty)}$ has no nonzero positive traces, we will use an argument similar to that of Proposition 2.10. Suppose for contradiction that τ is a nonzero positive trace on $\mathcal{L}_0^{(1,\infty)}$. Let $T\in\mathcal{L}_0^{(1,\infty)}$, $T\ge 0$ be such that $\tau(T)>0$, and let $\lambda_n=s_n(T)$ be its s–numbers. By Proposition 2.8, τ vanishes on the finite rank operators. Hence, to obtain a contradiction it will suffice to find

$$1=c_1\le c_2\le c_3\le\cdots$$

such that

$$\lim_{n\to\infty} c_n = \infty, \tag{17}$$

$$c_1\lambda_1 \geq c_2\lambda_2 \geq c_3\lambda_3 \geq \cdots, \tag{18}$$

$$\sum_{k=1}^{n} c_k\lambda_k = o(\log n), \tag{19}$$

since this would imply that $\tau(T_1) \geq c_n\tau(T)$ for every n, where $T_1 = \operatorname{diag}(c_1\lambda_1, c_2\lambda_2, \ldots)$. Let

$$\gamma_n' = \inf_{m\geq n}\left(\frac{\log m}{\sum_{k=1}^{m}\lambda_k}\right)^{1/2}$$

and let $\gamma_n = \max(1, \gamma_n')$. Then γ_n is increasing in n and tends to ∞ as $n \to \infty$. Define c_n recursively by $c_1 = 1$ and

$$c_{n+1} = \min(\frac{\lambda_n}{\lambda_{n+1}}c_n, \gamma_{n+1}).$$

Then $c_{n+1} \geq \min(c_n, \gamma_n) = c_n$. To see that (17) holds, let $M > 0$ and let $N \in \mathbf{N}$ be such that $\gamma_N \geq M$. Then $\forall k \geq 1$

$$c_{N+k} \geq \min(c_N\frac{\lambda_N}{\lambda_{N+k}}, M)$$

so $c_{N+k} \geq M$ for large enough k. Moreover, (18) holds because $c_{n+1} \leq \lambda_n c_n/\lambda_{n+1}$. Finally, (19) holds because, if n is large enough so that $\gamma_n' \geq 1$ then

$$\sum_{k=1}^{n} c_k\lambda_k \leq c_n\sum_{k=1}^{n}\lambda_k \leq \gamma_n'\sum_{k=1}^{n}\lambda_k \leq (\log n)^{1/2}\big(\sum_{k=1}^{n}\lambda_k\big)^{1/2}$$

and because $\sum_{k=1}^{n}\lambda_k = o(\log n)$.

□

§3. How many commutators.

In this section we give results related to the question of, given $T \in [\mathcal{I}, \mathcal{J}]$, how many commutators are needed to write T as a sum of commutators. One sees directly from Lemma 2.4 that $[\mathcal{I}, B(\mathcal{H})] \subseteq C_6(\mathcal{I}, B(\mathcal{H}))$. Moreover, using the characterization in Theorem 2.5 and by considering the real and imaginary parts of $T \in [\mathcal{I}, B(\mathcal{H})]$ simultaneously, in a proof analogous to that of Lemma 2.4, one can show $[\mathcal{I}, B(\mathcal{H})] \subseteq C_4(\mathcal{I}, B(\mathcal{H}))$. In a similar way, one can show $[\mathcal{I}, \mathcal{J}] \subseteq C_4(\mathcal{I}, \mathcal{J})$. We will use other techniques to prove and to improve upon these results, showing that $[\mathcal{I}, B(\mathcal{H})] \subseteq C_3(\mathcal{I}, B(\mathcal{H}))$.

Our convention will be that all entries of matrices are zero, except as otherwise indicated.

Lemma 3.1. *In any unital ring, if $A_1 + \cdots + A_n = 0$ then in the $n \times n$ matrices over the ring,*

$$\left[\begin{pmatrix} 0 & B_1 & & & \\ & 0 & B_2 & & \\ & & 0 & \ddots & \\ & & & \ddots & B_{n-1} \\ & & & & 0 \end{pmatrix}, \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & 0 & \\ & & & 1 & 0 \end{pmatrix}\right] = \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_n \end{pmatrix}$$

where $B_j = A_1 + \cdots + A_j$.

Lemma 3.2. *Let $\mathcal{I}$ and $\mathcal{J}$ be proper ideals of $B(\mathcal{H})$. If $A \in \mathcal{I}\mathcal{J}$ then $A = XY$ for some $X \in \mathcal{I}$ and $Y \in \mathcal{J}$.*

Proof. We have $A = X_1Y_1 + \cdots + X_mY_m$ for $X_j \in \mathcal{I}$ and $Y_j \in \mathcal{J}$. Choose an isomorphism $S : \mathcal{H} \to \mathcal{H}^{\oplus m}$, let Y be the composition

$$\mathcal{H} \xrightarrow{\Delta} \mathcal{H}^{\oplus m} \xrightarrow{\oplus Y_j} \mathcal{H}^{\oplus m} \xrightarrow{S^{-1}} \mathcal{H}$$

and let X be the composition

$$\mathcal{H} \xrightarrow{S} \mathcal{H}^{\oplus m} \xrightarrow{\oplus X_j} \mathcal{H}^{\oplus m} \xrightarrow{\Sigma} \mathcal{H},$$

where

$$\Delta(v) = \begin{pmatrix} v \\ \vdots \\ v \end{pmatrix} \in \mathcal{H}^{\oplus m} \quad \text{and} \quad \Sigma \begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix} = v_1 + \cdots + v_m.$$

Then $X \in \mathcal{I}$, $Y \in \mathcal{J}$ and $A = XY$.

□

Lemma 3.3. *Let $\mathcal{I}$ and $\mathcal{J}$ be ideals of $B(\mathcal{H})$ and suppose $A \in [\mathcal{I}, \mathcal{J}]$ and has an infinite dimensional reducing subspace on which it is zero. Then*

$$A \in C(\mathcal{I}, \mathcal{J}) + C(\mathcal{I}\mathcal{J}, B(\mathcal{H})). \tag{20}$$

More particularly,

$$A = [X, Y] + [B, Z] \tag{21}$$

with $X \in \mathcal{I}$, $Y \in \mathcal{J}$, $B \in \mathcal{I}\mathcal{J}$ and with $Z \in B(\mathcal{H})$ of the form $Z = Z_3 = \begin{pmatrix} 0&0&0 \\ 1&0&0 \\ 0&1&0 \end{pmatrix}$ with respect to some decomposition $\mathcal{H} \cong \mathcal{H}^{\oplus 3}$.

Proof. Let $n \in \mathbf{N}$ and $A = \sum_{j=1}^{n} [X_j, Y_j]$ for $X_j \in \mathcal{I}$ and $Y_j \in \mathcal{J}$. If $n = 1$ we are done, so assume $n \geq 2$. Let $\mathcal{H} \cong \mathcal{H}^{\oplus n}$ be an identification. Writing elements of $B(\mathcal{H})$ as $n \times n$ matrices with respect to this identification, we may assume

$$A = \begin{pmatrix} A & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} = \begin{pmatrix} \sum_1^n [X_j, Y_j] & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}$$

and we have from Lemma 3.1 that

$$\begin{pmatrix} A & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} - \begin{pmatrix} [X_1, Y_1] & & & \\ & [X_2, Y_2] & & \\ & & \ddots & \\ & & & [X_n, Y_n] \end{pmatrix} = [B, Z]$$

with $B \in \mathcal{I}\mathcal{J}$ and

$$Z = Z_n = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & 0 & \\ & & & 1 & 0 \end{pmatrix} \in B(\mathcal{H}),$$

an $n \times n$ matrix with respect to the above identification. But

$$\begin{pmatrix} [X_1, Y_1] & & \\ & \ddots & \\ & & [X_n, Y_n] \end{pmatrix} = \left[\begin{pmatrix} X_1 & & \\ & \ddots & \\ & & X_n \end{pmatrix}, \begin{pmatrix} Y_1 & & \\ & \ddots & \\ & & Y_n \end{pmatrix} \right],$$

so (21) holds as required but with $Z = Z_n$. However, by using (20), Lemma 3.2 and (15) we can repeat the proof but with $n = 3$.

□

Theorem 3.4. *Let $\mathcal{I}$ and $\mathcal{J}$ be ideals of $B(\mathcal{H})$. Then*

$$[\mathcal{I}, \mathcal{J}] \subseteq C_2(\mathcal{I}, \mathcal{J}) + C(\mathcal{I}\mathcal{J}, B(\mathcal{H})).$$

Proof. Let $A \in [\mathcal{I}, \mathcal{J}]$. By Lemma 3.2, $A = X_0 Y_0$ for some $X_0 \in \mathcal{I}$ and $Y_0 \in \mathcal{J}$. Let V be an isometry from $\mathcal{H}$ onto a subspace $V\mathcal{H}$ such that $(V\mathcal{H})^\perp$ is infinite dimensional. Then

$$A = [X_0 V^*, V Y_0] + V Y_0 X_0 V^*.$$

But $V Y_0 X_0 V^* = V(A - [X_0, Y_0])V^*$ and writing $A = \sum_{j=1}^n [X_j, Y_j]$ for some $n \in \mathbf{N}$, $X_j \in \mathcal{I}$ and $Y_j \in \mathcal{J}$, we have $V Y_0 X_0 V^* = -[V X_0 V^*, V Y_0 V^*] + \sum_{j=1}^n [V X_j V^*, V Y_j V^*]$ so by Lemma 3.3

$$V Y_0 X_0 V^* \in C(\mathcal{I}, \mathcal{J}) + C(\mathcal{I}\mathcal{J}, B(\mathcal{H})).$$

□

Now using (15) we immediately see

Corollary 3.5. *For ideals $\mathcal{I}$ and $\mathcal{J}$ of $B(\mathcal{H})$,*

$$[\mathcal{I}, B(\mathcal{H})] \subseteq C_3(\mathcal{I}, B(\mathcal{H})), \tag{22}$$

$$[\mathcal{I}, \mathcal{J}] \subseteq C_4(\mathcal{I}, \mathcal{J}). \tag{23}$$

We do not know if (22) and (23) are the best possible, but we shall see from Corollary 4.9 that it is not always possible to replace C_3 with C_1 in (22).

Lemma 3.6. *Let R be a unital ring, let $n \in \mathbf{N}$ and let Z be nilpotent in $M_n(R)$. Then for every proper ideal $I \subset R$ and every invertible element $r \in R$ satisfying $rZ = Zr$, the map $\phi_{n,r} : M_n(I) \to M_n(I)$ given by*

$$\phi_{n,r}(A) = rA + [A, Z]$$

is one–to–one and onto $M_n(I)$.

Proof. Let $m \in \mathbf{N}$ be such that $Z^m = 0$. Denote by ρ, L, and R the endomorphisms of the additive group $M_n(I)$ defined by left multiplication by r, left multiplication by Z and right multiplication by Z, respectively. These three endomorphisms are mutually commuting, and $\phi_{n,r} = \rho - L + R$. Since $L^m = R^m = 0$ we have $(L - R)^{2m} = 0$ and hence $\phi_{n,r}^{-1}$ is given by

$$\sum_{i=0}^{2m-1} \rho^{-i-1}(L - R)^i.$$

□

Lemma 3.7. *Let $\mathcal{I}$ and $\mathcal{J}$ be ideals of $B(\mathcal{H})$ and let $A \in \mathcal{I}\mathcal{J}$. Suppose there is an identification of $\mathcal{H}$ with $\mathcal{H}^{\oplus n}$ for some $n \in \mathbf{N}$ such that, writing $A = (A_{ij})_{1 \le i,j \le n}$ as a matrix with respect to this identification, we have for every $1 \le i \le n$ that $A_{ii} \in [\mathcal{I}, \mathcal{J}]$ and that A_{ii} has an infinite dimensional reducing subspace on which it is zero. Then $A \in C(\mathcal{I}, \mathcal{J}) + C(\mathcal{I}\mathcal{J}, B(\mathcal{H}))$.*

Proof. From Lemma 3.3, $A_{ii} = [X_i, Y_i] + [B_i, Z]$ for $X_i \in \mathcal{I}$, $Y_i \in \mathcal{J}$, $B_i \in \mathcal{I}\mathcal{J}$ and $Z \in B(\mathcal{H}_i)$, where we can take the same Z for each i and $Z^3 = 0$. Let $A'_{ii} = B_i$. Of course, $A_{ij} \in \mathcal{I}\mathcal{J}$ for each i, j. By Lemma 3.6, for each $i \neq j$ there is $A'_{ij} \in \mathcal{I}\mathcal{J}$ such that $A_{ij} = (i-j)A'_{ij} + [A'_{ij}, Z]$. Letting $A' = (A'_{ij})_{1 \le i,j \le n}$, we have

$$A = \left[A', \begin{pmatrix} 1+Z & & & \\ & 2+Z & & \\ & & \ddots & \\ & & & n+Z \end{pmatrix} \right] + \left[\begin{pmatrix} X_1 & & & \\ & X_2 & & \\ & & \ddots & \\ & & & X_n \end{pmatrix}, \begin{pmatrix} Y_1 & & & \\ & Y_2 & & \\ & & \ddots & \\ & & & Y_n \end{pmatrix} \right].$$

□

Theorem 3.8. *Let $\mathcal{I}$,$\mathcal{J}$ and $\mathcal{L}$ be ideals of $B(\mathcal{H})$. If $\mathcal{L} \subseteq [\mathcal{I}, \mathcal{J}]$ then*

$$\mathcal{L} \subseteq C(\mathcal{I}, \mathcal{J}) + C(\mathcal{I}\mathcal{J}, B(\mathcal{H})).$$

Proof. Let $A \in \mathcal{L}$. By Theorem 2 in [**3**] and the proof of Theorem 3 in [**3**], A is similar to $A_0 \in \mathcal{L}$ such that, with respect to an identification of $\mathcal{H}$ with $\mathcal{H}^{\oplus 4}$, writing $A_0 = (A_{ij})_{1 \le i,j \le 4}$, each A_{ii} has an infinite dimensional reducing subspace on which it is zero and $A_{ii} \in \mathcal{L}$. Now apply Lemma 3.7.

□

Corrollary 3.9. *Let $\mathcal{I}$, $\mathcal{J}$ and $\mathcal{L}$ be ideals of $B(\mathcal{H})$.*

(i) *If $\mathcal{L} \subseteq [\mathcal{I}, B(\mathcal{H})]$ then $\mathcal{L} \subseteq C_2(\mathcal{I}, B(\mathcal{H}))$*
(ii) *If $\mathcal{L} \subseteq [\mathcal{I}, \mathcal{J}]$ then $\mathcal{L} \subseteq C_3(\mathcal{I}, \mathcal{J})$.*

It will follow from Corollary 4.9 that C_2 is, in general, the best possible in (i).

Corollary 3.10. *If $\mathcal{I}$ and $\mathcal{J}$ are ideals of $B(\mathcal{H})$ and if T is an operator such that $|T| \in [\mathcal{I}, \mathcal{J}]$, then*

$$T \in C(\mathcal{I}, \mathcal{J}) + C(\mathcal{I}\mathcal{J}, B(\mathcal{H})).$$

Proof. Let $\mathcal{L}$ be the principal ideal generated by T. Since $|T| \in [\mathcal{I}, \mathcal{J}]$ it follows from Theorem 2.5 that $\mathcal{L} \subseteq [\mathcal{I}, \mathcal{J}]$.

□

§4. Single Commutators

In this section, we give some results about the class $C(\mathcal{I}, \mathcal{J})$ of commutators of elements of operator ideals.

First, some notation. Let $\mathcal{I}$ and $\mathcal{J}$ be operator ideals. Then

$$\mathcal{I} + \mathcal{J} \stackrel{\text{def}}{=} \{A + B \mid A \in \mathcal{I},\ B \in \mathcal{J}\}$$

is the operator ideal generated by $\mathcal{I} \cup \mathcal{J}$. For $t > 0$, $\mathcal{I}^t$ is the ideal whose characteristic set is the set of all sequences $(s_j^t)_{j=1}^{\infty}$ such that $(s_j)_{j=1}^{\infty}$ is in the characteristic set of $\mathcal{I}$. See Definition 1.2 for the tensor product ideal, $\mathcal{I} \diamondsuit \mathcal{J}$. As usual, we will let ω denote the sequence $(1, 1/2, 1/3, \ldots, 1/n, \ldots)$ and Ω denote the principal ideal generated by the diagonal operator whose diagonal entries form the sequence ω. Thus, for $t > 0$, Ω^t is the principal ideal generated by $\operatorname{diag}(1, 1/2^t, 1/3^t, \ldots, 1/n^t, \ldots)$.

Lemma 4.1. *Let $\mathcal{I}$ be a proper ideal of $B(\mathcal{H})$ and suppose $R \in \mathcal{I}$ has an infinite dimensional reducing subspace on which it is zero. Then*

$$R \in C(\mathcal{I} \diamondsuit \Omega^{1/2}, B(\mathcal{H})). \tag{24}$$

Proof. Let P denote a projection of rank one. If C and Z are as in [**1**, p. 129], but with $t = 0$, then $P = [C, Z]$ with $C \in \Omega^{1/2}$ and $Z \in B(\mathcal{H})$. Indeed, these are

$$C = \begin{pmatrix} 0 & A_1 & & \\ B_1 & 0 & A_2 & \\ & B_2 & 0 & \ddots \\ & & \ddots & \ddots \end{pmatrix} \quad \text{and} \quad Z = \begin{pmatrix} 0 & X_1 & & \\ Y_1 & 0 & X_2 & \\ & Y_2 & 0 & \ddots \\ & & \ddots & \ddots \end{pmatrix},$$

where the nth diagonal in C and Z is the $n \times n$ zero matrix, so that A_n and X_n are $n \times (n+1)$ matrices and B_n and Y_n are $(n+1) \times n$ matrices, and where these are given by

$$A_n = \begin{pmatrix} \frac{1}{n} & 0 & & & 0 \\ 0 & \frac{1}{n} & & & 0 \\ & & \ddots & & \vdots \\ & & & \frac{1}{n} & 0 \end{pmatrix}, \qquad X_n = \begin{pmatrix} 0 & \frac{1}{n} & 0 & & \\ 0 & 0 & \frac{2}{n} & & \\ \vdots & & & \ddots & \\ 0 & & & & \frac{n}{n} \end{pmatrix},$$

$$B_n = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ -\frac{1}{n+1} & 0 & & \\ 0 & -\frac{1}{n+1} & & \\ & & \ddots & \\ & & & -\frac{1}{n+1} \end{pmatrix}, \quad Y_n = \begin{pmatrix} \frac{n}{n+1} & 0 & & \\ 0 & \frac{n-1}{n+1} & & \\ & & \ddots & \\ & & & \frac{1}{n+1} \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Clearly $Z \in B(\mathcal{H})$, and since $\left(1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \dots\right) \leq \omega^{1/2}$, also $C \in \Omega^{1/2}$. Taking an appropriate identification $\mathcal{H} \cong \mathcal{H} \otimes \mathcal{H}$ gives $R \cong R \otimes P$. But $R \otimes P = [R \otimes C, 1 \otimes Z]$, so (24) holds.

□

Theorem 4.2. *Let $\mathcal{I}$ be a proper ideal of $B(\mathcal{H})$ and let R be a compact operator. If $s(R) \otimes \omega^{1/2}$ is in the characteristic set of $\mathcal{I}$, then $R \in C(\mathcal{I}, B(\mathcal{H}))$.*

Proof. By Theorem 2 in [**3**] and the proof of Theorem 3 in [**3**], R is similar to an operator, A, having the property that, for some identification $\mathcal{H} \cong \mathcal{H}^{\oplus 4}$, writing $A = (A_{ij})_{1 \leq i,j \leq 4}$ with respect to this identification, for each $i = 1, 2, 3, 4$, A_{ii} has an infinite dimensional reducing subspace on which it is zero. It will suffice to show that $A \in C(\mathcal{I}, B(\mathcal{H}))$. Clearly each $A_{ij} \in \langle A \rangle = \langle R \rangle$. Thus, by hypothesis, $s(A_{ii}) \otimes \omega^{1/2}$ is in the characteristic set of $\mathcal{I}$. Hence, by Lemma 4.1, there are $C_{ii} \in \mathcal{I}$ and $B_i \in B(\mathcal{H})$ such that $A_{ii} = [C_{ii}, B_i]$.

Now we show that there is an operator ideal, $\mathcal{J}$, with a complete norm and such that $A \in \mathcal{J} \subseteq \mathcal{I}$. Consider the sequence $\mu = s(A) \otimes \omega^{1/2}$. By hypothesis, μ is in the characteristic set of $\mathcal{I}$. Let $\mathcal{J}$ be the principal ideal generated by the diagonal operator whose diagonal sequence is the sequence μ. Then $A \in \mathcal{J} \subseteq \mathcal{I}$. We claim that $\mu \otimes \omega \asymp \mu$. Indeed, applying Proposition 1.5 gives

$$\mu \otimes \omega = (s(A) \otimes \omega^{1/2}) \otimes \omega = s(A) \otimes (\omega^{1/2} \otimes \omega) \asymp s(A) \otimes \omega^{1/2} = \mu.$$

Then by Theorem 1.3, $\mu^{(a)} \asymp \mu$. This implies that for some $c > 0$

$$\forall n \in \mathbf{N} \qquad \frac{\mu_1 + \cdots + \mu_n}{n} \leq c\mu_n,$$

hence that $\sum_1^n \mu_j = O(n\mu_n)$. This latter condition is the definition [**9**, III.§14] that μ is *regular*. Varga [**15**] showed that the principal ideal, $\mathcal{J}$, is a symmetrically normed ideal if and only if μ is regular. (In fact, for our purposes it is enough to see that μ regular implies that $\mathcal{J}$ is a symmetrically normed ideal. However, this follows by considering the characteristic set of the principal ideal $\mathcal{J}$ and Theorems III.14.2 and III.14.1 of [**9**].) Hence $\mathcal{J}$ is a complete, normed ideal.

By adding scalars, if necessary, we may assume that the spectra, $\sigma(B_1)$, $\sigma(B_2)$, $\sigma(B_3)$ and $\sigma(B_4)$, of $B_1, \ldots, B_4$ (as Hilbert space operators) are pairwise disjoint. Since the norm makes $\mathcal{J}$ a Banach space, since B_i acting by left or right multiplication on $\mathcal{J}$ is a bounded operator whose spectrum is contained $\sigma(B_i)$ and since left multiplication by B_i commutes with right multiplication by B_j, an application of the spectral mapping theorem (see [**13**, Lemma 0.11]) implies that for $i \neq j$ the operator on $\mathcal{J}$ given by

$$\mathcal{J} \ni D \mapsto DB_j - B_iD$$

is invertible. Hence, for $i \neq j$ let $C_{ij} \in \mathcal{J}$ be such that $C_{ij}B_j - B_iC_{ij} = A_{ij}$, let $C = (C_{ij})_{1\leq i,j\leq 4}$ and let $B = \operatorname{diag}(B_1, B_2, B_3, B_4)$. Then $C \in \mathcal{J} \subseteq \mathcal{I}$, $B \in B(\mathcal{H})$ and $[C, B] = A$. □

Corollary 4.3. *Let $\mathcal{I}$ and $\mathcal{J}$ be proper ideals of $B(\mathcal{H})$. If $\mathcal{I} \subseteq \mathcal{J} \diamondsuit \Omega^{1/2}$ then $\mathcal{I} \subseteq C(\mathcal{J}, B(\mathcal{H}))$.*

Compare the above result to Corollary 2.6(ii).

Corollary 4.4. *The previous corollary immediately implies the following.*

(a) $\mathcal{K} = C(\mathcal{K}, B(\mathcal{H}))$. *(This was proved in* [**1**]*).*
(b) *If $p > 2$ then $\mathcal{C}_p = C(\mathcal{C}_p, B(\mathcal{H}))$.*
(c) *If $\mathcal{J}$ is an ideal of $B(\mathcal{H})$ and if $\omega^{1/2} \in \mathcal{J}$ then $\mathcal{C}_2 \subseteq C(\mathcal{J}, B(\mathcal{H}))$.*

We now begin proving a partial converse to Corollary 4.4(c).

Lemma 4.5. *Let A be a compact operator on $\mathcal{H}$. Let $P_1, P_2, \dots$ be finite rank, mutually orthogonal projections in $B(\mathcal{H})$ and let $k \in \mathbf{N} \cup \{0\}$. Then there is a partial isometry, $V \in B(\mathcal{H})$, such that*

$$\forall n \in \mathbf{N} \qquad P_n V A P_n = |P_{n+k} A P_n|.$$

Proof. The polar decomposition yields $P_{n+k}AP_n = V_n|P_{n+k}AP_n|$, where V_n is a partial isometry on $\mathcal{H}$ such that $V_n = P_{n+k}V_nP_n$. Let $V \in B(\mathcal{H})$ be a partial isometry such that

$$\forall n, m \in \mathbf{N} \qquad P_n V P_m = \begin{cases} V_n^* & \text{if } m = n+k \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$P_n V A P_n = V_n^* P_{n+k} A P_n = V_n^* V_n |P_{n+k} A P_n| = |P_{n+k} A P_n|.$$

□

Theorem 4.6. *Let F be a finite rank operator having nonzero trace and suppose $F = [A, B]$ for A a compact operator and $B \in B(\mathcal{H})$. Let $s(A)$ denote the s–number sequence of A. Then $s(A) \otimes \omega \geq c\,\omega^{1/2}$ for some $c > 0$.*

Proof. By the lemma in [**5**], there are mutually orthogonal, nonzero projections, $P_1, P_2, P_3, \dots$, such that $F = P_1 F P_1$, there is $C > 0$ such that, for every k, $\operatorname{rank}(P_k) \leq Ck$ and

$$P_n A P_m = 0 = P_n B P_m \qquad \text{whenever } n > m+1.$$

Writing

$$\begin{aligned} F = P_1[A,B]P_1 &= (P_1AP_1)(P_1BP_1) - (P_1BP_1)(P_1AP_1) \\ &\quad + (P_1AP_2)(P_2BP_1) - (P_1BP_2)(P_2AP_1) \end{aligned}$$

and for every $k \geq 2$ writing

$$\begin{aligned} 0 = P_k[A,B]P_k &= (P_kAP_{k-1})(P_{k-1}BP_k) - (P_kBP_{k-1})(P_{k-1}AP_k) \\ &\quad + (P_kAP_k)(P_kBP_k) - (P_kBP_k)(P_kAP_k) \\ &\quad + (P_kAP_{k+1})(P_{k+1}BP_k) - (P_kBP_{k+1})(P_{k+1}AP_k), \end{aligned}$$

we have $\forall n \in \mathbf{N}$,

$$\operatorname{Tr}(F) = \sum_{k=1}^{n} \operatorname{Tr}(P_k[A,B]P_k) = \operatorname{Tr}(P_n A P_{n+1} B P_n) - \operatorname{Tr}(P_n B P_{n+1} A P_n),$$

hence

$$|\operatorname{Tr}(F)| \leq (|P_n A P_{n+1}|_1 + |P_{n+1} A P_n|_1)||B||, \tag{25}$$

where $|D|_1 \stackrel{\text{def}}{=} \operatorname{Tr}((D^*D)^{1/2}) = \operatorname{Tr}((DD^*)^{1/2})$. By Lemma 4.5 there are partial isometries $V, W \in B(\mathcal{H})$ such that

$$\forall n \in \mathbf{N} \qquad P_n V A P_n = |P_{n+1} A P_n| \quad \text{and} \quad P_n W A^* P_n = |P_{n+1} A^* P_n|.$$

Let $T = ||B||(VA + WA^*)|\operatorname{Tr}(F)|^{-1}$. Then for some $c_1 > 0$, $s(T) \leq c_1 s(A)$ and $\forall n \in \mathbf{N}$

$$P_n T P_n = ||B||(|P_{n+1} A P_n| + |P_{n+1} A^* P_n|)|\operatorname{Tr}(F)|^{-1},$$

so $P_n T P_n \geq 0$ and, from (25), $\operatorname{Tr}(P_n T P_n) \geq 1$. Hence by diagonalizing each $P_n T P_n$ we can find an orthonormal set $e_1, e_2, \ldots$ in $\mathcal{H}$ such that, letting $N_k = \sum_{j=1}^{k} \operatorname{rank}(P_j)$, we have $\sum_{i=1}^{N_k} \langle T e_i, e_i \rangle \geq k$. Thus, by a theorem of Fan, [**9**, Lemma II.4.1],

$$\sum_{i=1}^{N_k} s_i(T) \geq \sum_{i=1}^{N_k} \langle T e_i, e_i \rangle \geq k.$$

But $N_k \leq \sum_{j=1}^{k} Cj = \frac{Ck(k+1)}{2} < Ck^2$. Letting $m = [C] + 1$ we then get $\sum_{j=1}^{mk^2} s_j(T) \geq k$. Hence if $mk^2 \leq n < m(k+1)^2$ then

$$\sum_{j=1}^{n} s_j(T) \geq \sum_{j=1}^{mk^2} s_j(T) \geq k \geq \left(\frac{n}{m}\right)^{1/2} - 1.$$

Therefore there is $c_2 > 0$ such that $\forall n \in \mathbf{N}$, $\sum_{j=1}^{n} s_j(T) \geq c_2 \sqrt{n}$, hence $\sum_{j=1}^{n} s_j(A) \geq c_1 c_2 \sqrt{n}$. Dividing both sides by n we get for the arithmetic mean sequence, $s^{(a)}(A)$, of $s(A)$ that $s^{(a)}(A)_n \geq c_1 c_2 n^{-1/2}$. Now using Theorem 1.3 finishes the proof.

□

Corollary 4.7. *If $\mathcal{J}$ is one of the Gohberg–Krein ideals, $\mathfrak{S}_\Phi$* [**9**, III.§4]*, generated by a symmetric norming function Φ, and if $F \in C(\mathcal{J}, B(\mathcal{H}))$ for F a finite rank operator having nonzero trace, then $\omega^{1/2} \in \mathcal{J}$.*

Proof. From the proof of Theorem 4.6 there is $T \in \mathcal{J}$ such that

$$\forall n \in \mathbf{N} \quad \sum_{j=1}^{n} s_j(T) \geq \sum_{j=1}^{n} \frac{1}{\sqrt{j}}.$$

The corollary then follows immediately from the definition of $\mathfrak{S}_\Phi$ and the dominance property of Φ, [**9**, III.§4].

□

Corollary 4.8. *Let $0 < p \le 2$. Then $C(\mathcal{C}_p, B(\mathcal{H}))$ contains no finite rank operator of nonzero trace; hence*

$$C(\mathcal{C}_p, B(\mathcal{H})) \cap \mathcal{F} = \mathcal{F}^0.$$

Consequently $C_p \neq C(C_p, B(\mathcal{H}))$ whenever $0 < p \le 2$ and hence Corollary 4.4(b) is sharp.

Proof. If $0 < p \le 1$ then this follows because all elements of $C(\mathcal{C}_p, B(\mathcal{H}))$ must have zero trace. Assume $1 < p \le 2$. By Theorem 4.6, if $C(\mathcal{C}_2, B(\mathcal{H}))$ contains a finite rank operator of nonzero trace, then there is a p–summable sequence, $\lambda \in c_0^{++}$, such that $\lambda \otimes \omega \ge \omega^{1/2}$. But $\lambda \otimes \omega$ is p–summable, while $\omega^{1/2}$ is not p–summable, which is a contradiction.

□

Corollary 4.9. *If Φ is a symmetric norming function such that $\Phi(1, 1/2, 1/3, \dots) < \infty$, but $\Phi(1, 1/\sqrt{2}, 1/\sqrt{3}, \dots) = \infty$ then for the operator ideal $\mathfrak{S}_\Phi$,*

$$\mathcal{F} \subseteq C_2(\mathfrak{S}_\Phi, B(\mathcal{H}))$$

but whenever F is a finite rank operator of nonzero trace,

$$F \notin C_1(\mathfrak{S}_\Phi, B(\mathcal{H})).$$

Proof. Combine Theorem 3.8 and Theorem 2.5(i) for the inclusion and then use Corollary 4.7.

□

An example of a symmetric norming function satisfying the properties of the above corollary is

$$\Phi(\xi_1, \xi_2, \dots) = \sup_{n \ge 1} \frac{|\xi_1| + \dots + |\xi_n|}{1 + \frac{1}{2} + \dots + \frac{1}{n}}.$$

The following analogue of Lemma 4.1 for single commutators of two proper ideals follows readily from the techniques of [**1**].

Proposition 4.10. *Let $\mathcal{I}$ and $\mathcal{J}$ be proper ideals of $B(\mathcal{H})$ and suppose $R \in \mathcal{I}\mathcal{J}$ has an infinite dimensional reducing subspace on which it is zero. Then for every $0 < t < 1$,*

$$R \in C(\mathcal{L}_{t/2}, \mathcal{L}_{(1-t)/2})$$

where $\mathcal{L}_\alpha = (\mathcal{I} + \mathcal{J}) \diamondsuit \Omega^\alpha$.

Proof. By Lemma 3.2, $R = ST$ for some $S \in \mathcal{I}$ and $T \in \mathcal{J}$. Let $0 < t < 1$. If K_t and L_t are as in the proof of [**1**, Theorem 1] then $R = [K_t, L_t]$ and

$$K_t \in (\mathcal{I} + \mathcal{J}) \diamondsuit \mathcal{P}_t$$

$$L_t \in (\mathcal{I} + \mathcal{J}) \diamondsuit \mathcal{P}_{1-t},$$

where $\mathcal{P}_t$ is the ideal generated by an operator whose sequence of s–numbers is the nonincreasing re–arrangement of

$$\underbrace{\frac{1^t}{1},\frac{1^t}{2},\frac{1^t}{3},\ldots}, \underbrace{\frac{2^t}{2},\frac{2^t}{3},\frac{2^t}{4},\ldots}, \underbrace{\frac{3^t}{3},\frac{3^t}{4},\frac{3^t}{5},\ldots},\ldots, \underbrace{\frac{k^t}{k},\frac{k^t}{k+1},\frac{k^t}{k+2},\ldots},\ldots .$$

By elementary arguments, this nonincreasing re–arrangement is equivalent to $\omega^{(1-t)/2}$, and hence $\mathcal{P}_t = \Omega^{(1-t)/2}$.

□

§5. Open Questions.

5.1. Our characterization of $[\mathcal{I}, B(\mathcal{H})]$ is in terms of self–adjoint operators. Kalton's theorem [**10**], on the other hand, characterizes all $T \in [\mathcal{C}_1, B(\mathcal{H})]$ in terms of the eigenvalues of T, where $\mathcal{C}_1$ is the ideal of trace–class operators. Does this hold for general operator ideals, $\mathcal{I}$? More specifically, for every $T \in \mathcal{I}$, writing the nonzero points of its spectrum repeated according to algebraic multiplicity as $\lambda_1, \lambda_2, \ldots$, ordered so $|\lambda_1| \geq |\lambda_2| \geq \cdots$, is it true that a necessary and sufficient condition for $T \in [\mathcal{I}, B(\mathcal{H})]$ is that

$$\left(\frac{\lambda_1 + \cdots + \lambda_n}{n}\right)_{n=1}^{\infty} \in \mathcal{I} \quad ?$$

By *algebraic multiplicity* of a nonzero point λ of the spectrum of T we mean

$$\max_{n\geq 1} \dim \ker(T - \lambda I)^n .$$

Of course, if T has finite spectrum then we allow the list $\lambda_1, \lambda_2, \ldots$ to have a tail of zeros.

5.2. We have in general

$$[\mathcal{I}, B(\mathcal{H})] = C_3(\mathcal{I}, B(\mathcal{H}))$$

and examples where

$$[\mathcal{I}, B(\mathcal{H})] \nsubseteq C_1(\mathcal{I}, B(\mathcal{H})).$$

Is it true in general that

$$[\mathcal{I}, B(\mathcal{H})] = C_2(\mathcal{I}, B(\mathcal{H}))?$$

5.3. Is it true in general that $F \in C(\mathcal{J}, B(\mathcal{H}))$ implies $\omega^{1/2} \in \mathcal{J}$, where F is a finite rank operator having nonzero trace? (See Corollary 4.4(c) and Theorem 4.6.) More generally, what is a characterization of $C(\mathcal{J}, B(\mathcal{H}))$? For an operator ideal $\mathcal{J}$, is the sufficient condition of Theorem 4.2 for R to be in $C(\mathcal{J}, B(\mathcal{H}))$ also necessary when $R \geq 0$? For T a compact operator, does $|T| \in C(\mathcal{J}, B(\mathcal{H}))$ imply $T \in C(\mathcal{J}, B(\mathcal{H}))$?

5.4. Is there a strictly positive operator in $C(\mathcal{K},\mathcal{K})$, i.e. without any reducing subspace on which it is zero? (Compare [**1**, Theorem 1].)

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