Reprinted from JOURNAL OF FUNCTIONAL ANALYSIS

Vol. 69, No. 1, October 15, 1986
Printed in Belgium

The Relative Dixmier Property in Discrete Crossed Products

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Communicated by C. Foias

Received February 1, 1986; revised February 26, 1986

1. INTRODUCTION

The problem of determining the pure states of the algebra $B(H)$ of all bounded operators on a separable Hilbert space H can be reduced to determining all pure states of a maximal abelian *-subalgebra (masa) N of $B(H)$ provided every pure state of N has a unique extension to a state of $B(H)$. It is known that the only candidate for the extension property is an atomic masa N [18], which can be considered as the algebra of diagonal operators in an infinite matrix algebra. The extension property is equivalent to the property that given any x in $B(H)$, the norm closure of the convex hull of the set $\{vxv^* \mid v \text{ unitary in } N\}$(notation: $K(x)=\overline{\text{co}}\{vxv^* \mid v\in U(N)\}$) has a nonvoid intersection with the commutant N' (cf. [5, 2.4]).

There is a more general formulation for the question concerning the intersection of the convex hull with N'. For this, let M be a von Neumann algebra; then Dixmier [9] showed that the intersection of $\overline{\text{co}}\{vxv^* \mid v\in U(M)\}$ and M' is nonvoid for every x in M, i.e., the closed convex hull contains an element of the center. Several authors [1–5, 8, 9, 11–13, 15, 16, 20, 22] have studied this or other similar intersections under various hypotheses. For example, for a von Neumann algebra, the intersection can be viewed as a type of essential spectrum for x in a setting appropriate to von Neumann algebras (cf. [8, 12, 13, 24]). On the other hand, when M is a C^*-algebra, it is not always true that the intersection of $\overline{\text{co}}\{vxv^* \mid v\in U(M)\}$ with the center of M is nonvoid. Algebras in which all such intersections are nonvoid are said to possess the *Dixmier property*. Several algebras have been shown to have the Dixmier property; others do not have the property.

In this paper, we study a more general property which we call the relative Dixmier property. We fix a certain subalgebra N of the von

* Research partially supported by NSF Grant DMS 8503390.

0022-1236/86 $3.00

Neumann algebra M and consider for $x \in M$ the intersection of $K(x) = \overline{\text{co}}\{vxv^* \mid v \in U(N)\}$ with the commutant N' of N. This formulation is a general setting for the extension problem for $B(H) = M$ [18] as well as for some recent work on compact operators on type III_λ $(0 \leqslant \lambda < 1)$ factors [14, 15]. Here the algebra N will be a von Neumann algebra and M will be the crossed product of N by the action of an automorphism θ. In Section 3 we prove that $K(au)$ has a nonvoid intersection with N' for every a in N and every unitary operator u in M such that $\text{ad}\, u$ induces an automorphism of N. In particular, if $\text{ad}\, u$ is properly outer $K(au)$ contains 0. In Section 4 we assume that N is identified with its canonical image in the crossed product $M = N \times_\theta Z$ of N by the action θ of Z on N, where θ is an automorphism of N. We show that $K(x)$ has a nonvoid intersection with N' for every x in the C^*-algebra generated by the group $U(M; N)$ of unitaries u in M with $uNu^* = N$. In particular, we show that $K(x)$ will have a nonvoid intersection with N' whenever x is continuous under the dual automorphism $\hat{\theta}$ of θ. In Section 5 we study two special cases: (1) $M = B(H)$ and N is the atomic masa of diagonal operators with respect to some fixed orthonormal basis, and (2) M is a type III_λ factor $(0 \leqslant \lambda < 1)$ and N a type II_∞ algebra. We identify the set $U(M; N)$ and study its relationship to the elements continuous under the dual automorphism $\hat{\theta}$. In case (1) we show the relationship of $U(M; N)$ to the work of Kadison and Singer [18].

2. The Relative Dixmier Property

Let H be a Hilbert space, let M and N be von Neumann algebras on H, and let N be a subalgebra of M. Let F be the set of all functions of finite support of the set $U(N)$ of unitary operators of N into $[0, 1]$ such that $\sum \{f(v) \mid v \in U(N)\} = 1$. For x in M, let $f \cdot x = \sum \{f(v)\, vxv^* \mid v \in U(N)\}$ and let $K(x)$ be the closed convex set $\overline{\text{co}}\{vxv^* \mid v \in U(N)\}$; then

$$K(x) = \text{norm closure } \{f \cdot x \mid f \in F\}.$$

We note that $\| f \cdot x \| \leqslant \| x \|$ for every f in F. Also the relation

$$f \cdot (g \cdot x) = (f * g) \cdot x$$

holds, where $f * g$ is the usual convolution

$$f * g(w) = \sum \{f(v)\, g(wv^*) \mid v \in U(N)\}$$

of the functions f and g on the discrete group $U(N)$. We note that

$$f \cdot K(x) \subset K(x),$$

and hence, if $y \in K(x)$, then $K(y) \subset K(x)$. We remark that f leaves $N' \cap M$ pointwise invariant. Also in the sequel we use the fact that $K(u) \subset Nu$ for every $u \in U(M)$ with $uNu^* = N$; indeed $vuv^* = v(uv^*u^*)u$ for every $v \in U(N)$.

DEFINITION 2.1. An element x in M is said to possess the *relative Dixmier property* (RDP) with regard to N if $K(x) \cap N' = K'(x)$ is nonvoid. Let N_d be the set of all elements with the relative Dixmier property.

It is easy to see that the set N_d is a norm closed set. If $N = M$, then $K'(x)$ is always nonvoid. This is the theorem of Dixmier [9]. In this case $K'(x)$ acts as a numerical range. For example, if M is semifinite, then $K'(x) = \{x_0\}$ if and only if $x - x_0$ is in the strong radical of M [8, 12, 24].

LEMMA 2.2 [5]. *Let M and N be von Neumann algebras with $N \subset M$, let x be in M, and let $p_1, p_2, ..., p_n$ be orthogonal projections in N of sum* 1; *then $\sum p_k x p_k$ is in $K(x)$.*

Proof. By induction we may assume that

$$y = p_1 x p_1 + \cdots + p_{n-2} x p_{n-2} + (p_{n-1} + p_n)\, x (p_{n-1} + p_n)$$

is in $K(x)$. But $v = p_1 + \cdots + p_{n-1} - p_n$ is selfadjoint unitary in N and

$$\sum p_k x p_k = 2^{-1}(y + v y v^*),$$

is contained in $K(y)$ and hence in $K(x)$. Q.E.D.

Remark. We have actually shown that $\sum p_k x p_k$ is in $\operatorname{co}\{vxv^* \mid v = v^* \in U(N)\}$.

If N is abelian, several types of convex sets coincide as the next proposition demonstrates.

PROPOSITION 2.3. *Let N be an abelian von Neumann subalgebra of M and let x be in M. Let*

$$K_s(x) = \overline{\operatorname{co}}\{vxv^* \mid v = v^* \in U(N)\},$$

and let

$$K_p(x) = \operatorname{clos}\left\{\sum p_k x p_k \mid p_1, ..., p_n \text{ orthogonal projections in } N \text{ of sum } 1\right\},$$

then

$$K_p(x) \cap N' = K_s(x) \cap N' = K(x) \cap N' = K'(x).$$

Proof. By Lemma 2.2 and the following remark, we have that

$$K_p(x) \subset K_s(x) \subset K(x).$$

Hence, it is sufficient to show that $K'(x)$ is contained in $K_p(x)$. If a is in $K'(x)$, then 0 is in $K'(x-a)$; and if 0 is in $K_p(x-a)$ for a in $M \cap N'$, then a is in $K_p(x)$. Thus, there is no loss of generality in showing that 0 is in $K_p(x)$ whenever 0 is in $K(x)$. Given $\varepsilon > 0$, there are positive numbers $\alpha_1, ..., \alpha_n$ of sum 1 and corresponding unitary operators $v_1, ..., v_n$ in N such that $\| \sum \alpha_i v_i x v_i^* \| < \varepsilon$. Since N is abelian, the v_i have a joint spectral resolution. Hence, there is no loss in generality in the assumption that each v_i is of the form $v_i = \sum \{\alpha_{ij} p_j \mid 1 \leqslant j \leqslant m\}$, where p_j are mutually orthogonal projections in N of sum 1 and α_{ij} are numbers of modulus 1. Then for all i, j, we have that $p_j v_i x v_i^* p_j = p_j x p_j$, and hence, $\| \sum p_j x p_j \| = \| \sum_j p_j (\sum_i \alpha_i v_i x v_i^*) p_j \| < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, 0 is in $K_p(x)$.

Q.E.D.

The von Neumann algebra $N = l^\infty(Z)$ is a maximal abelian subalgebra of the algebra M of bounded operators on $l^2(Z)$. The algebra N is identified with the algebra of diagonal infinite (in two directions) matrices in the space of (bounded) infinite (in two directions) matrices. An element x in M is said to be *paveable* if given $\varepsilon > 0$, there are orthogonal projections $p_1, ..., p_n$ in N of sum 1 with $\| \sum p_i x p_i - E(x) \| < \varepsilon$. Here $E(x)$ is the diagonal of x. By Proposition 2.3 an element x in M is paveable if and only if x has the RDP relative to N.

Anderson has shown that every element x in M is paveable if and only if every pure state on N has a unique extension to a (pure) state on M [1, 2].

When N is abelian, the Markov–Kakutani theorem implies that the σ-weak closure of $K(x)$ has a nonvoid intersection with N'. Hence, there is a conditional expectation E of M onto $N' \cap M$ such that, for all x in M, $E(x)$ is in the σ-weak closure of $K(x)$ due to the result of Schwartz [23]. If E is any conditional expectation of M onto $N' \cap M$, and if $N' \cap M = N$, then the set $K'(x)$ is either void or equal to the singleton set $\{E(x)\}$. Thus, if $N' \cap M = N$, there exists a unique conditional expectation of M onto $N' \cap M$ whenever every x in M has the RDP with respect to N. We state a nonabelian version of this in Proposition 4.8.

We now study a subset N_0 of the set N_d of all elements with the relative Dixmier property.

PROPOSITION 2.4. *Let N and M be von Neumann algebras with $N \subset M$ and let*

$$N_0 = \{x \in M \mid f \cdot x \in N_d \text{ for every } f \in F\}.$$

Then N_0 is a norm closed self-adjoint subspace of M, which contains N, and is a two-sided module over $N' \cap M$.

Proof. Given x_1, x_2 in N_0, f in F, and $\varepsilon > 0$, there are functions g_1, g_2 in F and elements a_1, a_2 in $N' \cap M$ such that $\| g_1 \cdot f \cdot x_1 - a_1 \| < \varepsilon$ and $\| g_2 \cdot g_1 \cdot f \cdot x_2 - a_2 \| < \varepsilon$, whence

$$\| g_2 \cdot g_1 \cdot f \cdot (x_1 + x_2) - (a_1 + a_2) \| \leqslant \| g_2 \cdot (g_1 \cdot f \cdot x_1 - a_1) \| + \| g_2 \cdot g_1 \cdot f \cdot x_2 - a_2 \| < 2\varepsilon.$$

Here we used the fact that $g_2 \cdot a_1 = a_1$. This means that $K'(f \cdot (x_1 + x_2))$ is nonvoid because the distance of $K(f \cdot (x_1 + x_2))$ to N' is arbitrarily small. Consequently, we see that $x_1 + x_2$ is in N_0. The other algebraic properties are clear. Finally, N_0 is norm closed as is easy to verify using the fact that $\| f \cdot x \| \leqslant \| x \|$ for every x in M and f in F. Q.E.D.

COROLLARY 2.5. *If N is abelian, then $N_0 = N_d$.*

Proof. We have that $N_0 \subset N_d$ in general. Conversely, let $x \in N_d$, let $a \in K'(x)$ and let $f \in F$. There is a sequence $\{f_n\}$ in F such that $\lim f_n \cdot x = a$. However, we have that $f_n * f = f * f_n$ and $\lim f_n \cdot (f \cdot x) = \lim f \cdot (f_n \cdot x) = f \cdot a = a$. Thus $K'(f \cdot x) = K'(x)$ is nonvoid, whence $x \in N_0$. Q.E.D.

3. ALGEBRAS WITH SPATIAL AUTOMORPHISMS

Let N be a von Neumann algebra with center C on the *separable* Hilbert space H and let θ be an automorphism of N. There is a largest projection $p(\theta)$ in the fixed point algebra N^θ of N such that the restriction of θ to $N_{p(\theta)}$ is inner. The projection $p(\theta)$ actually is in C [7]. The automorphism θ is said to be *properly outer* if $p(\theta) = 0$ and it is said to be *aperiodic* if $p(\theta^n) = 0$ for all $n \neq 0$. Connes [7] has shown that the automorphism θ is properly outer if and only if, given a nonzero projection p in N and given $\varepsilon > 0$, there is a nonzero projection q in N, $q \leqslant p$, with $\| q\theta(q) \| < \varepsilon$ (cf. [25, 17.9]).

For a properly outer automorphism θ on N implemented by a unitary operator u on H, we show that $K(au)$ contains 0 for every a in N. We break the proof into several steps, each of which uses some form of the following proposition, which can be viewed as a generalization of the previously mentioned result of Connes [7].

PROPOSITION 3.1. *Let N be a von Neumann algebra and let θ be a properly outer automorphism on N; then, for every a in N, every nonzero pro-*

jection p in N, and every $\varepsilon > 0$, there is a nonzero projection $q \leqslant p$ in N such that $\| qa\theta(q) \| < \varepsilon$.

Proof. By passing to a nonzero subprojection of p if necessary, there is no loss of generality in the assumption that pa can be written as $pa = bw$, where $b \in N^+$ and $w \in U(N)$. The polar decomposition plus some manipulations with finite and purely infinite projections will produce this. There is also no loss of generality in the assumption that the range support of b is p; otherwise, there is a nonzero projection majorized by p that left annihilates pa and consequently trivially satisfies the conclusion of our proposition. There is a scalar $0 < \alpha < \| b \|$ and a nonzero spectral projection q' of b majorized by p such that

$$\| q'(b - \alpha) \| < \varepsilon/2.$$

Notice that ad $w \cdot \theta$ is still a properly outer automorphism of N. So there is a nonzero projection q majorized by q' such that

$$\| q \text{ ad } w \cdot \theta(q) \| < \varepsilon/2 \, \| b \|.$$

Then we have that

$$\begin{aligned} \| qa\theta(q) \| &= \| qbw\theta(q) \| \\ &\leqslant \| q(b - \alpha) \text{ ad } w \cdot \theta(q) \| + \alpha \, \| q \text{ ad } w \cdot \theta(q) \| \\ &\leqslant \varepsilon. \end{aligned}$$

Q.E.D.

Consideration of a properly outer automorphism θ on an algebra N with center C can be split into three cases: (1) $p(\theta \mid C) = 0$; (2) $p(\theta \mid C) = 1$ and N is finite; and (3) $p(\theta \mid C) = 1$ and N is properly infinite. We consider these three cases separately using a maximality argument based on Proposition 3.1.

PROPOSITION 3.2. *If the automorphism θ of N is properly outer on the center C of N, then $0 \in K(au)$ for every $a \in N$.*

Proof. Let $\{p_n\}$ be a maximal set of nonzero orthogonal projections in C such that $p_m \theta(p_n) = 0$ for all m, n. Setting $p = \sum p_n$, we have that $p\theta(p) = 0$. We show that $p_0 = \text{lub}\{p, \theta(p), \theta^{-1}(p)\} = 1$. On the contrary, suppose that $p_0 \neq 1$. Then there is a nonzero projection $q \leqslant 1 - p_0$ in C such that $\| q\theta(q) \| < 1$. Since $q\theta(q)$ is a projection, we have $q\theta(q) = 0$. But we have that

$$\theta(q) \leqslant \theta(1 - p_0) \leqslant 1 - p$$

and hence $p\theta(q)=0$. Likewise, we have that $\theta(p)q=0$. Thus, the existence of q contradicts the maximality of $\{p_n\}$. So we must have that $\text{lub}\{p, \theta(p), \theta^{-1}(p)\}=1$.

We have that $p\theta(p)=p\theta^{-1}(p)=0$. Consequently, the four projections given by $r_1=\theta^{-1}(p)\,\theta(p)$, $r_2=\theta(p)-r_1$, $r_3=\theta^{-1}(p)-r_1$, and $r_4=p$ are orthogonal central projections of sum 1. We have that

$$r_m u r_m = r_m \theta(r_m) u = 0$$

for $1 \leqslant m \leqslant 4$, whence

$$\sum r_m a u r_m = 0.$$

However, the sum $\sum r_m a u r_m$ is in $K(au)$ (Lemma 2.2) and so 0 is in $K(au)$. Q.E.D.

We now assume that $p(\theta \mid C)=1$. This means that θ is the identity on C.

PROPOSITION 3.3. *If N is finite and $p(\theta \mid C)=1$, then $0 \in K(au)$ for every $a \in N$.*

Proof. Let $\varepsilon>0$, let p be a nonzero projection of N and let $\{q_n\}$ be a maximal set of mutually orthogonal nonzero projections of N majorized by p such that (1) $\|q_n a\theta(q_n)\|<\varepsilon$ and (2) $q_n a\theta(q_m)=0$ for $n \neq m$. Let $q=\sum q_n$; then $q \leqslant p$ and $\|qa\theta(q)\|<\varepsilon$. Let r be any central projection and let

$$q'=\text{lub}\{l(rpa\theta(q)), l(rp\theta^{-1}(a^*q)), rq\},$$

where $l(x)$ denotes the left support of the operator x, i.e., the range projection of x. Then we have $q'=rp$. Indeed otherwise $q''=rp-q' \neq 0$. But then $q'' \leqslant p$,

$$q''a\theta(q)=q''rpa\theta(q)=q''q'rpa\theta(q)=0,$$

and likewise $qa\theta(q'')=0$ and $qq''=0$. If q'' is replaced by a smaller projection, the preceding relations also hold. But by Proposition 3.1 there is a nonzero projection $s \leqslant q''$, such that $\|sa\theta(s)\|<\varepsilon$. This contradicts the maximality of $\{q_n\}$. Therefore $q'=rp$.

Let now ϕ be the canonical center valued trace on N [10, III.5]. Since $\{\phi(x)\}=K(x)\cap C$ for all $x \in N$ and since it is easy to verify that $\theta(K(x))=K(\theta(x))$, we have $\phi(\theta(x))=\theta(\phi(x))=\phi(x)$. From

$$l(rpa\theta(q)) \sim l(r\theta(q)\,a^*p) \leqslant r\theta(q)=\theta(rq)$$

and

$$l(rp\theta^{-1}(a^*q)) \sim l(r\theta^{-1}(q)\,\theta^{-1}(a)\,p) \leqslant r\theta^{-1}(q) = \theta^{-1}(rq),$$

we have $\phi(rp) = \phi(q') \leqslant 3\phi(rq)$. Since this inequality holds for every central projection r, we conclude that $\phi(q) \geqslant 3^{-1}\phi(p)$. Therefore, by induction we can construct a sequence $\{p_n\}$ of mutually orthogonal projections in N such that (1) $\| p_n a\theta(p_n)\| < \varepsilon$, (2) $\phi(p_n) \geqslant 3^{-1}(1 - \phi(\sum \{p_m \mid 0 \leqslant m \leqslant n-1\}))$, where we choose $p_0 = 0$. Then we have $\phi(1 - \sum \{p_m \mid 0 \leqslant m \leqslant n\}) \leqslant (\frac{2}{3})^n$. Choose integers n, k so that $(\frac{2}{3})^n < 1/k < \varepsilon$ and set $p' = 1 - \sum \{p_m \mid 0 \leqslant m \leqslant n\}$. Since $\phi(p') \leqslant \phi(1 - p')$, p' is unitarily equivalent to a subprojection of $1 - p'$. By iteration we can find k unitary operators $w_j \in N$ such that the projections $w_j p' w_j^*$ are mutually orthogonal. Then by Lemma 2.2 the operator $\sum k^{-1} w_j(p_1 aup_1 + \cdots + p_n aup_n + p'aup')w_j^*$ belongs to $K(au)$ and has norm not larger than

$$\begin{aligned}
&\sum k^{-1} \| w_j(p_1 aup_1 + \cdots + p_n aup_n) w_j^* \| \\
&\qquad + \left\| \sum k^{-1} w_j p' w_j^* (w_j auw_j^*)\, w_j p' w_j^* \right\| \\
&\leqslant \| p_1 aup_1 + \cdots + p_n aup_n \| + k^{-1} \operatorname{lub} \| w_j auw_j^* \| \\
&\leqslant \operatorname{lub} \| p_j a\theta(p_j) \| + k^{-1} \| a \| \\
&\leqslant \varepsilon(1 + \| a \|).
\end{aligned}$$

Since ε is arbitrary, we conclude that $0 \in K(au)$. Q.E.D.

PROPOSITION 3.4. *If N is properly infinite and if $p(\theta \mid C) = 1$, then $0 \in K(au)$ for every $a \in N$.*

Proof. Let $\varepsilon > 0$ and let $\{q_n\}$ be a maximal set of mutually orthogonal nonzero projections of N such that (1) $\| q_n a\theta(q_n) \| < \varepsilon$ for all n and (2) $q_n a\theta(q_m) = 0$ for all $n \neq m$. Then the projection $q = \sum q_n$ is properly infinite and has central support $c(q) = 1$. On the contrary, there would be a nonzero central projection r such that qr is finite or zero. Setting

$$q' = \operatorname{lub}\{l(ra\theta(q)),\ l(r(\theta^{-1}(a^*q))),\ rq\},$$

we would have that q' is finite or zero because rq, $r\theta(q)$, and $r\theta^{-1}(q)$ are all finite or zero. As in the proof of Proposition 3.3, we could find a nonzero projection q'' with $q'' \leqslant r - q'$ that satisfies relations (1) and (2). However, this would contradict the maximality of the set $\{q_n\}$. Therefore, the projection q is properly infinite of central support 1. This means that $q \sim 1$

because N acts on a separable Hilbert space. We also have that $\| qa\theta(q) \| < \varepsilon$.

By passing if necessary to a subprojection of q, we may assume that q is unitarily equivalent to $1-q$. Now by recursion we can find k unitary operators w_j in N such that the projections $w_j(1-q)w_j^*$ are mutually orthogonal. Here k is chosen so that $1/k < \varepsilon$. Then the operator

$$\sum k^{-1} w_j(qauq + (1-q)\, au(1-q))w_j^*$$

is in $K(au)$ (Lemma 2.2) and has norm not exceeding

$$\| qa\theta(q) \| + k^{-1} \operatorname{lub} \| w_j(1-q)\, au(1-q)w_j^* \| < \varepsilon(1 + \| a \|).$$

Since $\varepsilon > 0$ is arbitrary, we conclude that 0 is in $K(au)$. Q.E.D.

Now we combine the previous propositions and obtain one of our main results.

THEOREM 3.5. *Let N be a von Neumann algebra on a separable Hilbert space H and let u be a unitary operator on H such that* ad u *induces an automorphism of N. Then, for every $a \in N$, the element au is in $N_0 = \{x \in B(H) \mid f \cdot x \in N_d \text{ for every } f \in F\}$ and in particular $0 \in K(au)$ if* ad u *is properly outer.*

Proof. Let θ be the restriction of ad u to N, let C be the center of N, and let $p_1 = p(\theta)$ and $p = 1 - p_1$. Then $p_1 \in C$ and $\theta(p_1) = p_1$ [6, 1.5.1]. Let $p_2 = p - p(\theta \mid C_p)$. We also have that $p_2 \in C$ and $\theta(p_2) = p_2$. Let $p(\theta \mid C_p) = p_3 + p_4$ be the canonical decomposition of $p(\theta \mid C_p)$ into the sum of a finite central projection p_3 and a properly infinite central projection p_4 for N. Since $\theta(p_3)$ is finite, we have that $\theta(p_3) = p_3$ and consequently that $\theta(p_4) = p_4$. Thus, we have the decomposition $N = \Sigma \oplus N_i$, $C = \Sigma \oplus C_i$, $\theta = \Sigma \oplus \theta_i$, where $N_i = N_{p_i}$, $C_i = C_{p_i} = N_i \cap N_i'$, $\theta_i = \theta \mid N_i =$ ad $up_i \mid N_i$ for $1 \leqslant i \leqslant 4$.

Let F_i be the set of nonnegative functions f with finite support on $U(N_i)$ such that $\sum \{f(v) \mid v \in U(N_i)\} = 1$. Embed F_i in F by setting

$$f_i(v) = f_i(vp_i)$$

if $v(1-p_i) = 1 - p_i$ and $f_i(v) = 0$ otherwise. For every a in N and $f \in F$, let $a(f)$ be the element in N given by $a(f) = \sum \{f(v)\, va\theta(v^*) \mid v \in U(N)\}$. Then $f \cdot (au) = a(f)u$. As a consequence of this, it is sufficient to show $au \in N_d$ for all $a \in N$ in order to show au is in N_0.

Now let $a \in N$ and let $\varepsilon > 0$. Since θ_1 is an inner automorphism on N_1, there is a unitary operator w in N_1' such that uwp_1 is in N_1. There is an f_1 in F_1 and a c in C_1 such that $\| f_1 \cdot (auwp_1) - c \| < \varepsilon/4$

by the Dixmier property. Thus, we get $\| f_1 \cdot (aup_1) - cw^* \| = \| f_1 \cdot (auwp_1) - c \| < \varepsilon/4$ due to the fact that multiplication by w commutes with the action of f_1. Since $\{N_i, \theta_i\}$ $(i=2, 3, 4)$ satisfy Propositions 3.2, 3.3, and 3.4, respectively, we can find $f_i \in F_i \subset F$ such that

$$\| f_2 \cdot a(f_1)\, up_2 \| < \varepsilon/4,$$

$$\| f_3 \cdot a(f_2 * f_1)\, up_3 \| < \varepsilon/4,$$

and

$$\| f_4 \cdot a(f_3 * f_2 * f_1)\, up_4 \| < \varepsilon/4.$$

Then, for $f = f_4 * f_3 * f_2 * f_1$, we have

$$\begin{aligned} \| f \cdot au - cw^* \| &\leqslant \| f_4 \cdot a(f_3 * f_2 * f_1)\, up_4 \| \\ &\quad + \| f_4 \cdot (f_3 \cdot a(f_2 * f_1)\, up_3) \| \\ &\quad + \| f_4 * f_3 \cdot (f_2 \cdot a(f_1)\, up_2) \| \\ &\quad + \| f_4 * f_3 * f_2 \cdot (f_1 \cdot aup_1 - cw^*) \| \\ &\leqslant \varepsilon. \end{aligned}$$

Since ε is arbitrary and since cw^* is in N_1', we have that $K'(au)$ is nonvoid. If u is properly outer, then $p_1 = 0$ and thus 0 is in $K'(au)$. Q.E.D.

We conclude with the following characterization of those $x \in N_0$ with $K'(x) = \{0\}$. In particular, this applies to $x = au$ for $a \in N$, and ad u properly outer.

PROPOSITION 3.6. *If $0 \in K(f \cdot x)$ for all $f \in F$, then $K'(x) = \{0\}$.*

Proof. Let $z \in K'(x)$ and let $\varepsilon > 0$. Then there is an $f \in F$ such that $\| f \cdot x - z \| < \varepsilon/2$ and a $g \in F$ such that $\| g \cdot (f \cdot x) \| < \varepsilon/2$. Thus

$$\| z \| \leqslant \| g \cdot (f \cdot x - z) \| + \| g \cdot (f \cdot x) \| < \varepsilon.$$

Since ε is arbitrary we conclude that $z = 0$. Q.E.D.

4. CROSSED PRODUCTS

Let N be a von Neuman algebra with center C on the separable Hilbert space K and let θ be an automorphism of N. Let H be the separable Hilbert space $H = L^2(K, Z)$ of square summable functions of Z into K and let the crossed product $M = N \times_\theta Z$ of N by the action θ of Z be represented on H. Let $\pi = \pi_\theta$ be the canonical embedding of N into M and let $u = u_\theta$ be the unitary operator on H given by $(u\zeta)(n)=\zeta(n-1)$. In the sequel, we identify N with its image $\pi(N)$. Let E be the *canonical expectation* of M onto N. Each element x in M is uniquely determined by the totality of the values $E(xu^{-n})$ and can be represented as a generalized Fourier series $\sum E(xu^{-n})u^n$. The series converges in the Bures topology which is weaker than the weak convergence of the finite partial sums [19]. Let $U(M; N)$ be the group of all unitary operators v in M with $vNv^*=N$. We note that $N'\cap M=N'\cap N=C$ if θ is aperiodic [25; 22.3]. The *normalizer* of E is the group of all unitary operators v in M such that $E(vxv^*)=vE(x)v^*$ for every x in M. If $N'\cap M=C$, the normalizer of E coincides with the group $U(M; N)$ [25, 10.17]. We note that $U(N)\subset U(M; N)$.

We can now state one of our main results.

THEOREM 4.1. *Let N be a von Neumann algebra on the separable Hilbert space H, let θ be an automorphism of N, and let $M=N\times_\theta Z$. Then the C^*-algebra A generated by the normalizer $U(M; N)$ is contained in N_0. In particular, every element $x\in A$ has the relative Dixmier property and if θ is aperiodic, $K(x)$ has a nonvoid intersection with the center C of N.*

Proof. The set of linear combinations of elements of $U(M; N)$ is dense in A because $U(M; N)$ is a group. Since N_0 is a Banach space (Proposition 2.4), it is sufficient to show $U(M; N)\subset N_0$. This has already been shown in a more abstract setting in Theorem 3.5. Finally, if θ is aperiodic, we have that $K'(x)=K(x)\cap N'\cap M=K(x)\cap C$. Q.E.D.

For every t in the torus T, identified with the dual group of Z, there is an automorphism $\hat{\theta}_t$ on M uniquely determined by $\hat{\theta}_t(x)=x$ $(x\in N)$ and $\hat{\theta}_t(u)=t^{-1}u$. This so-called *dual* action $t\to\hat{\theta}_t$ on M is strongly continuous. The canonical expectation E is then given by integration with respect to the normalized Haar measure on T as $E(x)=\int\hat{\theta}_t(x)\,dt$.

DEFINITION 4.2. Let M be the crossed product of N by the action of the automorphism θ. An element x in M is said to be *continuous* if $t\to\hat{\theta}_t(x)$ is continuous in the norm topology. The set of all continuous elements will be denoted by M_c.

The set M_c is a weakly dense norm closed *-subalgebra of M [21, 7.5.1]. We now describe M_c. We state this in a more general context.

PROPOSITION 4.3. *Let σ be the action of a locally compact abelian group G on the von Neumann algebra A; then the C^*-algebra A_c of all x in A such that $t\to\sigma_t(x)$ is continuous in the norm is the set $\{\sigma(\phi)\,x\mid\phi\in L^1(G),\ x\in A\}$. Here $\sigma(\phi)x$ is given by $\sigma(\phi)x=\int\phi(t)\sigma_t(x)\,dt$.*

Proof. It is known that the norm closure of $L'(G)A=\{\sigma(\phi)x\mid\phi\in L^1(G),$

$x \in A\}$ is equal to the set A_c of continuous elements (cf. [21, 7.5.1]). However, the set A_c forms a left Banach module over $L^1(G)$, since $\|\sigma(\phi)x\| \leqslant \|\phi\|_1 \|x\|$ for ϕ in $L^1(G)$. The approximate identity of $L^1(G)$ is an approximate identity for the Banach module A_c. Therefore, Cohen's factorization theorem is applicable (cf. [17, 32.22]) so that $L^1(G)A_c$ is already closed and the three spaces $L^1(G)A$, $L^1(G)A_c$, and A_c thus coincide.

Q.E.D.

In our setting, we have $M_c = \{\hat{\theta}(\phi)x \mid \phi \in L^1(T), x \in M\}$.

LEMMA 4.4. *Let* $\phi \in L^1(T)$ *and* $x = \sum a_n u^n \in M$, *then* $\hat{\theta}(\phi)x = \sum (\hat{\phi}(n)a_n)u^n$, *where the series are the generalized Fourier expansions of* x *and* $\hat{\theta}(\phi)x$, *respectively.*

Proof. Let $x = \sum a_n u^n$, i.e., $a_n = E(xu^{-n})$. Recall that E is σ-weakly continuous and that $E \circ \hat{\theta}_t = \hat{\theta}_t \circ E = E$ for all $t \in T$. Then

$$\begin{aligned} E((\hat{\theta}(\phi)x)u^{-n}) &= E\left(\int \phi(t)\,\hat{\theta}_t(x)\,u^{-n}\,dt\right) \\ &= E\left(\int \phi(t)\,t^{-n}\hat{\theta}_t(xu^{-n})\,dt\right) \\ &= \int \phi(t)t^{-n}\,E(\hat{\theta}_t(xu^{-n}))\,dt \\ &= \left(\int \phi(t)t^{-n}\,dt\right)E(xu^{-n}) \\ &= \hat{\phi}(n)a_n. \end{aligned}$$

Q.E.D.

Using this lemma on Cesaro summability, we have a simple proof of the fact that M_c coincides with the C^*-crossed product of N by θ, i.e., with the C^*-algebra generated by N and u, which is the norm closure of $\operatorname{span}\{au^n \mid a \in N, n \in Z\}$ (cf. [20]).

PROPOSITION 4.5. $\operatorname{span}\{au^n \mid a \in N, n \in Z\}$ *is norm dense in* M_c.

Proof. For every $a \in N$, $n \in Z$ we have that $au^n \in M_c$, since $\hat{\theta}_t(au^n) = t^{-n}au^n$. Conversely, let $\phi \in L^1(T)$, $x \in M$, and $\varepsilon > 0$, then there is a $\psi \in L^1(T)$ such that $\|\psi*\phi - \phi\|_1 < \varepsilon$ and such that the support of $\hat{\psi}$ is finite [17, 33.12]. Since

$$\|\hat{\theta}(\psi * \phi)(x) - \hat{\theta}(\phi)x\| \leqslant \|\psi * \phi - \phi\|_1 \|x\| \leqslant \varepsilon \|x\|$$

$\hat{\theta}(\phi)x$ is approximated by the finite sum

$$\hat{\theta}(\psi * \phi)x = \sum \hat{\psi}(n)\,\hat{\phi}(n)\,a_n u^n.$$

Q.E.D.

THEOREM 4.6. *Let N be a von Neumann algebra on a separable Hilbert space, let θ be an automorphism of N, let $M = N \times_\theta Z$, and let M_c be the set of elements of M continuous under the dual automorphism $\hat{\theta}$; then $M_c \subset N_0$. In particular, every continuous element of M has the relative Dixmier property.*

Proof. By Proposition 4.5, the algebra M_c is the C^*-algebra generated by N and u, and is thus contained in the C^*-algebra generated by $U(M; N)$. By Theorem 4.1, the algebra M_c is contained in N_0. Q.E.D.

We assume henceforth that M, θ are as in Theorem 4.6 and that θ is aperiodic. We shall see in Section 5 that there are elements with the RDP that are not in M_c, however, the continuous elements have a connection with the relative Dixmier property. We state this in the following form:

PROPOSITION 4.7. *The element $x \in M$ has the relative Dixmier property if and only if $K(x) \cap M_c$ is nonvoid.*

Proof. If x has the RDP, there is a $z \in K'(x)$; but then $z \in M \cap N' = N \cap N'$ is fixed under $\hat{\theta}_t$, hence is in M_c. Conversely, if $y \in K(x) \cap M_c$ then by Theorem 4.6, $K'(y) \subset K'(x)$ is nonvoid and thus x has the RDP. Q.E.D.

If $x \in M_c$, or more generally if $x \in N_0$, then $K'(x)$ coincides with the essential central range $K'(E(x))$ of $E(x)$ [8; 12; 13].

PROPOSITION 4.8. (a) *$K'(x) \subset K'(E(x))$ for every $x \in M$; and* (b) *$K'(x) = K'(E(x))$ for every $x \in N_0$.*

Proof. (a) Let $z \in K'(x)$, let $\varepsilon > 0$, and let $f \in F$ be such that $\| f \cdot x - z \| < \varepsilon$. Then $\| E(f \cdot x - z) \| = \| f \cdot E(x) - z \| < \varepsilon$. Since ε is arbitrary, $z \in K(E(x))$.

(b) *Let* $z \in K'(E(x))$, let $\varepsilon > 0$, and let $f \in F$ be such that $\| f \cdot E(x) - z \| < \varepsilon$. Then there is a $z' \in K'(f \cdot x)$ and hence a $g \in F$ such that $\| g * f \cdot x - z' \| < \varepsilon$. Therefore,

$$\begin{aligned} \| z - z' \| &\leqslant \| z - g * f \cdot E(x) \| + \| g * f \cdot E(x) - z' \| \\ &= \| g \cdot (z - f \cdot E(x)) \| + \| E(g * f \cdot x - z') \| \\ &< 2\varepsilon. \end{aligned}$$

This shows that $K'(x)$ is dense in $K'(E(x))$ and hence coincides with it. Q.E.D.

If N is a properly infinite algebra we can strengthen Theorem 4.6: we show that if $x \in M$ has a "large piece" in M_c then x has the RDP.

PROPOSITION 4.9. *Let N be properly infinite and let $x \in M$. If there is a projection $p \in N$, $p \sim I$ such that $pxp \in M_c$, then x has the relative Dixmier property.*

Proof. For every projection $p' \leqslant p$ in N we have that $t \to \hat{\theta}_t(p'xp') = p'\hat{\theta}_t(pxp)\,p'$ is norm continuous and hence $p'xp' \in M_c$. Thus we can assume without loss of generality that $1 - p \sim p \sim 1$. Then reasoning as in [13, Theorem 4.12], for every $\varepsilon > 0$ and $k > \varepsilon^{-1}\|x\|$ we can find k unitary operators w_j such that $w_j(1-p)w_j^*$ are mutually orthogonal. Let $f(v) = k^{-1}$ for $v = w_j$, $j = 1, 2,..., k$, and zero otherwise; then $\|f \cdot (1-p)\,x(1-p)\| = k^{-1}\,\|\sum w_j(1-p)\,x(1-p)\,w_j^*\| < \varepsilon$. Since $pxp \in M_c \subset N_0$, $f \cdot pxp \in N_0$ and hence there is a $z \in N \cap N'$ and a $g \in F$ such that $\|g * f \cdot pxp - z\| < \varepsilon$. But then $g * f \cdot (pxp + (1-p)\,x(1-p))$ belongs to $K(x)$ by Lemma 2.2 and has distance from z and hence N' not greater than 2ε. Therefore x has the RDP. Q.E.D.

5. $B(H)$ AND TYPE III_λ $(0 \leqslant \lambda < 1)$ FACTORS

Let $\{\zeta_n \mid n \in Z\}$ be the canonical basis of the Hilbert space $l^2 = l^2(Z)$ given by $\zeta_n(m) = \delta_{mn}$ (Kronecker delta) and let u_{mn} be the partial isometries on l^2 given by $u_{mn}(\zeta) = (\zeta, \zeta_n)\zeta_m$. The von Neumann algebra N of diagonal operators with respect to the basis $\{\zeta_n\}$ is isomorphic to $l^\infty(Z)$ under the identification $\phi \to \sum \phi(n)u_{nn}$. If u denotes the bilateral shift on $l^2(Z)$ given by $u(\sum \alpha_n\zeta_n) = \sum \alpha_n\zeta_{n+1}$, then ad $u = \theta$ is a properly outer automorphism on N. In fact, given any nonzero projection p in N, any 1-dimensional subprojection q of p satisfies $q\theta(q) = 0$. Similarly, each automorphism θ^n $(n \neq 0)$ is properly outer and so θ is aperiodic.

The algebra N and the bilateral shift generate the algebra $B(l^2)$ of all bounded linear operators on l^2. On the other hand, the bilateral shift generates the (von Neumann) algebra $L(Z)$ of Laurent operators. The algebra $L(Z)$ is isomorphic to $L^\infty(T)$ under the map $\phi \to \sum \hat{\phi}(n)u^n = L_\phi$ so that $L_\phi\zeta = \phi * \zeta$ for ζ in l^2. The function ϕ is called the symbol of the Laurent operator L_ϕ.

For t in T, let w_t be the unitary operator on l^2 given by $(w_t\zeta)(n) = t^{-n}\zeta(n)$. The map $t \to w_t$ is a strongly continuous unitary representation of T on $l^2(Z)$ with generator d (i.e., $w_t = \exp(\log td)$) with d equal to an unbounded selfadjoint operator affiliated with N. If the algebra $B(l^2)$ is identified with the crossed product $N \times_\theta Z$ under the isomorphism that sends x in N into $\pi_\theta(x)$ and u into u_θ, then ad w_t is the action $\hat{\theta}_t$ dual to θ.

The algebra $B_c = B(l^2)_c$ is precisely the C^*-algebra generated by N and u (cf. Proposition 4.5). The algebra $B_c \cap L(Z)$ is by Proposition 4.5 the C^*-algebra generated by u, i.e., the algebra of all Laurent operators with continuous symbol. By Theorem 4.6 all the elements of B_c have the relative Dixmier property. In another place [16, 4.2] we show that the Laurent operators with Riemann integrable symbol have the RDP. It is an open question [2; 18] whether all operators in $B(H)$ or even in $L(Z)$ have the relative Dixmier property.

The compact operators on l^2 are also in B_c. In fact, we have a more general result. Let T^∞ be the compact group $T^\infty = \mathsf{X}\{T_n \mid n \in Z\}$, where $T_n = T$. We embed T in T^∞ by identifying t in T with $\{t^n\}$ in T^∞. The map w of T^∞ into N given by $(w_\alpha \zeta)(n) = \overline{\alpha(n)}\, \zeta(n)$ for $\alpha \in T^\infty$ is a strongly continuous unitary representation of T^∞ on l^2 since

$$\alpha \to (w_\alpha \zeta, \xi) = \sum \overline{\alpha(n)}\, \zeta(n)\, \overline{\xi(n)}$$

is continuous if ζ, ξ have finite support and thus is continuous for any ζ, ξ in l^2. We note that w_t for t in T has the meaning originally assigned to it. Notice that the range of w is actually $U(N)$ and that T^∞ and $U(N)$ with its strong (equivalently, weak) topology are algebraically and topologically isomorphic. Let ω be the action of T^∞ on $B(l^2)$ given by $\omega_\alpha = \operatorname{ad} w_\alpha$.

Proposition 5.1. *An operator x in $B(l^2)$ is continuous under $\alpha \to \omega_\alpha(x)$ in the norm topology if and only if $x - E(x)$ is compact.*

Proof. Let χ_n be the character of T^∞ given by $\chi_n(\alpha) = \langle \chi_n, \alpha \rangle = \alpha(n)$. For the previously defined matrix units u_{mn}, we have that

$$\omega_\alpha(u_{mn}) = \overline{\alpha(m)}\, \alpha(n) u_{mn}.$$

Then, for any ϕ in $L^1(T^\infty)$ and $x \in B(l^2)$ let the operator $\omega(\phi)x$ be given by

$$\omega(\phi)x = \int \phi(\alpha)\, \omega_\alpha(x)\, d\alpha.$$

Thus,

$$\omega(\phi) u_{mn} = \hat{\phi}(\chi_m \chi_n^{-1}) u_{mn}.$$

However, linear combinations of the u_{mn} are dense in $B(l^2)$ in the σ-weak topology. Thus, the formula

$$\omega(\phi)x = \sum \hat{\phi}(\chi_m \chi_n^{-1})(x\zeta_n, \zeta_m) u_{mn}$$

holds for all x in $B(l^2)$. The sum is the limit of the net of the finite partial sums in the σ-weak topology.

Now let $\alpha \to \omega_\alpha(x)$ be continuous on T^∞ in the norm topology. Then

$$\alpha \to \omega_\alpha(x - E(x)) = \omega_\alpha(x) - E(x)$$

is also continuous. We have already used the fact that for preassigned $\varepsilon > 0$ there is a function ϕ in $L^1(T^\infty)$, whose Fourier transform $\hat{\phi}$ has finite support, such that

$$\| \omega(\phi)(x - E(x)) - (x - E(x)) \| < \varepsilon.$$

However, the operator $x - E(x)$ has zero diagonal, whence $\omega(\phi)(x - E(x)) = \sum \{ \hat{\phi}(\chi_m \chi_n^{-1})(x\zeta_n, \zeta_m) \, u_{mn} \mid m \neq n \}$ is a finite sum. Indeed, we note that $\chi_i \chi_j^{-1} = \chi_m \chi_n^{-1}$ implies either $i = j$ and $m = n$ or $i = m$ and $j = n$. Thus $x - E(x)$ is approximated in norm by the finite rank operator $\omega(\phi)(x - E(x))$, and consequently, $x - E(x)$ is a compact operator.

Conversely, suppose that $x - E(x)$ is compact. Then, given $\varepsilon > 0$, there is a finite linear combination $\sum \gamma_{mn} u_{mn}$ approximating $x - E(x)$ within ε. Because

$$\alpha \to \omega_\alpha\left(\sum \gamma_{mn} u_{mn} \right) = \sum \gamma_{mn} \overline{\alpha(m)} \, \alpha(n) u_{mn}$$

is continuous in norm and because

$$\left\| \omega_\alpha\left(\sum \gamma_{mn} u_{mn} - (x - E(x)) \right) \right\| < \varepsilon,$$

for all $\alpha \in T^\infty$, the function $\alpha \to \omega_\alpha(x - E(x))$ is the uniform limit of continuous functions and hence continuous. Finally, the function $\omega_\alpha(E(x)) = E(x)$ for all α and thus, the function $\alpha \to \omega_\alpha(x)$ is continuous.

Q.E.D.

COROLLARY 5.2. *If $x - E(x)$ is compact, then $x \in B_c$.*

The normalizer $U(B(l^2); N)$ is easy to describe.

PROPOSITION 5.3. *Let $P(Z)$ be the set of all permutations of Z; then, for every $\sigma \in P(Z)$, the equation $(u_\sigma \zeta)(n) = \zeta(\sigma^{-1}(n))$ defines a unitary operator on $l^2(Z)$ and $U(B(l^2); N) = \{ w_\alpha u_\sigma \mid \sigma \in P(Z), \alpha \in T^\infty \}$.*

Proof. We clearly have that $u_\sigma^* = u_{\sigma-1}$ and that

$$\| u_\sigma \zeta \|^2 = \sum | \zeta(\sigma^{-1}(n)) |^2 = \sum | \zeta(n) |^2 = \| \zeta \|^2$$

for every $\zeta \in l^2(Z)$, whence u_σ is unitary. Furthermore, we have that

$$(u_\sigma^* w_\alpha \zeta)(n) = \overline{\alpha(\sigma(n))} \, \zeta(\sigma(n))$$
$$= (w_{\alpha \cdot \sigma} u_\sigma^* \zeta)(n)$$

so that $u_\sigma^* w_\alpha u_\sigma = w_{\alpha \cdot \sigma}$. Thus, we get ad u_σ maps $U(N)$ into $U(N)$ and thus u_σ is in the normalizer. Conversely, let v be in the normalizer; then each projection ad $v(u_{nn})$ is a 1-dimensional projection in N. Hence, there is a σ in $P(Z)$ such that ad $v(u_{nn}) = u_{\sigma(n)\,\sigma(n)}$, and consequently, the automorphism $\mathrm{ad}(vu_\sigma^*)$ is the identity on N. Because N is maximal abelian, the unitary operator vu_σ^* is in N and $vu_\sigma^* = w_\alpha$ for some α in T^∞. Q.E.D.

The bilateral shift u is represented as u_σ for that σ in $P(Z)$ with $\sigma(n) = n+1$. Now we show the C^*-algebra A generated by the normalizer is strictly larger than B_c.

PROPOSITION 5.4. *The unitary operator u_σ is in B_c if and only if $Z_\sigma = \{\sigma(n) - n \mid n \in Z\}$ is a finite set.*

Proof. For every α in T^∞ we have that

$$w_\alpha u_\sigma w_\alpha^* = w_\alpha w_\beta u_\sigma,$$

where β in T^∞ is given by $\beta(n) = \alpha(\sigma^{-1}(n))^-$. Thus, we have that

$$\begin{aligned}\| w_\alpha u_\sigma w_\alpha^* - u_\sigma \| &= \| (w_\alpha w_\beta - 1) u_\sigma \| = \| w_\alpha w_\beta - 1 \| \\ &= \mathrm{lub}\{ |\alpha(n)\, \alpha(\sigma^{-1}(n))^- - 1| \mid n \in Z\}.\end{aligned}$$

In particular, we have that

$$\| w_t u_\sigma w_t^* - u_\sigma \| = \mathrm{lub}\{ |t^p - 1| \mid p \in Z_\sigma\}.$$

Since

$$\lim_{\varepsilon \to 0} (\mathrm{lub}\{ |t^p - 1| \mid p \in Z_\sigma, |t-1| < \varepsilon\}) = 0$$

if and only if Z_σ is a finite set, we see that $u_\sigma \in B_c$ if and only if Z_σ is a finite set. Q.E.D.

We can now state the extension property of Kadison and Singer [18, Theorem 3].

PROPOSITION 5.5. *Every pure state of the algebra N of diagonal operators in $B(l^2)$ has a unique extension to a pure state on the C^*-algebra A generated by the normalizer $U(B(l^2); N)$.*

Proof. The set $K'(x)$ is nonvoid for every x in A. So Theorem 2.4 [5] may be applied. Q.E.D.

We note that Proposition 3.2 gives directly that $0 \in K(au)$ since N is abelian. Also we see that $E(x) \in K(x)$ whenever $x \in B_c$ follows directly from the fact that there are Riemann sums for $\int \hat{\theta}_t(x)\, dt = E(x)$ that lie in $\overline{\text{co}}\{\omega_t(x) \mid t \in T\}$ and converge uniformly to $E(x)$. Also $\{w_t \mid t \in T\}$ is a compact subgroup of $U(N)$ that generates N; moreover $\omega_t(x) = \text{ad } w_t(x) \in N$ for all t in T implies $x \in N$.

We now treat a second example. Let N be a type II_∞ von Neumann algebra with center C on a separable Hilbert space, let τ be a faithful normal semifinite trace on N, and let θ be an automorphism of N such that $\tau \cdot \theta \leqslant \lambda\tau$ for some $0 < \lambda < 1$. Then the automorphism θ is aperiodic on N. In fact, given any nonzero projection p in N, there is a nonzero θ-wandering projection q (i.e., $q\theta^n(q) = 0$ for all $n \neq 0$) majorized by p [15, 3.3]. The construction of type III_λ factors $(0 < \lambda < 1)$ is based on the existence of such automorphisms θ on certain type II_∞ factors, and the construction of type III_0 factors is based on the existence of such automorphisms on certain type II_∞ algebras with diffuse center on which θ acts ergodically.

The elements v of the normalizer $U(M; N)$ of N in M have a specific description [6, 1.5.5] because M is generated by N and the subgroup $\{u^n \mid n \in Z\}$ of the normalizer. If v is in $U(M; N)$, then there is a doubly infinite sequence $\{p_n\}$ of projections in C of sum 1 and a doubly infinite sequence $\{u_n\}$ of partial isometries in N with $u_n^* u_n = u_n u_n^* = p_n$ such that $u^{-n} v p_n x p_n v^* u^n = u_n x u_n^*$ for every x in N. Because $N' \cap M = C$, the unitary v can be written as $v = \sum \{u^n w_n p_n \mid n \in Z\}$, where $\{w_n\}$ is a doubly infinite sequence of unitaries in N with $u_n^* w_n \in C p_n$. Conversely, if $\{p_n\}$ is a doubly infinite sequence of orthogonal projections in C of sum 1 and $\{w_n\}$ is a doubly infinite sequence of unitary operators in N such that $u^n w_n p_n w_n^* u^{-n} = p_n$ for all n, then $v = \sum u^n w_n p_n$ is in $U(N; M)$. In terms of the preceding sum decomposition of elements of $U(M; N)$, we can describe the continuous elements.

PROPOSITION 5.6. *Let $\{p_n \mid n \in Z\}$ be a sequence of orthogonal projections in C of sum* 1 *and let $\{w_n\}$ be a corresponding sequence of unitary operators in N such that $u^n w_n p_n (u^n w_n)^* = p_n$ for all n. Then the unitary operator $v = \sum u^n w_n p_n$ in $U(M; N)$ is in M_c if and only if the set $Z_v = \{n \in Z \mid p_n \neq 0\}$ is a finite set.*

Proof. We have that

$$\hat{\theta}_t(v) = \sum t^{-n} u^n w_n p_n,$$

and consequently, that

$$\| \hat{\theta}_t(v) - v \| = \text{lub}_n \, | t^n - 1 | \, \| u^n w_n p_n \|.$$

As in Proposition 5.4, $\lim_{t \to 1} \| \hat{\theta}_t(v) - v \| = 0$ if and only if Z_v is a finite set. Q.E.D.

Thus, if N is a factor, the C^*-algebra A generated by $U(M; N)$ is equal to M_c. If N has a diffuse center C and θ acts ergodically on C, then A is strictly larger than M_c. In fact, there is a sequence $\{p_n\}$ of orthogonal projections of sum 1 in C such that $\{\theta^n(p_n)\}$ are also orthogonal of sum 1 and such that $\{n \mid p_n \neq 0\}$ is infinite (cf. [25, 29.2]). The operator $\sum u^n p_n$ is in $U(M; N)$ by direct calculation and is not in M_c by Proposition 5.6.

We now state some results of an earlier paper in the context of the RDP.

PROPOSITION 5.7. *Let N be a type II_∞ factor (resp. type II_∞ algebra with diffuse center) on a separable Hilbert space, let τ be a faithful normal semifinite trace on N, and let θ be an automorphism of N such that $\tau \cdot \theta = \lambda\tau$ for some $0 < \lambda < 1$ (resp. such that $\tau \cdot \theta \leqslant \lambda\tau$ for some $0 < \lambda < 1$ and such that θ is ergodic on the center of N). Let x be in $(N \times_\theta Z)^+ = M^+$. If $E(x)$ is compact in N, i.e., is in the strong radical of N, then there is a cofinite projection p in N such that $pxp \in M_c$ and $K'(x) = \{0\}$.*

Proof. In [14, 6.1] we have proved the existence of a cofinite projection p such that the generalized Fourier expansion of pxp converges uniformly. This means that pxp is in M_c. The second statement follows from Proposition 4.8(a) and 4.9 together with the fact that $K'(E(x)) = \{0\}$ if $E(x)$ is compact [13, 4.12]. Q.E.D.

Let us consider now $1 \times_\theta Z \subset M$ which we can identify with $1 \otimes L(Z)$, where $L(Z)$ is the previously mentioned class of Laurent operators in l^2. While we do not know whether the elements of $L(Z)$ have the RDP in $B(l^2)$, we have the following:

PROPOSITION 5.8. *Let $x \in 1 \otimes L(Z) \subset M$, where M is a type III_λ factor with $0 < \lambda < 1$. Then x has the RDP and $K'(x) = \{E(x)\}$.*

Proof. Let $x = \sum a_n u^n$ be the generalized Fourier expansion of x. Then all the a_n are scalar multiples of 1 and $E(x) = a_0$. Thus $K'(x) \subset K'(E(x)) = \{E(x)\}$ by Proposition 4.8(a). Without loss of generality assume that $E(x) = 0$. By [15, 3.6(b)] we can find an infinite wandering projection $p \in N$. Thus

$$pxp = \sum a_n pu^n p = \sum a_n p\theta^n(p)u^n = 0.$$

Since $p \sim 1$, by Proposition 4.9 we see that $0 \in K(x)$. Then $K'(x) = \{0\}$. Q.E.D.

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Printed by the St. Catherine Press Ltd., Tempelhof 41, Bruges, Belgium