Integr. equ. oper. theory 99 (9999), 1–57
0378-620X/99000-0, DOI 10.1007/s00020-003-0000

Integral Equations and Operator Theory

Soft Ideals and Arithmetic Mean Ideals

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Abstract. This article investigates the soft-interior (se) and the soft-cover (sc) of operator ideals. These operations, and especially the first one, have been widely used before, but making their role explicit and analyzing their interplay with the arithmetic mean operations is essential for the study in [10] of the multiplicity of traces. Many classical ideals are "soft", i.e., coincide with their soft interior or with their soft cover, and many ideal constructions yield soft ideals. Arithmetic mean (am) operations were proven to be intrinsic to the theory of operator ideals in [6, 7] and arithmetic mean operations at infinity (am-∞) were studied in [10]. Here we focus on the commutation relations between these operations and soft operations. In the process we characterize the am-interior and the am-∞ interior of an ideal.

Mathematics Subject Classification (2000). Primary 47B47, 47B10, 47L20; Secondary 46A45, 46B45.

Keywords. Arithmetic means, operator ideals, countably generated ideals, Lorentz ideals, Orlicz ideals, Marcinkiewicz ideals, Banach ideals.

1. Introduction

Central to the theory of operator ideals (the two-sided ideals of the algebra $B(H)$ of bounded operators on a separable Hilbert space H) are the notions of the commutator space of an ideal I (the linear span of the commutators $TA - AT$,

Both authors were partially supported by grants of the Charles Phelps Taft Research Center; the second named author was partially supported by NSF Grants DMS 95-03062 and DMS 97-06911.

$A \in I$, $T \in B(H)$) and of a trace supported by the ideal. In this context, the arithmetic (Cesaro) mean of monotone sequences first appeared implicitly in [21], then played in [15] an explicit and key role for determining the commutator space of the trace class, and more recently entered center stage in [6, 7] by providing the framework for the characterization of the commutator space of arbitrary ideals. This prompted [7] to associate more formally to a given ideal I the arithmetic mean ideals I_a, ${}_aI$, $I^o = ({}_aI)_a$ (the am-interior of I) and $I^- = {}_a(I_a)$ (the am-closure of I). (See Section 2 for definitions.) In particular, the arithmetically mean closed ideals (those equal to their am-closure) played an important role in the study of single commutators in [7].

This paper and [10]-[13] are part of an ongoing program announced in [9] dedicated to the study of arithmetic mean ideals and their applications.

In [10] we investigated the question: "How many traces (up to scalar multiples) can an ideal support?" We found that for the following two classes of ideals which we call "soft" the answer is always zero, one or uncountably many: the soft-edged ideals that coincide with their soft-interior $\operatorname{se} I := IK(H)$ and the soft-complemented ideals that coincide with their soft complement $\operatorname{sc} I := I : K(H)$ ($K(H)$ is the ideal of compact operators on H and for quotients of ideals see Section 3).

Softness properties have often played a role in the theory of operator ideals, albeit that role was mainly implicit and sometimes hidden. Taking the product of I by $K(H)$ corresponds at the sequence level to the "little o" operation, which figures so frequently in operator ideal techniques. M. Wodzicki employs explicitly the notion of soft interior of an ideal (although he does not use this terminology) to investigate obstructions to the existence of positive traces on an ideal (see [22, Lemma 2.15, Corollary 2.17]. A special but important case of quotient is the celebrated Köthe dual of an ideal and general quotients have been studied among others by Salinas [18]. But to the best of our knowledge the power of combining these two soft operations has gone unnoticed along with their investigation and a systematic use of their properties. Doing just that permitted us in [10] to considerably extend and simplify our study of the codimension of commutator spaces.

In particular, we depended in a crucial way on the interplay between soft operations and arithmetic mean operations.

Arithmetic mean operations on ideals were first introduced in [7] and further studied in [10]. For summable sequences, the arithmetic mean must be replaced by the arithmetic mean at infinity (am-∞ for short), see for instance [1, 7, 14, 22]. In [10] we defined am-∞ ideals and found that their theory is in a sense dual to the theory of am-ideals, including the role of ∞-regular sequences studied in [10, Theorem 4.12]). In [10] we considered only the ideals ${}_aI$, I_a, ${}_{a_\infty}I$, and I_{a_∞}, and so in this paper we focus mostly on the other am and am-∞ ideals.

In Section 2 we prove that the sum of two am-closed ideals is am-closed (Theorem 2.5) by using the connection between majorization of infinite sequences and infinite substochastic matrices due to Markus [16]. (Recent outgrowths from [ibid] from the classical theory for finite sequences and stochastic matrices to the infinite is one focus of [11].) This leads naturally to defining a largest am-closed ideal $I_- \subset I$. We prove that $I_- = {}_aI$ for countably generated ideals (Theorem 2.9) while in general the inclusion is proper. An immediate consequence is that a countably generated ideal is am-closed ($I = I^-$) if and only if it is am-stable ($I = I_a$) (Theorem 2.11). This generalizes a result from [2, Theorem 3.11]. Then we prove that arbitrary intersections of am-open ideals must be am-open (Theorem 2.17) by first obtaining a characterization of the am-interior of a principal ideal (Lemma 2.14) and then of an arbitrary ideal (Corollary 2.16). This leads naturally to defining the smallest am-open ideal $I^{oo} \supset I$.

In Section 3 we obtain analogous results for the am-∞ case. But while the statement are similar, the techniques employed in proving them are often substantially different. For instance, the proof that the sum of two am-∞ closed ideals is am-∞ closed (Theorem 3.2) depends on a w^*-compactness argument rather than a matricial one.

In Section 4 we study soft ideals. The soft-interior $\operatorname{se} I$ and the soft-cover $\operatorname{sc} I$ are, respectively, the largest soft-edged ideal contained in I and the smallest soft-complemented ideal containing I. The pair $\operatorname{se} I \subset \operatorname{sc} I$ is the generic example of what we call a soft pair. Many classical ideals, i.e., ideals whose characteristic set is a classical sequence space, turn out to be soft. Among soft-edged ideals are minimal Banach ideals $\mathfrak{S}_\phi^{(o)}$ for a symmetric norming function ϕ, Lorentz ideals $\mathcal{L}(\phi)$, small Orlicz ideals $\mathcal{L}_M^{(o)}$, and idempotent ideals.

To prove soft-complementedness of an ideal we often find it convenient to prove instead a stronger property which we call strong soft-complementedness (Definition 4.4, Proposition 4.5). Among strongly soft complemented ideals are principal and more generally countably generated ideals, maximal Banach ideals ideals $\mathfrak{S}_\phi$, Lorentz ideals $\mathcal{L}(\phi)$, Marcinkiewicz ideals $_a(\xi)$, and Orlicz ideals $\mathcal{L}_M^{(o)}$. Köthe duals and idempotent ideals are always soft-complemented but can fail to be strongly soft-complemented.

Employing the properties of soft pairs for the embedding $\mathfrak{S}_\phi^{(o)} \subset \mathfrak{S}_\phi$ in the principal ideal case, we present a simple proof of the fact that if a principal ideal is a Banach ideal then its generator must be regular, which is due to Allen and Shen [2, Theorem 3.23] and was also obtained by Varga [20] (see Remark 4.8(iv) and [7, Theorem 5.20]). The same property of the embedding yields a simpler proof of part of a result by Salinas in [18, Theorem 2.3]. Several results relating small Orlicz and Orlicz ideals given in theorems in [7] follow immediately from the fact that $\mathcal{L}_M^{(o)} \subset \mathcal{L}_M$ are also soft pairs (see remarks after Proposition 4.11).

Various operations on ideals produce additional soft ideals. Powers of soft-edged ideals, directed unions (by inclusion) of soft-edged ideals, finite intersections and finite sums of soft-edged ideals are all soft-edged. Powers of soft-complemented ideals and arbitrary intersections of soft-complemented ideals are also soft-complemented (Section 4). As consequences follow the softness properties of the am and am-∞ stabilizers of the trace-class $\mathcal{L}_1$ (see Sections 2 (¶4) and 3(¶2) for the definitions) which play an important role in [9]-[10]. However, whether the sum of two soft-complemented ideals or even two strongly soft-complemented ideals is always soft-complemented remains unknown. We prove that it is under the additional hypothesis that one of the ideals is countably generated and the other is strongly soft-complemented (Theorem 5.7).

Some of the commutation relations between the soft-interior and soft-cover operations and the am and am-∞ operations played a key role in [10]. We investigate the commutation relations with the remaining operations in Section 6 (Theorems 6.1, 6.4, 6.9, and 6.10). As a consequence we obtain which operations preserve soft-complementedness and soft-edgedness. Some of the relations remain open, e.g., we do not know if $\operatorname{sc} I_a = (\operatorname{sc} I)_a$ (see Proposition 6.8).

Following this paper in the program outlined in [9] will be [11] where we clarify the interplay between arithmetic mean operations, infinite convexity, and diagonal invariance and [12] where we investigate the lattice properties of several classes of operator ideals proving results of the kind: between two proper ideals, at least one of which is am-stable (resp., am-∞ stable) lies a third am-stable (resp., am-∞ stable) principal ideal and applying them to various arithmetic mean cancellation and inclusion properties (see [9, Theorem 11 and Propositions 12–14]. Example, for which ideals I does the $I_a = J_a$ (resp., $I_a \subset J_a$, $I_a \supset J_a$) imply $I = J$ (resp., $I \subset J$, $I \supset J$) and in the latter cases, is there an "optimal" J?

2. Preliminaries and Arithmetic Mean Ideals

Calkin [5] established a correspondence between *two-sided ideals* of bounded operators on a complex separable infinite dimensional Hilbert space and *characteristic sets*, i.e., hereditary (i.e., solid) cones $\Sigma \subset c_o^*$ (the collection of sequences decreasing to 0), that are invariant under ampliations. For each $m \in \mathbb{N}$, the m-fold ampliation D_m is defined by:

$$c_o^* \ni \xi \longrightarrow D_m\xi := \langle \xi_1, \dots, \xi_1, \xi_2, \dots, \xi_2, \xi_3, \dots, \xi_3, \dots \rangle$$

with each entry ξ_i repeated m times. The Calkin correspondence $I \to \Sigma(I)$ induced by $I \ni X \to s(X) \in \Sigma(I)$, where $s(X)$ denotes the sequence of the s-numbers of X, is a lattice isomorphism between ideals and characteristic sets and its inverse is the map from a characteristic set Σ to the ideal generated by the collection of the diagonal operators $\{\operatorname{diag} \xi \mid \xi \in \Sigma\}$. For a sequence $0 \le \xi \in c_o$, denote by $\xi^* \in c_o^*$ the decreasing rearrangement of ξ, and for each $\xi \in c_o^*$ denote by (ξ) the principal ideal generated by $\operatorname{diag} \xi$, so that $(s(X))$ denotes the principal ideal generated by the operator $X \in K(H)$ (the ideal of compact operators on the Hilbert space H). Recall that $\eta \in \Sigma((\xi))$ precisely when $\eta = O(D_m\xi)$ for some m. Thus the equivalence between ξ and η ($\xi \asymp \eta$ if $\xi = O(\eta)$ and $\eta = O(\xi)$) is only sufficient for $(\xi) = (\eta)$. It is also necessary if one of the two sequences (and hence both) satisfy the $\Delta_{1/2}$-condition. Following the notations of [22], we say that ξ satisfies the $\Delta_{1/2}$-condition if $\sup \frac{\xi_n}{\xi_{2n}} < \infty$, i.e., $D_2\xi = O(\xi)$, which holds if and only if $D_m\xi = O(\xi)$ for all $m \in \mathbb{N}$.

Dykema, Figiel, Weiss and Wodzicki [6, 7] showed that the (Cesaro) *arithmetic mean* plays an essential role in the theory of operator ideals by using it to characterize the normal operators in the commutator space of an ideal. (The commutator space $[I, B(H)]$ of an ideal I, also called the commutator ideal of I, is the span of the commutators of elements of I with elements of $B(H)$). This led them to introduce and study the arithmetic mean and pre-arithmetic mean of an ideal and the consequent notions of am-interior and am-closure of an ideal.

The arithmetic mean of any sequence η is the sequence $\eta_a := \langle \frac{1}{n}\sum_{i=1}^n \eta_i\rangle$. For every ideal I, the pre-arithmetic mean ideal ${}_aI$ and the arithmetic mean ideal I_a are the ideals with characteristic sets

$$\Sigma({}_aI) = \{\xi \in c_o^* \mid \xi_a \in \Sigma(I)\}$$
$$\Sigma(I_a) = \{\xi \in c_o^* \mid \xi = O(\eta) \text{ for some } \eta \in \Sigma(I)\}.$$

A consequence of one of the main results in [7, Theorem 5.6] is that the positive part of the commutator space $[I, B(H)]$ coincides with the positive part of the pre-arithmetic mean ideal ${}_aI$, that is:

$$[I, B(H)]^+ = ({}_aI)^+$$

In particular, ideals that fail to support any nonzero trace, i.e., ideals for which $I = [I, B(H)]$, are precisely those for which $I = {}_aI$ (or, equivalently, $I = I_a$) and are called arithmetically mean stable (am-stable for short). The smallest nonzero am-stable ideal is the upper stabilizer of the trace-class ideal $\mathcal{L}_1$ (in the notations of [7])

$$st^a(\mathcal{L}_1) := \bigcup_{m=0}^{\infty} (\omega)_{a^m} = \bigcup_{m=0}^{\infty} (\omega \log^m)$$

where $\omega = \langle 1/n \rangle$ denotes the harmonic sequence (see [10, Proposition 4.18]). There is no largest proper am-stable ideal. Am-stability for many classical ideals was studied in [7, Sections 5.11–5.27].

Arithmetic mean operations on ideals were introduced in [7, Sections 2.8 and 4.3] and employed, in particular, in the study of single commutators [7, Section 7]: the arithmetic mean closure I^- and the arithmetic mean interior I^o of an ideal I are defined respectively as $I^- := {}_a(I_a)$ and $I^o := ({}_aI)_a$. The following 5-chain inclusion holds:

$$_aI \subset I^o \subset I \subset I^- \subset I_a$$

Ideals that coincide with their am-closure (resp., am-interior) are called am-closed (resp., am-open), and I^- is the smallest am-closed ideal containing I (resp., I^o is the largest am-open ideal contained in I). We list here some of the elementary properties of am-closed and am-open ideals, and since there is a certain symmetry between them, we shall consider both in parallel.

An ideal I is am-closed (resp., am-open) if and only if $I = {}_aJ$ (resp., $I = J_a$) for some ideal J. The necessity follows from the definition of I^- (resp., I^o) and the sufficiency follows from the identities $I_a = ({}_a(I_a))_a$ and ${}_aI = {}_a(({}_aI)_a)$ that are simple consequences of the 5-chain of inclusions listed above.

The characteristic set $\Sigma(\mathcal{L}_1)$ of the trace-class ideal is ℓ_1^*, the collection of monotone nonincreasing nonnegative summable sequences. It is elementary to show $\mathcal{L}_1 = {}_a(\omega)$, $\mathcal{L}_1$ is the smallest nonzero am-closed ideal, $(\omega) = F_a = (\mathcal{L}_1)_a$, and so (ω) is the smallest nonzero am-open ideal (F denotes the finite rank ideal.)

In terms of characteristic sets:

$$\Sigma(I^-) = \{\xi \in c_o^* \mid \xi_a \leq \eta_a \text{ for some } \eta \in \Sigma(I)\}$$

$$\Sigma(I^o) = \{\xi \in c_o^* \mid \xi \leq \eta_a \in \Sigma(I) \text{ for some } \eta \in c_o^*\}$$

Here and throughout, the relation between sequences "$\leq$" denotes pointwise, i.e., for all n. The relation $\xi \prec \eta$ defined by $\xi_a \leq \eta_a$ is called majorization and plays an important role in convexity theory (e.g., see [16, 17]). We will investigate it further in this context in [11] (see also [9]). But for now, notice that I is am-closed if and only if $\Sigma(I)$ is hereditary (i.e., solid) under majorization.

The two main results in this section are that the (finite) sum of am-closed ideals is am-closed and that intersections of am-open ideals are am-open. These will lead to two additional natural arithmetic mean ideal operations, I_- and I^{oo}, see Corollary 2.6 and Definition 2.18.

We start by determining how the arithmetic mean operations distribute with respect to direct unions and intersections of ideals and with respect to finite sums. Recall that the union of a collection of ideals that is directed by inclusion and the intersection of an arbitrary collection of ideals are ideals. The proofs of the following three lemmas are elementary, with the exception of one of the inclusions in Lemma 2.2(iii) which is a simple consequence of Theorem 2.17 below.

Lemma 2.1. *For $\{I_\gamma, \gamma \in \Gamma\}$ a collection of ideals directed by inclusion:*

(i) ${}_a(\bigcup_\gamma I_\gamma) = \bigcup_\gamma {}_a(I_\gamma)$

(ii) $(\bigcup_\gamma I_\gamma)_a = \bigcup_\gamma (I_\gamma)_a$

(iii) $(\bigcup_\gamma I_\gamma)^o = \bigcup_\gamma (I_\gamma)^o$

(iv) $(\bigcup_\gamma I_\gamma)^- = \bigcup_\gamma (I_\gamma)^-$

(v) *If all I_γ are am-stable, (resp., am-open, am-closed) then $\bigcup_\gamma I_\gamma$ is am-stable, (resp., am-open, am-closed).*

Lemma 2.2. *For $\{I_\gamma, \gamma \in \Gamma\}$ a collection of ideals:*

(i) ${}_a(\bigcap_\gamma I_\gamma) = \bigcap_\gamma {}_a(I_\gamma)$

(ii) $(\bigcap_\gamma I_\gamma)_a \subset \bigcap_\gamma (I_\gamma)_a$ *(inclusion can be proper by Example 2.4(i))*

(iii) $(\bigcap_\gamma I_\gamma)^o = \bigcap_\gamma (I_\gamma)^o$ *(equality holds by Theorem 2.17)*

(iv) $(\bigcap_\gamma I_\gamma)^- \subset \bigcap_\gamma (I_\gamma)^-$ *(inclusion can be proper by Example 2.4(i))*

(v) *If all I_γ are am-stable, (resp., am-open, am-closed) then $\bigcap_\gamma I_\gamma$ is am-stable, (resp., am-open, am-closed).*

Lemma 2.3. *For all ideals I, J:*

(i) $I_a + J_a = (I + J)_a$

(ii) ${}_aI + {}_aJ \subset {}_a(I + J)$ *(the inclusion can be proper by Example 2.4(ii))*

(iii) $I^o + J^o \subset (I + J)^o$ *(the inclusion can be proper by Example 2.4(ii))*

(iv) $I^- + J^- \subset (I + J)^-$ *(equality is Theorem 2.5)*

(v) *If I and J are am-open, so is $I + J$.*

Example 2.4. (i) In general, equality does not hold in Lemma 2.2(ii) or, equivalently, in (iv) even when Γ is finite. Indeed it is easy to construct two nonsummable sequences ξ and η in c_o^* such that $\min(\xi, \eta)$ is summable. But then, as it is elementary to show, $(\xi) \cap (\eta) = (\min(\xi, \eta))$ and hence $((\xi) \cap (\eta))_a = (\omega)$ while $(\xi)_a \cap (\eta)_a = (\xi_a) \cap (\eta_a) = (\min(\xi_a, \eta_a)) \supsetneqq (\omega)$, the inclusion since $\omega = o(\xi_a)$, $\omega = o(\eta_a)$, hence $\omega = o(\min(\xi_a, \eta_a))$, and the inequality since ω satisfies the $\Delta_{1/2}$-condition and then equality leads to a contradiction.

(ii) In general, equality does not hold in Lemma 2.3(ii) or (iii). Indeed take the principal ideals generated by two sequences ξ and η in c_o^* such that $\xi + \eta = \omega$ but $\omega \neq O(\xi)$ and $\omega \neq O(\eta)$, which implies that

$$ {}_a(\xi) = {}_a(\eta) = \{0\} \neq \mathcal{L}_1 = {}_a(\omega) = {}_a((\xi) + (\eta)). $$

The same example shows that

$$ (\xi)^o + (\eta)^o = \{0\} \neq (\omega) = ((\xi) + (\eta))^o. $$

That the sum of finitely many am-open ideals is am-open (Lemma 2.3(v)), is an immediate consequence of Lemma 2.3(iii). Less trivial is the fact that the sum of finitely many am-closed ideals is am-closed, or, equivalently, that equality holds in Lemma 2.3(iv). This result was announced in [9]. The proof we present here exploits the role of substochastic matrices in majorization theory ([16], see also [11]). Recall that a matrix P is called substochastic if $P_{ij} \geq 0$, $\sum_{i=1}^\infty P_{ij} \leq 1$ for all j and $\sum_{j=1}^\infty P_{ij} \leq 1$ for all i. By extending the well-known result for finite sequence majorization (e.g., see [17]), Markus showed in [16, Lemma 3.1] that if η, $\xi \in c_o^*$, then $\eta_a \leq \xi_a$ if and only if there is a substochastic matrix P such that $\eta = P\xi$. Finally, recall also the Calkin [5] isomorphism between proper two sided ideals of $B(H)$ and ideals of ℓ_∞ that associates to an ideal J the symmetric sequence space $S(J)$ defined by $S(J) := \{\eta \in c_o \mid \operatorname{diag} \eta \in J\}$ (e.g., see [5] or [7, Introduction]). It is immediate to see that $S(J) = \{\eta \in c_o \mid |\eta|^* \in \Sigma(J)\}$ and that for any two ideals, $S(I + J) = S(I) + S(J)$.

Theorem 2.5. *$(I + J)^- = I^- + J^-$ for all ideals I, J.*

In particular, the sum of two am-closed ideals is am-closed.

Proof. The inclusion $I^- + J^- \subset (I+J)^-$ is elementary and was stated in Lemma 2.3(iv). Let $\xi \in \Sigma((I+J)^-)$, then $\xi_a \in \Sigma((I+J)_a)$ so that $\xi_a \leq (\rho+\eta)_a$ for some $\rho \in \Sigma(I)$ and $\eta \in \Sigma(J)$. Then by Markus' lemma [16, Lemma 3.1], there is a substochastic matrix P such that $\xi = P(\rho+\eta)$. Let Π be a permutation matrix monotonizing $P\rho$, i.e., $(P\rho)^* = \Pi P\rho$, then ΠP too is substochastic and hence by the same result, $((P\rho)^*)_a \leq \rho_a$, i.e., $(P\rho)^* \in \Sigma(I^-)$, or equivalently, $P\rho \in S(I^-)$. Likewise, $P\eta \in S(J^-)$, whence $\xi \in S(I^-) + S(J^-) = S(I^- + J^-)$ and hence $\xi \in \Sigma(I^- + J^-)$. Thus $(I+J)^- \subset I^- + J^-$, concluding the proof. $\square$

As a consequence, the collection of all the am-closed ideals contained in an ideal I is directed and hence its union is an am-closed ideal by Lemma 2.1(v).

Corollary 2.6. *Every ideal I contains a largest am-closed ideal denoted by I_-, which is given by*

$$I_- := \bigcup\{J \mid J \subset I \textit{ and } J \textit{ is am-closed}\}.$$

Thus $I_- \subset I \subset I^-$ and I is am-closed if and only if one of the inclusions and hence both of them are equalities. Since ${}_aI \subset I$ and ${}_aI$ is am closed, ${}_aI \subset I_-$. The inclusion can be proper as seen by considering any am-closed but not am-stable ideal I, e.g, $I = \mathcal{L}_1$ where ${}_a(\mathcal{L}_1) = \{0\}$. If equality holds, we have the following equivalences:

Lemma 2.7. *For every ideal I, the following conditions are equivalent.*

(i) $I_- = {}_aI$

(ii) *If $J^- \subset I$ for some ideal J, then $J^- \subset {}_aI$.*

(iii) *If $J^- \subset I$ for some ideal J, then $J_a \subset I$.*

(iv) *If ${}_aJ \subset I$ for some ideal J, then $J^o \subset I$.*

We leave the proof to the reader. Notice that the converses (ii)–(iv) hold trivially for any pair of ideals I and J.

Theorem 2.9 below will show that for countably generated ideals the equality ${}_aI = I_-$ always holds, i.e., ${}_aI$ is the largest am-closed ideal contained in I.

We first need the following lemma.

Lemma 2.8. *If I is a countably generated ideal and $\mathcal{L}_1 \subset I$, then $(\omega) \subset I$. In particular, (ω) is the smallest principal ideal containing $\mathcal{L}_1$.*

Proof. Let $\rho^{(k)}$ be a sequence of generators for the characteristic set $\Sigma(I)$, i.e., for every $\xi \in \Sigma(I)$ there are $m, k \in \mathbb{N}$ for which $\xi = O(D_m\rho^{(k)})$. By adding if necessary to this sequence of generators all their ampliations and then by passing to the sequence $\rho^{(1)} + \rho^{(2)} + \cdots + \rho^{(k)}$, we can assume that $\rho^{(k)} \leq \rho^{(k+1)}$ and that then $\xi \in \Sigma(I)$ if and only if $\xi = O(\rho^{(m)})$ for some $m \in \mathbb{N}$. Thus if $\omega \notin \Sigma(I)$ there is an increasing sequence of indices n_k such that $(\frac{\omega}{\rho^{(k)}})_{n_k} \geq k^3$ for all $k \geq 1$. Set $n_o := 0$ and define $\xi_j := \frac{1}{k^2 n_k}$ for $n_{k-1} < j \leq n_k$ and $k \geq 1$. Then it is immediate that $\xi \in \ell_1^*$. On the other hand, $\xi \neq O(\rho^{(m)})$ for any $m \in \mathbb{N}$ since for every $k \geq m$,

$$\left(\frac{\xi}{\rho^{(m)}}\right)_{n_k} \geq \left(\frac{\xi}{\rho^{(k)}}\right)_{n_k} = \frac{1}{k^2 n_k \rho^{(k)}_{n_k}} \geq k.$$

and hence $\xi \notin \Sigma(I)$, against the hypothesis $\mathcal{L}_1 \subset I$. □

Theorem 2.9. *If I is a countably generated ideal, then $I_- = {}_aI$.*

Proof. Let $\eta \in \Sigma(I_-)$. Then $(\eta)^- \subset I_- \subset I$. We claim that $\eta_a \in \Sigma(I)$, i.e., $I_- \subset {}_aI$ and hence equality holds from the maximality of I_-. If $0 \neq \eta \in \ell_1^*$, then $(\eta)^- = \mathcal{L}_1$, hence by Lemma 2.8, $(\omega) \subset I$ and thus $\eta_a \asymp \omega \in \Sigma(I)$. If $\eta \notin \ell_1^*$, assume by contradiction that $\eta_a \notin \Sigma(I)$. As in the proof of Lemma 2.8, choose a sequence of generators $\rho^{(k)}$ for $\Sigma(I)$ with $\rho^{(k)} \leq \rho^{(k+1)}$ and such that for every $\xi \in \Sigma(I)$ there is an $m \in \mathbb{N}$ for which $\xi = O(\rho^{(m)})$. Then there is an increasing sequence of indices n_k such that $(\frac{\eta_a}{\rho^{(k)}})_{n_k} \geq k$ for every k. Exploiting the non-summability of η we can further require that $\frac{1}{n_k - n_{k-1}} \sum_{i=n_{k-1}+1}^{n_k} \eta_i \geq \frac{1}{2}(\eta_a)_{n_k}$ for every k. Set $n_o := 0$ and define $\xi_j = (\eta_a)_{n_k}$ for $n_{k-1} < j \leq n_k$. We prove by induction that $(\xi_a)_j \leq (2\eta_a)_j$. The inequality holds trivially for $j \leq n_1$ and assume it holds also for all $j \le n_{k-1}$. If $n_{k-1} < j \le n_k$, it follows that

$$\begin{aligned}
\sum_{i=1}^{j} \xi_i &= n_{k-1}(\xi_a) n_{k-1} + (j - n_{k-1})(\eta_a)_{n_k} \\
&\le 2n_{k-1}(\eta_a)_{n_{k-1}} + (j - n_{k-1})(\eta_a)_{n_k} \\
&\le 2\sum_{i=1}^{n_{k-1}} \eta_i + 2\frac{j-n_{k-1}}{n_k - n_{k-1}} \sum_{i=n_{k-1}+1}^{n_k} \eta_i \\
&\le 2\sum_{i=1}^{n_{k-1}} \eta_i + 2\sum_{i=n_{k-1}+1}^{j} \eta_i = 2j(\eta_a)_j
\end{aligned}$$

where the last inequality follows because $\frac{1}{j-n}\sum_{i=n+1}^{j} \eta_i$ is monotone nonincreasing in j for $j > n$. Thus $\xi \in \Sigma((\eta)^-) \subset \Sigma(I)$. On the other hand, for every $m \in \mathbb{N}$ and $k \ge m$, $(\frac{\xi}{\rho^{(m)}})_{n_k} \ge (\frac{\xi}{\rho^{(k)}})_{n_k} = (\frac{\eta_a}{\rho^{(k)}})_{n_k} \ge k$ and thus $\xi \notin \Sigma(I)$, a contradiction. $\square$

By Theorem 2.5, $I_- + J_-$ is am-closed for any pair of ideal I and J and it is contained in $I + J$. Hence $I_- + J_- \subset (I + J)_-$ and this inclusion can be proper by Theorem 2.9 and Example 2.4(ii).

Corollary 2.10. *If I is a countably generated ideal, then I_a is the smallest countably generated ideal containing I^-.*

Proof. By the five chain inclusion, $I^- \subset I_a$ and if $I^- \subset J$ for some countably generated ideal J, then $I^- \subset J_- = {}_aJ$ and hence $I_a = (I^-)_a \subset J^o \subset J$. $\square$

As a consequence of Theorem 2.9 we obtain also an elementary proof of the following, which was obtained for the principal ideal case by [2, Theorem 3.11].

Theorem 2.11. *A countably generated ideal is am-closed if and only if it is am-stable.*

Proof. If I is a countably generated am-closed ideal, then $I = I_-$ and hence $I = {}_aI$ by Theorem 2.9, i.e., I is am-stable. On the other hand, every am-stable ideal is am-closed by the five chain inclusion. $\square$

$\mathcal{L}_1$ is an example of a non countably generated ideal which is am-closed (and also am-∞ closed) but is neither am-stable nor am-∞ stable.

Now we pass to the second main result of this section, namely that the intersection of am-open ideals is am-open (Theorem 2.17). To prove it and to provide tools for our study in Section 6 of the commutation relations between the se and sc operations and the am-interior operation, we need the characterization of the am-interior I^o of an ideal I given in Corollary 2.16 below. This in turn will lead naturally to a characterization of the smallest am-open ideal I^{oo} containing I (Definition 2.18 and Proposition 2.21). Both characterizations depend on the principal ideal case.

As recalled earlier, an ideal I is am-open if $I = J_a$ for some ideal J (e.g., $J = I^- = {}_a(I_a)$). In terms of sequences, I is am-open if and only if for every $\xi \in \Sigma(I)$, one has $\xi \leq \eta_a \in \Sigma(I)$ for some $\eta \in c_o^*$. Remark 2.15(iii) show that there is a minimal $\eta_a \geq \xi$. First we note when a sequence is equal to the arithmetic mean of a c_o^*-sequence. The proof is elementary and is left to the reader.

Lemma 2.12. *A sequence ξ is the arithmetic mean η_a of some sequence $\eta \in c_o^*$ if and only if $0 \leq \xi \to 0$ and $\frac{\xi}{\omega}$ is monotone nondecreasing and concave, i.e., $(\frac{\xi}{\omega})_{n+1} \geq \frac{1}{2}((\frac{\xi}{\omega})_n + (\frac{\xi}{\omega})_{n+2})$ for all $n \in \mathbb{N}$ and $\xi_1 = (\frac{\xi}{\omega})_1 \geq \frac{1}{2}(\frac{\xi}{\omega})_2$.*

It is elementary to see that for every $\eta \in c_o^*$, $(\eta)_a = (\eta_a)$ and that η_a satisfies the $\Delta_{1/2}$-condition because $\eta_a \leq D_m\eta_a \leq m\eta_a$ for every $m \in \mathbb{N}$. In particular, all the generators of the principal ideal (η_a) are equivalent.

Lemma 2.13. *If I is a principal ideal, then the following are equivalent.*

(i) *I is am-open*

(ii) *$I = (\eta_a)$ for some $\eta \in c_o^*$*

(iii) *$I = (\xi)$ for some $\xi \in c_o^*$ for which $\frac{\xi}{\omega}$ is monotone nondecreasing.*

Proof. (i) $\Leftrightarrow$ (ii). Assume that $I = (\xi)$ for some $\xi \in c_o^*$ and that I is am-open and hence $I = J_a$ for some ideal J. Then $\xi \leq \eta_a$ for some $\eta_a \in \Sigma(I)$ and hence $\eta_a \leq MD_m\xi$ for some $M > 0$ and $m \in \mathbb{N}$. Since $\eta_a \asymp D_m\eta_a$, it follows that $\xi \asymp \eta_a$ and hence (ii) holds. The converse holds since $(\eta_a) = (\eta)_a$.

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (ii). $\frac{\xi}{\omega}$ is quasiconcave, i.e., $\frac{\xi}{\omega}$ is monotone nondecreasing and $\omega\frac{\xi}{\omega}$ is monotone nonincreasing. Adapting to sequences the proof of Proposition 5.10

in Chapter 2 of [4] (see also [7, Section 2.18]) shows that if ψ is the smallest concave sequence that majorizes $\frac{\xi}{\omega}$, then $\frac{\xi}{\omega} \leq \psi \leq 2\frac{\xi}{\omega}$ and hence $\psi \asymp \frac{\xi}{\omega}$. Moreover, $\psi_1 = \xi_1 = (\frac{\xi}{\omega})_1$ since otherwise we could lower ψ_1 and still maintain the concavity of ψ. And since the sequence $\frac{\xi_1}{\omega}$ is concave and $\frac{\xi_1}{\omega} \geq \frac{\xi}{\omega}$, it follows by the minimality of ψ that $\psi \leq \frac{\xi_1}{\omega}$ and so, in particular, $\psi_1 = \xi_1 \geq \frac{1}{2}\psi_2$. Since ψ is concave and nonnegative, it follows that it is monotone nondecreasing. But then, by Lemma 2.12 applied to $\omega\psi$, one has $\omega\psi = \eta_a$ for some sequence $\eta \in c_o^*$ and thus $(\xi) = (\omega\psi) = (\eta_a)$. $\square$

We need now the following notations from [7, Section 2.3]. The upper and lower monotone nondecreasing and monotone nonincreasing envelopes of a real-valued sequence ϕ are:

$$\operatorname{und}\phi := \Big\langle \max_{i\leq n} \phi_i \Big\rangle, \quad \operatorname{lnd}\phi := \Big\langle \inf_{i\geq n} \phi_i \Big\rangle, \quad \operatorname{uni}\phi := \Big\langle \sup_{i\geq n} \phi_i \Big\rangle, \quad \operatorname{lni}\phi := \Big\langle \min_{i\leq n} \phi_i \Big\rangle.$$

Lemma 2.14. *For every $\xi \in c_o^*$:*
(i) $(\xi)^o = (\omega \operatorname{lnd} \frac{\xi}{\omega})$
(ii) $(\omega \operatorname{und} \frac{\xi}{\omega})$ *is the smallest am-open ideal containing* (ξ).

Proof. (i) We first prove that $\omega \operatorname{lnd} \frac{\xi}{\omega}$ is monotone nonincreasing. Indeed, in the case when $(\operatorname{lnd} \frac{\xi}{\omega})_n = (\operatorname{lnd} \frac{\xi}{\omega})_{n+1}$, then $(\omega \operatorname{lnd} \frac{\xi}{\omega})_{n+1} \leq (\omega \operatorname{lnd} \frac{\xi}{\omega})_n$, but if on the other hand $(\operatorname{lnd} \frac{\xi}{\omega})_n \neq (\operatorname{lnd} \frac{\xi}{\omega})_{n+1}$, then $(\operatorname{lnd} \frac{\xi}{\omega})_n = (\frac{\xi}{\omega})_n$ and hence also $(\omega \operatorname{lnd} \frac{\xi}{\omega})_{n+1} \leq \xi_{n+1} \leq \xi_n = (\omega \operatorname{lnd} \frac{\xi}{\omega})_n$. Moreover, $\omega \operatorname{lnd} \frac{\xi}{\omega} \to 0$ since $\omega \operatorname{lnd} \frac{\xi}{\omega} \leq \xi$. Thus $(\omega \operatorname{lnd} \frac{\xi}{\omega}) \subset (\xi)$. By Lemma 2.13(i) and (iii), $(\omega \operatorname{lnd} \frac{\xi}{\omega})$ is am-open and hence $(\omega \operatorname{lnd} \frac{\xi}{\omega}) \subset (\xi)^o$. For the reverse inclusion, if $\mu \in \Sigma((\xi)^o)$, then $\mu \leq \zeta_a$ for some $\zeta_a \in \Sigma(\xi)$, i.e., $\zeta_a \leq MD_m\xi$ for some $M > 0$ and $m \in \mathbb{N}$. Then $D_m\zeta_a \leq m\zeta_a \leq mMD_m\xi$, whence $\frac{\zeta_a}{\omega} \leq mM\frac{\xi}{\omega}$. As $\frac{\zeta_a}{\omega}$ is monotone nondecreasing, also $\frac{\zeta_a}{\omega} \leq mM \operatorname{lnd} \frac{\xi}{\omega}$ so that $\mu \leq mM\omega \operatorname{lnd} \frac{\xi}{\omega}$. Thus $(\xi)^o \subset (\omega \operatorname{lnd} \frac{\xi}{\omega})$.

(ii) A similar proof as in (i) shows that $\omega \operatorname{und} \frac{\xi}{\omega} \in c_o^*$. Since by definition $\xi \leq \omega \operatorname{und} \frac{\xi}{\omega}$, we have that $(\xi) \subset (\omega \operatorname{und} \frac{\xi}{\omega})$, and the latter ideal is am-open by Lemma 2.13. If I is any am-open ideal containing (ξ), then $\xi \leq \zeta_a$ for some $\zeta_a \in \Sigma(I)$ and again, since $\frac{\zeta_a}{\omega}$ is monotone nondecreasing, $\omega \operatorname{und} \frac{\xi}{\omega} \leq \zeta_a$, hence $(\omega \operatorname{und} \frac{\xi}{\omega}) \subset I$. $\square$

Remark 2.15. (i) Lemma 2.14(i) shows that the am-interior $(\xi)^o$ of a principal ideal (ξ) is always principal and its generator $\omega \operatorname{lnd} \frac{\xi}{\omega}$ is unique up to equivalence by Lemma 2.13 and preceding remarks. Notice that (ω) being the smallest nonzero am-open ideal, $(\xi)^o = \{0\}$ if and only if $(\omega) \not\subset (\xi)$. In terms of sequences, this corresponds to the fact that that $\operatorname{lnd} \frac{\xi}{\omega} = 0$ if and only if $(\omega) \not\subset (\xi)$.

(ii) While $(\omega \operatorname{lnd} \frac{\xi}{\omega})$ is the largest am-open ideal contained in (ξ) by Lemma 2.14(i), it is easy to see that there is no (pointwise) nonzero largest arithmetic mean sequence majorized by ξ unless ξ is itself an arithmetic mean. However, there is an arithmetic mean sequence η_a majorized by ξ which is the largest in the O-sense (actually up to a factor of 2). Indeed, let ψ be the smallest concave sequence that majorizes the quasiconcave sequence $\frac{1}{2} \operatorname{lnd} \frac{\xi}{\omega}$. Then, as in the proof of Lemma 2.13(iii) $\Rightarrow$ (ii), $\psi = \frac{\eta_a}{\omega}$ for some $\eta \in c_o^*$ and $\psi \leq \operatorname{lnd} \frac{\xi}{\omega}$ and hence $\eta_a \leq \xi$. Moreover, for every $\rho \in c_o^*$ with $\rho_a \leq \xi$, since $\frac{\rho_a}{\omega}$ is monotone nondecreasing, it follows that $\frac{\rho_a}{\omega} \leq \operatorname{lnd} \frac{\xi}{\omega} \leq 2\frac{\eta_a}{\omega}$ and hence $\rho_a \leq 2\eta_a$.

(iii) Lemma 2.14(ii) shows that $(\omega \operatorname{und} \frac{\xi}{\omega})$ is the smallest am-open ideal containing (ξ), and moreover, from the proof of Lemma 2.13(iii) we see that $(\omega \operatorname{und} \frac{\xi}{\omega}) = (\eta_a)$ where $\frac{\eta_a}{\omega}$ is the smallest concave sequence that majorizes the quasiconcave sequence $\operatorname{und} \frac{\xi}{\omega}$. In contrast to (ii), η_a is also the (pointwise) smallest arithmetic mean that majorizes ξ. Indeed, if $\rho_a \geq \xi$ then $\frac{\rho_a}{\omega} \geq \operatorname{und} \frac{\xi}{\omega}$ because $\frac{\rho_a}{\omega}$ is monotone nondecreasing and moreover $\frac{\rho_a}{\omega} \geq \frac{\eta_a}{\omega}$ because $\frac{\rho_a}{\omega}$ is concave.

(iv) By [7, Section 2.33], $\omega \operatorname{lnd} \frac{\xi}{\omega}$ is the reciprocal of the fundamental sequence of the Marcinkiewicz norm for ${}_a(\xi)$.

Corollary 2.16. *For every ideal I:*

(i) $\Sigma(I^o) = \{\xi \in c_o^* \mid \omega \operatorname{und} \frac{\xi}{\omega} \in \Sigma(I)\} = \{\xi \in c_o^* \mid \xi \leq \omega \operatorname{lnd} \frac{\eta}{\omega}$ *for some* $\eta \in \Sigma(I)\}$.

(ii) *If I is an am-open ideal, then $\xi \in \Sigma(I)$ if and only if $\omega \operatorname{und} \frac{\xi}{\omega} \in \Sigma(I)$.*

Proof. If $\xi \in \Sigma(I^o)$, then $(\xi) \subset (\omega \operatorname{und} \frac{\xi}{\omega}) \subset I^o$ by Lemma 2.14(ii), whence $\omega \operatorname{und} \frac{\xi}{\omega} \in \Sigma(I)$. If $\omega \operatorname{und} \frac{\xi}{\omega} \in \Sigma(I)$, then $\xi \leq \omega \operatorname{und} \frac{\xi}{\omega} = \omega \operatorname{lnd}(\frac{\omega \operatorname{und} \frac{\xi}{\omega}}{\omega})$. Finally, if $\xi \leq \omega \operatorname{lnd} \frac{\eta}{\omega}$ for some $\eta \in \Sigma(I)$, then $\omega \operatorname{lnd} \frac{\eta}{\omega} \in \Sigma((\eta)^o) \subset \Sigma(I^o)$ by Lemma 2.14(i) and hence $\xi \in \Sigma(I^o)$. Thus all three sets are equal. This proves (i) and (ii) is a particular case. $\square$

An immediate consequence of Corollary 2.16(ii) is the following result.

Theorem 2.17. *Intersections of am-open ideals are am-open.*

Since $I \subset I_a$, the collection of all am-open ideals containing I is always nonempty. By Theorem 2.17 its intersection is am-open, hence it is the smallest am-open ideal containing I.

Definition 2.18. For each ideal I, denote $I^{oo} := \bigcap\{J \mid J \supset I \text{ and } J \text{ is am-open}\}$.

Remark 2.19. Lemma 2.14 affirms that if I is principal, so are I^o and I^{oo}.

Notice that $I^o \subset I \subset I^{oo}$ and I is am-open if and only if one of the inclusions and hence both of them are equalities. Since $I \subset I_a$ and I_a is am-open, $I^{oo} \subset I_a$. The inclusion can be proper even for principal ideals. Indeed if $\xi \in c_o^*$ and ξ_a is irregular, i.e., $\xi_{a^2} \neq O(\xi_a)$, then $I = (\xi_a)$ is am-open and hence $I = I^{oo}$, but $I_a = (\xi_{a^2}) \neq (\xi_a) = I^{oo}$. Of course, if I is am-stable then $I = I^{oo} = I_a$, and if $\{0\} \neq I \subset \mathcal{L}_1$ then $(\omega) = I^{oo} = I_a$, but as the following example shows, the equality $I^{oo} = I_a$ can hold also in other cases.

Example 2.20. Let $\xi_j = \frac{1}{k!}$ for $((k-1)!)^2 < j \leq (k!)^2$. Then direct computations show that ξ is irregular, indeed does not even satisfy the $\Delta_{1/2}$-condition, is not summable, but $\xi_a = O(\omega \operatorname{und} \frac{\xi}{\omega})$ and hence by Lemma 2.14 (ii), $(\xi)^{oo} = (\xi)_a$.

The characterization of $I^{oo} = (\omega \operatorname{und} \frac{\eta}{\omega})$ provided by Lemma 2.14(ii) for principal ideals $I = (\xi)$ extends to general ideals.

Proposition 2.21. *For every ideal I, the characteristic set of I^{oo} is given by*

$$\Sigma(I^{oo}) = \left\{\xi \in c_o^* \mid \xi \leq \omega \operatorname{und} \frac{\eta}{\omega} \text{ for some } \eta \in \Sigma(I)\right\}.$$

Proof. Let $\Sigma = \{\xi \in c_o^* \mid \xi \leq \omega \operatorname{und} \frac{\eta}{\omega}$ for some $\eta \in \Sigma(I)\}$. First we show that Σ is a characteristic set. Let $\xi, \rho \in \Sigma$, i.e., $\xi \leq \omega \operatorname{und} \frac{\eta}{\omega}$ and $\rho \leq \omega \operatorname{und} \frac{\mu}{\omega}$ for some $\eta, \mu \in \Sigma(I)$. Since $\omega \operatorname{und} \frac{\eta}{\omega} + \omega \operatorname{und} \frac{\mu}{\omega} \leq 2\omega \frac{\eta+\mu}{\omega}$ and $\eta + \mu \in \Sigma(I)$, it follows that $\xi + \rho \in \Sigma$. Moreover, if $\xi \leq \omega \operatorname{und} \frac{\eta}{\omega}$, then for all m,

$$D_m\xi \leq D_m\omega D_m \operatorname{und} \frac{\eta}{\omega} = D_m\omega \operatorname{und} D_m\frac{\eta}{\omega} \leq m\omega \operatorname{und} \frac{D_m\eta}{\omega}$$

and hence $D_m\xi \in \Sigma$, i.e., Σ is closed under ampliations. Clearly, Σ is also closed under multiplication by positive scalars and it is hereditary. Thus Σ is a characteristic set and hence $\Sigma = \Sigma(J)$ for some ideal J. Then $J \supset I$ follows from the inequality $\xi \leq \omega \,\mathrm{und}\, \frac{\xi}{\omega}$. If $\eta \in \Sigma(J)$, i.e., $\eta \leq \omega \,\mathrm{und}\, \frac{\xi}{\omega}$ for some $\xi \in \Sigma(I)$, then also $\omega \,\mathrm{und}\, \frac{\eta}{\omega} \leq \omega \,\mathrm{und}\, \frac{\xi}{\omega}$ and hence $\omega \,\mathrm{und}\, \frac{\eta}{\omega} \in \Sigma(J)$. By Corollary 2.16, this implies that J is am-open and hence $J \supset I^{oo}$. For the reverse inclusion, if $\eta \in \Sigma(J)$, i.e., $\eta \leq \omega \,\mathrm{und}\, \frac{\xi}{\omega}$ for some $\xi \in \Sigma(I)$, then $\omega \,\mathrm{und}\, \frac{\xi}{\omega} \in \Sigma((\xi)^{oo}) \subset \Sigma(I^{oo})$ by Lemma 2.14(ii). Thus $\eta \in \Sigma(I^{oo})$, hence $J \subset I^{oo}$, and we have equality. □

As a consequence of this proposition and by the subadditivity of "und", we see that $(I + J)^{oo} = I^{oo} + J^{oo}$ for any two ideals I and J.

For completeness' sake we collect in the following lemma the distributivity properties of the I^{oo} and I_- operations.

Lemma 2.22. *For all ideals I, J:*

(i) $I^{oo} + J^{oo} = (I + J)^{oo}$ *(paragraph after Proposition 2.21)*

(ii) $I_- + J_- \subset (I + J)_-$ *and the inclusion can be proper (remarks after Theorem 2.9).*

Let $\{I_\gamma, \gamma \in \Gamma\}$ be a collection of ideals. Then

(iii) $(\bigcap_\gamma I_\gamma)^{oo} \subset \bigcap_\gamma (I_\gamma)^{oo}$ *(the inclusion can be proper by Example 2.23(i))*

(iv) $(\bigcap_\gamma I_\gamma)_- = \bigcap_\gamma (I_\gamma)_-$ *(by Lemma 2.2(v))*

If $\{I_\gamma, \gamma \in \Gamma\}$ is directed by inclusion, then

(v) $(\bigcup_\gamma I_\gamma)^{oo} = \bigcup_\gamma (I_\gamma)^{oo}$ *(by Lemma 2.1(v))*

(vi) $(\bigcup_\gamma I_\gamma)_- \supset \bigcup_\gamma (I_\gamma)_-$ *(the inclusion can be proper by Example 2.23(ii))*

Example 2.23.

(i) The inclusion in (iii) can be proper even if Γ is finite. Indeed for the same construction as in Example 2.4(i), $((\xi) \cap (\eta))^{oo} = (\min(\xi, \eta))^{oo} = (\omega)$ since $\min(\xi, \eta)$ is summable, while $\omega = o(\omega \,\mathrm{und}\, \frac{\xi}{\omega})$ and $\omega = o(\omega \,\mathrm{und}\, \frac{\eta}{\omega})$ since ξ and η are not summable. Thus $\omega = o(\min(\omega \,\mathrm{und}\, \frac{\xi}{\omega}, \omega \,\mathrm{und}\, \frac{\eta}{\omega}))$ and hence

$$(\omega) \not\subset \left(\omega \,\mathrm{und}\, \frac{\xi}{\omega}\right) \cap \left(\omega \,\mathrm{und}\, \frac{\eta}{\omega}\right) = (\xi)^{oo} \cap (\eta)^{oo}$$

(ii) The inclusion in (vi) can be proper. $\mathcal{L}_1$ as every ideal with the exception of $\{0\}$ and F, is the directed union of distinct ideals I_γ. Since $\mathcal{L}_1$ is the smallest am-closed ideal, $(I_\gamma)_- = \{0\}$ for every γ. Thus $\mathcal{L}_1 = (\bigcup_\gamma I_\gamma)_-$ while $\bigcup_\gamma (I_\gamma)_- = \{0\}$.

3. Arithmetic Mean Ideals at Infinity

The arithmetic mean is not adequate for distinguishing between nonzero ideals contained in the trace-class since they all have the same arithmetic mean (ω) and the same pre-arithmetic mean $\{0\}$. The "right" tool for ideals in the trace-class is the arithmetic mean at infinity which was employed for sequences in [1, 7, 14, 22] among others. For every summable sequence η,

$$\eta_{a_\infty} := \langle \frac{1}{n} \sum_{n+1}^{\infty} \eta_j \rangle.$$

Many of the properties of the arithmetic mean and of the am-ideals have a dual form for the arithmetic mean at infinity but there are also substantial differences often linked to the fact that contrary to the am-case, the arithmetic mean at infinity ξ_{a_∞} of a sequence $\xi \in \ell_1^*$ may fail to satisfy the $\Delta_{1/2}$ condition and also may fail to majorize ξ (in fact, ξ_{a_∞} satisfies the $\Delta_{1/2}$ condition if and only if $\xi = O(\xi_{a_\infty})$, see [10, Corollary 4.4]). Consequently the results and proofs tend to be harder.

In [10] we defined for every ideal $I \neq \{0\}$ the am-∞ ideals ${}_{a_\infty}I$ (pre-arithmetic mean at infinity) and I_{a_∞} (arithmetic mean at infinity) with characteristic sets:

$$\Sigma({}_{a_\infty}I) = \{\xi \in \ell_1^* \mid \xi_{a_\infty} \in \Sigma(I)\}$$

$$\Sigma(I_{a_\infty}) = \{\xi \in c_o^* \mid \xi = O(\eta_{a_\infty}) \text{ for some } \eta \in \Sigma(I \cap \mathcal{L}_1)\}$$

Notice that $\xi_{a_\infty} = o(\omega)$ for all $\xi \in \ell_1^*$. Let $\mathrm{se}(\omega)$ denote the ideal with characteristic set $\{\xi \in c_o^* \mid \xi = o(\omega)\}$ (see Definition 4.1 for the soft-interior $\mathrm{se}\, I$ of a general ideal I). Thus

$${}_{a_\infty}I = {}_{a_\infty}(I \cap \mathrm{se}(\omega)) \subset \mathcal{L}_1 \qquad \text{and} \qquad I_{a_\infty} = (I \cap \mathcal{L}_1)_{a_\infty} \subset \mathrm{se}(\omega).$$

In [10, Corollary 4.10] we defined an ideal I to be am-∞ stable if $I = {}_{a_\infty}I$ (or, equivalently, if $I \subset \mathcal{L}_1$ and $I = I_{a_\infty}$). There is a largest am-∞ stable ideal, namely the lower stabilizer at infinity of $\mathcal{L}_1$, $st_{a_\infty}(\mathcal{L}_1) = \bigcap_{n=0}^{\infty} {}_{a_\infty^n}(\mathcal{L}_1)$, which together with the smallest nonzero am-stable ideal $st^a(\mathcal{L}_1)$ defined earlier plays an important role in [10].

Natural analogs to the am-interior and am-closure are the am-∞ interior of an ideal I

$$I^{o\infty} := ({}_{a_\infty}I)_{a_\infty} = (I \cap \mathrm{se}(\omega))^{o\infty}$$

and the am-∞ closure of an ideal I

$$I^{-\infty} := {}_{a_\infty}(I_{a_\infty}) = (I \cap \mathcal{L}_1)^{-\infty}.$$

We call an ideal I am-∞ open (resp., am-∞ closed) if $I = I^{o\infty}$ (resp., $I = I^{-\infty}$).

In [10, Proposition 4.8] we proved the analogs of the 5-chain of inclusions for am-ideals (see Section 2 paragraph 5 and [10, Section 2]):

$${}_{a_\infty}I \subset I^{o\infty} \subset I \cap \mathrm{se}(\omega)$$

and

$$I \cap \mathcal{L}_1 \subset I^{-\infty} \subset I_{a_\infty} \cap \mathcal{L}_1$$

and the idempotence of the maps $I \to I^{o\infty}$ and $I \to I^{-\infty}$, a consequence of the more general identities

$${}_{a_\infty}I = {}_{a_\infty}(({}_{a_\infty}I)_{a_\infty}) \qquad \text{and} \qquad I_{a_\infty} = ({}_{a_\infty}(I_{a_\infty}))_{a_\infty}.$$

Thus, like in the am-case, an ideal I is am-open (resp., am-∞ closed) if and only if there is an ideal J such that $I = J_{a_\infty}$ (resp., $I = {}_{a_\infty}J$). As $(\mathcal{L}_1)_{a_\infty} = \mathrm{se}(\omega)$ and ${}_{a_\infty}\mathrm{se}(\omega) = \mathcal{L}_1$ (see [10, Lemma 4.7, Corollary 4.9]), $\mathrm{se}(\omega)$ and $\mathcal{L}_1$ are, respectively, the largest am-∞ open and the largest am-∞ closed ideals. The finite rank ideal F is am-∞ stable and hence it is the smallest nonzero am-∞ open ideal and the smallest nonzero am-∞ closed ideal. Moreover, every nonzero ideal with the exception of F contains a nonzero principal am-∞ stable ideal (hence both am-∞ open and am-∞ closed) distinct from F [12]. Contrasting these properties for the am-∞ case with the properties for the am case, (ω) is the smallest nonzero am-open ideal, while $\mathcal{L}_1$ is the smallest nonzero am-closed ideal, and every principal ideal is contained in an am-stable principal ideal (hence both am-open and am-closed) and so there are no proper largest am-closed or am-open ideals.

We leave to the reader to verify that the exact analogs of Lemmas 2.1, 2.2 and 2.3 hold for the am-∞ case. Here Theorem 3.2 plays the role of Theorem 2.5 for the equality in Lemma 2.3(iv) and Theorem 3.11 plays the role of Theorem 2.17 for the equality in Lemma 2.2(iii). The same counterexample to equality in Lemma 2.2.(ii) given in Example 2.4(i) provides a counterexample to the equality in the analog am-∞ case: by [10, Lemma 4.7], $((\xi) \cap (\eta))_{a_\infty} = (\min(\xi, \eta))_{a_\infty} = ((\min(\xi, \eta))_{a_\infty})$ while $(\xi)_{a_\infty} = (\eta)_{a_\infty} = \mathrm{se}(\omega)$. The counterexample to the equality in Lemma 2.3(iii) and hence (ii) given in Example 2.4(ii) provides also a counterexample to the same equalities in the am-∞ analogs, but we postpone verifying that until after Lemma 3.9.

The distributivity of the am-∞ closure over finite sums, i.e., the am-∞ analog of Theorem 2.5, also holds, but for its proof we no longer can depend on the theory of substochastic matrices. Instead we will use the following finite dimensional lemma and then we will extend it to the infinite dimensional case via the w^* compactness of the unit ball of ℓ_1.

Lemma 3.1. *Let ξ, η, and $\mu \in [0, \infty)^n$ for some $n \in \mathbb{N}$. If for all $1 \le k \le n$, $\sum_{j=1}^k \eta_j + \sum_{j=1}^k \mu_j \le \sum_{j=1}^k \xi_j$, then there exist $\tilde{\eta}$ and $\tilde{\mu} \in [0, \infty)^n$ for which $\xi = \tilde{\eta} + \tilde{\mu}$, $\sum_{j=1}^k \eta_j \le \sum_{j=1}^k \tilde{\eta}_j$, and $\sum_{j=1}^k \mu_j \le \sum_{j=1}^k \tilde{\mu}_j$ for all $1 \le k \le n$.*

Proof. The proof is by induction on n. The case $n = 1$ is trivial, so assume the property is true for all integers less than equal to $n - 1$. Assume without loss of generality that $\sum_{j=1}^k \xi_j > 0$ for all $1 \le k \le n$ and let

$$\gamma = \max_{1 \le k \le n} \frac{\sum_{j=1}^k \eta_j + \sum_{j=1}^k \mu_j}{\sum_{j=1}^k \xi_j},$$

which maximum $\gamma \le 1$ is achieved for some k. Then

$$\sum_{j=1}^m \eta_j + \sum_{j=1}^m \mu_j \le \gamma \sum_{j=1}^m \xi_j \quad \text{for all } 1 \le m \le k,$$

with equality holding for $m = k$, so also

$$\sum_{j=k+1}^m \eta_j + \sum_{j=k+1}^m \mu_j \le \gamma \sum_{j=k+1}^m \xi_j \quad \text{for all } k+1 \le m \le n.$$

Thus if we apply the induction hypothesis separately to the truncated sequences $\gamma\xi\chi_{[1,k]}$, $\eta\chi_{[1,k]}$ and $\mu\chi_{[1,k]}$ and to $\gamma\xi\chi_{[k+1,n]}$, $\eta\chi_{[k+1,n]}$, and $\mu\chi_{[k+1,n]}$ we obtain that $\gamma\xi\chi_{[1,k]} = \rho+\sigma$ for two sequences $\rho, \sigma \in [0,\infty)^k$ for which $\sum_{j=1}^m \eta_j \leq \sum_{j=1}^m \rho_j$ and $\sum_{j=1}^m \mu_j \leq \sum_{j=1}^m \sigma_j$ for all $1 \leq m \leq k$. Similarly $\gamma\xi\chi_{[k+1,n]} = \rho' + \sigma'$ for $\rho', \sigma' \in [0,\infty)^{n-k}$ and $\sum_{j=k+1}^m \eta_j \leq \sum_{j=k+1}^m \rho'_j$, $\sum_{j=k+1}^m \mu_j \leq \sum_{j=k+1}^m \sigma'_j$ for all $k+1 \leq m \leq n$. But then it is enough to define $\tilde{\eta} = \frac{1}{\gamma}\langle \rho, \rho'\rangle$ and $\tilde{\mu} = \frac{1}{\gamma}\langle \rho, \rho'\rangle$ and verify that it satisfies the required condition. □

Theorem 3.2. $(I+J)^{-\infty} = I^{-\infty} + J^{-\infty}$ *for all ideals* I, J.

In particular, the sum of two am-∞ *closed ideals is am-*∞ *closed.*

Proof. Let $\xi \in \Sigma((I+J)^{-\infty})$, i.e., $\xi_{a_\infty} \leq (\eta+\mu)_{a_\infty} = \eta_{a_\infty} + \mu_{a_\infty}$ for some $\eta \in \Sigma(I \cap \mathcal{L}_1)$ and $\mu \in \Sigma(J \cap \mathcal{L}_1)$. By increasing if necessary the values of ξ_1 or η_1, we can assume that $\sum_{j=1}^\infty \xi_j = \sum_{j=1}^\infty \eta_j + \sum_{j=1}^\infty \mu_j$ and hence $\eta_a + \mu_a \leq \xi_a$. By applying Lemma 3.1 to the truncated sequences $\xi\chi_{[1,n]}$, $\eta\chi_{[1,n]}$, and $\mu\chi_{[1,n]}$, we obtain two sequences

$$\eta^{(n)} := \langle \eta_1^{(n)}, \eta_2^{(n)}, \ldots, \eta_n^{(n)}, 0, 0, \ldots\rangle \quad \text{and} \quad \mu^{(n)} := \langle \mu_1^{(n)}, \mu_2^{(n)}, \ldots, \mu_n^{(n)}, 0, 0, \ldots\rangle$$

for which $\xi_j = \eta_j^{(n)} + \mu_j^{(n)}$ for all $1 \leq j \leq n$ and

$$\sum_{j=1}^m \eta_j \leq \sum_{j=1}^m \eta_j^{(n)} \quad \text{and} \quad \sum_{j=1}^m \mu_j \leq \sum_{j=1}^m \mu_j^{(n)} \quad \text{for all } m \leq n.$$

Since $0 \leq \eta^{(n)}$ and $\mu^{(n)} \leq \xi$, by the sequential compacteness of the unit ball of ℓ_1 in the w^*-topology (as dual of c_o), we can find converging subsequences $\eta^{(n_k)} \underset{w^*}{\to} \tilde{\eta}$, $\mu^{(n_k)} \underset{w^*}{\to} \tilde{\mu}$. It is now easy to verify that $\xi = \tilde{\eta} + \tilde{\mu}$, that $\tilde{\eta} \geq 0$, $\tilde{\mu} \geq 0$, and that $\sum_{j=1}^n \eta_j \leq \sum_{j=1}^n \tilde{\eta}_j$ and $\sum_{j=1}^n \mu_j \leq \sum_{j=1}^n \tilde{\mu}_j$ for all n. It follows from $\sum_{j=1}^\infty \xi_j = \sum_{j=1}^\infty \eta_j + \sum_{j=1}^\infty \mu_j$ that $\sum_{j=1}^\infty \tilde{\eta}_j = \sum_{j=1}^\infty \eta_j$ and $\sum_{j=1}^\infty \tilde{\mu}_j = \sum_{j=1}^\infty \mu_j$, and hence $\sum_{j=n}^\infty \tilde{\eta}_j \leq \sum_{j=n}^\infty \eta_j$ and $\sum_{j=n}^\infty \tilde{\mu}_j \leq \sum_{j=n}^\infty \mu_j$ for all n. Let $\tilde{\eta}^*$, $\tilde{\mu}^*$ be the decreasing rearrangement of $\tilde{\eta}$ and $\tilde{\mu}$. Since $\sum_{j=n}^\infty \tilde{\eta}_j^* \leq \sum_{j=n}^\infty \tilde{\eta}_j$ for every n, it follows that $(\tilde{\eta}^*)_{a_\infty} \leq \eta_{a_\infty}$, i.e., $\tilde{\eta}^* \in \Sigma(I^{-\infty})$. Thus $\tilde{\eta} \in S(I^{-\infty})$. Similarly, $\tilde{\mu} \in S(J^{-\infty})$. But then $\xi \in S(I^{-\infty}) + S(J^{-\infty}) = S(I^{-\infty} + J^{-\infty})$, which proves that $\xi \in \Sigma(I^{-\infty} + J^{-\infty})$ and hence $(I+J)^{-\infty} \subset I^{-\infty} + J^{-\infty}$. Since the am-$\infty$ closure operation preserves inclusions, $I^{-\infty} + J^{-\infty} \subset (I+J)^{-\infty}$, concluding the proof. □

As a consequence, as in the am-case the collection of all the am-∞ closed ideals contained in an ideal I is directed and hence its union is an am-∞ closed ideal by the am-∞ analog of Lemma 2.1(v).

Corollary 3.3. *For every ideal I, $I_{-\infty} := \bigcup\{J \mid J \subset I$ and J is am-∞ closed$\}$ is the largest am-closed ideal contained in I.*

Notice that $I_{-\infty} \subset I \cap \mathcal{L}_1 \subset I^{-\infty}$ and I is am-∞ closed if and only if $I_{-\infty} = I$ if and only if $I = I^{-\infty}$. Moreover, ${}_{a_\infty}I$ is am-∞ closed, so ${}_{a_\infty}I \subset I_{-\infty}$. The inclusion can be proper: consider any ideal I that is am-∞ closed but not am-∞ stable, e.g., $\mathcal{L}_1$. Analogously to the am-case, we can identify $I_{-\infty}$ for I countably generated.

Theorem 3.4. *If I is a countably generated ideal, then $I_{-\infty} = {}_{a_\infty}I$.*

Proof. Let $\eta \in \Sigma(I_{-\infty})$. Since $I_{-\infty} \subset \mathcal{L}_1$, the largest am-$\infty$ closed ideal, $\eta \in \ell_1^*$. We claim that $\eta_{a_\infty} \in \Sigma(I)$, i.e., $\eta \in \Sigma({}_{a_\infty}I)$. This will prove that $I_{-\infty} \subset {}_{a_\infty}I$ and hence the equality. Assume by contradiction that $\eta_{a_\infty} \notin \Sigma(I)$ and as in the proof of Lemma 2.8, choose a sequence of generators $\rho^{(k)}$ for $\Sigma(I)$ with $\rho^{(k)} \leq \rho^{(k+1)}$ and so that for every $\xi \in \Sigma(I)$, $\xi = O(\rho^{(m)})$ for some $m \in \mathbb{N}$. Then there is an increasing sequence of indices n_k such that $(\frac{\eta_{a_\infty}}{\rho^{(k)}})_{n_k} \geq k$ for every $k \in \mathbb{N}$. By the summability of η, we can further request that $\sum_{j=n_{k-1}+1}^{n_k} \eta_j \geq \frac{1}{2}\sum_{j=n_{k-1}+1}^{\infty} \eta_j$. Set $n_o := 0$ and define $\xi_j = (\eta_{a_\infty})_{n_k}$ for $n_{k-1} < j \leq n_k$. Then

$$\begin{aligned}
\sum_{i=j+1}^{\infty} \xi_i &= (n_k - j)\xi_{n_k} + \sum_{i=k+1}^{\infty} (n_i - n_{i-1})\xi_{n_i} \\
&= \frac{n_k - j}{n_k} \sum_{i=n_k+1}^{\infty} \eta_i + \sum_{i=k+1}^{\infty} \frac{n_i - n_{i-1}}{n_i} \sum_{m=n_i+1}^{\infty} \eta_m \\
&\leq \sum_{i=n_k+1}^{\infty} \eta_i + 2 \sum_{i=k+1}^{\infty} \sum_{m=n_i+1}^{n_{i+1}} \eta_m \\
&\leq 3 \sum_{i=n_k+1}^{\infty} \eta_i \leq 3 \sum_{i=j+1}^{\infty} \eta_i.
\end{aligned}$$

Thus $\xi \in \Sigma((\eta)^{-\infty}) \subset \Sigma(I_{-\infty}) \subset \Sigma(I)$. On the other hand, for every $m \in \mathbb{N}$ and for every $k \geq m$, $(\frac{\xi}{\rho^{(m)}})_{n_k} \geq (\frac{\xi}{\rho^{(k)}})_{n_k} = (\frac{\eta_{a_\infty}}{\rho^{(k)}})_{n_k} \geq k$, whence $\xi \notin \Sigma(I)$, a contradiction. $\square$

Precisely as for the am-case we have:

Theorem 3.5. *A countably generated ideal is am-∞ closed if and only if it is am-∞ stable.*

Now we investigate the operations $I \to I^{o\infty}$ and $I \to I^{oo\infty}$, where $I^{oo\infty}$ is the am-∞ analog of I^{oo} and will be defined in Definition 3.12. While the statements are analogous to the statements in Section 2, the proofs are sometimes substantially different. The analog of Lemma 2.12 is given by:

Lemma 3.6. *A sequence ξ is the arithmetic mean at infinity η_{a_∞} of some sequence $\eta \in \ell_1^*$ if and only if $\frac{\xi}{\omega} \in c_o^*$ and is convex, i.e., $(\frac{\xi}{\omega})_{n+1} \leq \frac{1}{2}((\frac{\xi}{\omega})_n + (\frac{\xi}{\omega})_{n+2})$ for all n.*

The analog of Lemma 2.13 is given by:

Lemma 3.7. *For every principal ideal I, the following are equivalent.*
(i) *I is am-∞ open.*
(ii) *$I = (\eta_{a_\infty})$ for some $\eta \in \ell_1^*$.*
(iii) *$I = (\xi)$ for some ξ for which $\frac{\xi}{\omega} \in c_o^*$.*

Proof. (i) $\Leftrightarrow$ (ii). Assume I is am-∞ open and that $\xi \in c_o^*$ is a generator of I. Then $I = J_{a_\infty}$ for some ideal J, i.e., $\xi \leq \eta_{a_\infty}$ for some $\eta \in \ell_1^*$ such that $\eta_{a_\infty} \in \Sigma(I)$ and thus $(\xi) = (\eta_{a_\infty})$. The other implication is a direct consequence of the equality $(\eta_{a_\infty}) = (\eta)_{a_\infty}$ obtained in [10, Lemma 4.7].

(ii) $\Rightarrow$ (iii). Obvious as $\frac{\eta_{a_\infty}}{\omega} \downarrow 0$.

(iii) $\Rightarrow$ (ii). Since $F = (\langle 1, 1, 0, 0, \ldots \rangle)_{a_\infty}$, we can assume without loss of generality that $\xi_j > 0$ for all j. Let ψ be the largest (pointwise) convex sequence majorized by $\frac{\xi}{\omega}$. It is easy to see that such a sequence ψ exists, that $\psi > 0$, and that being convex, ψ is decreasing, hence $\psi \in c_o^*$ and by Lemma 3.6 , $\omega\psi = \eta_{a_\infty}$ for some $\eta \in \ell_1^*$. By definition, $\xi \geq \eta_{a_\infty}$ and hence $(\eta_{a_\infty}) \subset (\xi) = I$. To prove the reverse inclusion, first notice that the graph of ψ (viewed as the polygonal curve through the points $\{(n, \psi_n) \mid n \in \mathbb{N}\}$) must have infinitely many corners since $\psi_n > 0$ for all n. Let $\{k_p\}$ be the strictly increasing sequence of all the integers where the corners occur, starting with $k_1 = 1$, i.e., for all $p > 1$, $\psi_{k_p - 1} - \psi_{k_p} > \psi_{k_p} - \psi_{k_p + 1}$.

By the pointwise maximality of the convex sequence ψ, $\psi_{k_p} = (\frac{\xi}{\omega})_{k_p}$ for every $p \in \mathbb{N}$ (including $p = 1$) since otherwise we could contradict maximality by increasing ψ_{k_p} and still maintain convexity and majorization by $\frac{\xi}{\omega}$. Denote by $D_{\frac{1}{2}}$ the operator $(D_{\frac{1}{2}}\zeta)_j = \zeta_{2j}$ for $\zeta \in c_o^*$. We claim that for every j, $(D_{\frac{1}{2}}\frac{\xi}{\omega})_j < 2\psi_j$. Assume otherwise that there is a $j \geq 1$ such that $(\frac{\xi}{\omega})_{2j} \geq 2\psi_j$ and let p be the integer for which $k_p \leq j < k_{p+1}$. Then $k_p < 2j$ and also $2j < k_{p+1}$ because otherwise we would have the contradiction $2\psi_j \leq (\frac{\xi}{\omega})_{2j} \leq (\frac{\xi}{\omega})_{k_{p+1}} = \psi_{k_{p+1}} \leq \psi_j$. Moreover, since k_p and k_{p+1} are consecutive corners, between them ψ is linear, i.e.,

$$\psi_j = \psi_{k_p} + \frac{\psi_{k_{p+1}} - \psi_{k_p}}{k_{p+1} - k_p}(j - k_p) = \frac{k_{p+1} - j}{k_{p+1} - k_p}\psi_{k_p} + \frac{j - k_p}{k_{p+1} - k_p}\psi_{k_{p+1}}$$

and hence

$$\psi_j \geq \frac{k_{p+1} - j}{k_{p+1} - k_p}\psi_{k_p} > \left(1 - \frac{j}{k_{p+1}}\right)\psi_{k_p} > \frac{1}{2}\left(\frac{\xi}{\omega}\right)_{k_p} \geq \frac{1}{2}\left(\frac{\xi}{\omega}\right)_{2j} \geq \psi_j.$$

This contradiction proves that $D_{\frac{1}{2}}\frac{\xi}{\omega} < 2\psi$. It is now easy to verify that $(\frac{\xi}{\omega})_j \leq (D_3 D_{\frac{1}{2}}\frac{\xi}{\omega})_j < 2(D_3\psi)_j$ for $j > 1$, and hence $I = (\xi) \subset (\eta_{a_\infty})$ because $\xi_j < 2\omega_j(D_3\psi)_j \leq 2(D_3\omega)_j(D_3\psi)_j = 2(D_3(\omega\psi))_j = 2D_3(\eta_{a_\infty})_j$. □

Example 3.8. In the proof of the implication (iii) ⇒ (ii), one cannot conclude that $\xi = O(\eta_{a_\infty})$. Indeed consider $\xi_j = \frac{1}{jk!}$ for $k! \leq j < (k+1)!$ where it is elementary to compute $\psi_j = \frac{1}{k!}(1 - \frac{j-k!}{(k+1)!})$ for $k! \leq j < (k+1)!$. Also, this example shows that while in the am-case the smallest concave sequence $\frac{\eta_a}{\omega}$ that majorizes $\frac{\xi}{\omega}$ (when $\frac{\xi}{\omega}$ is monotone nondecreasing) provides also the smallest arithmetic mean η_a that majorizes ξ (see Remark 2.15(iii)), this is no longer true for the am-∞ case.

We have seen in Lemma 2.14 that the am-interior of a nonzero principal ideal is always principal and it is nonzero if and only if the ideal is large enough (that is, it contains (ω)). Furthermore, there is always a smallest am-open ideal containing it and it too is principal. The next lemma shows that the am-∞ interior of a nonzero principal ideal is principal if only if the ideal is small enough (that is, it does not contain (ω)). Furthermore, if the principal ideal is contained in se(ω), which is the largest am-∞ open ideal, then there is a smallest am-∞ open ideal containing it and it is principal.

Lemma 3.9. *For every $\xi \in c_o^*$:*

(i) $(\xi)^{o\infty} = \begin{cases} (\omega \,\mathrm{lni}\, \frac{\xi}{\omega}) & \text{if } \omega \not\subset (\xi) \\ \mathrm{se}(\omega) & \text{if } \omega \subset (\xi) \end{cases}$

(ii) *If $(\xi) \subset \mathrm{se}(\omega)$, then $(\omega \,\mathrm{uni}\, \frac{\xi}{\omega})$ is the smallest am-∞ open ideal containing (ξ).*

Proof. (i) If $(\xi) = F$, then also $(\omega \,\mathrm{lni}\, \frac{\xi}{\omega}) = (\omega \,\mathrm{uni}\, \frac{\xi}{\omega}) = F$, so assume that $\xi \notin \Sigma(F)$. If $(\omega) \subset (\xi)$, then $\mathrm{se}(\omega) = (\xi)^{o\infty}$ because $\mathrm{se}(\omega)$ is the largest am-∞ open ideal. If $(\omega) \not\subset (\xi)$, in particular $\omega \neq O(\xi)$ and hence $\mathrm{lni}\, \frac{\xi}{\omega} \in c_o^*$. But then by Lemma 3.7, $(\omega \,\mathrm{lni}\, \frac{\xi}{\omega}) = (\eta_{a_\infty})$ for some $\eta \in \ell_1^*$, and since $(\eta_{a_\infty}) = (\eta)_{a_\infty}$ by [10, Lemma 4.7], it follows that $(\omega \,\mathrm{lni}\, \frac{\xi}{\omega})$ is am-∞ open. Since $\omega \,\mathrm{lni}\, \frac{\xi}{\omega} \leq \xi$ and hence $(\omega \,\mathrm{lni}\, \frac{\xi}{\omega}) \subset (\xi)$, it follows that $(\omega \,\mathrm{lni}\, \frac{\xi}{\omega}) \subset (\xi)^{o\infty}$. For the reverse inclusion, if $\zeta \in \Sigma((\xi)^{o\infty})$, then $\zeta \leq \rho_{a_\infty} \leq M D_m \xi$ for some $\rho \in \ell_1^*$, $M > 0$ and $m \in \mathbb{N}$. But then $\frac{\rho_{a_\infty}}{\omega} \leq \mathrm{lni}\, M \frac{D_m \xi}{\omega}$ because $\frac{\rho_{a_\infty}}{\omega}$ is monotone nonincreasing, and from this and $\omega \leq D_m \omega \leq m\omega$, it follows that

$$\begin{aligned} \rho_{a_\infty} &\leq M\omega \,\mathrm{lni}\, \frac{D_m \xi}{\omega} \leq mM\omega \,\mathrm{lni}\, D_m\left(\frac{\xi}{\omega}\right) = mM\omega D_m \,\mathrm{lni}\, \frac{\xi}{\omega} \\ &\leq mM(D_m\omega)\left(D_m \,\mathrm{lni}\, \frac{\xi}{\omega}\right) = mMD_m\left(\omega \,\mathrm{lni}\, \frac{\xi}{\omega}\right) \end{aligned}$$

where the first equality follows by an elementary computation. Thus $\zeta \in \Sigma(\omega \,\mathrm{lni}\, \frac{\xi}{\omega})$, i.e., $(\xi)^{o\infty} \subset (\omega \,\mathrm{lni}\, \frac{\xi}{\omega})$ and the equality of these ideals is established.

(ii) If $(\xi) \subset \mathrm{se}(\omega)$, then $\mathrm{uni}\, \frac{\xi}{\omega} \in c_o^*$, hence $(\omega \,\mathrm{uni}\, \frac{\xi}{\omega})$ is am-∞ open by Lemma 3.7. Clearly, $(\xi) \subset (\omega \,\mathrm{uni}\, \frac{\xi}{\omega})$ and if $(\xi) \subset I$ for an am-∞ open ideal I, then $\xi \leq \rho_{a_\infty}$ for some $\rho_{a_\infty} \in \Sigma(I)$. Since $\frac{\rho_{a_\infty}}{\omega}$ is monotone nonincreasing, by the minimality of "uni", $\omega \,\mathrm{uni}\, \frac{\xi}{\omega} \leq \rho_{a_\infty}$ and hence $(\omega \,\mathrm{uni}\, \frac{\xi}{\omega}) \subset I$. □

As a consequence of this lemma we see that if $(\omega) = (\xi) + (\eta)$ but $(\omega) \not\subset (\xi)$ and $(\omega) \not\subset (\eta)$ as in Example 2.4(ii), then $(\omega)^{o\infty} = \mathrm{se}(\omega)$ is not principal but

$$(\xi)^{o\infty} + (\eta)^{o\infty} = \left(\omega \,\mathrm{lni}\, \frac{\xi}{\omega}\right) + \left(\omega \,\mathrm{lni}\, \frac{\eta}{\omega}\right) = \left(\omega \,\mathrm{lni}\, \frac{\xi}{\omega} + \omega \,\mathrm{lni}\, \frac{\eta}{\omega}\right)$$

which is principal. By the same token, ${}_{a_\infty}(\xi) + {}_{a_\infty}(\eta) \neq {}_{a_\infty}((\xi) + (\eta))$ and in view of Theorem 3.4, $(\xi)_{-\infty} + (\eta)_{-\infty} \neq ((\xi) + (\eta))_{-\infty}$.

¿From this lemma we obtain an analog of Corollary 2.16.

Corollary 3.10. *Let I be an ideal. Then*

(i)

$$\begin{aligned}\Sigma(I^{o\infty}) &= \{\xi \in \Sigma(\mathrm{se}(\omega)) \mid \omega \,\mathrm{uni}\, \frac{\xi}{\omega} \in \Sigma(I)\}\\ &= \{\xi \in c_o^* \mid \xi \leq \omega \,\mathrm{lni}\, \frac{\eta}{\omega} \text{ for some } \eta \in \Sigma(I \cap \mathrm{se}(\omega))\}.\end{aligned}$$

If I is am-∞ open and $\xi \in c_o^*$, then

(ii) $\xi \in \Sigma(I)$ *if and only if* $\omega \,\mathrm{uni}\, \frac{\xi}{\omega} \in \Sigma(I)$.

Proof. (i) If $\xi \in \Sigma(I^{o\infty})$, then $\xi \in \Sigma(\mathrm{se}(\omega))$ and hence $\omega \,\mathrm{uni}\, \frac{\xi}{\omega} \in \Sigma(I^{o\infty}) \subset \Sigma(I)$ by Lemma 3.9(ii). If $\xi \in \Sigma(\mathrm{se}(\omega))$ and $\omega \,\mathrm{uni}\, \frac{\xi}{\omega} \in \Sigma(I)$, then $\omega \,\mathrm{uni}\, \frac{\xi}{\omega} \in \Sigma(I \cap \mathrm{se}(\omega))$ and $\xi \leq \omega \,\mathrm{uni}\, \frac{\xi}{\omega} = \omega \,\mathrm{lni}\, \frac{\omega \,\mathrm{uni}\, \frac{\xi}{\omega}}{\omega}$. Thus

$$\begin{aligned}\Sigma(I^{o\infty}) &\subset \{\xi \in \Sigma(\mathrm{se}(\omega)) \mid \omega \,\mathrm{uni}\, \frac{\xi}{\omega} \in \Sigma(I)\}\\ &\subset \{\xi \in c_o^* \mid \xi \leq \omega \,\mathrm{lni}\, \frac{\eta}{\omega} \text{ for some } \eta \in \Sigma(I \cap \mathrm{se}(\omega))\}.\end{aligned}$$

Finally, let $\xi \in c_o^*$, $\xi \leq \omega \,\mathrm{lni}\, \frac{\eta}{\omega}$ for some $\eta \in \Sigma(I \cap \mathrm{se}(\omega))$. From the inequality $\xi \leq \omega \,\mathrm{uni}\, \frac{\xi}{\omega} \leq \omega \,\mathrm{lni}\, \frac{\eta}{\omega}$, it follows by by Lemma 3.9(i) that $\xi \in \Sigma((\eta)^{o\infty}) \subset \Sigma(I^{o\infty})$, which concludes the proof.

(ii) Just notice that $\xi \leq \omega \,\mathrm{uni}\, \frac{\xi}{\omega} \in \Sigma(I) \subset \Sigma(\mathrm{se}(\omega))$. ◻

Now Theorem 2.17, Definition 2.18 and Proposition 2.21 extend to the am-∞ case with proofs similar to the am-case.

Theorem 3.11. *The intersection of am-∞ open ideals is am-∞ open.*

Definition 3.12. For every ideal I, define

$$I^{oo\infty} := \bigcap \{J \mid I \cap \mathrm{se}(\omega) \subset J \ \text{ and } J \text{ is am-}\infty \text{ open}\}$$

.

Remark 3.13. Lemma 3.9 affirms that if I is principal then $I^{o\infty}$ is principal if and only if $(\omega) \not\subset (\xi)$ and $I^{oo\infty}$ is principal if and only if $(\xi) \subset \mathrm{se}(\omega)$.

The next proposition generalizes to general ideals the characterization of $I^{oo\infty}$ given by Lemma 3.9 in the case of principal ideals.

Proposition 3.14. *For every ideal I, the characteristic set of $I^{oo\infty}$ is given by:*

$$\Sigma(I^{oo\infty}) = \left\{\xi \in c_o^* \mid \eta \leq \omega \operatorname{uni} \frac{\eta}{\omega} \textit{ for some } \eta \in \Sigma(I \cap \operatorname{se}(\omega))\right\}.$$

Notice that $I^{o\infty} \subset I \cap \operatorname{se}(\omega) \subset I^{oo\infty}$ and I is am-∞ open if and only if one of the inclusions and hence both of them are equalities. Also, $I \cap \operatorname{se}(\omega) \subset I_{a_\infty}$ and I_{a_∞} is am-∞ open so $I^{oo\infty} \subset I_{a_\infty}$. As for the am-case, we see by considering an am-∞ open principal ideal that is not am-∞ stable that the inclusion may be proper, and by considering am-∞ stable ideals that it may become an equality.

Example 3.15. Let $\xi_j = \frac{1}{2^k k!}$ for $(k-1)! < j \leq k!$ for $k > 1$. Then a direct computation shows that $(\operatorname{uni} \frac{\xi}{\omega})_j = \frac{1}{2^k}$ for $(k-1)! < j \leq k!$ and that $\operatorname{uni} \frac{\xi}{\omega} \asymp \frac{\xi_{a_\infty}}{\omega}$. Thus by Lemma 3.9, $(\xi)^{oo\infty} = (\xi)_{a_\infty}$. On the other hand, $(\frac{\omega \operatorname{uni} \frac{\xi}{\omega}}{\xi})_{(k-1)!} = k$ and hence $\xi_{a_\infty} \neq O(\xi)$. By [10, Theorem 4.12], ξ is ∞-irregular, i.e., $(\xi) \neq (\xi)_{a_\infty}$.

A consequence of Proposition 3.14 and the subadditivity of "uni" is that $I^{oo\infty} + J^{oo\infty} = (I+J)^{oo\infty}$ for any two ideals I and J.

Proposition 3.14 also permits us to determine simple sufficient conditions on I under which $I^{-\infty}$ (resp., $I^{oo\infty}$) is the largest am-∞ closed ideal $\mathcal{L}_1$ (resp., the largest am-∞ open ideal $\operatorname{se}(\omega)$).

Lemma 3.16. *Let I be an ideal.*

(i) *If $I \not\subset \mathcal{L}_1$, then $I^{-\infty} = \mathcal{L}_1$.*

(ii) *If $I \not\subset \operatorname{se}(\omega)$, then $I^{oo\infty} = \operatorname{se}(\omega)$.*

Proof. (i) Let $\xi \in \Sigma(I) \setminus \ell_1^*$. Then $\operatorname{se}(\omega) = (\xi)_{a_\infty} \subset I_{a_\infty}$ by [10, Lemma 4.7]. Since $I_{a_\infty} \subset \operatorname{se}(\omega)$ holds generally, $I_{a_\infty} = \operatorname{se}(\omega)$ and thus $I^{-\infty} = {}_{a_\infty}\operatorname{se}(\omega) = \mathcal{L}_1$.

(ii) Let $\eta \in \Sigma(\operatorname{se}(\omega))$, set $\alpha := \operatorname{uni} \frac{\eta}{\omega}$, $\alpha_o = \alpha_1$, and choose an arbitrary $\xi \in \Sigma(I) \setminus \Sigma(\operatorname{se}(\omega))$. Then there is an increasing sequence of integers n_k with $n_0 = 0$ and an $\varepsilon > 0$ such that $\xi_{n_k} \geq \varepsilon\omega_{n_k}$ for all $k \geq 1$. Set $\mu_j = \frac{1}{n_k}$ and $\rho_j = \frac{\alpha_{n_{k-1}}}{n_k}$ for $n_{k-1} < j \leq n_k$ and $k \geq 1$. Then $\mu, \rho \in c_o^*$, $\mu \leq \frac{1}{\varepsilon}\xi$, hence $\mu \in \Sigma(I)$ and $\rho = o(\omega)$, $\rho \leq \alpha_1\mu$, hence $\rho \in \Sigma(I \cap \operatorname{se}(\omega))$. Moreover, $\max\{(\frac{\rho}{\omega})_j \mid n_{k-1} < j \leq n_k\} = \alpha_{n_{k-1}}$ and thus $(\operatorname{uni} \frac{\rho}{\omega})_j = \alpha_{n_{k-1}}$ for $n_{k-1} < j \leq n_k$. But then, $\alpha \leq \operatorname{uni} \frac{\rho}{\omega}$ and hence $\eta \leq \alpha\omega \leq \omega \operatorname{uni} \frac{\rho}{\omega}$. By Proposition 3.14, $\eta \in \Sigma(I^{oo\infty})$, which proves the claim. $\square$

Finally, it is easy to see that the exact analog of Lemma 2.22 holds.

4. Soft Ideals

It is well-known that the product $IJ = JI$ of two ideals I and J is the ideal with characteristic set

$$\Sigma(IJ) = \{\xi \in c_o^* \mid \xi \leq \eta\rho \text{ for some } \eta \in \Sigma(I) \text{ and } \rho \in \Sigma(J)\}$$

and that for all $p > 0$, the ideal I^p is the ideal with characteristic set

$$\Sigma(I^p) = \{\xi \in c_o^* \mid \xi^{1/p} \in \Sigma(I)\}$$

(see [7, Section 2.8] as but one convenient reference). Recall also from [7, Sections 2.8 and 4.3] that the quotient $\Sigma(I) : X$ of a characteristic set $\Sigma(I)$ by a nonempty subset $X \subset [0, \infty)^{\mathbb{N}}$ is defined to be the characteristic set

$$\big\{\xi \in c_o^* \mid \big((D_m\xi)x\big)^* \in \Sigma(I) \text{ for all } x \in X \text{ and } m \in \mathbb{N}\big\}.$$

Whenever $X = \Sigma(J)$, denote the associated ideal by $I : J$. A special important case is the Köthe dual $X^\times$ of a set X, which is the ideal with characteristic set $\ell_1^* : X$.

In [9] and [10] we introduced the following definitions of soft ideals.

Definition 4.1. The soft interior of an ideal I is the product $\operatorname{se} I := IK(H)$, i.e., the ideal with characteristic set

$$\Sigma(\operatorname{se} I) = \{\xi \in c_o^* \mid \xi \leq \alpha\eta \text{ for some } \alpha \in c_o^*,\ \eta \in \Sigma(I)\}.$$

The soft cover of an ideal I is the quotient $\operatorname{sc} I := I : K(H)$, i.e., the ideal with characteristic set

$$\Sigma(\operatorname{sc} I) = \{\xi \in c_o^* \mid \alpha\xi \in \Sigma(I) \text{ for all } \alpha \in c_o^*\}.$$

An ideal is called soft-edged if $\operatorname{se} I = I$ and soft-complemented if $\operatorname{sc} I = I$.
A pair of ideals $I \subset J$ is called a soft pair if $\operatorname{se} J = I$ and $\operatorname{sc} I = J$.

This terminology is motivated by the fact that I is soft-edged if and only if, for every $\xi \in \Sigma(I)$, one has $\xi = o(\eta)$ for some $\eta \in \Sigma(I)$. Analogously, an ideal I is soft-complemented if and only if, for every $\xi \in c_o^* \setminus \Sigma(I)$, one has $\eta = o(\xi)$ for some $\eta \in c_o^* \setminus \Sigma(I)$.

Below are some simple properties of the soft interior and soft cover operations that we shall use frequently throughout this paper.

Lemma 4.2. *For all ideals I, J:*

(i) se *and* sc *are inclusion preserving, i.e.,* $\operatorname{se} I \subset \operatorname{se} J$ *and* $\operatorname{sc} I \subset \operatorname{sc} J$ *whenever* $I \subset J$.

(ii) se *and* sc *are idempotent, i.e.,* $\operatorname{se}(\operatorname{se} I) = \operatorname{se} I$ *and* $\operatorname{sc}(\operatorname{sc} I) = \operatorname{sc} I$ *and so* $\operatorname{se} I$ *and* $\operatorname{sc} I$ *are, respectively, soft-edged and soft-complemented.*

(iii) $\operatorname{se} I \subset I \subset \operatorname{sc} I$

(iv) $\operatorname{se}(\operatorname{sc} I) = \operatorname{se} I$ *and* $\operatorname{sc}(\operatorname{se} I) = \operatorname{sc} I$

(v) $\operatorname{se} I$ *and* $\operatorname{sc} I$ *form a soft pair.*

(vi) *If* $I \subset J$ *form a soft pair and* L *is an intermediate ideal,* $I \subset L \subset J$, *then* $I = \operatorname{se} L$ *and* $J = \operatorname{sc} L$.

(vii) *If* $I \subset J$, $I = \operatorname{se} I$, *and* $J = \operatorname{sc} J$, *then* I *and* J *form a soft pair if and only if* $\operatorname{sc} I = J$ *if and only if* $\operatorname{se} J = I$.

Proof. (i) and (iii) follow easily from the definitions. From $K(H) = K(H)^2$ follows the idempotence of se in the first part of (ii) and the inclusion $\operatorname{sc}(\operatorname{sc} I) \subset \operatorname{sc} I$, while the equality here follows from (iii) and (i). That $\operatorname{se}(\operatorname{sc} I) \subset I \subset \operatorname{sc}(\operatorname{se} I)$ is immediate by Definition 4.1. Applying se to the first inclusion, by (i)–(iii) follows the first equality in (iv) and the second equality follows similarly. (v), (vi) and (vii) are now immediate. □

An easy consequence of this proposition and of Definition 4.1 is:

Corollary 4.3. *For every ideal I,*

(i) $\operatorname{se} I$ *is the largest soft-edged ideal contained in* I *and it is the smallest ideal whose soft cover contains* I

(ii) $\operatorname{sc} I$ *is the smallest soft-complemented ideal containing* I *and it is the largest ideal whose soft interior is contained in* I.

The rest of this section is devoted to showing that many ideals in the literature are soft-edged or soft-complemented (or both) and that soft pairs occur naturally. Rather than proving directly soft-complementedness, it is sometimes easier to prove a stronger property:

Definition 4.4. An ideal I is said to be strongly soft-complemented (ssc for short) if for every countable collection of sequences $\{\eta^{(k)}\} \subset c_o^* \setminus \Sigma(I)$ there is a sequence of indices $n_k \in \mathbb{N}$ such that $\xi \notin \Sigma(I)$ whenever $\xi \in c_o^*$ and $\xi_i \geq \eta_i^{(k)}$ for all k and for all $1 \leq i \leq n_k$.

Proposition 4.5. *Strongly soft-complemented ideals are soft-complemented.*

Proof. Let I be an ssc ideal, let $\eta \notin \Sigma(I)$, and for each $k \in \mathbb{N}$, set $\eta^{(k)} := \frac{1}{k}\eta$. Since $\{\eta^{(k)}\} \subset c_o^* \setminus \Sigma(I)$, there is an associated sequence of indices n_k which, without loss of generality, can be taken to be strictly increasing. Set $n_o = 0$ and define $\alpha_i := \frac{1}{k}$ for $n_{k-1} < i \leq n_k$. Then $\alpha \in c_o^*$ and $(\alpha\eta)_i \geq \eta_i^{(k)}$ for all $1 \leq i \leq n_k$ and all k. Therefore $\alpha\eta \notin \Sigma(I)$ and hence, by the remark following Definition 4.1, I is soft-complemented. □

Example 4.15 and Proposition 5.3 provide soft-complemented ideals that are not strongly soft-complemented.

Proposition 4.6. (i) *Countably generated ideals are strongly soft-complemented and hence soft-complemented.*

(ii) *If I is a countably generated ideal and if $\{\rho^{(k)}\}$ is a sequence of generators for its characteristic set $\Sigma(I)$, then I is soft-edged if and only if for every $k \in \mathbb{N}$ there are $m, k' \in \mathbb{N}$ for which $\rho^{(k)} = o(D_m\rho^{(k')})$. In particular, a principal ideal (ρ) is soft-edged if and only if $\rho = o(D_m\rho)$ for some $m \in \mathbb{N}$. If a principal ideal (ρ) is soft-edged, then $(\rho) \subset \mathcal{L}_1$.*

Proof. (i) As in the proof of Lemma 2.8, choose a sequence of generators $\rho^{(k)}$ for $\Sigma(I)$ with $\rho^{(k)} \leq \rho^{(k+1)}$ and such that $\xi \in \Sigma(I)$ if and only if $\xi = O(\rho^{(m)})$ for some $m \in \mathbb{N}$. Let $\{\eta^{(k)}\} \subset c_o^* \setminus \Sigma(I)$. Then, in particular, $\eta^{(k)} \neq O(\rho^{(k)})$ for every k. Thus there is a strictly increasing sequence of indices $n_k \in \mathbb{N}$ such that $\eta_{n_k}^{(k)} \geq k\rho_{n_k}^{(k)}$ for all k. If $\xi \in c_o^*$ and for each k, $\xi_i \geq \eta_i^{(k)}$ for all $1 \leq i \leq n_k$, then for all $k \geq m$, $\xi_{n_k} \geq \eta_{n_k}^{(k)} \geq k\rho_{n_k}^{(k)} \geq k\rho_{n_k}^{(m)}$. Hence $\xi \neq O(\rho^{(m)})$ for each m and thus $\xi \notin \Sigma(I)$, establishing that I is ssc.

(ii) Assume that I is soft-edged and let $k \in \mathbb{N}$. By the remarks following Definition 4.1, $\rho^{(k)} = o(\xi)$ for some $\xi \in \Sigma(I)$. But also $\xi = O(D_m\rho^{(k')})$ for some m

and k' and hence $\rho^{(k)} = o(D_m\rho^{(k')})$. Conversely, assume that the condition holds and let $\xi \in \Sigma(I)$. Then $\xi = O(D_m\rho^{(k)})$ for some m and k and $\rho^{(k)} = o(D_p\rho^{(k')})$ for some p and k'. Since $D_mD_p = D_{mp}$, one has

$$\lim_n \frac{\big(D_m\rho^{(k)}\big)_n}{\big(D_{mp}\rho^{(k')}\big)_n} = \lim_n \left(D_m\left(\frac{\rho^{(k)}}{D_p\rho^{(k')}}\right)\right)_n = \lim_j \left(\frac{\rho^{(k)}}{D_p\rho^{(k')}}\right)_j = 0,$$

i.e., $D_m\rho^{(k)} = o(D_{mp}\rho^{(k')})$, whence $\xi \in \Sigma(\operatorname{se} I)$ and I is soft-edged. Thus, if I is a soft-edged principal ideal with a generator ρ, then $\rho = o(D_m\rho)$ for some $m \in \mathbb{N}$. As a consequence, $\rho_{m^k} \le \frac{1}{m^2}\rho_{m^{k-1}}$ for k large enough, from which it follows that ρ is summable. $\square$

Next we consider **Banach ideals**. These are ideals that are complete with respect to a symmetric norm (see for instance [7, Section 4.5]) and were called uniform-cross-norm ideals by Schatten [19], symmetrically normed ideals by Gohberg and Krein [8], and symmetric norm ideals by other authors. Recall that the norm of I induces on the finite rank ideal F (or, more precisely, on $S(F)$, the associated space of sequences of c_o with finite support) a symmetric norming function ϕ, and the latter permits one to construct the so-called minimal and maximal Banach ideals $\mathfrak{S}_\phi^{(o)} = \operatorname{cl}(F)$ contained in I (the closure taken in the norm of I) and $\mathfrak{S}_\phi$ containing I where

$$\Sigma\big(\mathfrak{S}_\phi\big) = \big\{\xi \in c_o^* \mid \phi(\xi) := \sup \phi\big(\langle \xi_1, \xi_2, \dots, \xi_n, 0, 0, \dots\rangle\big) < \infty\big\}$$
$$\Sigma\big(\mathfrak{S}_\phi^{(o)}\big) = \big\{\xi \in \Sigma\big(\mathfrak{S}_\phi\big) \mid \phi\big(\langle \xi_n, \xi_{n+1}, \dots\rangle\big) \longrightarrow 0\big\}.$$

As the following proposition implies, the ideals $\mathfrak{S}_\phi^{(o)}$ and $\mathfrak{S}_\phi$ can be obtained from I through a "soft" operation, i.e., $\mathfrak{S}_\phi^{(o)} = \operatorname{se} I$ and $\mathfrak{S}_\phi = \operatorname{sc} I$, and the embedding $\mathfrak{S}_\phi^{(o)} \subset \mathfrak{S}_\phi$ is a natural example of a soft pair. In particular, if I is a Banach ideal, then so also are $\operatorname{se} I$ and $\operatorname{sc} I$.

Proposition 4.7. *For every symmetric norming function ϕ, $\mathfrak{S}_\phi^{(o)}$ is soft-edged, $\mathfrak{S}_\phi$ is ssc, and $\mathfrak{S}_\phi^{(o)} \subset \mathfrak{S}_\phi$ is a soft pair.*

Proof. We first prove that $\mathfrak{S}_\phi^{(o)}$ is soft-edged. For every $\xi \in \Sigma(\mathfrak{S}_\phi^{(o)})$, that is, $\phi(\langle \xi_n, \xi_{n+1}, \dots\rangle) \to 0$, choose a strictly increasing sequence of indices n_k with $n_o = 0$ for which $\phi(\langle \xi_{n_k+1}, \xi_{n_k+2}, \dots\rangle) \le 2^{-k}$ and $k\xi_{n_k} \downarrow 0$. Set $\beta_i := k$ for all

$n_{k-1} < i \leq n_k$ and $\eta := \text{lni}\,\beta\xi$. Then $\eta \in c_o^*$ since $\eta_{n_k} \leq \beta_{n_k}\xi_{n_k} = k\xi_{n_k} \to 0$ and $\xi = o(\eta)$ because for every k and $n_{k-1} < n \leq n_k$,

$$\begin{aligned}
\eta_n &= \min\{\beta_i\xi_i \mid i \leq n\} \\
&= \min\Big\{\big\{\min\big\{j\xi_i \,|\, n_{j-1} < i \leq n_j\big\} \mid 1 \leq j \leq k-1\big\},\ \min\big\{k\xi_i \mid n_{k-1} < i \leq n\big\}\Big\} \\
&= \min\big\{\big\{j\xi_{n_j} \mid 1 \leq j \leq k-1\big\},\ k\xi_n\big\} \\
&= \min\big\{(k-1)\xi_{n_{k-1}},\ k\xi_n\big\} \\
&\geq (k-1)\xi_n.
\end{aligned}$$

Furthermore, $\eta \in \Sigma(\mathfrak{S}_\phi^{(o)})$ which establishes that $\mathfrak{S}_\phi^{(o)}$ is soft-edged. Indeed, for all $h > k > 1$,

$$\begin{aligned}
\phi\big(\langle \eta_{n_k+1}, \ldots, \eta_{n_h}, 0, 0, \ldots\rangle\big) &\leq \sum_{j=k}^{h-1} \phi\big(\langle \eta_{n_j+1}, \ldots, \eta_{n_{j+1}}, 0, 0, \ldots\rangle\big) \\
&\leq \sum_{j=k}^{h-1} (j+1)\phi\big(\langle \xi_{n_j+1}, \ldots, \xi_{n_{j+1}}, 0, 0, \ldots\rangle\big) \\
&\leq \sum_{j=k}^{h-1} (j+1)\phi\big(\langle \xi_{n_j+1}, \xi_{n_j+2}, \ldots\rangle\big) \\
&\leq \sum_{j=k}^{h-1} \frac{j+1}{2^j}.
\end{aligned}$$

Thus

$$\begin{aligned}
\phi\big(\langle \eta_{n_k+1}, \eta_{n_k+2}, \ldots\rangle\big) &= \sup_n \phi\big(\langle \eta_{n_k+1}, \ldots, \eta_n, 0, 0, \ldots\rangle\big) \\
&\leq \sum_{j=k}^{\infty} \frac{j+1}{2^j} \longrightarrow 0 \quad \text{as } k \longrightarrow \infty.
\end{aligned}$$

from which it follows that $\phi(\langle \eta_n, \eta_{n+1}, \ldots\rangle) \to 0$ as $n \to \infty$.

Next we prove that $\mathfrak{S}_\phi$ is ssc. For every $\{\eta^{(k)}\} \subset c_o^* \setminus \Sigma(\mathfrak{S}_\phi)$, that is, $\sup_n \phi(\langle \eta_1^{(k)}, \eta_2^{(k)}, \ldots, \eta_n^{(k)}, 0, 0, \ldots\rangle) = \infty$ for each k, choose a strictly increasing sequence of indices $n_k \in \mathbb{N}$ for which $\phi(\langle \eta_1^{(k)}, \ldots, \eta_{n_k}^{(k)}, 0, 0, \ldots\rangle) \geq k$. Thus, if $\xi \in c_o^*$ and for each $k, \xi_i \geq \eta_i^{(k)}$ for all $1 \leq i \leq n_k$, then $\phi(\langle \xi_1, \xi_2, \ldots, \xi_{n_k}, 0, 0, \ldots\rangle) \geq k$ and hence $\xi \notin \Sigma(\mathfrak{S}_\phi)$, which shows that $\mathfrak{S}_\phi$ is ssc.

Finally, to prove that $\mathfrak{S}_\phi^{(o)} \subset \mathfrak{S}_\phi$ form a soft pair, in view of Lemma 4.2(vii), Corollary 4.3(i) and the first two results in this proposition, it suffices to show that $\mathrm{se}(\mathfrak{S}_\phi) \subset \mathfrak{S}_\phi^{(o)}$. Let $\xi \in \Sigma(\mathrm{se}(\mathfrak{S}_\phi))$, i.e., $\xi \leq \alpha\eta$ for some $\alpha \in c_o^*$ and $\eta \in \Sigma(\mathfrak{S}_\phi)$. Then $\phi(\langle \xi_n, \xi_{n+1}, \ldots \rangle) \leq \alpha_n \phi(\langle \eta_n, \eta_{n+1}, \ldots \rangle) \leq \alpha_n \phi(\eta) \to 0$, i.e., $\xi \in \Sigma(\mathfrak{S}_\phi^{(o)})$. □

Remark 4.8.

(i) In the notations of [7] and of this paper, Gohberg and Krein [8] showed that the symmetric norming function $\phi(\eta) := \sup \frac{\eta_a}{\xi_a}$ induces a complete norm on the am-closure $(\xi)^-$ of the principal ideal (ξ) and for this norm

$$\mathrm{cl}(F) = \mathfrak{S}_\phi^{(o)} \subset \mathrm{cl}(\xi) \subset \mathfrak{S}_\phi = (\xi)^-.$$

(ii) The fact that $\mathfrak{S}_\phi$ is soft-complemented was obtained in [18, Theorem 3.8], but Salinas proved only that (in our notations) $\mathrm{se}\,\mathfrak{S}_\phi \subset \mathfrak{S}_\phi^{(o)}$ [18, Remark 3.9]. Varga reached the same conclusion in the case of the am-closure of a principal ideal with a non-trace class generator [20, Remark 3].

(iii) By Lemma 4.2(vi), if I is a Banach ideal such that $\mathfrak{S}_\psi^{(o)} \subset I \subset \mathfrak{S}_\psi$ for some symmetric norming function Ψ and if ϕ is the symmetric norming function induced by the norm of I on $\Sigma(F)$, then $\mathfrak{S}_\phi^{(o)} = \mathfrak{S}_\psi^{(o)}$ and $\mathfrak{S}_\phi = \mathfrak{S}_\psi$ and hence ϕ and ψ are equivalent (cf. [8, Chapter 3, Theorem 2.1]).

(iv) The fact that $\mathfrak{S}_\phi^{(o)} \subset \mathfrak{S}_\phi$ is always a soft pair yields immediately the equivalence of parts (a)–(c) in [18, Theorem 2.3] without the need to consider norms and hence establish (d) and (e).

That $\mathfrak{S}_\phi^{(o)} \subset \mathfrak{S}_\phi$ is a soft pair can help simplify the classical analysis of principal ideals. In [2, Theorem 3.23] Allen and Shen used Salinas' results [18] on (second) Köthe duals to prove that $(\xi) = \mathrm{cl}(\xi)$ if and only if ξ is regular (i.e., $\xi \asymp \xi_a$, or in terms of ideals, if and only if (ξ) is am-stable). In [20, Theorem 3] Varga gave an independent proof of the same result. This result is also a special case of [7, Theorem 2.36], obtained for countably generated ideals by yet independent methods. A still different and perhaps simpler proof of the same result follows immediately from Theorem 2.11 and the fact that $\mathfrak{S}_\phi^{(o)} \subset \mathfrak{S}_\phi$ form a soft pair.

Proposition 4.9. $(\xi) = \mathrm{cl}(\xi)$ *if and only if* ξ *is regular.*

Proof. The inclusion $\mathfrak{S}_\phi^{(o)} \subset (\xi) = \mathrm{cl}(\xi) \subset \mathfrak{S}_\phi = (\xi)^-$ and the fact that (ξ) is soft complemented by Proposition 4.6(i), $\mathfrak{S}_\phi$ is soft complemented by Proposition 4.7, and $\mathfrak{S}_\phi^{(o)} \subset \mathfrak{S}_\phi$ is a soft pair (ibid), proves by applying the sc operation to the above inclusion that $(\xi) = (\xi)^-$. The conclusion now follows from Theorem 2.11. $\square$

Remark 4.10. If $(\xi)^-$ is countably generated, so in particular if it is principal, by Theorem 2.11 it is am-stable and hence $(\xi)^- = ((\xi)^-)_a = (\xi)_a = (\xi_a)$, so that ξ_a is regular. This implies that ξ itself is regular, as was proven in [7, Theorem 3.10] and as is implicit in [20, Theorem IRR]. This conclusion fails for general ideals: we construct in [12] a non am-stable ideal with an am-closure that is countably generated and hence am-stable by Theorem 2.11.

Next we consider **Orlicz ideals** which provide another natural example of soft pairs. Recall from [7, Sections 2.37 and 4.7] that if M is a monotone nondecreasing function on $[0,\infty)$ with $M(0) = 0$, then the small Orlicz ideal $\mathfrak{L}_M^{(o)}$ is the ideal with characteristic set $\{\xi \in c_o^* \mid \sum_n M(t\xi_n) < \infty$ for all $t > 0\}$ and the Orlicz ideal $\mathfrak{L}_M$ is the ideal with characteristic set $\{\xi \in c_o^* \mid \sum_n M(t\xi_n) < \infty$ for some $t > 0\}$. If the function M is convex, then $\mathfrak{L}_M^{(o)}$ and $\mathfrak{L}_M$ are respectively the ideals $\mathfrak{S}_\phi^{(o)}$ and $\mathfrak{S}_\phi$ for the symmetric norming function defined by

$$\phi\big(\langle \xi_1, \xi_2, \ldots, \xi_n, 0, 0, \ldots\rangle\big) := \inf_{t>0}\left\{\frac{1}{t} \mid \sum_{i=1}^n M(t\xi_i) \le 1\right\}.$$

Thus, when M is convex, $\mathfrak{L}_M^{(o)} \subset \mathfrak{L}_M$ form a soft pair by Proposition 4.7. In fact, the same can be proven directly without assuming convexity for M.

Proposition 4.11. *Let M be a monotone nondecreasing function on $[0,\infty)$ with $M(0) = 0$. Then $\mathfrak{L}_M^{(o)}$ is soft-edged, $\mathfrak{L}_M$ is ssc, and $\mathfrak{L}_M^{(o)} \subset \mathfrak{L}_M$ is a soft pair.*

Proof. Take $\xi \in \Sigma(\mathfrak{L}_M^{(o)})$ and choose a strictly increasing sequence of indices $n_k \in \mathbb{N}$ such that $\sum_{i=n_{k-1}+1}^\infty M(k^2\xi_i) \le 2^{-k}$ and $k\xi_{n_k} \downarrow 0$. As in the proof of Proposition 4.7, set $n_0 = 0$ and $\beta_i := k$ for all $n_{k-1} < i \le n_k$ and $\eta := \mathrm{lni}\,\beta\xi$. Then $\eta \in c_o^*$ and $\xi = o(\eta)$. Let $t > 0$ be arbitrary and fix an integer $k \ge t$. Then since $\eta \le \beta\xi$

and M is monotone nondecreasing, it follows that

$$\sum_{i=n_k+1}^{\infty} M(t\eta_i) \leq \sum_{i=n_k+1}^{\infty} M(k\beta_i\xi_i) = \sum_{j=k+1}^{\infty} \sum_{i=n_{j-1}+1}^{n_j} M(kj\xi_i)$$
$$\leq \sum_{j=k+1}^{\infty} \sum_{i=n_{j-1}+1}^{\infty} M(j^2\xi_i) \leq \sum_{j=k+1}^{\infty} 2^{-j} < \infty.$$

Therefore $\eta \in \Sigma(\mathcal{L}_M^{(o)})$, which proves that $\mathcal{L}_M^{(o)}$ is soft-edged.

Next we prove that $\mathcal{L}_M$ is ssc. For every countable collection of sequences $\eta^{(k)} \in c_o^* \setminus \Sigma(\mathcal{L}_M)$, since $\sum_i M(\frac{1}{k}\eta_i^{(k)}) = \infty$ for all k, we can choose a strictly increasing sequence of indices $n_k \in \mathbb{N}$ such that $\sum_{i=1}^{n_k} M(\frac{1}{k}\eta_i^{(k)}) \geq k$. If $\xi \in c_o^*$ and $\xi_i \geq \eta_i^{(k)}$ for all $1 \leq i \leq n_k$, then for all m and all $k \geq m$ it follows that

$$\sum_{i=1}^{n_k} M(\frac{1}{m}\xi_i) \geq \sum_{i=1}^{n_k} M(\frac{1}{k}\xi_i) \geq \sum_{i=1}^{n_k} M(\frac{1}{k}\eta_i^{(k)}) \geq k$$

and hence $\sum_i M(t\xi_i) = \infty$ for all $t > 0$. Thus $\xi \notin \Sigma(\mathcal{L}_M)$, which proves that $\mathcal{L}_M$ is ssc.

To prove that $\mathcal{L}_M^{(o)} \subset \mathcal{L}_M$ is a soft pair, since $\mathcal{L}_M^{(o)}$ is soft-edged and $\mathcal{L}_M$ is soft-complemented, by Lemma 4.2(vii) it suffices to prove that $\operatorname{se}\mathcal{L}_M \subset \mathcal{L}_M^{(o)}$. Let $\xi \in \Sigma(\mathcal{L}_M)$, let $t_o > 0$ be such that $\sum_n M(t_o\xi_n) < \infty$, and let $\alpha \in c_o^*$. For each $t > 0$ choose N so that $t\alpha_n \leq t_o$ for $n \geq N$. By the monotonicity of M, $\sum_{n=N}^{\infty} M(t\alpha_n\xi_n) < \infty$ and hence $\alpha\xi \in \Sigma(\mathcal{L}_M^{(o)})$. □

The fact that $\mathcal{L}_M^{(o)} \subset \mathcal{L}_M$ forms a soft pair can simplify proofs of some properties of Orlicz ideals. Indeed, together with [10, Proposition 3.4] that states that for an ideal I, $\operatorname{se} I$ is am-stable if and only if $\operatorname{sc} I$ is am-stable if and only if $I_a \subset \operatorname{sc} I$, and combined with Lemma 4.16 below it yields an immediate proof of the following results in [7]: the equivalence of (a), (b), (c) in Theorem 4.21 and hence the equivalence of (a), (b), (c) in Theorem 6.25, the equivalence of (b), (c), and (d) in Corollary 2.39, the equivalence of (b) and (c) in Corollary 2.40, and the equivalence of (a), (b), and (c) in Theorem 3.21.

Next we consider **Lorentz ideals**. If ϕ is a monotone nondecreasing nonnegative sequence satisfying the Δ_2-condition, i.e., $\sup \frac{\phi_{2n}}{\phi_n} < \infty$, then in the notations of [7, Sections 2.25 and 4.7] the Lorentz ideal $\mathcal{L}(\phi)$ corresponding to the sequence space $\ell(\phi)$ is the ideal with characteristic set

$$\Sigma(\mathcal{L}(\phi)) := \left\{ \xi \in c_o^* \mid \|\xi\|_{\ell(\phi)} := \sum_n \xi_n(\phi_{n+1} - \phi_n) < \infty \right\}.$$

A special case of Lorentz ideal is the trace class $\mathcal{L}_1$ which corresponds to the sequence $\phi = \langle n \rangle$ and the sequence space $\ell(\phi) = \ell_1$.

Notice that $\mathcal{L}(\phi)$ is also the Köthe dual $\{\langle \phi_{n+1} - \phi_n \rangle\}^{\times} = \ell_1^* : \{\langle \phi_{n+1} - \phi_n \rangle\}$ of the singleton set consisting of the sequence $\langle \phi_{n+1} - \phi_n \rangle$ (cf. [7, Section 2.8(iv)]). $\mathcal{L}(\phi)$ is a Banach ideal with norm induced by the cone norm $\| \cdot \|_{\ell(\phi)}$ on $\ell(\phi)^*$ if and only if the sequence ϕ is concave (cf. [7, Lemma 2.29 and Section 4.7]), and it is easy to verify that in this case $\ell(\phi)^* = \mathfrak{S}_{\psi}^{(o)} = \mathfrak{S}_{\psi}$ where ψ is the restriction of $\| \cdot \|_{\ell(\phi)}$ to $\Sigma(F)$. Thus by Proposition 4.7, $\mathcal{L}(\phi)$ is both strongly soft-complemented and soft-edged. In fact, the same holds without the concavity assumption for ϕ as we see in the next proposition.

Proposition 4.12. *If ϕ be a monotone nondecreasing nonnegative sequence satisfying the Δ_2-condition, then $\mathcal{L}(\phi)$ is both soft-edged and strongly soft-complemented.*

Proof. For $\xi \in \Sigma(\mathcal{L}(\phi))$, choose a strictly increasing sequence of indices $n_k \in \mathbb{N}$ with $k\xi_{n_k} \downarrow 0$ and $\sum_{i=n_k}^{\infty} \xi_i(\phi_{i+1} - \phi_i) \le 2^{-k}$. As in Proposition 4.7(proof), set $n_o = 0$, $\beta_i := k$ for all $n_{k-1} < i \le n_k$, hence $\eta = \operatorname{lni} \beta\xi \in c_o^*$ and $\xi = o(\eta)$. Then

$$\sum_i^{\infty} \eta_i(\phi_{i+1} - \phi_i) \le \sum_i^{\infty} \beta_i \xi_i(\phi_{i+1} - \phi_i) = \sum_{k=1}^{\infty} \sum_{i=n_{k-1}+1}^{n_k} k\xi_i(\phi_{i+1} - \phi_i) \le \sum_{k=1}^{\infty} k2^{-k+1} < \infty,$$

whence $\eta \in \Sigma(\mathcal{L}(\phi))$. Thus $\xi \in \Sigma(\operatorname{se} \mathcal{L}(\phi))$ and hence $\mathcal{L}(\phi)$ is soft-edged.

Finally, for every sequence of sequences $\{\eta^{(k)}\} \subset c_o^* \setminus \Sigma(\mathcal{L}(\phi))$, choose a strictly increasing sequence $n_k \in \mathbb{N}$ such that for all k, $\sum_{i=1}^{n_k} \eta_i^{(k)}(\phi_{i+1} - \phi_i) \ge k$. Thus if $\xi \in c_o^*$ and $\xi_i \ge \eta_i^{(k)}$ for all $1 \le i \le n_k$, then $\sum_{i=1}^{n_k} \xi_i(\phi_{i+1} - \phi_i) \ge k$ and hence $\xi \notin \Sigma(\mathcal{L}(\phi))$, thus proving that $\mathcal{L}(\phi)$ is ssc. □

In particular, we use frequently that $\mathcal{L}_1$ is both soft-edged and soft-complemented.

As the next proposition shows, any quotient with a soft-complemented ideal as numerator is always soft-complemented (cf. first paragraph of this section for the definition of quotient), but as Example 4.15 shows, even a Köthe dual of a singleton can fail to be strongly soft-complemented.

Proposition 4.13. *Let I be a soft-complemented ideal and let X be a nonempty subset of $[0,\infty)^{\mathbb{N}}$. Then the ideal with characteristic set $\Sigma(I) : X$ is soft-complemented.*

Proof. Let $\xi \in c_o^* \setminus (\Sigma(I) : X)$, i.e., $((D_m\xi)x)^* \notin \Sigma(I)$ for some $m \in \mathbb{N}$ and $x \in X$. As I is soft-complemented, there exists $\alpha \in c_o^*$ such that $\alpha((D_m\xi)x)^* \notin \Sigma(I)$. Let π be an injection that monotonizes $(D_m\xi)x$, i.e., $(((D_m\xi)x)^*)_i = ((D_m\xi)x)_{\pi(i)}$ for all i. Define

$$\gamma_j := \begin{cases} \alpha_{\pi^{-1}(j)} & \text{if } j \in \pi(N) \\ 0 & \text{if } j \notin \pi(N) \end{cases}.$$

Then $\gamma \to 0$ and hence $\operatorname{uni}\gamma \in c_o^*$. Thus for all i,

$$\begin{aligned} (\alpha((D_m\xi)x)^*)_i &= \gamma_{\pi(i)}(D_m\xi)_{\pi(i)}x_{\pi(i)} \\ &\le (\operatorname{uni}\gamma)_{\pi(i)}(D_m\xi)_{\pi(i)}x_{\pi(i)} \\ &\le (D_m((\operatorname{uni}\gamma)\xi))_{\pi(i)}x_{\pi(i)}. \end{aligned}$$

From this inequality, and from the elementary fact that for two sequence ρ and μ, $0 \le \rho \le \mu$ implies $\rho^* \le \mu^*$, it follows that $\alpha((D_m\xi)x)^* \le ((D_m((\operatorname{uni}\gamma)\xi))x)^*$. Thus $((D_m((\operatorname{uni}\gamma)\xi))x)^* \notin \Sigma(I)$, i.e., $(\operatorname{uni}\gamma)\xi \notin \Sigma(I) : X$, proving the claim. □

Remark 4.14. If X is itself a characteristic set, the above result follows by the simple identities for ideals I, J, L analogous to the numerical quotient operation "$\div$":

$$(I : J) : L = I : (JL) = (I : L) : J$$

Indeed if in these identities we set $L = K(H)$ (the ideal of compact operators), we obtain $\operatorname{sc}(I : J) = I : \operatorname{se} J = \operatorname{sc} I : J$. Thus if I is soft-complemented or J is soft-edged it follows that $I : J$ is soft-complemented. As an aside:

$$(I : J)J \subset I \subset (IJ : J) \subset I : J$$

and each of the embeddings can be proper (see also [18]).

Example 4.15. The Köthe dual $I := \{\langle e^n\rangle\}^\times$ of the singleton $\{\langle e^n\rangle\}$ is soft-complemented by Proposition 4.13 but it is not strongly soft-complemented. Indeed, by definition, $\xi \in \Sigma(I)$ if and only if $((D_m\xi)\langle e^n\rangle)^* \in \ell_1^*$ (or, equivalently, $(D_m\xi)\langle e^n\rangle \in \ell_1$) for every m, which in turns is equivalent to $\sum_n \xi_n e^{mn} < \infty$ for every m. Choose $\eta \in c_o^*$ such that $\sum_n \eta_n e^n < \infty$ but $\sum_n \eta_n e^{2n} = \infty$ and hence $\eta \notin \Sigma(I)$, and set $\eta^{(k)} := D_{1/k}\eta$, i.e., $\eta_i^{(k)} = \eta_{ki}$ for all i. As $(D_{2k}\eta^{(k)})_i \geq \eta_i$ for $i \geq k$, it follows that for every k, $D_{2k}\eta^{(k)}$ and hence $\eta^{(k)}$ are not in $\Sigma(I)$. Let $n_k \in \mathbb{N}$ be an arbitrary strictly increasing sequence of indices, set $n_o = 0$ and define $\xi_i := \eta_i^{(k)}$ for $n_{k-1} < i \leq n_k$. As $\eta^{(k+1)} \leq \eta^{(k)}$, it follows that ξ is monotone nonincreasing and for all k, $\xi_i \geq \eta_i^{(k)}$ for $1 \leq i \leq n_k$. On the other hand, for all m and for all $k \geq m$,

$$\sum_{i=n_{k-1}+1}^{n_k} \xi_i e^{mi} \leq \sum_{i=n_{k-1}+1}^{n_k} \eta_{ki} e^{ki} \leq \sum_{i=kn_{k-1}+1}^{kn_k} \eta_i e^i$$

and thus

$$\sum_{i=n_{m-1}+1}^{\infty} \xi_i e^{mi} \leq \sum_{i=n_{m-1}+1}^{\infty} \eta_i e^i < \infty,$$

which proves that $\xi \in \Sigma(I)$ and hence that I is not ssc.

Next we consider **idempotent ideals**, i.e., ideals for which $I = I^2$. Notice that an ideal is idempotent if and only if $I = I^p$ for some $p \neq 0, 1$, if and only if $I = I^p$ for all $p \neq 0$. The following lemma is an immediate consequence of Definition 4.1, the remarks following it, and of Definition 4.4.

Lemma 4.16. *For every ideal I and $p > 0$:*

(i) $\mathrm{se}(I^p) = (\mathrm{se}\, I)^p$ *and* $\mathrm{sc}(I^p) = (\mathrm{sc}\, I)^p$

In particular, if I is soft-edged or soft-complemented, then so respectively is I^p.

(ii) *If $I \subset J$ is a soft pair, then so is $I^p \subset J^p$.*

(iii) *If I is ssc, then so is I^p.*

Proposition 4.17. *Idempotent ideals are both soft-edged and soft-complemented.*

Proof. Let I be an idempotent ideal. That I is soft-edged follows from the inclusions $I = I^2 \subset K(H)I = \mathrm{se}\, I \subset I$. That I is soft-complemented follows from the inclusions

$$\operatorname{sc} I = \operatorname{sc}(I^2) = (\operatorname{sc} I)^2 \subset K(H) \operatorname{sc} I = \operatorname{se}(\operatorname{sc} I) = \operatorname{se} I \subset I \subset \operatorname{sc} I$$

which follows from Lemmas 4.16 and 4.2(iii),(iv). □

The remarks following Proposition 5.3 show that idempotent ideals may fail to be strongly soft-complemented.

Finally, we consider the **Marcinkiewicz ideals** namely, the pre-arithmetic means of principal ideals, and we consider also their am-∞ analogs. That these ideals are strongly soft-complemented follows from the following proposition combined with Proposition 4.6(i).

Proposition 4.18. *The pre-arithmetic mean and the pre-arithmetic mean at infinity of a strongly soft-complemented ideal is strongly soft-complemented.*

In particular, Marcinkiewicz ideals are strongly soft-complemented.

Proof. Let I be an ssc ideal. We first prove that ${}_aI$ is ssc. Let $\{\eta^{(k)}\} \subset c_o^* \setminus \Sigma({}_aI)$, i.e., $\{\eta_a^{(k)}\} \subset c_o^* \setminus \Sigma(I)$, and let $n_k \in \mathbb{N}$ be a strictly increasing sequence of indices for which if $\zeta \in c_o^*$ and $\zeta_i \geq (\eta_a^{(k)})_i$ for all $1 \leq i \leq n_k$ and all k, then $\zeta \notin \Sigma(I)$. Let $\xi \in c_o^*$ and $\xi_i \geq (\eta^{(k)})_i$ for all $1 \leq i \leq n_k$ and all k. But then $(\xi_a)_i \geq (\eta_a^{(k)})_i$ for all $1 \leq i \leq n_k$ and all k and hence $\xi_a \notin \Sigma(I)$, i.e., $\xi \notin \Sigma({}_aI)$.

We now prove that ${}_{a_\infty}I$ is ssc. Let $\{\eta^{(k)}\} \subset c_o^* \setminus \Sigma({}_{a_\infty}I)$. Assume first that infinitely many of the sequences $\eta^{(k)}$ are not summable. Since the trace class $\mathcal{L}_1$ is ssc by Proposition 4.12, there is an associated increasing sequence of indices $n_k \in \mathbb{N}$ so that if $\xi \in c_o^*$ and $\xi_i \geq \eta_i^{(k)}$ for all $1 \leq i \leq n_k$, then $\xi \notin \Sigma(\mathcal{L}_1)$ and hence $\xi \notin \Sigma({}_{a_\infty}I)$ since ${}_{a_\infty}I \subset \mathcal{L}_1$. Thus assume without loss of generality that all $\eta^{(k)}$ are summable and hence $\eta_{a_\infty}^{(k)} \notin \Sigma(I)$. Let $n_k \in \mathbb{N}$ be a strictly increasing sequence of indices for which $\zeta \notin \Sigma(I)$ whenever $\zeta \in c_o^*$ and $\zeta_i \geq (\eta_{a_\infty}^{(k)})_i$ for all $1 \leq i \leq n_k$ and all k. For every k and n choose an integer $p(k,n) \geq n$ for which $\sum_{i=n}^{p(k,n)} \eta_i^{(k)} \geq \frac{1}{2} \sum_{i=n}^{\infty} \eta_i^{(k)}$. Set $N_k := \max\{p(k,n) \mid 1 \leq n \leq n_k + 1\}$. For any $\xi \in c_o^*$ such that $\xi_i \geq \eta_i^{(k)}$ for all $1 \leq i \leq N_k$ consider two cases. If ξ is not summable then $\xi \notin \Sigma({}_{a_\infty}I)$ trivially. If ξ is summable, then for all $1 \leq n \leq n_k$ and for all k

$$\begin{aligned}\left(\xi_{a_\infty}\right)_n &= \frac{1}{n}\sum_{i=n+1}^{\infty}\xi_i \geq \frac{1}{n}\sum_{i=n+1}^{N_k}\xi_i \geq \frac{1}{n}\sum_{i=n+1}^{N_k}\eta_i^{(k)} \\ &\geq \frac{1}{n}\sum_{i=n+1}^{p(k,n+1)}\eta_i^{(k)} \geq \frac{1}{2n}\sum_{i=n+1}^{\infty}\eta_i^{(k)} = \frac{1}{2}\left(\eta_{a_\infty}^{(k)}\right)_n\end{aligned}$$

and hence $\xi_{a_\infty} \notin \Sigma(I)$, i.e., $\xi \notin \Sigma({}_{a_\infty}I)$. □

That Marcinkiewicz ideals are ssc can be seen also by the following consequence of Proposition 4.7. If I is a Marcinkiewicz ideal, then $I = {}_a(\xi) = {}_a((\xi)^o)$ for some $\xi \in c_o^*$. By Lemma 2.13, $(\xi)^o = (\eta_a) = (\eta)_a$ for some $\eta \in c_o^*$. Thus $I = {}_a((\eta)_a) = (\eta)^-$ and $(\eta)^-$ is ssc by Proposition 4.7 and Remark 4.8(i).

Corollary 6.7 and Proposition 6.11 below show that the pre-arithmetic mean (resp., the pre-arithmetic mean at infinity) also preserve soft-complementedness. They also show that the am-interior and the am-closure of a soft-edged ideal are soft-edged, that the am-interior of a soft-complemented ideal is soft-complemented by Proposition 6.11, and that the same holds for the corresponding am-∞ operations. However, as mentioned prior to Proposition 6.8, (resp., Proposition 6.11) we do not know whether the am-closure (resp., the am-∞ closure) of a soft-complemented ideal is soft-complemented. Likewise, we do not know whether the am-closure (resp., am-∞ closure) of an ssc ideal is ssc.

One non-trivial case in which we can prove directly that the am-closure of an ssc ideal is scc is the following. If I is countably generated, then I_a too is countably generated and hence, by Propositions 4.6(i) and 4.18(i), its am-closure I^- is also ssc, and then by Lemma 4.16 so is $(I^-)^p$ for any $p > 0$. The latter ideal is in general not countably generated (e.g., if $0 \neq \xi \in \ell_1^*$, then $(\xi)^- = \mathcal{L}_1$ is not countably generated) but Lemma 4.19 below shows that nevertheless its am-closure is ssc.

Lemma 4.19. *For every ideal I,*

$$\left((I^-)^p\right)^- = \begin{cases}(I^-)^p & \text{for } 0 < p \leq 1\\ (I^p)^- & \text{for } p \geq 1.\end{cases}$$

Proof. Let $\xi \in \Sigma(((I^-)^p)^-)$. By definition, $\xi_a \leq \eta_a$ for some $\eta \in \Sigma((I^-)^p)$, i.e., $\eta^{1/p} \in \Sigma(I^-)$, which in turns holds if and only if $(\eta^{1/p})_a \leq \rho_a$ for some $\rho \in \Sigma(I)$. Recall from [17, 3.C.1.b] that if μ and ν are monotone sequences and $\mu_a \leq \nu_a$, then $(\mu^q)_a \leq (\nu^q)_a$ for $q \geq 1$. Thus, if $p \leq 1$, $(\xi^{1/p})_a \leq (\eta^{1/p})_a \leq \rho_a$ and consequently $\xi^{1/p} \in \Sigma(I^-)$, i.e., $\xi \in \Sigma((I^-)^p)$. Thus $((I^-)^p)^- \subset (I^-)^p$, which then implies equality since the reverse inclusion is automatic. If $p > 1$, the inequality $(\eta^{1/p})_a \leq \rho_a$ implies for the same reason that $\eta_a \leq (\rho^p)_a$. Hence $\xi_a \leq (\rho^p)_a$, i.e., $\xi \in \Sigma((I^p)^-)$. Thus $((I^-)^p)^- \subset (I^p)^-$, which then implies equality since the reverse inclusion is again automatic. □

Proposition 4.20. *If I is countably generated and $0 < p < \infty$, then $((I^-)^p)^-$ is strongly soft-complemented.*

5. Operations on Soft Ideals

In this section we investigate the soft interior and soft cover of arbitrary intersections of ideals, unions of collections of ideals directed by inclusion, and finite sums of ideals.

Proposition 5.1. *For every collection of ideals $\{I_\gamma, \gamma \in \Gamma\}$:*

(i) $\bigcap_\gamma \operatorname{se} I_\gamma \supset \operatorname{se}(\bigcap_\gamma I_\gamma)$

(ii) $\bigcap_\gamma \operatorname{sc} I_\gamma = \operatorname{sc}(\bigcap_\gamma I_\gamma)$

In particular, the intersection of soft-complemented ideals is soft-complemented.

Proof. (i) and the inclusion $\bigcap_\gamma \operatorname{sc} I_\gamma \supset \operatorname{sc}(\bigcap_\gamma I_\gamma)$ are immediate consequences of Lemma 4.2(i). For the reverse inclusion in (ii), by (i) and Lemma 4.2 (i)–(iv) we have:

$$\operatorname{sc}\left(\bigcap_\gamma I_\gamma\right) \supset \bigcap_\gamma I_\gamma \supset \bigcap_\gamma \operatorname{se} I_\gamma = \bigcap_\gamma \operatorname{se}(\operatorname{sc} I_\gamma) \supset \operatorname{se}\left(\bigcap_\gamma \operatorname{sc} I_\gamma\right)$$

and hence

$$\operatorname{sc}\left(\bigcap_\gamma I_\gamma\right) \supset \operatorname{sc}\left(\operatorname{se}\left(\bigcap_\gamma \operatorname{sc} I_\gamma\right)\right) = \operatorname{sc}\left(\bigcap_\gamma \operatorname{sc} I_\gamma\right) \supset \bigcap_\gamma \operatorname{sc} I_\gamma.$$

□

It follows directly from Definition 4.1 that if Γ is finite, then equality holds in (i). In general, equality in (i) does not hold, as seen in Example 5.2 below, where the intersection of soft-edged ideals fails to be soft-edged, thus showing that the inclusion in (i) is proper.

Example 5.2. Let $\xi \in c_o^*$ be a sequence that satisfies the $\Delta_{1/2}$-condition, i.e., $\sup \frac{\xi_n}{\xi_{2n}} < \infty$, and let $\{I_\gamma\}_{\gamma\in\Gamma}$ be the collection of all soft-edged ideals containing the principal ideal (ξ). Then $I := \bigcap_\gamma I_\gamma$ is not soft-edged. Indeed, assume that it is and hence $\xi = o(\eta)$ for some $\eta \in \Sigma(I)$. By Lemma 6.3 of the next section, there is a sequence $\gamma \uparrow \infty$ for which $\gamma \leq \frac{\eta}{\xi}$ and $\mu := \gamma\xi \in c_o^*$. Then

$$(\xi) \subset \mathrm{se}(\mu) \subset (\mu) \subset (\eta) \subset I.$$

Then $\mathrm{se}(\mu) \in \{I_\gamma\}_{\gamma\in\Gamma}$, hence $I \subset \mathrm{se}(\mu)$, and thus $\mathrm{se}(\mu) = (\mu)$. By Proposition 4.6(ii), this implies that $\mu = o(D_m\mu)$ for some integer m. This is impossible since $\frac{\mu_n}{\mu_{2n}} = \frac{\gamma_n}{\gamma_{2n}}\frac{\xi_n}{\xi_{2n}} \leq \frac{\xi_n}{\xi_{2n}}$ which implies that μ too satisfies the $\Delta_{1/2}$-condition and hence $D_m\mu = O(\mu)$, a contradiction.

Notice that the conclusion that $\bigcap_\gamma I_\gamma$ is not soft-edged follows likewise if $\{I_\gamma\}$ is a maximal chain of soft-edged ideals that contain the principal ideal (ξ). Moreover, Example 5.2 shows that in general there is no smallest soft-edged cover of an ideal.

The next proposition shows that an intersection of strongly soft-complemented ideals, which is soft-complemented by Proposition 5.1(ii), can yet fail to be strongly soft-complemented.

Proposition 5.3. *The intersection of an infinite countable strictly decreasing chain of principal ideals is never strongly soft-complemented.*

Proof. Let $\{I_k\}$ be the chain of principal ideals with $I_k \supsetneqq I_{k+1}$ and set $I = \bigcap_k I_k$. First we find generators $\eta^{(k)} \in c_o^*$ for the ideals I_k such that $\eta^{(k)} \geq \eta^{(k+1)}$. Assuming the construction up to $\eta^{(k)}$, if ξ is a generator of I_{k+1} then $\xi \leq MD_m\eta^{(k)}$ for some $M > 0$ and $m \in \mathbb{N}$. Set $\eta^{(k+1)} := \frac{1}{M}D_{1/m}\xi$, where $(D_{1/m}\xi)_i = \xi_{mi}$. Then $\eta^{(k+1)} \in c_o^*$ and $\eta^{(k+1)} \leq \eta^{(k)}$ since $D_{1/m}D_m = id$. Moreover, $\eta^{(k+1)} \leq \frac{1}{M}\xi$ and by an elementary computation, $\xi_i \leq (D_{2m}D_{1/m}\xi)_i$ for

$i \geq m$ so that $(\xi) \subset (\eta^{(k+1)})$ and hence $I_{k+1} = (\xi) = (\eta^{(k+1)})$. By assumption, $\eta^{(k)} \notin \Sigma(I)$ for all k. For any given strictly increasing sequence of indices $n_k \in \mathbb{N}$, set $n_o = 0$ and $\xi_i := \eta_i^{(k)}$ for $n_{k-1} < i \leq n_k$. Since $\eta^{(k)} \geq \eta^{(k+1)}$ for all k, it follows that $\xi \in c_o^*$ and $\xi_i \geq \eta_i^{(k)}$ for $1 \leq i \leq n_k$. Yet, since $\xi_i \leq \eta_i^{(k)}$ for all $i \geq n_k$, one has $\xi \in \Sigma(\eta^{(k)})$ for all k and hence $\xi \in \Sigma(I)$. Thus I is not strongly soft-complemented. □

Notice that if in the above construction $\eta^{(k)} = \rho^k$ for some $\rho \in c_o^*$ that satisfies the $\Delta_{1/2}$-condition, then $I = \bigcap_k (\rho^k)$ is also idempotent. This shows that while idempotent ideals are soft-complemented by Proposition 4.17, they can fail to be strongly soft-complemented.

Proposition 5.4. *For* $\{I_\gamma\}_{\gamma \in \Gamma}$ *a collection of ideals directed by inclusion:*

(i) $\bigcup_\gamma \operatorname{se} I_\gamma = \operatorname{se}(\bigcup_\gamma I_\gamma)$

In particular, the directed union of soft edged ideals is soft-edged.

(ii) $\bigcup_\gamma \operatorname{sc} I_\gamma \subset \operatorname{sc}(\bigcup_\gamma I_\gamma)$

Proof. As in Proposition 5.1, (ii) and the inclusion $\bigcup_\gamma \operatorname{se} I_\gamma \subset \operatorname{se}(\bigcup_\gamma I_\gamma)$ in (i) are immediate. For the reverse inclusion in (i), from (ii) and Lemma 4.2(iii) and (iv) we have

$$\operatorname{se}\left(\bigcup_\gamma I_\gamma\right) \subset \operatorname{se}\left(\bigcup_\gamma \operatorname{sc}(\operatorname{se} I_\gamma)\right) \subset \operatorname{se}\left(\operatorname{sc}\left(\bigcup_\gamma \operatorname{se} I_\gamma\right)\right) = \operatorname{se}\left(\bigcup_\gamma \operatorname{se} I_\gamma\right) \subset \bigcup_\gamma \operatorname{se} I_\gamma.$$

□

It follows directly from Definition 4.1 that if Γ is finite, then equality holds in (ii), but in general, it does not. Indeed, any ideal I is the union of the collection of all the principal ideals contained in I and this collection is directed by inclusion since $(\eta) \subset I$ and $(\mu) \subset I$ imply that $(\eta), (\mu) \subset (\eta + \mu) \subset I$. By Proposition 4.6(i), principal ideals are ssc, hence soft-complemented. Notice that by assuming the continuum hypothesis, every ideal I is the union of an increasing nest of countably generated ideals [3], so then even nested unions of ssc ideals can fail to be soft-complemented.

The smallest nonzero am-stable ideal $st^a(\mathcal{L}_1) = \bigcup_{m=0}^{\infty} = (\omega)_{a^m}$ and the largest am-∞ stable ideal $st_{a_\infty}(\mathcal{L}_1) = \bigcap_{m=0}^{\infty} {}_{a_\infty^m}(\mathcal{L}_1)$ (see Section 2) play an important role in [9, 10].

Proposition 5.5. *The ideals $st^a(\mathcal{L}_1)$ and $st_{a_\infty}(\mathcal{L}_1)$ are both soft-edged and soft-complemented, $st^a(\mathcal{L}_1)$ is ssc, but $st_{a_\infty}(\mathcal{L}_1)$ is not ssc.*

Proof. For every natural number m, $(\omega)_{a^m} = (\omega_{a^m}) = (\omega \log^m)$ is principal, hence $\Sigma(st^a(\mathcal{L}_1))$ is generated by the collection$\{\omega \log^m\}_m$. Since $\omega \log^m = o(\omega \log^{m+1})$ for all m, by Proposition 4.6(i) and (ii), $st^a(\mathcal{L}_1)$ is both soft-edged and ssc. From [10, Proposition 4.17 (ii)], $st_{a_\infty}(\mathcal{L}_1) = \bigcap_{m=0}^{\infty} \mathcal{L}(\sigma(\log^m))$, where using the notations of [7, Sections 2.1, 2.25, 4.7], $\mathcal{L}(\sigma(\log^m))$ is the Lorentz ideal with characteristic set $\{\xi \in c_o^* \mid \xi(\log)^m \in \ell_1\}$. Thus if $\xi \in \Sigma(\bigcap_{m=0}^{\infty} \mathcal{L}(\sigma(\log^m)))$, then also $\xi \log \in \Sigma(\bigcap_{m=0}^{\infty} \mathcal{L}(\sigma(\log^m)))$ and hence $st_{a_\infty}(\mathcal{L}_1)$ is soft-edged. By Propositions 4.12 and 5.1(ii), $st_{a_\infty}(\mathcal{L}_1)$ is soft-complemented. However, $st_{a_\infty}(\mathcal{L}_1)$ is not ssc. Indeed, set $\eta^{(k)} := \omega(\log)^{-k}$. Then $\eta^{(k)} \notin \Sigma(st_{a_\infty}(\mathcal{L}_1))$ for all k, but $\eta^{(k)} \in \Sigma(\mathcal{L}(\sigma(\log^{k-2})))$ for each $k \geq 2$. For any arbitrary sequence of increasing indices n_k, set $n_o = 0$ and $\xi_j := (\eta^{(k)})_j$ for $n_{k-1} < j \leq n_k$. Then $\xi_j \geq (\eta^{(k)})_j$ for $1 \leq j \leq n_k$ but also $\xi_j \leq (\eta^{(k)})_j$ for $j \geq n_k$. Thus $\xi \in \Sigma(\mathcal{L}(\sigma(\log^{k-2})))$ for all $k \geq 2$, hence $\xi \in \Sigma(st_{a_\infty}(\mathcal{L}_1))$ which shows that $st_{a_\infty}(\mathcal{L}_1)$ is not ssc. □

Now consider finite sums of ideals. Clearly, $K(H)(I+J) = K(H)I + K(H)J$, i.e., $\mathrm{se}(I+J) = \mathrm{se}\, I + \mathrm{se}\, J$ and hence finite sums of soft-edged ideals are soft-edged.

The situation is far less simple for the soft-cover of a finite sum of ideals. The inclusion $\mathrm{sc}(I+J) \supset \mathrm{sc}\, I + \mathrm{sc}\, J$ is trivial, but so far we are unable to determine whether or not equality holds in general or, equivalently, whether or not the sum of two soft-complemented ideals is always soft-complemented. We also do not know if the sum of two ssc ideals is always soft-complemented. However, the following lemma permits us to settle the latter question in the affirmative when one of the ideals is countably generated. Recall that if $0 \leq \lambda \in c_o$, then λ^* denotes the decreasing rearrangement of λ.

Lemma 5.6. *For all ideals I, J and sequences $\xi \in c_o^*$:*

$$\xi \in \Sigma(I+J) \quad \text{if and only if} \quad (\max((\xi - \rho), 0))^* \in \Sigma(I) \text{ for some } \rho \in \Sigma(J).$$

Proof. If $\xi \in \Sigma(I+J)$, then $\xi \le \zeta + \rho$ for some $\zeta \in \Sigma(I)$ and $\rho \in \Sigma(J)$. (Actually, one can choose ζ and ρ so that $\xi = \zeta + \rho$ but equality is not needed here.) Thus $\xi - \rho \le \zeta$, and so $\max((\xi - \rho), 0) \le \zeta$. But then, by the elementary fact that if for two sequence $0 \le \nu \le \mu$, then $\nu^* \le \mu^*$, it follows that $\max((\xi-\rho), 0)^* \le \zeta^* = \zeta$ and hence $(\max((\xi-\rho), 0))^* \in \Sigma(I)$. Conversely, assume that $(\max((\xi-\rho), 0))^* \in \Sigma(I)$ for some $\rho \in \Sigma(J)$. Since $0 \le \xi \le \max((\xi-\rho), 0) + \rho$,

$$\xi = \xi^* \le (\max((\xi - \rho), 0) + \rho)^* \le D_2(\max((\xi-\rho), 0)^*) + D_2\rho \in \Sigma(I+J),$$

where the second inequality, follows from the fact that $(\rho + \mu)^* \le D_2\rho^* + D_2\mu^*$ for any two non-negative sequences ρ and μ, which fact is likely to be previously known but is also the commutative case of a theorem of K. Fan [8, II Corollary 2.2, Equation (2.12)]. □

Theorem 5.7. *The sum $I + J$ of an ssc ideal I and a countably generated ideal J is ssc and hence soft-complemented.*

Proof. As in the proof of Lemma 2.8 there is an increasing sequence of generators $\rho^{(k)} \le \rho^{(k+1)}$ for the characteristic set $\Sigma(J)$ such that $\mu \in \Sigma(J)$ if and only if $\mu = O(\rho^{(m)})$ for some integer m. By passing if necessary to the sequences $k\rho^{(k)}$, we can further assume that $\mu \in \Sigma(J)$ if and only if $\mu \le \rho^{(m)}$ for some integer m. Let $\{\eta^{(k)}\} \subset c_o^* \setminus \Sigma(I+J)$. By Lemma 5.6, for each k, $(\max((\eta^{(k)} - \rho^{(k)}), 0))^* \notin \Sigma(I)$ so, in particular, $\eta_i^{(k)} > \rho_i^{(k)}$ for infinitely many indices i. Let $\pi_k : \mathbb{N} \to \mathbb{N}$ be a monotonizing injection for $\max((\eta^{(k)} - \rho^{(k)}), 0)$, i.e., for all $i \in \mathbb{N}$,

$$\Big(\max\Big(\Big(\eta^{(k)} - \rho^{(k)}\Big), 0\Big)\Big)_i^* = \Big(\max\Big(\Big(\eta^{(k)} - \rho^{(k)}\Big), 0\Big)\Big)_{\pi_k(i)} = \Big(\eta^{(k)} - \rho^{(k)}\Big)_{\pi_k(i)} > 0.$$

Since I is ssc, there is a strictly increasing sequence of indices $n_k \in \mathbb{N}$ such that if $\zeta \in c_o^*$ and $\zeta_i \ge (\max((\eta^{(k)} - \rho^{(k)}), 0))_i^*$ for all $1 \le i \le n_k$, then $\zeta \notin \Sigma(I)$. Choose integers $N_k \ge \max\{\pi_k(i) \mid 1 \le i \le n_k\}$ so that N_k is increasing. We claim that if $\xi \in c_o^*$ and $\xi_i \ge \eta_i^{(k)}$ for all $1 \le i \le N_k$ and all k, then $\xi \notin \Sigma(I+J)$, which would conclude the proof. Indeed, for any given $m \in \mathbb{N}$ and for each $k \ge m$,

$1 \le j \le n_k$ and $1 \le i \le j$, it follows that $\pi_k(i) \le N_k$ and hence

$$\begin{aligned}\left(\xi - \rho^{(m)}\right)_{\pi_k(i)} &\ge \left(\eta^{(k)} - \rho^{(k)}\right)_{\pi_k(i)} = \left(\max\left(\left(\eta^{(k)} - \rho^{(k)}\right), 0\right)\right)^*_i \\ &\ge \left(\max\left(\left(\eta^{(k)} - \rho^{(k)}\right), 0\right)\right)^*_j.\end{aligned}$$

Thus there are at least j values of $(\xi - \rho^{(m)})_n$ that are greater than or equal to $(\max((\eta^{(k)} - \rho^{(k)}), 0))^*_j$ and hence $(\max((\xi - \rho^{(m)}), 0))^*_j \ge (\max((\eta^{(k)} - \rho^{(k)}), 0))^*_j$. By the defining property of the sequence $\{n_k\}$, $(\max((\xi - \rho^{(m)}), 0))^* \notin \Sigma(I)$ for every m. But then, for any $\mu \in \Sigma(J)$ there is an m such that $\mu \le \rho^{(m)}$ so that $(\max((\xi - \mu), 0))^* \ge (\max((\xi - \rho^{(m)}), 0))^*$ and hence $(\max((\xi - \mu), 0))^* \notin \Sigma(I)$. By Lemma 5.6, it follows that $\xi \notin \Sigma(I + J)$, which concludes the proof of the claim and thus of the theorem. □

6. Arithmetic Means and Soft Ideals

The proofs of the main results in [10, Theorems 7.1 and 7.2] depend in a crucial way on some of the commutation relations between the se and sc operations and the pre and post-arithmetic means and pre and post arithmetic means at infinity operations. In this section we shall investigate these relations. We start with the arithmetic mean and for completeness, we list the relations already obtained in [10, Lemma 3.3] as parts (i)–(ii′) of the next theorem.

Theorem 6.1. *Let I be an ideal.*

(i) $\operatorname{sc}{}_aI \subset {}_a(\operatorname{sc} I)$

(i′) $\operatorname{sc}{}_aI = {}_a(\operatorname{sc} I)$ *if and only if* $\omega \notin \Sigma(\operatorname{sc} I) \setminus \Sigma(I)$

(ii) $\operatorname{se} I_a \subset (\operatorname{se} I)_a$

(ii′) $\operatorname{se} I_a = (\operatorname{se} I)_a$ *if and only if* $I = \{0\}$ *or* $I \not\subset \mathcal{L}_1$

(iii) $\operatorname{sc} I_a \supset (\operatorname{sc} I)_a$

(iv) $\operatorname{se}{}_aI \supset {}_a(\operatorname{se} I)$

(iv′) $\operatorname{se}{}_aI = {}_a(\operatorname{se} I)$ *if and only if* $\omega \notin \Sigma(I) \setminus \Sigma(\operatorname{se} I)$.

The "missing" reverse inclusion of (iii) will be explored in Proposition 6.8. The proof of parts (iii)–(iv′) of Theorem 6.1 depend on the following two lemmas.

Lemma 6.2.

(i) $F_a = (\mathcal{L}_1)_a = (\omega)$ *and* ${}_a(\omega) = \mathcal{L}_1$

Consequently (ω) *and* $\mathcal{L}_1$ *are, respectively, the smallest nonzero am-open ideal and the smallest nonzero am-closed ideal.*

(ii) $\{0\} = {}_aI$ *if and only if* $\mathcal{L}_1 \not\subset {}_aI$ *if and only if* $\omega \notin \Sigma(I)$

(iii) $\mathcal{L}_1 = {}_aI$ *if and only if* $\omega \in \Sigma(I) \setminus \Sigma(\operatorname{se} I)$

(iv) $\mathcal{L}_1 \subsetneqq {}_aI$ *if and only if* $\omega \in \Sigma(\operatorname{se} I)$

Proof. Notice that $\eta_a \asymp \omega$ for every $0 \neq \eta \in \ell_1^*$ and that $\omega = o(\eta_a)$ for every $\eta \notin \ell_1^*$. Thus (ii) and the equalities in (i) follow directly from the definitions. Recall from the paragraphs preceding Lemma 2.1 that an ideal is am-open (resp., am-closed) if and only if it is the arithmetic mean of an ideal, in which case if it is nonzero, it contains $F_a = (\omega)$ (resp., if and only if it is the prearithmetic mean of an ideal, in which case by (ii), it contains $\mathcal{L}_1$). Thus the minimality of (ω) (resp., $\mathcal{L}_1$) are established. (iii) follows immediately from (ii) and (iv).

(iv) Assume first that $\mathcal{L}_1 \subsetneqq {}_aI$. Then $\mathcal{L}_1 \subset \operatorname{se}{}_aI$ since $\mathcal{L}_1$ is soft-edged (Proposition 4.12) and hence by (i),

$$(\omega) = (\mathcal{L}_1)_a \subset (\operatorname{se}{}_aI)_a = \operatorname{se}(({}_aI)_a) = \operatorname{se} I^o \subset \operatorname{se} I$$

where the second equality follows from Theorem 6.1(ii′) applied to ${}_aI$ which is not contained in $\mathcal{L}_1$. Conversely, assume that $\omega \in \Sigma(\operatorname{se} I)$, i.e., $\omega = o(\eta)$ for some $\eta \in \Sigma(I)$. Then $\mathcal{L}_1 \subset {}_aI$ by (ii). It follows directly from the definition of lnd (see paragraph preceding Lemma 2.14) that $\omega = o(\omega \operatorname{lnd} \frac{\eta}{\omega})$. By Lemma 2.14(i), $\omega \operatorname{lnd} \frac{\eta}{\omega} \in \Sigma(I^o)$, i.e., $\omega \operatorname{lnd} \frac{\eta}{\omega} \leq \rho_a \in \Sigma(I)$ for some $\rho \in \Sigma({}_aI)$. But $\rho \notin \ell_1^*$ since $\omega = o(\rho_a)$ and hence $\mathcal{L}_1 \neq {}_aI$. □

Lemma 6.3. *For* $\eta \in c_o^*$ *and* $0 < \beta \to \infty$*, there is a sequence* $0 < \gamma \leq \beta$ *with* $\gamma \uparrow \infty$ *for which* $\gamma\eta$ *is monotone nonincreasing.*

Proof. The case where η has finite support is elementary, so assume that for all i, $\eta_i > 0$. By replacing if necessary β with $\operatorname{lnd}\beta$ we can assume also that β is monotone nondecreasing. Starting with $\gamma_1 := \beta_1$, define recursively

$$\gamma_n := \frac{1}{\eta_n} \min(\gamma_{n-1}\eta_{n-1}, \beta_n\eta_n).$$

It follows immediately that $\gamma \leq \beta$ and that $\gamma\eta$ is monotone nonincreasing. Moreover, $\gamma_n \geq \gamma_{n-1}$ for all n since both $\beta_n \geq \beta_{n-1} \geq \gamma_{n-1}$ and $\gamma_{n-1}\frac{\eta_{n-1}}{\eta_n} \geq \gamma_{n-1}$. In the case that $\gamma_n = \beta_n$ infinitely often, then $\gamma \to \infty$. In the case that $\gamma_n \neq \beta_n$ for all $n > m$, then $\gamma_n\eta_n = \gamma_{n-1}\eta_{n-1}$ and so also $\gamma_n = \frac{\eta_m}{\eta_n}\gamma_m \to \infty$ since $\eta_n \to 0$ and $\eta_m\gamma_m \neq 0$. $\square$

Proof of Theorem 6.1. (i)–(ii′) See [10, Lemma 3.3].

(iii) If $\xi \in \Sigma((\operatorname{sc} I)_a)$, then $\xi \leq \eta_a$ for some $\eta \in \Sigma(\operatorname{sc} I)$. So for every $\alpha \in c_o^*$, $\alpha\eta \in \Sigma(I)$ and $\alpha\xi \leq \alpha\eta_a \leq (\alpha\eta)_a \in \Sigma(I_a)$, where the last inequality follows from the monotonicity of α. Thus $\xi \in \Sigma(\operatorname{sc} I_a)$.

(iv) Let $\xi \in \Sigma({}_a(\operatorname{se} I))$, i.e., $\xi_a \leq \alpha\eta$ for some $\alpha \in c_o^*$ and $\eta \in \Sigma(I)$. Since $(\frac{1}{\alpha}\xi)_a \leq \frac{1}{\alpha}\xi_a \leq \eta \in c_o^*$ where the first inequality follows from the monotonicity of α, by Lemma 6.3 there is a sequence $\gamma \uparrow \infty$ such that $\gamma \leq \frac{1}{\alpha}$ and $\gamma\xi$ is monotone nonincreasing. Thus $(\gamma\xi)_a \leq \eta \in \Sigma(I)$, i.e., $\gamma\xi \in \Sigma({}_aI)$, and hence $\xi \in \Sigma(\operatorname{se} {}_aI)$.

(iv′) There are three cases. If $\omega \notin \Sigma(I)$, then by Lemma 6.2(ii), both ${}_aI = \{0\}$ and ${}_a(\operatorname{se} I) = \{0\}$ and hence the equality holds. If $\omega \in \Sigma(I) \setminus \Sigma(\operatorname{se} I)$, then $\mathcal{L}_1 = {}_aI$ by Lemma 6.2(iii) and hence $\operatorname{se} {}_aI = \mathcal{L}_1$ since $\mathcal{L}_1$ is soft-edged by Proposition 4.12. But ${}_a(\operatorname{se} I) = \{0)$ by Lemma 6.2(ii), so the inclusion in (iv) fails. For the final case, if $\omega \in \Sigma(\operatorname{se} I)$, then by Lemma 6.2(iv), $\mathcal{L}_1 \subsetneqq {}_aI$. Let $\xi \in \Sigma(\operatorname{se} {}_aI)$, i.e., $\xi = o(\eta)$ for some $\eta \in \Sigma({}_aI)$. By adding to η, if necessary, a nonsummable sequence in $\Sigma({}_aI)$, we can assume that η is itself not summable. But then it is easy to verify that $\xi_a = o(\eta_a)$, i.e., $\xi_a \in \Sigma(\operatorname{se} I)$ and hence $\xi \in \Sigma({}_a(\operatorname{se} I))$. $\square$

Now we examine how the operations sc and se commute with the arithmetic mean operations of am-interior $I^o := ({}_aI)_a$ and am-closure $I^- := {}_a(I_a)$.

Theorem 6.4. *Let I be an ideal.*

(i) $\operatorname{sc} I^- \supset (\operatorname{sc} I)^-$

(ii) $\operatorname{se} I^- = (\operatorname{se} I)^-$

(iii) $\operatorname{sc} I^o \subset (\operatorname{sc} I)^o$

(iii′) $\operatorname{sc} I^o = (\operatorname{sc} I)^o$ *if and only if* $\omega \notin \Sigma(\operatorname{sc} I) \setminus \Sigma(I)$

(iv) $\operatorname{se} I^o \supset (\operatorname{se} I)^o$

(iv′) $\operatorname{se} I^o = (\operatorname{se} I)^o$ *if and only if* $\omega \notin \Sigma(I) \setminus \Sigma(\operatorname{se} I)$

Proof. (i) The case $I = \{0\}$ is obvious. If $I \neq \{0\}$, then $\omega \in \Sigma(I_a)$ and hence, by Theorem 6.1(i′) and (iii), it follows that

$$\operatorname{sc} I^- = \operatorname{sc}{}_a(I_a) = {}_a(\operatorname{sc} I_a) \supset {}_a((\operatorname{sc} I)_a) = (\operatorname{sc} I)^-.$$

(ii) There are three possible cases. The case when $I = \{0\}$ is again obvious. In the second case when $\{0\} \neq I \subset \mathcal{L}_1$, then $I^- = \mathcal{L}_1$ and $(\operatorname{se} I)^- = \mathcal{L}_1$ since $\mathcal{L}_1$ is the smallest nonzero am-closed ideal by Lemma 6.2(i). Since $\mathcal{L}_1$ is soft-edged by Proposition 4.12, $\operatorname{se} I^- = \mathcal{L}_1$, so equality in (ii) holds. In the third case, $I \not\subset \mathcal{L}_1$. Then $\mathcal{L}_1 \subsetneqq I^-$ and $\omega \in \Sigma(\operatorname{se} I_a)$ by Lemma 6.2(iv). Then

$$\operatorname{se} I^- = \operatorname{se}{}_a(I_a) = {}_a(\operatorname{se}(I_a)) = {}_a((\operatorname{se} I)_a) = (\operatorname{se} I)^-$$

where the second and third equalities follow from Theorem 6.1(iv′) and (ii′).

(iii) Let $\xi \in \Sigma(\operatorname{sc} I^o)$ and let $\alpha \in c_o^*$. By the definition of "und" (see the paragraph preceding Lemma 2.14) it follows easily that $\alpha\omega \operatorname{und} \frac{\xi}{\omega} \leq \omega \operatorname{und} \frac{\alpha\xi}{\omega}$ and by Corollary 2.16, that $\omega \operatorname{und} \frac{\alpha\xi}{\omega} \in \Sigma(I)$ since $\alpha\xi \in \Sigma(I^o)$. Thus $\alpha\omega \operatorname{und} \frac{\xi}{\omega} \in \Sigma(I)$ and hence $\omega \operatorname{und} \frac{\xi}{\omega} \in \Sigma(\operatorname{sc} I)$. But then, again by Corollary 2.16, $\xi \in \Sigma((\operatorname{sc} I)^o)$.

(iii′) If $\omega \notin \Sigma(\operatorname{sc} I) \setminus \Sigma(I)$, then

$$\operatorname{sc} I^o = \operatorname{sc}({}_aI)_a \supset (\operatorname{sc}({}_aI))_a = ({}_a(\operatorname{sc} I))_a = (\operatorname{sc} I)^o$$

by Theorem 6.1(iii) and (i′). If on the other hand $\omega \in \Sigma(\operatorname{sc} I) \setminus \Sigma(I)$, then by Lemma 6.2(ii) ${}_a(\operatorname{sc} I) \neq \{0\}$ and hence $(\operatorname{sc} I)^o \neq \{0)$, while ${}_a(I) = \{0\}$ and hence $\operatorname{sc}(I)^o = \{0\}$.

(iv) and (iv′). There are three possible cases. If $\omega \notin \Sigma(I)$, then $I^o = \{0\}$ by Lemma 6.2(ii) and so $\operatorname{se} I^o = \{0\}$ and $(\operatorname{se} I)^o = \{0\}$, i.e., (iv′) holds trivially. If $\omega \in \Sigma(I) \setminus \Sigma(\operatorname{se} I)$, then $I^o \neq \{0\}$ and $(\operatorname{se} I)^o = \{0\}$ again by Lemma 6.2(ii). But then $\operatorname{se} I^o \neq \{0\}$, so (iv) holds but (iv′) does not. Finally, when $\omega \in \Sigma(\operatorname{se} I)$, then $\mathcal{L}_1 \subsetneqq {}_aI$ by Lemma 6.2 (iv) and hence

$$\operatorname{se} I^o = \operatorname{se}({}_aI)_a = (\operatorname{se}({}_aI))_a = ({}_a(\operatorname{se} I))_a = (\operatorname{se} I)^o$$

by Theorem 6.1(ii′) and (iv′). □

We were unable to find natural conditions under which the reverse inclusion of Theorem 6.4(i) holds (see also Proposition 6.8), nor examples where it fails.

Corollary 6.5.

(i) *If I is an am-open ideal, then* $\operatorname{sc} I$ *is am-open while* $\operatorname{se} I$ *is am-open if and only if* $I \neq (\omega)$.

(ii) *If I is an am-closed ideal, then* $\operatorname{sc} I$ *and* $\operatorname{se} I$ *are am-closed.*

Proof. (ii) and the first implication in (i) are immediate from Theorem 6.4. For the second implication of (i), assume that I is am-open and that $0 \neq I \neq (\omega)$. Then by Lemma 6.2(i), $(\omega) \subsetneqq I$ and $\mathfrak{L}_1 = {}_a(\omega) \subset {}_aI$. But $\mathfrak{L}_1 \neq {}_aI$ follows from $(\mathfrak{L}_1)_a = (\omega) \neq I = ({}_aI)_a$. Then $\omega \in \Sigma(\operatorname{se} I)$ by Lemma 6.2(iv), hence $\operatorname{se} I = \operatorname{se} I^o = (\operatorname{se} I)^o$ by Theorem 6.4(iv$'$) and thus $\operatorname{se} I$ is am-open. If $I = \{0\}$, then $\operatorname{se} I = \{0\}$ too is am-open. If $I = (\omega)$, then $\operatorname{se} I \subsetneqq (\omega)$ cannot be am-open, again by Lemma 6.2(i). □

For completeness' sake we list also some se and sc commutation properties for the largest am-closed ideal I_- contained in I and the smallest am-open ideal I^{oo} containing I (see Corollary 2.6 and Definition 2.18).

Proposition 6.6. *For every ideal I:*

(i) $\operatorname{sc} I_- = (\operatorname{sc} I)_-$

(ii) $\operatorname{se} I_- \subset (\operatorname{se} I)_-$

(iii) $\operatorname{sc} I^{oo} \supset (\operatorname{sc} I)^{oo}$

(iv) $\operatorname{se} I^{oo} \subset (\operatorname{se} I)^{oo}$

(iv$'$) $\operatorname{se} I^{oo} = (\operatorname{se} I)^{oo}$ *if and only if either* $I = \{0\}$ *or* $I \not\subset (\omega)$

Proof. (i)–(iii) Corollary 6.5 and the maximality (resp., minimality) of I_- (resp., I^{oo}) yield the inclusions $\operatorname{sc} I_- \subset (\operatorname{sc} I)_-$, $\operatorname{se} I_- \subset (\operatorname{se} I)_-$, and $\operatorname{sc} I^{oo} \supset (\operatorname{sc} I)^{oo}$. From the second inclusion it follows that

$$\operatorname{se}((\operatorname{sc} I)_-) \subset (\operatorname{se}(\operatorname{sc} I))_- = (\operatorname{se} I)_- \subset I_-$$

and hence

$$(\operatorname{sc} I)_- \subset \operatorname{sc}(\operatorname{sc} I)_- = \operatorname{sc}(\operatorname{se}((\operatorname{sc} I)_-)) \subset \operatorname{sc} I_-$$

so that equality holds in (i).

(iv) If $\eta \in \Sigma((\operatorname{se} I^{oo})$, then by Proposition 2.21, $\eta \leq \alpha\omega$ und $\frac{\xi}{\omega}$ for some $\xi \in \Sigma(I)$ and $\alpha \in c_o^*$. As remarked in the proof of Theorem 6.4(iii), it follows that $\eta \leq \omega$ und $\frac{\alpha\xi}{\omega}$ and hence $\eta \in \Sigma((\operatorname{se} I)^{oo})$, again by Proposition 2.21.

(iv′) There are three cases. If $I = \{0\}$, (iv′) holds trivially. If $\{0\} \neq I \subset (\omega)$, then by the minimality of (ω) among nonzero am-open ideals, $I^{oo} = (\omega)$ and $(\operatorname{se} I)^{oo} = (\omega)$, so the inclusion in (iv′) fails. If $I \not\subset (\omega)$, then $I^{oo} \neq (\omega)$ and hence by Corollary 6.5(i), $\operatorname{se} I^{oo}$ is am-open and by minimality of $(\operatorname{se} I)^{oo}$, (iv′) holds. □

It is now an easy application of the above results to verify that the following am-operations preserve softness.

Corollary 6.7.

(i) *If I is soft-complemented, then so are ${}_aI$, I^o, and I_-.*

(ii) *If I is soft-edged, then so are ${}_aI$, I^o, and I^-.*

(iii) *If I is soft-edged, then I_a is soft-edged if and only if either $I = \{0\}$ or $I \not\subset \mathcal{L}_1$.*

(iv) *If I is soft-edged, then I^{oo} is soft-edged if and only if either $I = \{0\}$ or $I \not\subset (\omega)$.*

Several of the "missing" statements that remain open are equivalent as shown in the next proposition.

Proposition 6.8. *For every ideal I, the following conditions are equivalent.*

(i) $\operatorname{sc} I_a \subset (\operatorname{sc} I)_a$

(ii) $(\operatorname{sc} I)_a$ *is soft-complemented*

(iii) $(\operatorname{sc} I)^-$ *is soft-complemented*

(iv) $\operatorname{sc} I^- \subset (\operatorname{sc} I)^-$

Proof. Implications (i) ⇒ (ii) ⇒ (iii) ⇒ (iv) are easy consequences of Theorem 6.1 and Corollary 6.7. We prove that (iv) ⇒ (i). The case $I = \{0\}$ being trivial, assume $I \neq \{0\}$. Then $\omega \in \Sigma(I_a)$, hence $\operatorname{sc} I^- \supset {}_a\operatorname{sc}(I_a)$ by Theorem 6.1(i′). Moreover, since I_a is am-open, then so is $\operatorname{sc} I_a$ by Corollary 6.5, i.e., $\operatorname{sc} I_a = (\operatorname{sc} I_a)^o$. Then

$$\operatorname{sc} I_a = ({}_a(\operatorname{sc} I_a))_a \subset (\operatorname{sc} I^-)_a \subset ((\operatorname{sc} I)^-)_a = (\operatorname{sc} I)_a,$$

the latter equality following from the general identity $({}_a(J_a))_a = J_a$. □

Now we investigate the relations between arithmetic means at infinity and the se and sc operations and we list some results already obtained in [10, Lemma 4.19] as parts (i) and (ii) of the next theorem.

Theorem 6.9. *For every ideal $I \neq \{0\}$:*

(i) $\operatorname{sc}{}_{a_\infty} I = {}_{a_\infty}(\operatorname{sc} I)$

(ii) $\operatorname{se} I_{a_\infty} = (\operatorname{se} I)_{a_\infty}$

(iii) $\operatorname{sc} I_{a_\infty} \supset (\operatorname{sc} I)_{a_\infty}$

(iv) $\operatorname{se}{}_{a_\infty} I = {}_{a_\infty}(\operatorname{se} I)$

Proof. (i)–(ii) See [10, Lemma 4.19].

(iii) If $\xi \in \Sigma((\operatorname{sc} I)_{a_\infty})$, $\xi \leq \eta_{a_\infty}$ for some $\eta \in \Sigma(\operatorname{sc} I \cap \mathcal{L}_1)$. In [10, Lemma 4.19 (i)](proof) we showed that for every $\alpha \in c_o^*$, $\alpha\eta_{a_\infty} \leq (\alpha'\eta)_{a_\infty}$ for some $\alpha' \in c_o^*$. But then $\alpha'\eta \in \Sigma(I \cap \mathcal{L}_1)$ and so $\alpha\xi \leq (\alpha'\eta)_{a_\infty} \in \Sigma(I_{a_\infty})$, i.e., $\xi \in \Sigma(\operatorname{sc} I_{a_\infty})$.

(iv) Let $\xi \in \Sigma(\operatorname{se}{}_{a_\infty} I)$, then $\xi \leq \alpha\eta$ for some $\alpha \in c_o^*$ and $\eta \in \Sigma({}_{a_\infty} I)$. But then by the monotonicity of α, $\xi_{a_\infty} \leq (\alpha\eta)_{a_\infty} \leq \alpha\eta_{a_\infty} \in \Sigma(\operatorname{se} I)$. Thus $\xi \in \Sigma({}_{a_\infty}(\operatorname{se} I))$ which proves the inclusion $\operatorname{se}{}_{a_\infty} I \subset {}_{a_\infty}(\operatorname{se} I)$.

Now let $\xi \in \Sigma({}_{a_\infty}(\operatorname{se} I))$, i.e., $\xi_{a_\infty} \leq \alpha\eta$ for some $\alpha \in c_o^*$ and $\eta \in \Sigma(I)$. We construct a sequence $\gamma \uparrow \infty$ such that $\gamma\xi$ is monotone nonincreasing and $(\gamma\xi)_{a_\infty} \leq \eta$. Without loss generality assume that $\xi_n \neq 0$ and hence $\alpha_n \neq 0$ for all n. We choose a strictly increasing sequence of indices n_k (with $n_o = 0$) such that for $k \geq 1$, $\alpha_{n_k} \leq 2^{-k-2}$ and $\sum_{n_{k+1}+1}^\infty \xi_i \leq \frac{1}{4}\sum_{n_k+1}^\infty \xi_i$ for all k. Set $\beta_n = 2^k$ for $n_k < n \leq n_{k+1}$. Then for all $k \geq 0$ and $n_k < n+1 \leq n_{k+1}$ we have

$$\begin{aligned}
\sum_{n+1}^\infty \beta_i\xi_i &= 2^k \sum_{n+1}^{n_{k+1}} \xi_i + 2^{k+1} \sum_{n_{k+1}+1}^{n_{k+2}} \xi_i + 2^{k+2} \sum_{n_{k+2}+1}^{n_{k+3}} \xi_i + \cdots \\
&\leq 2^k \sum_{n+1}^{n_{k+1}} \xi_i + 2^{k+1} \left(\sum_{n_{k+1}+1}^{\infty} \xi_i + 2 \sum_{n_{k+2}+1}^{\infty} \xi_i + 2^2 \sum_{n_{k+3}+1}^{\infty} \xi_i + \cdots \right) \\
&\leq 2^k \sum_{n+1}^{n_{k+1}} \xi_i + 2^{k+2} \sum_{n_{k+1}+1}^{\infty} \xi_i \leq 2^{k+2} \sum_{n+1}^{\infty} \xi_i \\
&\leq \frac{1}{\alpha_{n_k}} \sum_{n+1}^{\infty} \xi_i \leq \frac{1}{\alpha_n} \sum_{n+1}^{\infty} \xi_i = \frac{n}{\alpha_n} (\xi_{a_\infty})_n \leq n\eta_n.
\end{aligned}$$

This proves that $(\beta\xi)_{a_\infty} \leq \eta$. Now Lemma 6.3 provides a sequence $\gamma \leq \beta$, with $\gamma \uparrow \infty$ and $\gamma\xi$ monotone nonincreasing, and hence $(\gamma\xi)_{a_\infty} \leq (\beta\xi)_{a_\infty} \leq \eta$. Thus $\gamma\xi \in \Sigma({}_{a_\infty}I)$ and hence $\xi = \frac{1}{\gamma}(\gamma\xi) \in \Sigma(\mathrm{se}\,{}_{a_\infty}I)$. $\square$

The reverse inclusion in Theorem 6.9(iii) does not hold in general. Indeed, whenever $I_{a_\infty} = \mathrm{se}(\omega)$ (which condition by [10, Corollary 4.9 (ii)] is equivalent to $I^{-\infty} = {}_{a_\infty}(I_{a_\infty}) = \mathcal{L}_1$ and in particular is satisfied by $I = \mathcal{L}_1$), it follows that $\mathrm{sc}\, I_{a_\infty} = (\omega)$ while $(\mathrm{sc}\, I)_{a_\infty} \subset \mathrm{se}(\omega)$. We do not know of any natural sufficient condition for the reverse inclusion in Theorem 6.9(iii) to hold.

Many of the other results obtained for the arithmetic mean case have an analog for the am-∞ case:

Theorem 6.10. *For every ideal I:*

(i) $\mathrm{sc}\, I^{-\infty} \supset (\mathrm{sc}\, I)^{-\infty}$

(ii) $\mathrm{se}\, I^{-\infty} = (\mathrm{se}\, I)^{-\infty}$

(iii) $\mathrm{sc}\, I^{o\infty} \supset (\mathrm{sc}\, I)^{o\infty}$

(iii′) $\mathrm{sc}\, I^{o\infty} = (\mathrm{sc}\, I)^{o\infty}$ *if and only if* $\mathrm{sc}\, I^{o\infty} \subset \mathrm{se}(\omega)$

(iv) $\mathrm{se}\, I^{o\infty} = (\mathrm{se}\, I)^{o\infty}$

Proof. (i), (ii), (iii), and (iv) follow immediately from Theorem 6.9.

(iii′) Since every am-∞ open ideal is contained in $\mathrm{se}(\omega)$, it follows that $\mathrm{sc}\, I^{o\infty} = (\mathrm{sc}\, I)^{o\infty} \subset \mathrm{se}(\omega)$. Assume now that $\mathrm{sc}\, I^{o\infty} \subset \mathrm{se}(\omega)$, let $\xi \in \Sigma(\mathrm{sc}\, I^{o\infty})$, and let $\alpha \in c_o^*$. Since $\xi = o(\omega)$, there is an increasing sequence of integers n_k with $n_o = 0$ for which $(\mathrm{uni}\, \frac{\xi}{\omega})_j = (\frac{\xi}{\omega})_{n_k}$ for $n_{k-1} < j \leq n_k$. Define $\tilde{\alpha}_j = \alpha_1$ for $1 < j \leq n_1$ and $\tilde{\alpha}_j = \alpha_{n_k}$ for $n_k < j \leq n_{k+1}$ for $k \geq 1$. Then $\tilde{\alpha} \in c_o^*$ and for all $k \geq 1$ and $n_{k-1} < j \leq n_k$

$$\left(\alpha\, \mathrm{uni}\, \frac{\xi}{\omega}\right)_j = \alpha_j \left(\frac{\xi}{\omega}\right)_{n_k} \leq \alpha_{n_{k-1}} \left(\frac{\xi}{\omega}\right)_{n_k} = \left(\frac{\tilde{\alpha}\xi}{\omega}\right)_{n_k} \leq \left(\mathrm{uni}\, \frac{\tilde{\alpha}\xi}{\omega}\right)_{n_k} \leq \left(\mathrm{uni}\, \frac{\tilde{\alpha}\xi}{\omega}\right)_j.$$

Since $\tilde{\alpha}\xi \in \Sigma(I^{o\infty})$ by hypothesis, it follows that $\omega\, \mathrm{uni}\, \frac{\tilde{\alpha}\xi}{\omega} \in \Sigma(I)$ by Corollary 3.10. But then $\alpha\omega\, \mathrm{uni}\, \frac{\xi}{\omega} \in \Sigma(I)$ for all $\alpha \in c_o^*$, i.e., $\omega\, \mathrm{uni}\, \frac{\xi}{\omega} \in \Sigma(\mathrm{sc}\, I)$. Hence, again by Corollary 3.10, $\xi \in \Sigma((\mathrm{sc}\, I)^{o\infty})$ and hence $\mathrm{sc}\, I^{o\infty} \subset (\mathrm{sc}\, I)^{o\infty}$. By (iii) we have equality. $\square$

The necessary and sufficient condition in Theorem 6.10 (iii′) is satisfied in the case of most interest, namely when $I \subset \mathcal{L}_1$. As in the am-case, we know of no natural conditions under which the reverse inclusion of (i) holds nor examples where it fails (see also Proposition 6.8). In the following proposition we collect the am-∞ analogs of Corollary 6.5, Proposition 6.6, and Corollary 6.7. Recall by Lemma 3.16 that $I^{oo\infty} = \mathrm{se}(\omega)$ for any ideal $I \not\subset \mathrm{se}(\omega)$.

Proposition 6.11. *Let $I \neq \{0\}$ be an ideal.*

(i) *If I is am-∞ open, then so is* $\mathrm{se}\, I$*.*

(i′) *If I is am-∞ open, then* $\mathrm{sc}\, I$ *is am-open if and only if* $\mathrm{sc}\, I \subset \mathrm{se}(\omega)$*.*

(ii) *If I is am-∞ closed, then so are* $\mathrm{se}\, I$ *and* $\mathrm{sc}\, I$*.*

(iii) $\mathrm{se}\, I^{oo\infty} = (\mathrm{se}\, I)^{oo\infty}$

(iv) $\mathrm{sc}\, I^{oo\infty} \supset (\mathrm{sc}\, I)^{oo\infty}$

(v) $\mathrm{se}\, I_{-\infty} \subset (\mathrm{se}\, I)_{-\infty}$

(vi) $\mathrm{sc}\, I_{-\infty} = (\mathrm{sc}\, I)_{-\infty}$

(vii) *If I is soft-edged, then so are $_{a_\infty}I$, I_{a_∞}, $I^{-\infty}$, $I^{o\infty}$, and $I^{oo\infty}$.*

(viii) *If I is soft-complemented, then so is $_{a_\infty}I$ and $I_{-\infty}$.*

(viii′) *If I is soft-complemented, then $I^{o\infty}$ is soft-complemented if and only if*

$$\mathrm{sc}\, I^{o\infty} \subset \mathrm{se}(\omega).$$

Proof. (i) Immediate from Theorem 6.10(iv).

(i′) If $\mathrm{sc}\, I \subset \mathrm{se}(\omega)$ then $\mathrm{sc}\, I = (\mathrm{sc}\, I)^{o\infty}$ by Theorem 6.10(iii′) and hence $\mathrm{sc}\, I$ is am-∞ open. The necessity is clear since $\mathrm{se}(\omega)$ is the largest am-∞ open ideal.

(ii) $\mathrm{se}\, I$ is am-∞ closed by Theorem 6.10(ii). By Theorem 6.10(i) and the am-∞ analog of the 5-chain of inclusions given in Section 2,

$$\mathrm{sc}\, I = \mathrm{sc}\, I^{-\infty} \supset (\mathrm{sc}\, I)^{-\infty} \supset \mathrm{sc}\, I \cap \mathcal{L}_1 = \mathrm{sc}\, I,$$

where the last equality holds because $\mathcal{L}_1$ is the largest am-∞ closed ideal so contains I, and being soft-complemented it contains $\mathrm{sc}\, I$.

(iii) By (i), $\mathrm{se}\, I^{oo\infty}$ is am-∞ open and by Definition 3.12 and Proposition 5.1 and following remark, it contains $\mathrm{se}(I \cap \mathrm{se}(\omega)) = \mathrm{se}\, I \cap \mathrm{se}(\omega)$, hence it must contain $(\mathrm{se}\, I)^{oo\infty}$. On the other hand, if $\xi \in \Sigma(\mathrm{se}\, I^{oo\infty})$, then by Proposition 3.14

there is an $\alpha \in c_o^*$ and $\eta \in \Sigma(I \cap \mathrm{se}(\omega))$ such that $\xi \leq \alpha\omega \,\mathrm{uni}\, \frac{\eta}{\omega}$. Then, by the proof in Theorem 6.10(iii′), there is an $\tilde{\alpha} \in c_o^*$ such that $\alpha\omega \,\mathrm{uni}\, \frac{\eta}{\omega} \leq \omega \,\mathrm{uni}\, \frac{\tilde{\alpha}\eta}{\omega}$. Since $\tilde{\alpha}\eta \in \Sigma(\mathrm{se}\, I \cap \mathrm{se}(\omega))$, Proposition 3.14 yields again $\xi \in \Sigma((\mathrm{se}\, I)^{oo\infty})$ which proves the equality in (iii).

(iv) Let $\xi \in \Sigma((\mathrm{sc}\, I)^{oo\infty})$. By Proposition 3.14 there is an $\eta \in \Sigma((\mathrm{sc}\, I) \cap \mathrm{se}(\omega))$ such that $\xi \leq \omega \,\mathrm{uni}\, \frac{\eta}{\omega}$. Then, by the proof in Theorem 6.10(iii′), for every $\alpha \in c_o^*$ there is an $\tilde{\alpha} \in c_o^*$ such that $\alpha\xi \leq \alpha\omega \,\mathrm{uni}\, \frac{\eta}{\omega} \leq \omega \,\mathrm{uni}\, \frac{\tilde{\alpha}\eta}{\omega}$. As $\tilde{\alpha}\eta \in \Sigma(I \cap \mathrm{se}(\omega))$, again by Proposition 3.14, $\alpha\xi \in \Sigma(I^{oo\infty})$ and hence $\xi \in \Sigma(\mathrm{sc}(I^{oo\infty}))$.

(v) This is an immediate consequence of (ii).

(vi) The inclusion $\mathrm{sc}\, I_{-\infty} \subset (\mathrm{sc}\, I)_{-\infty}$ is similarly an immediate consequences of (ii). The reverse inclusion follows from (v) applied to the ideal $\mathrm{sc}\, I$:

$$\mathrm{se}(\mathrm{sc}\, I)_{-\infty} \subset (\mathrm{se}\,\mathrm{sc}\, I)_{-\infty} = (\mathrm{se}\, I)_{-\infty} \subset I_{-\infty}$$

hence

$$(\mathrm{sc}\, I)_{-\infty} \subset \mathrm{sc}(\mathrm{sc}\, I)_{-\infty} = \mathrm{sc}(\mathrm{se}(\mathrm{sc}\, I)_{-\infty}) \subset \mathrm{sc}\, I_{-\infty}.$$

(vii) The first two statements follow from Theorem 6.9 (iv) and (ii), the next two from Theorem 6.10(ii) and (iv), and the last one from (iii).

(viii), (viii′) follow respectively from Theorem 6.9(i) and Theorem 6.10(iii′). $\square$

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Acknowledgments

We wish to thank Ken Davidson for his input on the initial phase of the research and Daniel Beltita for valuable suggestions on this paper.

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