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Quasitriangular Subalgebras of Semifinite von Neumann Algebras Are Closed

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Arveson has shown in [1] that if A is a nest algebra of operators acting on a separable Hilbert space H with nest of order type the extended natural numbers and if $K(H)$ is the ideal of compact operators on H, then the quasitriangular algebra $A + K(H)$ is norm closed. In [6] Fall, Arveson, and Muhly extended Arveson's result proving that in $B(H)$, general quasitriangular algebras (i.e., compact perturbations of nest algebras) are always norm closed. A key step in their proof, and a result of independent interest, first obtained by Erdos in [5], is the σ-weak density of $A \cap K(H)$ in A. From this fact they deduced that quasitriangular algebras were norm closed by using an argument depending on the duality $K(H)^{**} = B(H)$.

Every semifinite von Neumann algebra M has an ideal of compact operators that behaves like $K(H)$, namely the norm-closed two-sided ideal K generated by the finite projections of the algebra. It is a natural question,

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which has circulated informally for several years, whether the above two results and especially the closure result can be extended to the von Neumann algebra setting.

In this paper we settle both questions in the affirmative. However, the closure result does not seem to be obtainable from the density result by using duality arguments as in $B(H)$, since in general $K^{**} \neq M$. Instead our proof is matricial and is based on approximating Hilbert–Schmidt elements of M (relative to a semifinite trace) by Hilbert–Schmidt elements in the nest algebra, retaining joint control of both the operator norm and the Hilbert–Schmidt norm. In the case $M = B(H)$, this provides a new proof of the closure of quasitriangular algebras independent of the density result.

In addition, as an application of our matricial technique, we obtain a joint norm control Nehari type theorem in which we approximate L^∞ functions on the unit circle by H^∞ functions, while retaining control of both the L^∞ and L^2-norms. As a consequence of our construction, we obtain another proof of Sarason's theorem on the closure of $H^\infty + C$ [21]. Recently, G. Pisier showed us how to deduce a qualitative version of the joint norm control Nehari type theorem from some deep interpolation theoretic work of Jones based on Carleson measure techniques [10].

We use the following notations: M denotes a semifinite (but not necessarily σ-finite) von Neumann algebra with center Z and predual M_*, τ is a faithful semifinite normal trace on M, and $C_2(M)$ (resp. $C_1(M)$) is the Hilbert–Schmidt class with norm $\|\cdot\|_2$ (resp. the trace class with norm $\|\cdot\|_1$). That is, $C_2(M)$ (resp. $C_1(M)$) is the ideal of the elements $x \in M$ such that $\|x\|_2^2 = \tau(x^*x) < \infty$ (resp. $\|x\|_1 = \tau(|x|) < \infty$). Clearly $C_1(M)$ and $C_2(M)$ are contained in the ideal K of compact operators defined above.

A nest N in M is a family of projections in M containing 0 and I which is totally ordered by inclusion and is complete in the sense that it contains the intersections and the joins of arbitrary subfamilies. For every nest N in M, let $A_N = \{x \in M \mid p^\perp x p = 0 \text{ for } p \in N\}$ be the corresponding nest subalgebra [8], let I_N be the set of intervals in N, (i.e., the set of the projections of the form $p - p'$ with $p, p' \in N$, $p \geqslant p'$), let C_N be the core of N, (i.e., $C_N = N''$), and let D_N be the diagonal of A_N, (i.e., $D_N = C'_N \cap M = A_N \cap A_N^*$).

Power [19, Sect. 7], Kraus [unpublished], Pop [16], and Pai [13] noted that Power's proof [17] of Arveson's distance formula for nest algebras in $B(H)$ [1] adapts to nest subalgebras of von Neumann algebras, and we shall use this fact without further reference. That is, $d(x, A_N) = \sup\{\|p^\perp x p\| \mid p \in N\}$ for every $x \in M$, where $d(\cdot,\cdot)$ denotes distance in the operator norm. Likewise $d_2(\cdot,\cdot)$ will denote distance in the Hilbert–Schmidt norm.

A 2×2 operator matrix construction in [14, 2] is essential in Power's proof of the Arveson distance formula [17] as well as in many related results (e.g., see [4, 18, 3]). We quote here a special case which we use repeatedly in our paper.

LEMMA 1. *Let* $y = \begin{pmatrix} b & y_0 \\ a & c \end{pmatrix}$ *be a matrix with bounded operator entries. If* $\max\{\|\binom{b}{a}\|, \|(ac)\|\} \leqslant \eta$, $\|a\| < \eta$, *and* $y_0 = -b(\eta^2 I - a^*a)^{-1}a^*c$, *then* $\|y\| \leqslant \eta$.

Note that $b(\eta^2 I - a^*a)^{-1/2}$ is a contraction. Therefore if $\|a\| \leqslant d < \eta$, (e.g., if $d = \max\{\|\binom{b}{a}\|, \|(ac)\|\} < \delta$), we have

$$\|-b(\eta^2 I - a^*a)^{-1}a^*\| \leqslant \|-b(\eta^2 I - a^*a)^{-1/2}\| \, \|(\eta^2 I - a^*a)^{-1/2}a^*\| \leqslant d(\eta^2 - d^2)^{-1/2}.$$

In the case that $c \in C_2(M)$, we get from the above inequality that $\|y_0\|_2 \leqslant d(\eta^2 - d^2)^{-1/2}\|c\|_2$. Thus in the usual "filling in process" of a matrix used to find its best upper triangular approximant, if we give up the exactness of the operator norm approximation, we gain control of the Hilbert–Schmidt norm of the approximant. This idea is the key to several of our results.

PROPOSITION 2. *Let* N *be a finite nest. Then for every* $x \in C_2(M)$ *and* $\delta > 1$, *there exists* $z \in A_N \cap C_2(M)$ *such that* $\|x - z\| \leqslant \delta d(x, A_N)$ *and* $\|x - z\|_2 \leqslant \delta(\delta^2 - 1)^{-1/2}\|x\|_2$.

Proof. Let $N = \{p_n\}_{n=0,\dots,m}$, where $p_0 = 0$ and $p_m = I$, and set $d = d(x, A_N)$. We shall construct $y = x - z$ inductively by replacing the upper triangular terms $p_n x(p_n - p_{n-1})$ in the expansion

$$x = \sum_{n=1}^{m} \{p_n x(p_n - p_{n-1}) + p_n^{\perp} x(p_n - p_{n-1})\}$$

with operators $y_n = p_n y_n(p_n - p_{n-1}) \in M$ such that for each $1 \leqslant k \leqslant m$, we have

(i) $\|\sum_{n=1}^{k} (y_n + p_n^{\perp} x(p_n - p_{n-1}))\| \leqslant \delta d$,

(ii) $\|y_k\|_2 \leqslant (\delta^2 - 1)^{-1/2} \, \|p_k^{\perp} x(p_k - p_{k-1})\|_2$.

Choose $y_1 = 0$, assume the construction done for $n \leqslant k$, and set

$$b = p_{k+1} \sum_{n=1}^{k} (y_n + p_n^{\perp} x(p_n - p_{n-1}))$$

$$a = p_{k+1}^{\perp} \sum_{n=1}^{k} (y_n + p_n^{\perp} x(p_n - p_{n-1})) = p_{k+1}^{\perp} x p_k$$

$$c = p_{k+1}^{\perp} x(p_{k+1} - p_k).$$

Then "fill in" the block matrix $\left(\begin{smallmatrix} b & * \\ a & c\end{smallmatrix}\right)$ by choosing

$$y_{k+1} = -b(\delta^2 d^2 I - a^*a)^{-1}a^*c.$$

The induction hypothesis guarantees that $\|\left(\begin{smallmatrix} b \\ a\end{smallmatrix}\right)\| \leqslant \delta d$ and Arveson's distance formula applied to x guarantees that $\|(a\ c)\| \leqslant d$ and hence that $\|a\| \leqslant d$. Apply Lemma 1 choosing $\eta = \delta d$. Then (i) follows from

$$\left\| \sum_{n=1}^{k+1} (y_n + p_n^\perp x(p_n - p_{n-1})) \right\| = \left\| \begin{pmatrix} b & y_{k+1} \\ a & c \end{pmatrix} \right\| \leqslant \delta d.$$

Moreover, from the remark following Lemma 1, we have

$$\| y_{k+1} \|_2 \leqslant d(\eta^2 - d^2)^{-1/2} \|c\|_2 = (\delta^2 - 1)^{-1/2} \|c\|_2.$$

In the (final) step $k = m$, the a and c parts are absent and we choose $y_m = 0$. Thus if we set $y = \sum_{n=1}^m y_n + p_n^\perp x(p_n - p_{n-1})$, then $z = x - y \in A_N$ and $\|x - z\| = \|y\| \leqslant \delta d$. Since y is a finite sum of elements in $C_2(M)$, it belongs to $C_2(M)$ (and hence so does z) and we have

$$\begin{aligned}
\|x - z\|_2^2 &= \sum \|y(p_n - p_{n-1})\|_2^2 \\
&= \sum (\|p_n y(p_n - p_{n-1})\|_2^2 + \|p_n^\perp x(p_n - p_{n-1})\|_2^2 \\
&\leqslant \sum (1 + (\delta^2 - 1)^{-1}) \|p_n^\perp x(p_n - p_{n-1})\|_2^2 \\
&\leqslant \delta^2(\delta^2 - 1)^{-1} \|x\|_2^2. \quad \blacksquare
\end{aligned}$$

Clearly, the same proof also holds for an infinite sequence $\{p_n\}$.

Note that for δ "large," e.g., $\delta d \geqslant \|x\|$, the construction in Proposition 2 does not give the best joint norm approximation (e.g., $y = x$ gives a better Hilbert–Schmidt norm approximation). The case $\delta = 1$ will be treated in Example 8.

Note also that in the next to last inequality in the proof, we have actually obtained the stronger inequality $\|x - z\|_2 \leqslant \delta(\delta^2 - 1)^{-1/2} \|LT(x)\|_2$, where $LT(x)$ is the strictly lower triangular part of x and $\|LT(x)\|_2 = d_2(x, A_N)$.

Now we extend this result to the case of general nest algebras.

THEOREM 3. *For every $x \in C_2(M)$ and $\delta > 1$, there exists $a \in A_N \cap C_2(M)$ such that $\|x - a\| \leqslant \delta d(x, A_N)$ and $\|x - a\|_2 \leqslant \delta(\delta^2 - 1)^{-1/2} d_2(x, A_N \cap C_2(M))$.*

Proof. Choose $b_n \in A_N \cap C_2(M)$ so that

$$\|x - b_n\|_2 \leqslant d_2(x, A_N \cap C_2(M)) + 1/n$$

and set $d=d(x, A_N)$. Consider the directed set Γ of all the finite subnests N_γ of N ordered by inclusion. For every $N_\gamma=\{0, p_1^\gamma, p_2^\gamma, ..., I\}\in\Gamma$ and for every $z\in M$ define

$$\Phi_\gamma(z)=\sum_{j=1}^{n(\gamma)}(p_j^\gamma-p_{j-1}^\gamma)\,z(p_j^\gamma-p_{j-1}^\gamma).$$

Since $A_N\subset A_{N_\gamma}$, we have $d(x-b_n, A_{N_\gamma})\leqslant d(x-b_n, A_N)=d$. By Proposition 2 applied to the finite nest N_γ, we can find $a_{\gamma,n}\in A_{N_\gamma}$ and $y_{\gamma,n}\in C_2(M)$ such that $x-b_n=a_{\gamma,n}+y_{\gamma,n}$, $\|y_{\gamma,n}\|\leqslant\delta d$, and $\|y_{\gamma,n}\|_2\leqslant\delta(\delta^2-1)^{-1/2}\|x-b_n\|_2$. Now consider the decomposition

$$x-b_n=(a_{\gamma,n}-\Phi_\gamma(a_{\gamma,n}))+\Phi_\gamma(x-b_n)-\Phi_\gamma(y_{\gamma,n})+y_{\gamma,n}.$$

For fixed n, each of the four nets in this decomposition is norm bounded. Thus by using the σ-weak compactness of the unit ball of M and passing successively to converging subnets, we can assume without loss of generality that each net converges σ-weakly. Then

$$x-b_n=\lim_\gamma\,(a_{\gamma,n}-\Phi_\gamma(a_{\gamma,n}))+\lim_\gamma\,\Phi_\gamma(x-b_n)$$
$$-\lim_\gamma\,\Phi_\gamma(y_{\gamma,n})+\lim_\gamma\,y_{\gamma,n}.$$

It is now easy to verify that $a_{\gamma,n}-\Phi_\gamma(a_{\gamma,n})\in A_N$ and hence $\lim_\gamma(a_{\gamma,n}-\Phi_\gamma(a_{\gamma,n}))\in A_N$. For every $p\in N$ and every finite nest $N_\gamma\supset\{0, p, I\}$, p commutes with $\Phi_\gamma(y_{\gamma,n})$ and hence $\lim_\gamma\Phi_\gamma(y_{\gamma,n})\in A_N$. By the same reasoning $\lim_\gamma\Phi_\gamma(x-b_n)\in A_N$. Set $y_n=\lim_\gamma y_{\gamma,n}$; then $x-b_n-y_n\in A_N$ and so $a_n=x-y_n\in A_N$. By the σ-weak lower semicontinuity of $\|\cdot\|$ and $\|\cdot\|_2$, we obtain $\|y_n\|\leqslant\delta d$, $y_n\in C_2(M)$, and $\|y_n\|_2\leqslant\delta(\delta^2-1)^{-1/2}\|x-b_n\|_2$. Since $\{y_n\}$ is norm bounded, $\{a_n\}$ is also norm bounded. Again by the σ-weak compactness of the unit ball, by passing if necessary to subnets, we can assume without loss of generality that both sequences converge σ-weakly. Let $a=\lim a_n$ and $y=\lim y_n$. From the σ-weak lower semicontinuity of both norms we obtain that $\|y\|\leqslant\delta d$, $y\in C_2(M)$, hence $a=x-y\in A_N\cap C_2(M)$ and

$$\|y\|_2\leqslant\delta(\delta^2-1)^{-1/2}d_2(x, A_N\cap C_2(M)). \quad ∎$$

Remarks. (i) In general one cannot find $b\in A_N\cap C_2(M)$ such that $\|x-b\|_2=d_2(x, A_N\cap C_2(M))$ (i.e., $A_N\cap C_2(M)$ is not proximinal in $C_2(M)$ in the $\|\cdot\|_2$-norm). See Example 9.

(ii) If only the first inequality in Theorem 3 were known, then Corollary 4 and Theorem 5 would still follow. However, we do not know of a way to obtain this directly without also proving the second "dual control" inequality, which is new even for the case $M = B(H)$. Indeed, the uniform boundedness of the Hilbert-Schmidt norms of $y_{\gamma,n}$ (resp. y_n) was essential in proving that their σ-weak limits y_n (resp. y) belonged to $C_2(M)$.

An immediate consequence of Theorem 3 is that

$$d(x, A_N) = d(x, A_N \cap C_2(M))$$

for every $x \in C_2(M)$. This property extends to K:

COROLLARY 4. *$d(k, A_N) = d(k, A_N \cap K)$ for every $k \in K$.*

Proof. Clearly, $d(k, A_N) \leqslant d(k, A_N \cap K)$. For the opposite inequality, let $\varepsilon > 0$ and choose a finite rank operator h such that $\|k - h\| < \varepsilon$. Note that h may fail to belong to $C_2(M)$, but there is a decomposition of the identity into mutually orthogonal central projections e_λ such that $he_\lambda \in C_2(M)$ for each λ [15]. Since the center of M is contained in A_N, we have $d(he_\lambda, A_N) \leqslant d(h, A_N)$. Applying Theorem 3 with $1 < \delta \leqslant 1 + \varepsilon/d(h, A_N)$, we can decompose $he_\lambda = a_\lambda + y_\lambda$ so that $a_\lambda \in A_N$, $y_\lambda \in C_2(M)$, and

$$\|y_\lambda\| \leqslant \delta d(he_\lambda, A_N) \leqslant d(h, A_N) + \varepsilon \leqslant d(k, A_N) + 2\varepsilon.$$

Assume without loss of generality that $a_\lambda e_\lambda = a_\lambda$ and $y_\lambda e_\lambda = y_\lambda$, and let $a = \sum a_\lambda$ and $y = \sum y_\lambda$. Then $a \in A_N$ because A_N is σ-weakly closed, and $y \in K$ because $y_\lambda \in K$ and $\sum y_\lambda$ is a central direct sum. Moreover, $\|y\| \leqslant d(k, A_N) + 2\varepsilon$. Now $a = h - y \in A_N \cap K$ and $k - a = y + (k - h)$, hence $d(k, A_N \cap K) \leqslant d(k, A_N) + 3\varepsilon$. Since ε is arbitrary, $d(k, A_N \cap K) \leqslant d(k, A_N)$. ∎

As in the Fall, Arveson, and Muhly proof [6, Theorem 1.1], from the isometry of the map $\alpha(k + A_N \cap K) = k + A_N$ from $K/A_N \cap K$ into M/A_N, it follows that $\alpha(K/A_N \cap K) = (K + A_N)/A_N$ is norm closed, and hence $A_N + K = \pi^{-1}((K + A_N)/A_N)$ is also norm closed (where $\pi\colon M \to M/A_N$ is the quotient map). This proves main result.

THEOREM 5. *$A_N + K$ is norm closed.*

We now apply the techniques used in Proposition 2 and Theorem 3 to obtain a joint norm control Nehari type theorem. Let $L^\infty = L^\infty(\mathbb{T})$ (where $\mathbb{T}$ is the unit circle with normalized Lebesgue measure) and similarly let

$L^2 = L^2(\mathbb{T})$, $C = C(\mathbb{T})$. Let H^∞ (resp. H^2) be the classical Hardy space, which we identify with the closed subspace of L^∞(resp. L^2) of the functions with vanishing negative Fourier coefficients. For every $\varphi \in L^\infty$, let M_φ be multiplication by φ on L^2. It is well known that the map $\varphi \to M_\varphi$ is an isometry onto the algebra $\mathscr{L}$ of Laurent matrices (with respect to the orthonormal basis $\{z^n\}_{n\in\mathbb{Z}}$). Moreover, the image of H^∞ under this isometry is $(T(\mathbb{Z}) \cap \mathscr{L})^*$, where $T(\mathbb{Z})$ is the algebra of the upper triangular matrices (i.e., the nest algebra corresponding to the nest $\{p_n\}_{n\in\mathbb{Z}}$, where p_n is the projection onto span $\{z^k \mid k \leqslant n\}$).

An interesting application of Arveson's distance formula and the existence of a conditional expectation $E\colon B(L^2) \to \mathscr{L}$ mapping $T(\mathbb{Z})$ onto $T(\mathbb{Z}) \cap \mathscr{L}$ [1, Proposition 5.1] is Parrott's proof [14, see also 18, Theorem 1.5] of Nehari's theorem [12],

$$d(\varphi, H^\infty) = \|H_\varphi\| \qquad \text{for every} \quad \varphi \in L^\infty,$$

where the distance is computed in the L^∞-norm and $H_\varphi = p_0 M_\varphi p_0^\perp$ denotes the Hankel matrix with symbol φ.

The technique in Proposition 2 enables us to construct directly an approximating element $\psi \in H^\infty$ and to simultaneously control the L^2-norm of $\varphi - \psi$. The price we pay is to give up exact approximation in the L^∞-norm, but we show in Remark (iii) after Theorem 6 that this is unavoidable.

Let $d_2(\varphi, H^\infty)$ denote the distance of φ to H^∞ in L^2-norm and note that

$$d_2(\varphi, H^\infty) = d_2(\varphi, H^2) = \|p_{-1}\varphi\|_2.$$

THEOREM 6. *For every $\varphi \in L^\infty$ and $\delta > 1$, there exists $\psi \in H^\infty$ such that $\|\varphi - \psi\|_\infty \leqslant \delta d(\varphi, H^\infty)$ and $\|\varphi - \psi\|_2 \leqslant \delta(\delta^2 - 1)^{-1/2} d_2(\varphi, H^\infty)$.*

Proof. Set $d = d(M_\varphi^*, T(\mathbb{Z}))$. We are going to construct a matrix y so that $M_\varphi^* - y \in T(\mathbb{Z}) \cap \mathscr{L}$ and $\|y\| \leqslant \delta d$. First we determine $p_{-1}^\perp y$, the "bottom half" of the matrix y. Set

$$p_{-1}^\perp y = \begin{pmatrix} \cdots & \varphi_2 & \varphi_1 & y_{0,0} & y_{0,1} & \cdots & y_{0,n-1} & y_{0,n} & \cdots \\ \cdots & \varphi_3 & \varphi_2 & \varphi_1 & y_{1,1} & \cdots & y_{1,n-1} & y_{1,n} & \cdots \\ \cdots & \varphi_4 & \varphi_3 & \varphi_2 & \varphi_1 & \cdots & y_{2,n-1} & y_{2,n} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & y_{n-1,n-1} & y_{n-1,n} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \varphi_1 & y_{n,n} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \varphi_2 & \varphi_1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix},$$

where $\varphi_j = \hat{\varphi}(j) = \bar{\varphi}(-j)$, $j \geqslant 1$ are the strictly lower triangular entries of the matrix M_φ^* and $y_{j,j}$ are the diagonal entries of y. Let

$$b_n = \begin{pmatrix} \cdots & \varphi_2 & \varphi_1 & y_{0,0} & y_{0,1} & \cdots & y_{0,n-1} \\ \cdots & \varphi_3 & \varphi_2 & \varphi_1 & y_{1,1} & \cdots & y_{1,n-1} \\ \cdots & \varphi_4 & \varphi_3 & \varphi_2 & \varphi_1 & \cdots & y_{2,n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & y_{n-1,n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \varphi_1 \end{pmatrix}, \qquad Y_n = \begin{pmatrix} y_{0,n} \\ y_{1,n} \\ y_{2,n} \\ \cdots \\ y_{n-1,n} \\ y_{n,n} \end{pmatrix},$$

$$a = \begin{pmatrix} \cdots & \varphi_4 & \varphi_3 & \varphi_2 \\ \cdots & \cdots & \varphi_4 & \varphi_3 \\ \cdots & \cdots & \cdots & \varphi_4 \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}, \qquad c = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \cdots \end{pmatrix},$$

where $b_0 = (..., \varphi_3, \varphi_2, \varphi_1)$ and $Y_0 = (y_{0,0})$. We determine inductively the columns Y_n by setting

$$Y_n = -b_n(\delta^2 d^2 I - a^* a)^{-1} a^* c.$$

For all $m \in \mathbb{Z}$

$$(a \;\; c) = p_{-1}^\perp M_\varphi^* p_{-1} = p_m^\perp M_\varphi^* p_m = H_\varphi^*,$$

hence by Arveson's distance formula, $\|(a \;\; c)\| = d < \delta d$. Moreover, for all $n \geqslant 0$,

$$p_{-1}^\perp y p_n = \begin{pmatrix} b_n & Y_n \\ a & c \end{pmatrix} = \begin{pmatrix} b_{n+1} \\ a \end{pmatrix}.$$

Thus from Lemma 1 with $\eta = \delta d$, we obtain that $\| p_{-1}^\perp y p_n \| \leqslant \delta d$.

It follows that $p_{-1}^\perp y$ is a bounded operator, that $\| p_{-1}^\perp y \| \leqslant \delta d$, and that $p_{-1}^\perp M_\varphi^* - p_{-1}^\perp y \in T(\mathbb{Z})$. Next, we show by a matrix computation that $p_{-1}^\perp y$ has constant diagonals. This property holds trivially for $p_{-1}^\perp y p_0$. Assume now that $p_{-1}^\perp y p_{n-1}$ and hence b_n has constant diagonals. Let $e = -(\delta^2 d^2 I - a^* a)^{-1} a^* c$. Then

$$Y_n = \begin{pmatrix} y_{0,n} \\ y_{1,n} \\ y_{2n} \\ \cdots \\ y_{n-1,n} \\ y_{n,n} \end{pmatrix} = b_n e = \begin{pmatrix} \cdots & \varphi_1 & y_{0,0} & y_{0,1} & y_{0,2} & \cdots & y_{0,n-1} \\ \cdots & \varphi_2 & \varphi_1 & y_{0,0} & y_{0,1} & \cdots & y_{0,n-2} \\ \cdots & \varphi_3 & \varphi_2 & \varphi_1 & y_{0,0} & \cdots & y_{0,n-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & y_{0,0} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \varphi_1 \end{pmatrix} e.$$

Thus

$$\begin{pmatrix} y_{1,n} \\ y_{2,n} \\ \cdots \\ y_{n-1,n} \\ y_{n,n} \end{pmatrix} = \begin{pmatrix} \cdots & \varphi_2 & \varphi_1 & y_{0,0} & y_{0,1} & \cdots & y_{0,n-2} \\ \cdots & \varphi_3 & \varphi_2 & \varphi_1 & y_{0,0} & \cdots & y_{0,n-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & y_{0,0} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \varphi_1 \end{pmatrix} e = b_{n-1} e = Y_{n-1},$$

which shows that $p_{-1}^{\perp} y p_n$ and hence $p_{-1}^{\perp} y$ have constant diagonals. Now extend $p_{-1}^{\perp} y$ to a Laurent matrix y by extending the constant diagonals upwards to the left. By construction, $M_{\varphi}^* - y \in T(\mathbb{Z})$. Therefore $M_{\varphi}^* - y = M_{\psi}^*$ for some $\psi \in H^{\infty}$, But then

$$\|\varphi - \psi\|_{\infty} = \|y\| = \|p_{-1}^{\perp} y\| \leqslant \delta d(M_{\varphi}^*, T(\mathbb{Z})) \leqslant \delta d(\varphi, H^{\infty}).$$

On the other hand, $\|\varphi - \psi\|_2 = \|(p_0 - p_{-1}) y\|$, where

$$(p_0 - p_{-1}) y = (..., \varphi_3, \varphi_2, \varphi_1, y_{0,0}, y_{0,1}, y_{0,2}, ...).$$

As in the proof of Proposition 2, we have

$$\begin{aligned} \left(\sum_{j=0}^{n} |y_{0,j}|^2 \right)^{1/2} &= \|Y_n\| \\ &\leqslant \|b_n(\delta^2 d^2 I - a^* a)^{-1} a^*\| \, \|c\| \\ &\leqslant (\delta^2 - 1)^{-1/2} \|p_{-1} \varphi\|_2 \end{aligned}$$

and hence

$$\begin{aligned} \|(p_0 - p_{-1}) y\| &\leqslant \delta(\delta^2 - 1)^{-1/2} \|p_{-1} \varphi\|_2 \\ &= \delta(\delta^2 - 1)^{-1/2} d_2(\varphi, H^{\infty}). \quad \blacksquare \end{aligned}$$

Remarks. (i) Our approach gives a constructive proof of the identities

$$d(\varphi, H^{\infty}) = d(M_{\varphi}^*, T(\mathbb{Z})) = d(M_{\varphi}^*, T(\mathbb{Z}) \cap \mathscr{L}) \qquad \text{for every} \quad \varphi \in L^{\infty}.$$

The proximinality of $T(\mathbb{Z}) \cap \mathscr{L}$ in $\mathscr{L}$ follows now by a natural compactness argument since $\mathscr{L}$ and $T(\mathbb{Z})$ are both σ-weakly closed.

(ii) Let $c = \inf_{\delta > 1} \max(\delta, \delta(\delta^2 - 1)^{-1/2}) = \sqrt{2}$. As a consequence of Theorem 6, we see that for every $\varphi \in L^{\infty}$ there exists a $\psi \in H^{\infty}$ such that $\|\varphi - \psi\|_{\infty} \leqslant cd(\varphi, H^{\infty})$ and $\|\varphi - \psi\|_2 \leqslant cd_2(\varphi, H^{\infty})$. As G. Pisier has pointed out to us, the existence of such a constant c can also be deduced from a deep interpolation theoretic result of Jones [10].

(iii) If we choose a best approximant $\psi \in H^\infty$, i.e., if $\|\varphi-\psi\|_\infty = d(\varphi, H^\infty)$, then we lose control of the ratio $(\|\varphi-\psi\|_2/d_2(\varphi, H^\infty))$. Indeed $\varphi(z)=\sum_{k=-n}^{-1} z^k$ has a unique best approximant $\psi \in H^\infty$ and furthermore $|\varphi-\psi|$ is constant (and hence equal to $\|\varphi-\psi\|_\infty$) [20, Theorem 5.6]. But then $\|\varphi-\psi\|_2 = d(\varphi, H^\infty) = \|H_\varphi\|$, which is well known to be asymptotic to $2n/\pi$, while $d_2(\varphi, H^\infty) = \|\varphi\|_2 = \sqrt{n}$.

A simple consequence of the construction in the proof of Theorem 6 is a new proof of Sarason's theorem [21].

COROLLARY 7. *$H^\infty + C$ is norm closed.*

Proof. Let φ be any polynomial in z and z^{-1}. Using the notations in the proof of Theorem 6, set $y_n = y_{0,n}$ for $n \geqslant 0$ and $y_n = \varphi_{-n}$ for $n \leqslant 0$. Assume that $\varphi_j = 0$ for $j > k$, then the vector $e = -(\delta^2 d^2 I - a^*a)^{-1} a^*c$ has at most $k-1$ nonzero entries, i.e., $e = (..., 0, 0, e_{k-1}, ..., e_2, e_1)^t$. From $Y_n = b_n e$, we get for all $n \geqslant 0$ the recurrence relation

$$y_n = (..., 0, y_{-k}, ..., y_{-2}, y_{-1}, y_0, y_1, ..., y_{n-1})\, e = \sum_{j=1}^{k-1} e_j y_{n-j}.$$

Since $M_{\overline{\varphi-\psi}} = M_\varphi^* - M_\psi^* = y$, we see that $\overline{\varphi-\psi} = \sum_{n=-k}^{\infty} y_n z^n$, where the series converges in L^2-norm. Let $\omega(z) = 1 - \sum_{j=1}^{k-1} e_j z^j$. From the recurrence relation on y_n for $n \geqslant 0$, we obtain that $\omega(\overline{\varphi-\psi})$ is a polynomial (with only negative powers of z), and hence that $\overline{\varphi-\psi}$ is a rational function. Thus ψ is rational, and hence it is continuous. Since δ is arbitrary, we obtain $d(\varphi, H^\infty) = d(\varphi, H^\infty \cap C)$ for every polynomial φ and hence for every $\varphi \in C$. By the same argument that yields Theorem 5, we obtain that $H^\infty + C$ is closed. ∎

Now we return to general nest algebras. A natural question (cf. also Remark (iii) after Theorem 6) is whether we can still retain control of the Hilbert–Schmidt norms if we choose best approximants in the nest algebra, i.e., to what extent do Proposition 2 and Theorem 3 hold for $\delta = 1$. The following finite dimensional example shows that if $\|x-a\| = d(x, A_N)$, then there is no upper bound for $(\|x-a\|_2/\|x\|_2)$.

EXAMPLE 8. Let $M = M_n(\mathbb{C})$, let p_k be the projection on $\operatorname{span}\{e_j \mid j \leqslant k\}$, where $\{e_j\}$ is the standard basis, and let $N = \{p_k\}$ so that A_N is the algebra of the upper triangular $n \times n$ matrices. Let v be any vector in $\mathbb{C}^n$ with all nonzero entries and set

$$\alpha_1 = \|p_1^\perp v\|^{-1}, \qquad \alpha_k = (\|p_k^\perp v\|^{-2} - \|p_{k-1}^\perp v\|^{-2})^{1/2} \qquad \text{for} \quad 2 \leqslant k \leqslant n-1.$$

Define the matrix x with columns:

$$x = (\alpha_1 p_1^\perp v, \alpha_2 p_2^\perp v, \alpha_3 p_3^\perp v, ..., 0).$$

Thus x is strictly lower triangular and it is easy to verify that all the corners $p_k^\perp x p_k$ have rank 1 and norm 1 for $1 \leqslant k \leqslant n-1$. In particular, $d(x, A_N) = 1$, and thus there exists $a \in A_N$ such that $\|x - a\| = 1$. We will show that ap_{n-1} is uniquely determined (clearly, $a(p_n - p_{n-1})$ is never unique). Set $y = x - a$ and for $2 \leqslant k \leqslant n-1$, define $b_k = p_k y p_{k-1}$, $Y_k = p_k y(p_k - p_{k-1})$, $a_k = p_k^\perp y p_{k-1}$, $c_k = p_k^\perp y(p_k - p_{k-1})$, so that

$$yp_1 = \begin{pmatrix} y_{1,1} \\ \alpha_1 p_1^\perp v \end{pmatrix} \quad \text{and} \quad yp_k = \begin{pmatrix} b_k & Y_k \\ a_k & c_k \end{pmatrix}.$$

Since $\|y\| = 1$ and $\|(a_k c_k)\| = \|p_k^\perp x p_k\| = 1$, we have $\|yp_k\| = 1$ for all k. For $k = 1$, it follows that $y_{1,1} = 0$. Also, for $2 \leqslant k \leqslant n-1$ it follows that the positive matrix

$$yp_k y^* = \begin{pmatrix} b_k b_k^* + Y_k Y_k^* & b_k a_k^* + Y_k c_k^* \\ a_k b_k^* + c_k Y_k^* & a_k a_k^* + c_k c_k^* \end{pmatrix}$$

has norm one and $a_k a_k^* + c_k c_k^*$ is a (rank 1) projection. Hence, as is elementary to show, $b_k a_k^* + Y_k c_k^* = 0$. By multiplying on the right by c_k (which is nonzero by the hypothesis on v) we obtain $Y_k = -\|c_k\|^{-2} b_k a_k^* c_k$. One can verify directly that this formula for Y_k coincides with that given by Lemma 1 (where we can choose $\eta = 1$ because $\|a_k\| < 1$). Now we determine Y_k explicitly for $2 \leqslant k \leqslant n-1$. A direct computation shows that

$$\|c_k\|^{-2} a_k^* c_k = \frac{1}{\alpha_k} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \cdots \\ \alpha_{k-1} \end{pmatrix}.$$

By solving the equation $Y_k = -\|c_k\|^{-2} b_k a_k^* c_k$, we obtain by induction on k that $y_{j,k} = 0$ for $j < k$ and $y_{k,k} = -((\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_{k-1}^2)/\alpha_k) v_k$. Furthermore, since

$$\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_k^2 = \|p_k^\perp v\|^{-2} \quad \text{and} \quad |v_k|^2 = \|p_{k-1}^\perp v\|^2 - \|p_k^\perp v\|^2,$$

we obtain for $k \geqslant 2$,

$$|y_{k,k}|^2 = \|p_k^\perp v\|^2 \|p_{k-1}^\perp v\|^{-2},$$

so that for $n \geqslant 3$,

$$\|y\|_2^2 \geqslant \|yp_{n-1}\|_2^2 = \|x\|_2^2 + \sum_{k=2}^{n-1} \|p_k^\perp v\|^2 \|p_{k-1}^\perp v\|^{-2} = n-1.$$

(Note that $\|y\|_2^2 \leqslant n\|y\|^2 = n$). From the last identity we also obtain

$$1 \leqslant \|x\|_2^2 \leqslant n - 1 - (n-2)\,|v_n|^2\,\|v\|^{-2}.$$

Thus if we choose the vector v so that $\|v\| = 1$ and $v_n = 1 - \varepsilon$, then

$$\frac{\|y\|_2^2}{\|x\|_2^2} \geqslant \frac{n-1}{1+2(n-2)\varepsilon},$$

which can be made as large as we wish.

Recall that if A_N is the algebra of upper triangular operators in a separable Hilbert space, then every compact operator (and, in particular, every Hilbert–Schmidt operator) has a finite rank best operator norm approximant in A_N [18, Theorem 1.4(i)]. Despite this fact, from Example 8 we see that best operator norm approximants (even for Hilbert–Schmidt operators) may be "very bad" Hilbert–Schmidt norm approximants.

EXAMPLE 9. We show that $A_N \cap C_2(M)$ need not be $\|\cdot\|_2$-proximinal in $C_2(M)$. In the special case $M = B(H)$ there is proximinality simply because $C_2(M)$ is a Hilbert space and $A_N \cap C_2(M)$ is a closed subspace. However the symbol for the infinite Hilbert matrix in $B(H)$ provides an example why H^∞ is not $\|\cdot\|_2$-proximinal in L^∞. The above reasoning does not apply because H^∞ and L^∞ are not $\|\cdot\|_2$-complete. If M is type II, then $C_2(M)$ is not $\|\cdot\|_2$-complete. This suggests the following construction.

Let M be a type II_1 factor and $q_k \in M$ be a decomposition of the identity into infinitely many mutually orthogonal projections. Decompose further each q_k into k equivalent mutually orthogonal projections $r_{k,j}$. Let N be the nest $\{0, r_{1,1}, r_{1,1} + r_{2,1}, r_{1,1} + r_{2,1} + r_{2,2}, \ldots\}$. By using the equivalence of the projections $r_{k,j}$, embed $M(\mathbb{C}^k)$ into $q_k M q_k$. Let $x_k \in q_k M q_k$ be the image under this embedding of the compression of the Hilbert matrix of size k and let $x = \sum_{k=1}^{\infty} x_k$. Then $x \in M = C_2(M)$. It is well known that $LT(x)$, the strictly lower triangular part of x, is an unbounded operator (affiliated with M). Define $\|LT(x)\|_2 = \{\sum_{k=1}^{\infty} \|LT(x_k)\|_2^2\}^{1/2}$ $(\leqslant \|x\|_2)$. (One can actually prove this identity.) It is clear that $\|x - a\|_2 \geqslant \|LT(x)\|_2$ for every $a \in A_N$. Moreover , if $a_n = \sum_{k=1}^{n} (x_k - LT(x_k))$, then $a_n \in A_N$ and $\|x - a_n\|_2 \to \|LT(x)\|_2$. Thus $d_2(x, A_N) = \|LT(x)\|_2$. Now it is easy to verify that if we had $a \in A_N$ and $\|x - a\|_2 = \|LT(x)\|_2$, then $x - a = LT(x)$, which is unbounded.

We finish with our density theorem. Note that its proof is actually independent from the other parts of this paper.

THEOREM 10. *$A_N \cap K$ is σ-weakly dense in A_N.*

Proof. Since $D_N \cap K$ is a two-sided ideal of the von Neumann algebra D_N, there is a projection e in the center of D_N such that $\overline{(D_N \cap K)_e}^{\sigma w} = \{0\}$ and $\overline{(D_N \cap K)_{e^\perp}}^{\sigma w} = (D_N)_{e^\perp}$. (In [11], $(D_N)_{e^\perp}$ and $(D_N)_e$ are called the M-semifinite and the M-type III parts of D_N). In particular, $e^\perp \in \overline{A_N \cap K}^{\sigma w}$ and thus for every $a \in A_N$

$$a - eae = eae^\perp + e^\perp ae + e^\perp ae^\perp \in \overline{A_N \cap K}^{\sigma w}.$$

Hence, without loss of generality, we can assume that $e = I$, i.e., that $D_N \cap K = \{0\}$.

By using an invariant mean on the (abelian) unitary group G of C_N, we can define a conditional expectation $E: M \to D_N = C'_N \cap M$ such that $E(x) \in \overline{co}^{\sigma w}\{uxu^* \mid u \in G\}$ for all $x \in M$. By the σ-weak lower semicontinuity of the trace τ, we have $\tau \circ E \leqslant \tau$. Thus for every $x \in C_1(M)^+$ we have $E(x) \in C_1(M) \subset K$ and $E(x) \in D_N$, hence $E(x) = 0$. So since τ is semifinite, we see that $M = \overline{\operatorname{Ker} E}^{\sigma w}$, (i.e., E is singular). Moreover, for every $x \in M$,

$$x - E(x) \in \overline{co}^{\sigma w}\{(xu^*)u - u(xu^*) \mid u \in G\} \subset [C_N, M]^{-\sigma w},$$

where $[C_N, M]$ denotes the span of the commutators of C_N and M. Therefore $\operatorname{Ker} E \subset [C_N, M]^{-\sigma w}$, and hence $M = [C_N, M]^{-\sigma w}$. But $[C_N, M]^{-\sigma w} = [N, M]^{-\sigma w}$ and, as is easy to verify, $[N, M] = \operatorname{span}\{pxp^\perp \mid p \in N \cup N^\perp, x \in M\}$, so we obtain

$$M = \overline{\operatorname{span}}^{\sigma w}\{pxp^\perp \mid p \in N \cup N^\perp, x \in M\}.$$

Set $S_N = \overline{\operatorname{span}}^{\sigma w}\{pxp^\perp \mid p \in N, x \in M\}$. Then from the last identity we see that $\overline{S_N + S_N^*}^{\sigma w} = M$. We claim that $S_N \supset D_N$.

Reasoning by contradiction, assume that there is an element $a \in D_N \backslash S_N$, and by the Hahn–Banach theorem, find a normal functional $\omega \in M_*$ that vanishes on S_N and for which $\omega(a) \neq 0$. Without loss of generality we may assume that M is represented on a Hilbert space, and by replacing M, if necessary, with its infinite ampliation $M \otimes I$, we may assume that ω is a vector functional. That is, there are two vectors $\xi, \eta \in H$ such that $\omega(x) = (x\xi, \eta)$ for all $x \in M$. Let $p, q \in B(H)$ be the (cyclic) projections on $\{(D_N + S_N)\xi\}$ and $\{S_N \xi\}$, respectively. Thus $q \leqslant p$. Since S_N is a two-sided module over D_N, for every $d \in D_N$, we have $dq = qdq$, and for every $s \in S_N$, we have $sp = qsp$. From the last identity, we obtain $q^\perp S_N p = \{0\}$ and hence $(p-q) S_N (p-q) = \{0\}$. By taking adjoints, we also get $(p-q) S_N^* (p-q) = \{0\}$, and hence $(p-q) \overline{S_N + S_N^*}^{\sigma w} (p-q) = \{0\}$, whence $p=q$. On the other hand, from $\omega(s)=(s\xi, \eta)=0$ for every $s \in S_N$ we get $q\eta=0$ and, since D_N is unital and $0 \in S_N$, we get $q\xi=p\xi=\xi$. But then

$$\omega(a)=(a\xi, \eta)=(aq\xi, q^{\perp}\eta)=(qaq\xi, q^{\perp}\eta)=0,$$

which contradicts the assumption that $\omega(a) \neq 0$. Thus we have proven that $S_N \supset D_N$.

From [7, Proposition 2.1] we know that $A_N = S_N + D_N$, hence $A_N = S_N$. Finally, K is σ-weakly dense in M and thus

$$A_N = S_N = \overline{\text{span}}^{\sigma w}\{pxp^{\perp} \mid p \in N, x \in K\}. \quad \blacksquare$$

We conclude with some questions.

(i) In [6] the term "local" was used to denote a norm-closed linear subspace S of $B(H)$ with the property that $S \subset \overline{S \cap K(H)}^{\sigma w}$ and the duality argument used in [6] proved that $S + K(H)$ is closed for every local subspace S. If we use the term local for the analogous property in the semifinite von Neumann algebra setting, then Theorem 10 states that nest subalgebras are local. However our proof of the closure of quasitriangular algebras required an argument separate from duality considerations. All of this raises the question: if $S \subset M$ is local, is $S + K$ closed?

(ii) In a related but different direction, we note that the closure result of [6] was generalized in [9] to show that if A is a finite width CSL algebra (i.e., an algebra generated by a finite number of commuting nests), then $A + K(H)$ is closed. Does this remain true when $B(H)$ and $K(H)$ are replaced by M and K?

(iii) The statement of the "dual control" result of Theorem 3 formally makes sense for arbitrary operator algebras (and in fact for linear spaces of operators). Do many operator algebras, and especially do many CSL algebras, satisfy at least a qualitative analog of this result (e.g., see Remark 2 after Theorem 6)?

(iv) We have used matricial techniques to obtain the Nehari type Theorem 6. As mentioned in Remark (ii) following Theorem 6, a qualitative version of this theorem can be deduced from interpolation results based on Carleson measures. This leads to the question of whether these matricial techniques can be further exploited to produce more elementary proofs of known interpolation results.

(v) Conversely, it is natural to ask which interpolation theoretic results have non-commutative analogs in nest algebras.

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