PAVING SMALL MATRICES AND THE KADISON-SINGER EXTENSION PROBLEM

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Abstract. We compute paving parameters for classes of small matrices and the matrices that yield them. The convergence to 1 or not of the sequence of these parameters is equivalent to the Kadison-Singer Extension Problem.

Contents

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1. Introduction

1.1. **The Kadison-Singer Extension Problem.** Let ℓ^2 be the Hilbert space of absolutely square-summable complex sequences and $B(\ell^2)$ denote the von Neumann algebra of all bounded linear operators on ℓ^2. Every $T \in B(\ell^2)$ has an infinite matrix representation with respect to the standard orthonormal basis $\mathcal{E} = \{e_n : n \in \mathbb{N}\}$ of ℓ^2, namely $\big[T\big]_{\mathcal{E}} = \big[\langle Te_j, e_i\rangle\big]_{i,j\in\mathbb{N}}$. Let ℓ^∞ stand for the set of all $T \in B(\ell^2)$ for which $\big[T\big]_{\mathcal{E}}$ is diagonal. Then ℓ^∞ is a maximal abelian von Neumann subalgebra (or MASA) of $B(\ell^2)$. A fundamental open problem in the theory of operator algebras is the *Kadison-Singer Extension Problem* (hereafter KS) [8]:

> Does every pure state on ℓ^∞ extend uniquely to a pure state on $B(\ell^2)$?

Existence is straightforward, the issue is uniqueness. Indeed, any Hahn-Banach extension of a state on ℓ^∞ is a state on $B(\ell^2)$. If the original state is pure, then the Krein-Milman Theorem implies the existence of a pure state extension. Alternatively, an explicit construction is available—the composition of a pure state on ℓ^∞ with the normal conditional expectation of $B(\ell^2)$ onto ℓ^∞ is a pure state on $B(\ell^2)$ [3]. An affirmative answer to KS would entail a complete description of those pure states on $B(\ell^2)$ which restrict to pure states on ℓ^∞. They would be precisely the states of the form $\Phi_{\mathcal{U}}(T) = \lim_{\mathcal{U}}\langle Te_n, e_n\rangle$, where $\mathcal{U}$ is an ultrafilter on $\mathbb{N}$. While this would not cover all pure states on $B(\ell^2)$ [1], it would be a substantial step in that direction. Kadison and Singer doubted the truth of KS [8], and that is also the prevailing opinion among experts today.

1.2. **Anderson's Paving Problem.** A major advance in the study of KS was made by Anderson, who reformulated the problem in terms of finite matrices [2]. We will state his result in terms of certain *paving parameters*. To define these we need the notion of a *paving*, which in turn relies on the idea of a *compression*.

Definition 1.2.1 (compression)**.** *For $A \in \mathbb{M}_n(\mathbb{C})$ ($n \times n$ complex matrices) and $\sigma \subseteq \{1, 2, ..., n\}$, the σ-compression of A is $A_\sigma := P_\sigma A P_\sigma$, where $P_\sigma \in \mathbb{M}_n(\mathbb{C})$ is the orthogonal projection onto span$\{\vec{e}_i : i \in \sigma\}$. By a p-compression of A we mean a compression A_σ with* $\mathrm{card}(\sigma) = p$.

More generally, if $\mu, \nu \subseteq \{1, 2, ..., n\}$, then $A_{\mu,\nu} := P_\mu A P_\nu$. Note that

$$\|A_{\mu,\nu}\| = \|P_\mu A P_\nu\| \le \|P_\mu\|\|A\|\|P_\nu\| \le \|A\|,$$

where $\|\cdot\|$ is the operator norm. In particular, $\|A_\sigma\| \le \|A\|$.

Definition 1.2.2 (paving). *For $A \in \mathbb{M}_n(\mathbb{C})$ and $\pi \in \Pi_k^n$ (the set of all k-partitions of $\{1, 2, ..., n\}$), the π-paving of A is $A^\pi := \sum_{\sigma\in\pi} A_\sigma$. By a k-paving of A we mean a paving A^π with $\operatorname{card}(\pi) = k$. By an $(n_1, n_2, ..., n_k)$-paving of A we mean a paving A^π with $\pi = \{\sigma_1, \sigma_2, ..., \sigma_k\}$, where $\operatorname{card}(\sigma_i) = n_i$ for all $1 \le i \le k$.*

Note that

$$\|A^\pi\| = \left\| \sum_{\sigma\in\pi} A_\sigma \right\| = \max\{\|A_\sigma\| : \sigma \in \pi\} \le \|A\|.$$

Definition 1.2.3 (paving parameters for a matrix). *For $0 \neq A \in \mathbb{M}_n(\mathbb{C})$, the k-paving parameter of A is*

$$\alpha_k(A) := \min\{\|A^\pi\| : \pi \in \Pi_k^n\} \in [0, \|A\|].$$

The normalized k-paving parameter of A is

$$\beta_k(A) := \frac{\alpha_k(A)}{\|A\|} \in [0, 1].$$

Definition 1.2.4 (paving parameters for matrix classes). *For $\emptyset \neq \mathcal{S} \subseteq \mathbb{M}_n(\mathbb{C})$, the (normalized) k-paving parameter of $\mathcal{S}$ is*

$$\beta_k(\mathcal{S}) := \sup\{\beta_k(A) : A \in \mathcal{S}\} \in [0, 1].$$

In this paper, $\mathcal{S}$ above will be one of the following classes:

i. $\mathbb{M}_n^0(\mathbb{C})$, the set of all $n \times n$ zero-diagonal complex matrices.
ii. $\mathbb{M}_n^0(\mathbb{R})$, the set of all $n \times n$ zero-diagonal real matrices.
iii. $\mathbb{M}_n^0(\mathbb{R}_+)$, the set of all $n \times n$ zero-diagonal non-negative (entried) matrices.
iv. $\mathbb{M}_n^0(\mathbb{C})_\triangle$, the set of all $n \times n$ strictly upper-triangular complex matrices.
v. $\mathbb{M}_n^0(\mathbb{C})_\circlearrowleft$, the set of all $n \times n$ zero-diagonal complex circulants (cf. Sect. 5.1).
vi. $\mathbb{M}_n^0(\mathbb{C})_{sa} = \{A \in \mathbb{M}_n^0(\mathbb{C}) : A^* = A\}$ (here A^* is the adjoint of A, i.e., the conjugate-transpose).
vii. $\mathbb{M}_n^0(\mathbb{R})_{sa} = \{A \in \mathbb{M}_n^0(\mathbb{R}) : A^* = A\}$.
viii. $\mathbb{M}_n^0(\mathbb{R}_+)_{sa} = \{A \in \mathbb{M}_n^0(\mathbb{R}_+) : A^* = A\}$.
ix. $\mathbb{M}_n^0(\mathbb{C})_{\circlearrowleft,sa} = \{A \in \mathbb{M}_n^0(\mathbb{C})_\circlearrowleft : A^* = A\}$.

Using the fact that $\beta_k(A\oplus 0) = \beta_k(A)$ [4], we deduce that $\beta_k(\mathbb{M}_n^0(\mathbb{C})) \le \beta_k(\mathbb{M}_{n+1}^0(\mathbb{C}))$, and so

$$\lim_{n\to\infty} \beta_k(\mathbb{M}_n^0(\mathbb{C})) = \sup_n \beta_k(\mathbb{M}_n^0(\mathbb{C})) \in [0, 1].$$

The same is true for all matrix classes considered above, with the exception of $\mathbb{M}_n^0(\mathbb{C})_\circlearrowleft$ and $\mathbb{M}_n^0(\mathbb{C})_{\circlearrowleft,sa}$ (the direct sum of a nonzero circulant with zero is never circulant). We can now state Anderson's theorem on the equivalence of KS and the so-called *Paving Problem*:

Theorem 1.2.5 ([2]). *The following are equivalent:*

(1) *Every pure state on ℓ^∞ extends uniquely to a pure state on $B(\ell^2)$, i.e., KS is true.*
(2) *There exists a $k \in \mathbb{N}$ such that $\lim_{n\to\infty} \beta_k(\mathbb{M}_n^0(\mathbb{C})) < 1$.*
(3) *For every $0 < \epsilon < 1$, there exists a $k \in \mathbb{N}$ such that $\lim_{n\to\infty} \beta_k(\mathbb{M}_n^0(\mathbb{C})) < \epsilon$.*

Using the formula $\alpha_{k^2}(A+B) \le \alpha_k(A)+\alpha_k(B)$ [4], one can show that $\mathbb{M}_n^0(\mathbb{C})$ in Theorem 1.2.5 may be replaced by $\mathbb{M}_n^0(\mathbb{R})$, $\mathbb{M}_n^0(\mathbb{C})_{sa}$, or $\mathbb{M}_n^0(\mathbb{R})_{sa}$. Owing to recent work of Paulsen and Raghupathi, $\mathbb{M}_n^0(\mathbb{C})_{\triangle}$ also works [10]. So solving the Paving Problem for any of these classes would settle KS.

1.3. **Paving Results.** Because of Theorem 1.2.5, it is of substantial interest to compute $\lim_{n\to\infty} \beta_k(\mathbb{M}_n^0(\mathbb{C}))$ for $k \in \mathbb{N}$, $k \ge 2$ (as well as the corresponding limits for other matrix classes). Nonetheless, heretofore this has only been accomplished for $k = 2$. Since

$$\beta_2\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\right) = 1,$$

we have (trivially) that

$$\lim_{n\to\infty} \beta_2(\mathbb{M}_n^0(\mathbb{C})) = \lim_{n\to\infty} \beta_2(\mathbb{M}_n^0(\mathbb{R})) = \lim_{n\to\infty} \beta_2(\mathbb{M}_n^0(\mathbb{R}_+)) = 1.$$

The self-adjoint case, which is much more delicate, was recently settled by Casazza, Edidin, Kalra, and Paulsen:

Theorem 1.3.1 ([5])**.** $\lim_{n\to\infty} \beta_2(\mathbb{M}_n^0(\mathbb{C})_{sa}) = 1.$

Remark 1.3.2. *The question of attainment in Theorem 1.3.1, i.e., whether or not there exists an $A \in \mathbb{M}_n^0(\mathbb{C})_{sa}$ with $\beta_2(A) = 1$, is still open and of considerable interest.*

Turning to $k = 3$, the only result in the literature is due to Halpern, Kaftal, and Weiss:

Theorem 1.3.3 ([7])**.** $\lim_{n\to\infty} \beta_3(\mathbb{M}_n^0(\mathbb{C})) \ge \lim_{n\to\infty} \beta_3(\mathbb{M}_n^0(\mathbb{R})) \ge \frac{2}{3}$.

On the other hand, the Paving Problem for non-negative matrices is known to have a positive answer, thanks to work of Berman, Halpern, Kaftal, and Weiss:

Theorem 1.3.4 ([4])**.** *For $k \in \mathbb{N}$,*

$$\lim_{n\to\infty} \beta_k(\mathbb{M}_n^0(\mathbb{R}_+)_{sa}) = \frac{1}{k} \text{ and } \lim_{n\to\infty} \beta_k(\mathbb{M}_n^0(\mathbb{R}_+)) \le \frac{2}{k}.$$

Unfortunately, KS seems not equivalent to the Paving Problem for non-negative matrices.

1.4. **Summary of the Paper.** The impetus for this paper was a question of Halpern, Kaftal, and Weiss concerning Theorem 1.3.3 [7]:

> Is $\lim_{n\to\infty} \beta_3(\mathbb{M}_n^0(\mathbb{C})) < 1$ or is $\lim_{n\to\infty} \beta_3(\mathbb{M}_n^0(\mathbb{C})) = 1$? At least is $\lim_{n\to\infty} \beta_3(\mathbb{M}_n^0(\mathbb{C})) > \frac{2}{3}$?

By computing $\beta_3(\mathbb{M}_n^0(\mathbb{C}))$ for small values of n, we are able to answer the second question affirmatively. We obtain the following 3-paving tables, which are the main results of our investigation:

Theorem 1.4.1 (3-Paving Table for Nonselfadjoint Matrices)**.**

n	$\beta_3(\mathbb{M}_n^0(\mathbb{C}))$	$\beta_3(\mathbb{M}_n^0(\mathbb{R}))$	$\beta_3(\mathbb{M}_n^0(\mathbb{R}_+))$	$\beta_3(\mathbb{M}_n^0(\mathbb{C})_\triangle)$	$\beta_3(\mathbb{M}_n^0(\mathbb{C})_\circlearrowleft)$
4	.6180	.6180	.5550	.5412	$[.6000, .6030]$
5	.6180	.6180	.5550	$[.5609, .5774]$	$[.6120, .6180]$
6	.7071	.7071	$[.5550, .5774]$	$[.5725, .5774]$	$[.5726, .6325]$
7	$[.8239, 1]$	$[.8029, 1]$	$[.5550, .6667]$	$[.6503, .9258]$	$[.8239, 1]$
10	$[.8540, 1]$	$[.8079, 1]$	$[.5550, .6667]$	$[.6703, 1]$	$[.8540, 1]$
13	$[.8615, 1]$	$[.8195, 1]$	$[.5550, .6667]$	$[.6800, 1]$	$[.8615, 1]$

Theorem 1.4.2 (3-Paving Table for Selfadjoint Matrices)**.**

n	$\beta_3(\mathbb{M}_n^0(\mathbb{C})_{sa})$	$\beta_3(\mathbb{M}_n^0(\mathbb{R})_{sa})$	$\beta_3(\mathbb{M}_n^0(\mathbb{C})_{\circlearrowleft,sa})$
4	.5774	.4472	.4142
5	.5774	.4472	.4472
6	.5774	.4851	$[.4069, .4495]$
7	$[.6872, .7559]$	$[.6667, .7559]$	$[.6544, .7559]$
8	$[.6872, .8819]$	$[.6667, .8819]$	$[.5797, .8819]$
9	$[.6872, .8889]$	$[.6667, .8889]$	$[.5539, .8889]$
10	$[.7536, 1]$	$[.7454, 1]$	$[.6686, 1]$
13	$[.7536, 1]$	$[.7454, 1]$	$[.6983, 1]$
16	$[.7574, 1]$	$[.7454, 1]$	$[.7019, 1]$

To support bootstrapping arguments for the 3-paving tables, as well as because of intrinsic interest, we also compute the following 2-paving table:

Theorem 1.4.3 (2-Paving Table)**.**

n	$\beta_2(\mathbb{M}_n^0(\mathbb{C})_\triangle)$	$\beta_2(\mathbb{M}_n^0(\mathbb{C})_\circlearrowleft)$	$\beta_2(\mathbb{M}_n^0(\mathbb{C})_{sa})$	$\beta_2(\mathbb{M}_n^0(\mathbb{R})_{sa})$	$\beta_2(\mathbb{M}_n^0(\mathbb{C})_{\circlearrowleft,sa})$
3	.6180	1	.5774	.5000	.5774
4	.7071	$[.6000, .6030]$	.5774	$[.5493, .5577]$	.4142
5	$[.7715, 1]$	1	.8944	.8944	.8944
6	$[.8337, 1]$	1	.8944	.8944	$[.7454, .8944]$
7	$[.8500, 1]$	1	$[.9225, 1]$		$[.9073, 1]$
8	$[.8866, 1]$	$[.9623, 1]$	$[.9225, 1]$		$[.7689, 1]$
9	$[.8965, 1]$	1	$[.9414, 1]$		$[.8920, 1]$
10	$[.9149, 1]$	1	$[.9414, 1]$		
11	$[.9207, 1]$	1	$[.9477, 1]$		
12		1	$[.9477, 1]$		
13		1	$[.9547, 1]$		
14		1	$[.9547, 1]$		
15		1	$[.9625, 1]$		
16		$[.9846, 1]$	$[.9625, 1]$		
17		1	$[.9692, 1]$		
18		1	$[.9692, 1]$		
19		1	$[.9742, 1]$		
20		1	$[.9742, 1]$		

The remainder of the paper, which is divided into seven sections, consists of a myriad of propositions, each of which establishes particular entries in our tables. Each section corresponds to a certain class of matrices. Although the arguments are structurally similar, the details vary from class to class. Each section begins with a subsection (or two) which gathers together the needed tools. We assume that the reader is familiar with basic operator theory and basic graph theory.

Remark 1.4.4 (exact vs. approximate)**.** *The numbers in our paving tables are decimal approximations. The corresponding exact expressions (when available) appear in the proposition statements.*

Remark 1.4.5 (computer-generated examples)**.** *For those table entries which consist of an interval (e.g. the $n = 7$ entry of the first column of Table 1.4.1), the lower bound is (almost always) the result of a computer-generated example. To our knowledge, these examples do not have closed-form expressions, and (with one exception) we do not include them in the paper. In the appendix we do show the worst-known 3-paver, a 13×13 complex circulant A such that $\beta_3(A) \approx .8615$.*

Remark 1.4.6 (open questions)**.** *This paper invites many questions. In particular, can any of the non-sharp table entries be improved? Here are some other interesting questions:*

(1) *Is $\lim_{n\to\infty} \beta_3(\mathbb{M}_n^0(\mathbb{C})) < 1$ or is $\lim_{n\to\infty} \beta_3(\mathbb{M}_n^0(\mathbb{C})) = 1$? At least is $\lim_{n\to\infty} \beta_3(\mathbb{M}_n^0(\mathbb{C})) > .8615$? This is the aforementioned question of Halpern, Kaftal, and Weiss, amended to reflect the information in Table 1.4.1.*
(2) *Does there exist $n \in \mathbb{N}$ and $A \in \mathbb{M}_n^0(\mathbb{C})_{sa}$ such that $\beta_2(A) = 1$. Table 1.4.3 suggests an affirmative answer with $n \approx 30$. Remember that it is known that $\lim_{n\to\infty} \beta_2(\mathbb{M}_n^0(\mathbb{C})_{sa}) = 1$.*
(3) *Is $\lim_{n\to\infty} \beta_2(\mathbb{M}_n^0(\mathbb{C})_\triangle) = 1$? Table 1.4.3 suggests an affirmative answer. In that case, is there an $n \in \mathbb{N}$ and $A \in \mathbb{M}_n^0(\mathbb{C})_\triangle$ such that $\beta_2(A) = 1$?*
(4) *Is $\lim_{n\to\infty} \beta_3(\mathbb{M}_n^0(\mathbb{R}_+)) < \frac{2}{3}$ or is $\lim_{n\to\infty} \beta_3(\mathbb{M}_n^0(\mathbb{R}_+)) = \frac{2}{3}$? Table 1.4.1 (as well as substantial computer experimentation) suggests that $\lim_{n\to\infty} \beta_3(\mathbb{M}_n^0(\mathbb{R}_+)) \approx .5550$.*
(5) *Is KS equivalent to the Paving Problem for circulants?*

2. Paving General Matrices

This section establishes the first and second columns of Table 1.4.1.

2.1. **Tools.**

Lemma 2.1.1 (Frobenius domination principle)**.** *For $A, B \in \mathbb{M}_n(\mathbb{R}_+)$, if $a_{ij} \leq b_{ij}$ for all $1 \leq i, j \leq n$, then $\|A\| \leq \|B\|$.*

Proof. Let $\vec{x}, \vec{y} \in \mathbb{C}^n$ be such that $\|\vec{x}\| = 1$, $\|\vec{y}\| = 1$, and $\|A\| = |\langle A\vec{x}, \vec{y}\rangle|$. Then

$$\begin{aligned} \|A\| &= |\langle A\vec{x}, \vec{y}\rangle| = \left| \sum_{i,j=1}^n a_{ij} x_j \overline{y_i} \right| \leq \sum_{i,j=1}^n a_{ij}|x_j||y_i| \\ &\leq \sum_{i,j=1}^n b_{ij}|x_j||y_i| = \langle B|\vec{x}|, |\vec{y}|\rangle \leq \|B\| \||\vec{x}|\| \||\vec{y}|\| = \|B\|. \end{aligned}$$

□

Lemma 2.1.2. *Let $a, b, d \in \mathbb{C}$. If $|a|, |b|, |d| \geq 1$, then*

$$\left\| \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \right\| \geq \frac{1+\sqrt{5}}{2}.$$

This inequality is sharp.

Proof. Let $\alpha, \beta, \delta \in \mathbb{R}$ be such that $a = |a|e^{i\alpha}$, $b = |b|e^{i\beta}$, and $d = |d|e^{i\delta}$. By Lemma 2.1.1,

$$\begin{aligned} \left\| \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \right\| &= \left\| \begin{bmatrix} e^{i\beta} & 0 \\ 0 & e^{i\delta} \end{bmatrix} \begin{bmatrix} |a| & |b| \\ 0 & |d| \end{bmatrix} \begin{bmatrix} e^{i(\alpha-\beta)} & 0 \\ 0 & 1 \end{bmatrix} \right\| \\ &= \left\| \begin{bmatrix} |a| & |b| \\ 0 & |d| \end{bmatrix} \right\| \geq \left\| \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\| = \frac{1+\sqrt{5}}{2}. \end{aligned}$$

□

Lemma 2.1.3. *If*

$$A = \begin{bmatrix} 0 & * & * & a_{14} & a_{15} \\ * & 0 & * & a_{24} & a_{25} \\ * & * & 0 & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & 0 & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & 0 \end{bmatrix},$$

where "$*$" indicates an arbitrary entry and for every $i \neq j$ either $|a_{ij}| \geq 1$ or $|a_{ji}| \geq 1$, then $\|A\| \geq \frac{1+\sqrt{5}}{2}$. This inequality is sharp.

Proof. Permuting the indices 4 and 5 if necessary, we may assume $|a_{45}| \geq 1$. If $|a_{51}|, |a_{52}|, |a_{53}| \geq 1$, then $\|A\| \geq \sqrt{3} > \frac{1+\sqrt{5}}{2}$, and we are done. Thus, we may assume that at least one of a_{51}, a_{52}, a_{53} has modulus less than 1, in which case at least one of a_{15}, a_{25}, a_{35} has modulus greater than or equal to 1. Permuting the indices 1, 2, and 3, if necessary, we may assume that $|a_{35}| \geq 1$. If $|a_{34}| \geq 1$, then (by Lemma 2.1.2)

$$\|A\| \geq \|A_{\{3,4\},\{4,5\}}\| = \left\| \begin{bmatrix} a_{34} & a_{35} \\ 0 & a_{45} \end{bmatrix} \right\| \geq \frac{1+\sqrt{5}}{2}.$$

If, on the other hand, $|a_{43}| \geq 1$, then (by Lemma 2.1.2 again)

$$\|A\| \geq \|A_{\{3,4\},\{3,5\}}\| = \left\| \begin{bmatrix} 0 & a_{35} \\ a_{43} & a_{45} \end{bmatrix} \right\| \geq \frac{1+\sqrt{5}}{2}.$$

Since

$$\left\| \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix} \right\| = \frac{1+\sqrt{5}}{2},$$

the inequality is sharp. □

2.2. **Computation of 3-Paving Parameters.** The following proposition establishes the first entries of the first and second columns of Table 1.4.1:

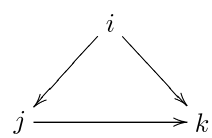
Proposition 2.2.1. $\beta_3(\mathbb{M}_4^0(\mathbb{C})) = \beta_3(\mathbb{M}_4^0(\mathbb{R})) = \frac{2}{1+\sqrt{5}} \approx .6180$.

Proof. Let $A \in \mathbb{M}_4^0(\mathbb{C})$ with $\alpha_3(A) = 1$. We aim to show that $\|A\| \geq \frac{1+\sqrt{5}}{2} \approx 1.6180$. Associate a digraph $D = (V, E)$ with A as follows: $V = \{1, 2, 3, 4\}$ and $(i, j) \in E$ if and only if $|a_{ij}| \geq 1$. We may assume that D has the following properties:

I. For every $i \neq j$, either $(i, j) \in E$ or $(j, i) \in E$. Otherwise, if $(i, j), (j, i) \notin E$, then
$$\|A_{\{i,j\}}\| = \left\| \begin{bmatrix} 0 & a_{ij} \\ a_{ji} & 0 \end{bmatrix} \right\| = \max\{|a_{ij}|, |a_{ji}|\} < 1,$$
which implies that A has a $(1, 1, 2)$-paving of norm less than 1, contradicting $\alpha_3(A) = 1$.

II. For every vertex i, the out-degree of i is less than or equal to two. Otherwise, if i has out-degree equal to three, then the ith row of A has three entries of modulus greater than or equal to 1, which implies that $\|A\| \geq \sqrt{3} > \frac{1+\sqrt{5}}{2}$ (and we are done). Likewise, the in-degree of i is less than or equal to two.

III. The digraph



is not a subgraph of D. Otherwise,
$$A_{\{i,j,k\}} = \begin{bmatrix} 0 & \blacksquare & \blacksquare \\ * & 0 & \blacksquare \\ * & * & 0 \end{bmatrix},$$
where "$\blacksquare$" indicates an entry of modulus greater than or equal to 1 and "$*$" indicates an arbitrary entry. By Lemma 2.1.2, $\|A\| \geq \|A_{\{i,j,k\}}\| \geq \frac{1+\sqrt{5}}{2}$ (and we are done).

Checking [11, pp. 293–297], there are no such digraphs D. Thus, $\|A\| \geq \frac{1+\sqrt{5}}{2}$ and $\beta_3(A) = \frac{\alpha_3(A)}{\|A\|} \leq \frac{2}{1+\sqrt{5}}$. Since the choice of A was arbitrary, $\beta_3(\mathbb{M}_4^0(\mathbb{C})) \leq \frac{2}{1+\sqrt{5}}$. Since
$$\beta_3\left(\begin{bmatrix} 0 & 1 & 1 & -\frac{2}{1+\sqrt{5}} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \right) = \frac{2}{1+\sqrt{5}},$$
we have $\frac{2}{1+\sqrt{5}} \leq \beta_3(\mathbb{M}_4^0(\mathbb{R})) \leq \beta_3(\mathbb{M}_4^0(\mathbb{C})) \leq \frac{2}{1+\sqrt{5}}$. □

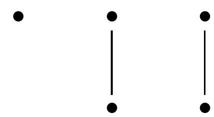
The following proposition establishes the second entries of the first and second columns of Table 1.4.1:

Proposition 2.2.2. $\beta_3(\mathbb{M}_5^0(\mathbb{C})) = \beta_3(\mathbb{M}_5^0(\mathbb{R})) = \frac{2}{1+\sqrt{5}} \approx .6180$.

Proof. By Proposition 2.2.1,
$$\frac{2}{1+\sqrt{5}} = \beta_3(\mathbb{M}_4^0(\mathbb{R})) \leq \beta_3(\mathbb{M}_5^0(\mathbb{R})) \leq \beta_3(\mathbb{M}_5^0(\mathbb{C})).$$

Now let $A \in \mathbb{M}_5^0(\mathbb{C})$ with $\alpha_3(A) = 1$. We aim to show that $\|A\| \geq \frac{1+\sqrt{5}}{2}$. Associate a graph $G = (V, E)$ with A as follows: $V = \{1, 2, ..., 5\}$ and $(i, j) \in E$ if and only if $\|A_{\{i,j\}}\| < 1$ (i.e. $|a_{ij}| < 1$ and $|a_{ji}| < 1$). We may assume that G has the following properties:

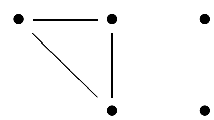
I. The graph



is not a subgraph of G. Otherwise, A has a $(1, 2, 2)$-paving of norm less than 1, contradicting $\alpha_3(A) = 1$.

II. By removing one vertex (and all associated edges) from G one cannot arrive at the "edgeless" graph $(\{i, j, k, l\}, \emptyset)$. Otherwise $\alpha_3(A_{\{i,j,k,l\}}) \geq 1$ (since every $(1, 1, 2)$-paving has norm greater than or equal to 1), which implies that $\|A_{\{i,j,k,l\}}\| \geq \frac{1+\sqrt{5}}{2}$ (by Proposition 2.2.1). Then $\|A\| \geq \|A_{\{i,j,k,l\}}\| \geq \frac{1+\sqrt{5}}{2}$ (and we are done).

Checking [11, p. 8], we see that G must be



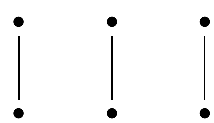
By Lemma 2.1.3, $\|A\| \geq \frac{1+\sqrt{5}}{2}$. Since the choice of A was arbitrary, $\beta_3(\mathbb{M}_5^0(\mathbb{C})) \leq \frac{2}{1+\sqrt{5}}$. Thus, $\beta_3(\mathbb{M}_5^0(\mathbb{C})) = \beta_3(\mathbb{M}_5^0(\mathbb{R})) = \frac{2}{1+\sqrt{5}}$. □

The following proposition establishes the third entries of the first and second columns of Table 1.4.1:

Proposition 2.2.3. $\beta_3(\mathbb{M}_6^0(\mathbb{C})) = \beta_3(\mathbb{M}_6^0(\mathbb{R})) = \frac{1}{\sqrt{2}} \approx .7071$.

Proof. Let $A \in \mathbb{M}_6^0(\mathbb{C})$ with $\alpha_3(A) = 1$. We aim to show that $\|A\| \geq \sqrt{2} \approx 1.4142$. Associate a graph $G = (V, E)$ with A as follows: $V = \{1, 2, ..., 6\}$ and $(i, j) \in E$ if and only if $\|A_{\{i,j\}}\| < 1$. We may assume that G has the following properties:

I. The graph



is not a subgraph of G. Otherwise A has a $(2, 2, 2)$-paving of norm less than 1, contradicting $\alpha_3(A) = 1$.

II. For every vertex i, the degree of i is at least three. Otherwise, if the degree of i is less than or equal to two, then there exist j, k, l distinct (and different from i) such that $(i, j), (i, k), (i, l) \notin E$. Then $\|A_{\{i,j\}}\|, \|A_{\{i,k\}}\|, \|A_{\{i,l\}}\| \geq 1$. It follows that either the ith row of A or the ith column of A has two entries of modulus greater than or equal to 1, which implies that $\|A\| \geq \sqrt{2}$ (and we are done).

Checking [11, pp. 9–11], there are no such graphs G. Thus $\|A\| \geq \sqrt{2}$. Since the choice of A was arbitrary, $\beta_3(\mathbb{M}_6^0(\mathbb{C})) \leq \frac{1}{\sqrt{2}}$. Now let

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ -\frac{1}{2} & 1 & \frac{1}{2} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & -\frac{1}{2} & 0 & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix} \in \mathbb{M}_6^0(\mathbb{R}).$$

Then A is $\sqrt{2}$ times a unitary and $\alpha_3(A) = 1$. Thus $\beta_3(A) = \frac{1}{\sqrt{2}}$. Hence $\frac{1}{\sqrt{2}} \leq \beta_3(\mathbb{M}_6^0(\mathbb{R})) \leq \beta_3(\mathbb{M}_6^0(\mathbb{C})) \leq \frac{1}{\sqrt{2}}$. $\square$

3. Paving Non-negative Matrices

In this section we establish the third column of Table 1.4.1.

3.1. **The Concentration Principle.**

In order to obtain a good lower bound on the operator norm of a matrix A, it is useful to know the configuration of the large-modulus entries of A. Indeed, this principle was the basis of our analysis in the previous section. Unfortunately, it is often the case that A has many large-norm submatrices but few large-modulus entries, which makes the analysis of $\|A\|$ much harder. In this section we prove a result which allows us to "concentrate" large-norm submatrices of a non-negative matrix A into large-modulus entries, such that the resulting matrix A' satisfies $\|A'\| \leq \|A\|$. Since A' has more large-modulus entries than A, it should be easier to analyze $\|A'\|$ than $\|A\|$. Of course, a lower bound for $\|A'\|$ is, a fortiori, a lower bound for $\|A\|$. Our result is based on the following minimization formula for the operator norm of a non-negative rectangular matrix, which is due to Mathias:

Theorem 3.1.1 ([9])**.** *For $A \in \mathbb{M}_{m\times n}(\mathbb{R}_+)$,*

$$\|A\| = \min\{r_{\max}(B)c_{\max}(C) : B, C \in \mathbb{M}_{m\times n}(\mathbb{R}_+) \text{ and } A = B \bullet C\},$$

where $r_{\max}(B)$ is the maximum row norm of B, $c_{\max}(C)$ is the maximum column norm of C, and $B \bullet C$ is the entrywise (i.e., Hadamard) product of B and C.

In order to introduce our result, we need some terminology. We say that a matrix $A \in \mathbb{M}_{m\times n}(\mathbb{R}_+)$ is *concentrated* at $(i,j) \in \{1,2,...,m\}\times\{1,2,...,n\}$ if $\|A\| = a_{ij}$. If $A \neq 0$ is concentrated at (i,j), then the only nonzero entry in the ith row of A and the jth column of A is a_{ij}. For $A \in \mathbb{M}_{m\times n}(\mathbb{R}_+)$ and $(i,j) \in \{1,2,...,m\}\times\{1,2,...,n\}$ we define the *concentration* of A at (i,j) to be the matrix

$$A^{(i,j)} := \begin{bmatrix} a_{1,1} & \dots & a_{1,j-1} & 0 & a_{1,j+1} & \dots & a_{1,n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{i-1,1} & \dots & a_{i-1,j-1} & 0 & a_{i-1,j+1} & \dots & a_{i-1,n} \\ 0 & \dots & 0 & \|A\| & 0 & \dots & 0 \\ a_{i+1,1} & \dots & a_{i+1,j-1} & 0 & a_{i+1,j+1} & \dots & a_{i+1,n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & \dots & a_{m,j-1} & 0 & a_{m,j+1} & \dots & a_{m,n} \end{bmatrix} \in \mathbb{M}_{m\times n}(\mathbb{R}_+).$$

Since $\|A^{(i,j)}\| = \|A\|$, we see that $A^{(i,j)}$ is concentrated at (i,j). Now suppose we concentrate a submatrix $A_{\mu,\nu}$ of a nonnegative matrix A, producing a new matrix A'. It can easily happen that $\|A'\| > \|A\|$. The following result asserts that by exercising care in how we concentrate $A_{\mu,\nu}$, we can achieve $\|A'\| \leq \|A\|$.

Theorem 3.1.2 (concentration principle). *For $A \in \mathbb{M}_n(\mathbb{R}_+)$ and $\mu, \nu \subseteq \{1, 2, ..., n\}$,*

$$\min \left\{ \left\| \begin{bmatrix} A_{\mu,\nu}^{(i,j)} & A_{\mu,\nu^c} \\ A_{\mu^c,\nu} & A_{\mu^c,\nu^c} \end{bmatrix} \right\| : (i,j) \in \mu \times \nu \right\} \leq \|A\|.$$

Proof. Without loss of generality, $\mu = \{1, 2, ..., s\}$ and $\nu = \{1, 2, ..., t\}$ for some $1 \leq s, t \leq n$. By Theorem 3.1.1, there exist $B, C \in \mathbb{M}_n(\mathbb{R}_+)$ such that $A = B \bullet C$ and $\|A\| = r_{\max}(B) c_{\max}(C)$. Clearly $A_{\mu,\nu} = B_{\mu,\nu} \bullet C_{\mu,\nu}$, and so $\|A_{\mu,\nu}\| \leq r_{\max}(B_{\mu,\nu}) c_{\max}(C_{\mu,\nu})$ by Theorem 3.1.1 again. Let $i \in \mu$ be such that the ith row of $B_{\mu,\nu}$ has norm $r_{\max}(B_{\mu,\nu})$ and $j \in \nu$ be such that the jth column of $C_{\mu,\nu}$ has norm $c_{\max}(C_{\mu,\nu})$. Let $B'_{\mu,\nu}$ equal $B_{\mu,\nu}$ with the ith row replaced by

$$\begin{bmatrix} 0_1 & \dots & 0_{j-1} & r_{\max}(B_{\mu,\nu}) & 0_{j+1} & \dots & 0_t \end{bmatrix}$$

and $C'_{\mu,\nu}$ equal $C_{\mu,\nu}$ with the jth column replaced by

$$\begin{bmatrix} 0_1 \\ \vdots \\ 0_{i-1} \\ c_{\max}(C_{\mu,\nu}) \\ 0_{i+1} \\ \vdots \\ 0_s \end{bmatrix}.$$

Note that every row norm of $B'_{\mu,\nu}$ agrees with the corresponding row norm of $B_{\mu,\nu}$, and that every column norm of $C'_{\mu,\nu}$ agrees with the corresponding column norm of $C_{\mu,\nu}$. Define $A'_{\mu,\nu} = B'_{\mu,\nu} \bullet C'_{\mu,\nu}$. Then every entry of $A'_{\mu,\nu}$ is greater than or equal to the matching entry of $A_{\mu,\nu}^{(i,j)}$. Indeed, all but the (i,j) entry agree, and the (i,j) entry of $A'_{\mu,\nu}$ equals $r_{\max}(B_{\mu,\nu}) c_{\max}(C_{\mu,\nu})$ while the (i,j) entry of $A_{\mu,\nu}^{(i,j)}$ equals $\|A_{\mu,\nu}\|$. Therefore,

$$\begin{aligned}
\|A\| &= r_{\max}\left(\begin{bmatrix} B_{\mu,\nu} & B_{\mu,\nu^c} \\ B_{\mu^c,\nu} & B_{\mu^c,\nu^c} \end{bmatrix} \right) c_{\max}\left(\begin{bmatrix} C_{\mu,\nu} & C_{\mu,\nu^c} \\ C_{\mu^c,\nu} & C_{\mu^c,\nu^c} \end{bmatrix} \right) \\
&= r_{\max}\left(\begin{bmatrix} B'_{\mu,\nu} & B_{\mu,\nu^c} \\ B_{\mu^c,\nu} & B_{\mu^c,\nu^c} \end{bmatrix} \right) c_{\max}\left(\begin{bmatrix} C'_{\mu,\nu} & C_{\mu,\nu^c} \\ C_{\mu^c,\nu} & C_{\mu^c,\nu^c} \end{bmatrix} \right) \\
&\overset{\text{Theorem 3.1.1}}{\geq} \left\| \begin{bmatrix} B'_{\mu,\nu} & B_{\mu,\nu^c} \\ B_{\mu^c,\nu} & B_{\mu^c,\nu^c} \end{bmatrix} \bullet \begin{bmatrix} C'_{\mu,\nu} & C_{\mu,\nu^c} \\ C_{\mu^c,\nu} & C_{\mu^c,\nu^c} \end{bmatrix} \right\| \\
&= \left\| \begin{bmatrix} A'_{\mu,\nu} & A_{\mu,\nu^c} \\ A_{\mu^c,\nu} & A_{\mu^c,\nu^c} \end{bmatrix} \right\| \overset{\text{Lemma 2.1.1}}{\geq} \left\| \begin{bmatrix} A_{\mu,\nu}^{(i,j)} & A_{\mu,\nu^c} \\ A_{\mu^c,\nu} & A_{\mu^c,\nu^c} \end{bmatrix} \right\|.
\end{aligned}$$

□

3.2. **Other Tools.**

Lemma 3.2.1. *Let $A \in \mathbb{M}_4^0(\mathbb{R}_+)$. Assume that*

(i) *For all $i < j$, either $a_{ij} \geq 1$ or $a_{ji} \geq 1$.*
(ii) *Some row or column of A has three entries greater than or equal to 1.*

Then $\|A\| \geq 2$. This inequality is sharp.

Proof. Replacing A by A^*, if necessary, we may assume that some row of A has three entries greater than or equal to 1. Permuting the indices, if necessary, we may assume that the first row of A has three entries greater than or equal to 1. Now by assumption, for all $2 \leq i < j \leq 4$, either $a_{ij} \geq 1$ or $a_{ji} \geq 1$. Thus, by Lemma 2.1.1,

$$\|A\| \geq \min \left\{ \left\| \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & \delta_{23} & \delta_{24} \\ 0 & 1-\delta_{23} & 0 & \delta_{34} \\ 0 & 1-\delta_{24} & 1-\delta_{34} & 0 \end{bmatrix} \right\| : \delta_{23}, \delta_{24}, \delta_{34} \in \{0,1\} \right\} = 2,$$

where the last equality follows by checking $2^3 = 8$ cases on a computer. Since

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \in \mathbb{M}_4^0(\mathbb{R}_+)$$

satisfies (i), (ii), and $\|A\| = 2$, the inequality is sharp. □

3.3. **Computation of 3-Paving Parameters.** For future reference, define

$$\kappa := \left\| \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \right\| = \sqrt{\frac{5 + 2\sqrt{7}\cos(\tan^{-1}(3\sqrt{3})/3)}{3}} \approx 1.8019.$$

The following proposition establishes the first entry of the third column of Table 1.4.1:

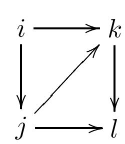
Proposition 3.3.1. *$\beta_3(\mathbb{M}_4^0(\mathbb{R}_+)) = \frac{1}{\kappa} \approx .5550$.*

Proof. Let $A \in \mathbb{M}_4^0(\mathbb{R}_+)$ with $\alpha_3(A) = 1$. We aim to show that $\|A\| \geq \kappa \approx 1.8019$. Associate a digraph $D = (V, E)$ with A as follows: $V = \{1, 2, 3, 4\}$ and $(i, j) \in E$ if and only if $a_{ij} \geq 1$. We may assume that D has the following properties:

I. For every $i \neq j$, either $(i, j) \in E$ or $(j, i) \in E$. Otherwise $\|A_{\{i,j\}}\| < 1$, which implies that A has a $(1, 1, 2)$-paving of norm less than 1, contradicting $\alpha_3(A) = 1$.

II. For every vertex i, the out-degree of i is less than three. Otherwise, the ith row of A has three entries greater than or equal to 1, which (by Lemma 3.2.1) implies that $\|A\| \geq 2 > \kappa$ (and we are done). Likewise, the in-degree of i is less than three.

III. The digraph



is not a subgraph of D. Otherwise, by Lemma 2.1.1,

$$\|A\| = \|A_{\{i,j,k,l\}}\| \geq \left\| \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right\| = \kappa$$

(and we are done).

Checking [11, pp. 293–297], there are no such digraphs D. Thus, $\|A\| \geq \kappa$. Since the choice of A was arbitrary, $\beta_3(\mathbb{M}_4^0(\mathbb{R}_+)) \leq \frac{1}{\kappa}$. Since

$$\beta_3\left(\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}\right) = \frac{1}{\kappa},$$

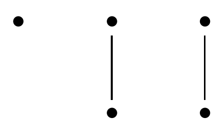
we have $\beta_3(\mathbb{M}_4^0(\mathbb{R}_+)) = \frac{1}{\kappa}$. □

The following proposition establishes the second entry of the third column of Table 1.4.1:

Proposition 3.3.2. $\beta_3(\mathbb{M}_5^0(\mathbb{R}_+)) = \frac{1}{\kappa} \approx 0.5550$.

Proof. By Proposition 3.3.1, $\beta_3(\mathbb{M}_5^0(\mathbb{R}_+)) \geq \beta_3(\mathbb{M}_4^0(\mathbb{R}_+)) = \frac{1}{\kappa}$. Now let $A \in \mathbb{M}_5^0(\mathbb{R}_+)$ with $\alpha_3(A) = 1$. We aim to show that $\|A\| \geq \kappa \approx 1.8019$. Associate a graph $G = (V, E)$ with A as follows: $V = \{1, 2, 3, 4, 5\}$ and $(i, j) \in E$ if and only if $\|A_{\{i,j\}}\| < 1$. We may assume that G satisfies the following properties:

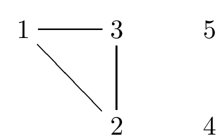
I. The graph



is not subgraph of G. Otherwise A has a $(1, 2, 2)$-paving of norm less than 1, contradicting $\alpha_3(A) = 1$.

II. By removing one vertex (and all associated edges) from G, one cannot arrive at the "edgeless" graph $(\{i, j, k, l\}, \emptyset)$. Otherwise $\alpha_3(A_{\{i,j,k,l\}}) \geq 1$ (since every $(1, 1, 2)$-paving has norm greater than or equal to 1), which implies that $\|A_{\{i,j,k,l\}}\| \geq \kappa$ (by Proposition 3.3.1). Then $\|A\| \geq \|A_{\{i,j,k,l\}}\| \geq \kappa$ (and we are done).

Checking [11, p. 9], we have that (after permuting vertices) G must be



Thus, $\|A_{\{4,5\}}\| \geq 1$ and $\|A_{\{k,l\}}\| \geq 1$ for all $(k, l) \in \{1, 2, 3\} \times \{4, 5\}$. Since $\alpha_3(A) = 1$, the $\{\{1, 2, 3\}, \{4\}, \{5\}\}$-paving of A has norm greater than or equal to

1, i.e., $\|A_{\{1,2,3\}}\| \geq 1$. Therefore,

$$
\begin{aligned}
\|A\| &\overset{\text{Theorem 3.1.2}}{\geq} \min\left\{ \left\| \left[\begin{array}{ccc|cc} & & & a_{14} & a_{15} \\ & A^{(i,j)}_{\{1,2,3\}} & & a_{24} & a_{25} \\ & & & a_{34} & a_{35} \\ \hline a_{41} & a_{42} & a_{43} & 0 & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & 0 \end{array}\right] \right\| : i,j \in \{1,2,3\} \right\} \\
&\overset{\text{Lemma 2.1.1}}{\geq} \min\left\{ \left\| \left[\begin{array}{ccc|cc} & & & \delta_{14} & \delta_{15} \\ & E_{ij} & & \delta_{24} & \delta_{25} \\ & & & \delta_{34} & \delta_{35} \\ \hline 1-\delta_{14} & 1-\delta_{24} & 1-\delta_{34} & 0 & \delta_{45} \\ 1-\delta_{15} & 1-\delta_{25} & 1-\delta_{35} & 1-\delta_{45} & 0 \end{array}\right] \right\| : \substack{i,j\in\{1,2,3\} \\ \delta_{kl}\in\{0,1\}} \right\} \\
&= \kappa.
\end{aligned}
$$

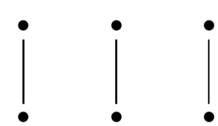
Here $E_{ij} \in \mathbb{M}_3(\mathbb{R}_+)$ is the matrix with 1 in the (i,j) entry and 0 elsewhere. The last equality follows by checking $3^2 \times 2^7 = 1152$ cases on a computer. □

The following proposition establishes the third entry of the third column of Table 1.4.1:

Proposition 3.3.3. $\beta_3(\mathbb{M}_6^0(\mathbb{R}_+)) \in \left[\frac{1}{\kappa}, \frac{1}{\sqrt{3}}\right] \approx [.5550, .5774]$.

Proof. By Proposition 3.3.2, $\beta_3(\mathbb{M}_6^0(\mathbb{R}_+)) \geq \beta_3(\mathbb{M}_5^0(\mathbb{R}_+)) = \frac{1}{\kappa}$. Now let $A \in \mathbb{M}_6^0(\mathbb{R}_+)$ with $\alpha_3(A) = 1$. We aim to show that $\|A\| \geq \sqrt{3} \approx 1.7321$. Associate a graph $G = (V, E)$ with A as follows: $V = \{1,2,3,4,5,6\}$ and $(i,j) \in E$ if and only if $\|A_{\{i,j\}}\| < 1$. We may assume that G has the following properties:

I. The graph

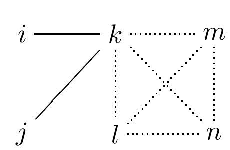


is not a subgraph of G. Otherwise, A has a $(2,2,2)$-paving of norm less than 1, contradicting $\alpha_3(A) = 1$.

II. G has no isolated vertices. Otherwise, if vertex i is isolated, then either the ith row of A or the ith column of A has at least three entries greater than or equal to 1, which implies $\|A\| \geq \sqrt{3}$ (and we are done).

III. By removing two vertices (and all associated edges) from G one cannot arrive at the "edgeless" graph $(\{i,j,k,l\}, \emptyset)$. Otherwise $\alpha_3(A_{\{i,j,k,l\}}) \geq 1$ (since every $(1,1,2)$ paving has norm greater than or equal to 1), which implies that $\|A_{\{i,j,k,l\}}\| \geq \kappa$ (by Proposition 3.3.1). Then $\|A\| \geq \|A_{\{i,j,k,l\}}\| \geq \kappa > \sqrt{3}$ (and we are done).

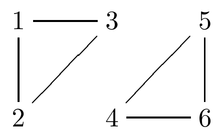
IV. G is not the graph



where a dotted line indicates an edge which may or may not be there. Otherwise, $\alpha_3(A_{\{i,j,l,m,n\}}) \geq 1$. Indeed, since $\|A_\sigma\| \geq 1$ whenever $\sigma \subseteq \{i,j,l,m,n\}$, $\text{card}(\sigma) \geq 2$, and $\sigma \cap \{i,j\} \neq \emptyset$, every $(1,2,2)$-paving of

$A_{\{i,j,l,m,n\}}$ has norm greater than or equal to 1 and every $(1,1,3)$-paving of $A_{\{i,j,l,m,n\}}$ except possibly the $\{\{i\},\{j\},\{l,m,n\}\}$-paving has norm greater than or equal to 1. Since $\alpha_3(A)=1$, the $\{\{i\},\{j,k\},\{l,m,n\}\}$-paving of A has norm greater than or equal to 1. Since $\|A_{\{j,k\}}\|<1$, $\|A_{\{l,m,n\}}\|\geq 1$. Thus the $\{\{i\},\{j\},\{l,m,n\}\}$-paving of $A_{\{i,j,l,m,n\}}$ has norm greater than or equal to 1 also. By Proposition 3.3.2, $\|A_{\{i,j,l,m,n\}}\|\geq\kappa$. Then $\|A\|\geq \|A_{\{i,j,l,m,n\}}\|\geq\kappa>\sqrt{3}$ (and we are done).

Checking [11, pp. 9–11] we see that (up to a permutation of the vertices) G must be



Thus $\|A_{\{k,l\}}\|\geq 1$ for all $(k,l)\in\{1,2,3\}\times\{4,5,6\}$. Since $\alpha_3(A)=1$, the $\{\{1,2,3\},\{4,5\},\{6\}\}$-paving of A has norm greater than or equal to 1. Since $\|A_{\{4,5\}}\|<1$, $\|A_{\{1,2,3\}}\|\geq 1$. Likewise, $\|A_{\{4,5,6\}}\|\geq 1$. Therefore,

$$
\begin{aligned}
\|A\| &\overset{\text{Theorem 3.1.2}}{\geq} \min\left\{\left\|\left[\begin{array}{ccc|ccc} & & & a_{14} & a_{15} & a_{16}\\ & A^{(i,j)}_{\{1,2,3\}} & & a_{24} & a_{25} & a_{26}\\ & & & a_{34} & a_{35} & a_{36}\\ \hline a_{41} & a_{42} & a_{43} & & & \\ a_{51} & a_{52} & a_{53} & & A^{(s,t)}_{\{4,5,6\}} & \\ a_{61} & a_{62} & a_{63} & & & \end{array}\right]\right\| : {\scriptstyle i,j\in\{1,2,3\} \atop s,t\in\{4,5,6\}}\right\}\\
&\overset{\text{Lemma 2.1.1}}{\geq} \min\left\{\left\|\left[\begin{array}{ccc|ccc} & & & \delta_{14} & \delta_{15} & \delta_{16}\\ & E_{ij} & & \delta_{24} & \delta_{25} & \delta_{26}\\ & & & \delta_{34} & \delta_{35} & \delta_{36}\\ \hline 1-\delta_{14} & 1-\delta_{24} & 1-\delta_{34} & & & \\ 1-\delta_{15} & 1-\delta_{25} & 1-\delta_{35} & & E_{st} & \\ 1-\delta_{16} & 1-\delta_{26} & 1-\delta_{36} & & & \end{array}\right]\right\| : {\scriptstyle i,j,s,t\in\{1,2,3\} \atop \delta_{kl}\in\{0,1\}}\right\}\\
&\approx 1.9419>\sqrt{3}.
\end{aligned}
$$

Here $E_{ij}\in\mathbb{M}_3(\mathbb{R}_+)$ (resp. $E_{st}\in\mathbb{M}_3(\mathbb{R}_+)$) is the matrix with 1 in the (i,j) entry (resp. the (s,t) entry) and 0 elsewhere. The last (approximate) equality follows by checking $3^4\times 2^9=41472$ cases on a computer. □

Remark 3.3.4. *By further analyzing the case when G has an isolated vertex, one can show that $\beta_3(\mathbb{M}^0_6(\mathbb{R}_+))\lesssim 0.5577$. We suspect that $\beta_3(\mathbb{M}^0_6(\mathbb{R}_+))=\kappa\approx 0.5550$. Could it be that $\beta_3(\mathbb{M}^0_n(\mathbb{R}_+))=\kappa$ for all $n\geq 4$?*

4. Paving Upper-Triangular Matrices

This section establishes the fourth column of Table 1.4.1 and the first column of Table 1.4.3.

4.1. **Tools.**

Lemma 4.1.1. *For* $|a|, |b|, |c|, |d| \geq 1$,

$$\left\| \begin{bmatrix} a & b \\ 0 & c \\ 0 & d \end{bmatrix} \right\| \geq \sqrt{2+\sqrt{2}} \approx 1.8478.$$

This inequality is sharp.

Proof. Let $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ be such that $a = |a|e^{i\alpha}$, $b = |b|e^{i\beta}$, $c = |c|e^{i\gamma}$, and $d = |d|e^{i\delta}$. Then

$$\begin{bmatrix} a & b \\ 0 & c \\ 0 & d \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{i(\gamma-\beta)} & 0 \\ 0 & 0 & e^{i(\delta-\beta)} \end{bmatrix} \begin{bmatrix} |a| & |b| \\ 0 & |c| \\ 0 & |d| \end{bmatrix} \begin{bmatrix} e^{i\alpha} & 0 \\ 0 & e^{i\beta} \end{bmatrix}.$$

Thus,

$$\left\| \begin{bmatrix} a & b \\ 0 & c \\ 0 & d \end{bmatrix} \right\| = \left\| \begin{bmatrix} |a| & |b| \\ 0 & |c| \\ 0 & |d| \end{bmatrix} \right\| \overset{\text{Lemma 2.1.1}}{\geq} \left\| \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \right\| = \sqrt{2+\sqrt{2}}.$$

Letting $a = b = c = d = 1$ shows that the inequality is sharp. □

4.2. **Computation of 2-Paving Parameters.** The following proposition establishes the first entry of the first column of Table 1.4.3:

Proposition 4.2.1. $\beta_2(\mathbb{M}_3^0(\mathbb{C})_\triangle) = \beta_2(\mathbb{M}_3^0(\mathbb{R})_\triangle) = \beta_2(\mathbb{M}_3^0(\mathbb{R}_+)_\triangle) = \frac{2}{1+\sqrt{5}} \approx .6180$.

Proof. Let

$$A = \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} \in \mathbb{M}_3^0(\mathbb{C})_\triangle,$$

with $\alpha_2(A) = 1$. We aim to show that $\|A\| \geq \frac{2}{1+\sqrt{5}}$. Since $\alpha_2(A) = 1$, every $(1,2)$-paving of A has norm greater than or equal to 1, i.e., every 2-compression of A has norm greater than or equal to 1, i.e., $|a|, |b|, |c| \geq 1$. By Lemma 2.1.2, $\|A\| \geq \frac{1+\sqrt{5}}{2}$. Since the choice of A was arbitrary, $\beta_2(\mathbb{M}_3^0(\mathbb{C})_\triangle) \leq \frac{2}{1+\sqrt{5}}$. Since

$$\beta_2\left(\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right) = \frac{2}{1+\sqrt{5}},$$

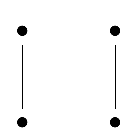
we have $\frac{2}{1+\sqrt{5}} \leq \beta_2(\mathbb{M}_3^0(\mathbb{R}_+)_\triangle) \leq \beta_2(\mathbb{M}_3^0(\mathbb{R})_\triangle) \leq \beta_2(\mathbb{M}_3^0(\mathbb{C})_\triangle) \leq \frac{2}{1+\sqrt{5}}$. □

The following proposition establishes the second entry in the first column of Table 1.4.3:

Proposition 4.2.2. $\beta_2(\mathbb{M}_4^0(\mathbb{C})_\triangle) = \beta_2(\mathbb{M}_4^0(\mathbb{R})_\triangle) = \frac{1}{\sqrt{2}} \approx 0.7071$.

Proof. Let $A \in \mathbb{M}_4^0(\mathbb{C})_\triangle$ with $\alpha_2(A) = 1$. We aim to show that $\|A\| \geq \sqrt{2}$. Associate a graph $G = (V, E)$ with A as follows: $V = \{1, 2, 3, 4\}$ and $(i, j) \in E$ if and only if $\|A_{\{i,j\}}\| < 1$. We may assume that G has the following properties:

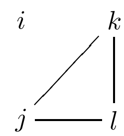
I. The graph



is not a subgraph of G. Otherwise, A has a $(2,2)$-paving of norm less than 1, contradicting $\alpha_2(A) = 1$.

II. By removing one vertex (and all associated edges) from G one cannot arrive at the "edgeless" graph $(\{i,j,k\},\emptyset)$. Otherwise, $\alpha_2(A_{\{i,j,k\}}) \geq 1$ (since every $(1,2)$-paving has norm greater than or equal to 1), which implies that $\|A_{\{i,j,k\}}\| \geq \frac{1+\sqrt{5}}{2}$ (by Proposition 4.2.1). Then $\|A\| \geq \|A_{\{i,j,k\}}\| \geq \frac{1+\sqrt{5}}{2} > \sqrt{2}$ (and we are done).

Checking [11, p. 8] we see that G must be



If $i = 1$ (resp. $i = 4$), then the first row (resp. fourth column) of A has three entries of modulus greater than or equal to 1, which implies $\|A\| \geq \sqrt{3} > \sqrt{2}$. Likewise, if $i = 2$ (resp. $i = 3$), then the second row (resp. third column) of A has two entries of modulus greater than or equal to 1, which implies $\|A\| \geq \sqrt{2}$. Since the choice of A was arbitrary, $\beta_2(\mathbb{M}_4^0(\mathbb{C})_\triangle) \leq \frac{1}{\sqrt{2}}$. Since

$$\beta_2\left(\begin{bmatrix} 0 & 0 & 1 & -\frac{1}{\sqrt{2}} \\ 0 & 0 & 1 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right) = \frac{1}{\sqrt{2}},$$

we have $\frac{1}{\sqrt{2}} \leq \beta_2(\mathbb{M}_4^0(\mathbb{R})_\triangle) \leq \beta_2(\mathbb{M}_4^0(\mathbb{C})_\triangle) \leq \frac{1}{\sqrt{2}}$. □

4.3. **Computation of 3-Paving Parameters.** The following proposition establishes the first entry of the fourth column of Table 1.4.1:

Proposition 4.3.1. $\beta_3(\mathbb{M}_4^0(\mathbb{C})_\triangle) = \beta_3(\mathbb{M}_4^0(\mathbb{R})_\triangle) = \frac{1}{\sqrt{2+\sqrt{2}}} \approx .5412$.

Proof. Let $A \in \mathbb{M}_4^0(\mathbb{C})_\triangle$ with $\alpha_3(A) = 1$. We aim to show that $\|A\| \geq \sqrt{2+\sqrt{2}}$. Since $\alpha_3(A) = 1$, every $(1,1,2)$-paving of A has norm greater than or equal to 1, i.e., every 2-compression of A has norm greater than or equal to 1. Hence,

$$A = \begin{bmatrix} 0 & \blacksquare & \blacksquare & \blacksquare \\ 0 & 0 & \blacksquare & \blacksquare \\ 0 & 0 & 0 & \blacksquare \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where "$\blacksquare$" indicates an entry of modulus greater than or equal to 1. By Lemma 4.1.1, $\|A\| \geq \|A_{\{1,2,3\},\{2,4\}}\| \geq \sqrt{2+\sqrt{2}}$. Since the choice of A was arbitrary, $\beta_3(\mathbb{M}_4^0(\mathbb{C})_\triangle) \leq \frac{1}{\sqrt{2+\sqrt{2}}}$. Since

$$\beta_3\left(\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & -\sqrt{2} & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right) = \frac{1}{\sqrt{2+\sqrt{2}}},$$

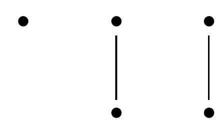
we have $\frac{1}{\sqrt{2+\sqrt{2}}} \leq \beta_3(\mathbb{M}_4^0(\mathbb{R})_\triangle) \leq \beta_3(\mathbb{M}_4^0(\mathbb{C})_\triangle) \leq \frac{1}{\sqrt{2+\sqrt{2}}}$. □

The following proposition establishes the second entry in the fourth column of Table 1.4.1:

Proposition 4.3.2. $\beta_3(\mathbb{M}_5^0(\mathbb{C})_\triangle) \leq \frac{1}{\sqrt{3}} \approx .5774$.

Proof. Let $A \in \mathbb{M}_5^0(\mathbb{C})_\triangle$ with $\alpha_3(A) = 1$. We aim to show that $\|A\| \geq \sqrt{3}$. Associate a graph $G = (V, E)$ with A as follows: $V = \{1, 2, ..., 5\}$ and $(i, j) \in E$ if and only if $\|A_{\{i,j\}}\| < 1$. We may assume that G has the following properties:

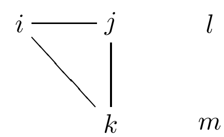
I. The graph



is not a subgraph of G. Otherwise A has a $(1, 2, 2)$-paving of norm less than 1, contradicting $\alpha_3(A) = 1$.

II. By removing one vertex (and all associated edges) from G one cannot arrive at the "edgeless" graph $(\{i, j, k, l\}, \emptyset)$. Otherwise $\alpha_3(A_{\{i,j,k,l\}}) \geq 1$ (since every $(1, 1, 2)$-paving has norm greater than or equal to 1), which implies $\|A_{\{i,j,k,l\}}\| \geq \sqrt{2 + \sqrt{2}}$ (by Proposition 4.3.1). Then $\|A\| \geq \|A_{\{i,j,k,l\}}\| \geq \sqrt{2 + \sqrt{2}} > \sqrt{3}$.

Checking [11, p. 8] we see that G must be



We may assume $l < m$.

Case 1: If $l = 1$ (resp. $m = 5$), then the first row (resp. fifth column) of A has four entries of modulus greater than or equal to 1, which implies $\|A\| \geq 2 > \sqrt{3}$.

Case 2: If $l = 2$ and $m = 4$, then

$$A = \begin{bmatrix} 0 & \blacksquare & \bullet & \blacksquare & \bullet \\ 0 & 0 & \blacksquare & \blacksquare & \blacksquare \\ 0 & 0 & 0 & \blacksquare & \bullet \\ 0 & 0 & 0 & 0 & \blacksquare \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where "$\blacksquare$" indicates an entry of modulus greater than or equal to 1 and "$\bullet$" indicates an entry of modulus less than 1. By Lemma 4.1.1, $\|A\| \geq \|A_{\{1,2,3\},\{2,4\}}\| \geq \sqrt{2 + \sqrt{2}} > \sqrt{3}$.

Case 3: If $l = 2$ and $m = 3$, then

$$A = \begin{bmatrix} 0 & \blacksquare & \blacksquare & \bullet & \bullet \\ 0 & 0 & \blacksquare & \blacksquare & \blacksquare \\ 0 & 0 & 0 & \blacksquare & \blacksquare \\ 0 & 0 & 0 & 0 & \bullet \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

Since the second row of A contains three entries of modulus greater than or equal to 1, $\|A\| \geq \sqrt{3}$.

Case 4: Finally, if $l = 3$ and $m = 4$, then (arguing as in Case 3), the fourth column of A contains three entries of modulus greater than or equal to 1, which implies $\|A\| \geq \sqrt{3}$.

Since the choice of A was arbitrary, $\beta_3(\mathbb{M}_5^0(\mathbb{C})_\triangle) \leq \frac{1}{\sqrt{3}}$. □

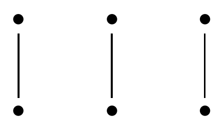
Remark 4.3.3. *A more careful analysis of Cases 3 and 4 above should yield a better bound than* $\|A\| \geq \sqrt{3}$*, which would improve the overall result.*

The following proposition establishes the third entry in the fourth column of Table 1.4.1:

Proposition 4.3.4. $\beta_3(\mathbb{M}_6^0(\mathbb{C})_\triangle) \leq \frac{1}{\sqrt{3}} \approx .5774.$

Proof. Let $A \in \mathbb{M}_6^0(\mathbb{C})_\triangle$ with $\alpha_3(A) = 1$. We aim to show that $\|A\| \geq \sqrt{3}$. Associate a graph $G = (V, E)$ with A as follows: $V = \{1, 2, ..., 6\}$ and $(i, j) \in E$ if and only if $\|A_{\{i,j\}}\| < 1$. We may assume that G has the following properties:

I. The graph

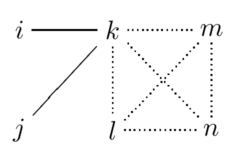


is not a subgraph of G. Otherwise A has a $(2, 2, 2)$-paving of norm less than 1, contradicting $\alpha_3(A) = 1$.

II. G has no isolated vertices. Otherwise, if vertex i is isolated, then either the ith row of A or the ith column of A has three entries of modulus greater than or equal to 1, which implies $\|A\| \geq \sqrt{3}$ (and we are done).

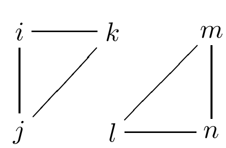
III. By removing two vertices (and all associated edges) from G one cannot arrive at the "edgeless" graph $(\{i, j, k, l\}, \emptyset)$. Otherwise, $\alpha_3(A_{\{i,j,k,l\}}) \geq 1$ (since every $(1, 1, 2)$-paving has norm greater than or equal to 1), which implies $\|A_{\{i,j,k,l\}}\| \geq \sqrt{2 + \sqrt{2}}$ (by Proposition 4.3.1). Then $\|A\| \geq \|A_{\{i,j,k,l\}}\| \geq \sqrt{2 + \sqrt{2}} > \sqrt{3}$ (and we are done).

IV. G does not equal the graph



where a dotted line indicates an edge which may or may not be present. Otherwise, $\alpha_3(A_{\{i,j,l,m,n\}}) \geq 1$ (see the proof of Proposition 3.3.3), which implies $\|A_{\{i,j,l,m,n\}}\| \geq \sqrt{3}$ (by Proposition 4.3.2). Then $\|A\| \geq \|A_{\{i,j,l,m,n\}}\| \geq \sqrt{3}$ (and we are done).

Checking [11, pp. 9-11], we see that G must be



It follows that the first row (and the sixth column) of A contains three entries of modulus greater than or equal to 1, which implies $\|A\| \geq \sqrt{3}$. Since the choice of A was arbitrary, $\beta_3(\mathbb{M}_6^0(\mathbb{C})_\triangle) \leq \frac{1}{\sqrt{3}}$. $\square$

The following proposition establishes the fifth entry in the fourth column of Table 1.4.1:

Proposition 4.3.5. $\beta_3(\mathbb{M}_7^0(\mathbb{C})_\triangle) \leq \sqrt{\frac{6}{7}} \approx .9258$.

Proof. Let $A \in \mathbb{M}_7^0(\mathbb{C})_\triangle$ with $\alpha_3(A) = 1$. We aim to show that $\|A\| \geq \sqrt{\frac{7}{6}} \approx 1.0801$.

Case 1: Suppose every 3-compression of A has norm greater than or equal to 1. Since there are $\binom{7}{3} = 35$ of these,

$$35 \leq \sum_{\text{card}(\sigma)=3} \|A_\sigma\|^2 \leq \sum_{\text{card}(\sigma)=3} \|A_\sigma\|_{HS}^2,$$

where $\|\cdot\|_{HS}$ stands for the Hilbert-Schmidt (or Frobenius) norm of a matrix. Now for each $i \neq j$, there exist five $\sigma \subseteq \{1, 2, ..., 7\}$ such that $\text{card}(\sigma) = 3$ and $\{i, j\} \subseteq \sigma$. It follows that

$$\sum_{\text{card}(\sigma)=3} \|A_\sigma\|_{HS}^2 = 5\|A\|_{HS}^2.$$

Also, since $\text{rank}(A) \leq 6$, $\|A\|_{HS}^2 \leq 6\|A\|^2$. Hence

$$35 \leq \sum_{\text{card}(\sigma)=3} \|A_\sigma\|_{HS}^2 = 5\|A\|_{HS}^2 \leq 30\|A\|^2,$$

which implies $\|A\| \geq \sqrt{\frac{7}{6}}$.

Case 2: Suppose $\|A_\sigma\| < 1$ for some $\sigma \subseteq \{1, 2, ..., 7\}$ with $\text{card}(\sigma) = 3$. Since $\alpha_3(A) = 1$, $\alpha_2(A_{\sigma^c}) \geq 1$. By Proposition 4.2.2, $\|A_{\sigma^c}\| \geq \sqrt{2}$. Thus, $\|A\| \geq \|A_{\sigma^c}\| \geq \sqrt{2} > \sqrt{\frac{7}{6}}$.

Since the choice of A was arbitrary, $\beta_3(\mathbb{M}_7^0(\mathbb{C})_\triangle) \leq \sqrt{\frac{6}{7}}$. $\square$

5. Paving Circulant Matrices

In this section we establish the fifth column of Table 1.4.1 and the second column of Table 1.4.3.

5.1. **A Few Words about Circulants.** A *circulant* is a matrix which is constant on "wrap-around diagonals" (cf. [6]). For example, the generic 6×6 circulant is

$$A = \begin{bmatrix} f & a & b & c & d & e \\ e & f & a & b & c & d \\ d & e & f & a & b & c \\ c & d & e & f & a & b \\ b & c & d & e & f & a \\ a & b & c & d & e & f \end{bmatrix},$$

where $a, b, c, d, e, f \in \mathbb{C}$. Clearly the adjoint of a circulant is also a circulant. Not so apparent initially is that any two circulants (of the same size) commute, and that their product is again a circulant. This follows from the fact that every circulant is a polynomial in the *shift matrix* (of the appropriate size), and conversely. Indeed, if

$$S_6 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

then A above equals $aS_6^5 + bS_6^4 + cS_6^3 + dS_6^2 + eS_6 + fI_6$, where I_6 is the 6×6 identity matrix. For the converse, note that $S_6^6 = I_6$, so that an arbitrary polynomial in S_6 may be written as a fifth-degree polynomial in S_6, i.e., a 6×6 circulant. From the preceding discussion we deduce that every circulant is normal. Since $\sigma(S_6) = \{z \in \mathbb{C} : z^6 = 1\}$, the Spectral Mapping Theorem implies that

$$\sigma(A) = \{az^5 + bz^4 + cz^3 + dz^2 + ez + f : z^6 = 1\}.$$

Since the operator norm of a normal matrix equals its spectral radius, we have the formula

$$\|A\| = \max\{|az^5 + bz^4 + cz^3 + dz^2 + ez + f| : z^6 = 1\}. \tag{1}$$

If one is searching for "bad" pavers (matrices for which the normalized paving parameter is near 1), it is natural to consider circulants. Obviously, $\mathbb{M}_n(\mathbb{C})_{\circlearrowleft}$ is a much smaller search space than $\mathbb{M}_n(\mathbb{C})$, n-dimensional instead of n^2-dimensional, which greatly speeds up the search. But there is also a heuristic argument as to why circulants should be bad pavers—the compressions of a circulant are rarely circulant. Thus at the "macro" level, a circulant has a nice structure which tends to produce a small operator norm in comparison with the size of the entries, whereas at the "micro" level this structure (and the corresponding operator norm benefits) disappears. On the other hand, there a couple of drawbacks to the circulant class, at least from the point of view of paving. First, the sequence $\{\beta_k(\mathbb{M}_n^0(\mathbb{C})_{\circlearrowleft}) : n \in \mathbb{N}\}$ need not be monotone increasing. Second, it is not known whether the Paving Problem for circulants is equivalent to KS. In our experience, the positives outweigh the negatives, and computing paving parameters for circulants has proven very fruitful.

5.2. **Tools.**

Lemma 5.2.1 (operator norm of $n \times 2$ matrix)**.** *For $\vec{x}, \vec{y} \in \mathbb{C}^n$,*

$$\left\| \begin{bmatrix} \vec{x} & \vec{y} \end{bmatrix} \right\| = \sqrt{\frac{\|\vec{x}\|^2 + \|\vec{y}\|^2 + \sqrt{(\|\vec{x}\|^2 - \|\vec{y}\|^2)^2 + 4|\langle \vec{x}, \vec{y}\rangle|^2}}{2}}.$$

Proof. By the C^*-identity,

$$\left\| \begin{bmatrix} \vec{x} & \vec{y} \end{bmatrix} \right\|^2 = \left\| \begin{bmatrix} \vec{x}^* \\ \vec{y}^* \end{bmatrix} \begin{bmatrix} \vec{x} & \vec{y} \end{bmatrix} \right\| = \left\| \begin{bmatrix} \vec{x}^*\vec{x} & \vec{x}^*\vec{y} \\ \vec{y}^*\vec{x} & \vec{y}^*\vec{y} \end{bmatrix} \right\| = \left\| \begin{bmatrix} \|\vec{x}\|^2 & \langle \vec{y}, \vec{x}\rangle \\ \langle \vec{x}, \vec{y}\rangle & \|\vec{y}\|^2 \end{bmatrix} \right\|.$$

A straightforward calculation shows that

$$\sigma\left(\begin{bmatrix} \|\vec{x}\|^2 & \langle \vec{y}, \vec{x}\rangle \\ \langle \vec{x}, \vec{y}\rangle & \|\vec{y}\|^2 \end{bmatrix} \right) = \left\{ \frac{\|\vec{x}\|^2 + \|\vec{y}\|^2 \pm \sqrt{(\|\vec{x}\|^2 - \|\vec{y}\|^2)^2 + 4|\langle \vec{x}, \vec{y}\rangle|^2}}{2} \right\},$$

and the result follows. □

Lemma 5.2.2. *For*

$$A = \begin{bmatrix} 0 & a & b & c \\ c & 0 & a & b \\ b & c & 0 & a \\ a & b & c & 0 \end{bmatrix} \in \mathbb{M}_4^0(\mathbb{C})_{\circlearrowleft},$$

we have

$$\alpha_3(A) = \alpha_2(A) = \min\{\max\{|a|, |c|\}, |b|\}.$$

Proof. We have that

$$\begin{aligned} \alpha_3(A) &= \min\{\|A^\pi\| : \pi \in \Pi_3^4\} \\ &= \min\{\|A_\sigma\| : \mathrm{card}(\sigma) = 2\} \\ &= \min\{\max\{|a|, |c|\}, |b|\}. \end{aligned}$$

¿From general considerations, $\alpha_3(A) \leq \alpha_2(A)$. Considering the $\{\{1,2\},\{3,4\}\}$-paving, we see that $\alpha_2(A) \leq \max\{|a|, |c|\}$. Considering the $\{\{1,3\},\{2,4\}\}$-paving, we see that $\alpha_2(A) \leq |b|$. Thus,

$$\min\{\max\{|a|, |c|\}, |b|\} = \alpha_3(A) \leq \alpha_2(A) \leq \min\{\max\{|a|, |c|\}, |b|\}.$$

□

Lemma 5.2.3 (norm of permutation matrix compression)**.** *Let $\phi : \{1, 2, ..., n\} \to \{1, 2, ..., n\}$ be a permutation, $U_\phi \in \mathbb{M}_n(\mathbb{R}_+)$ be the corresponding permutation matrix, and $\sigma \subseteq \{1, 2, ..., n\}$. Then*

$$\|(U_\phi)_\sigma\| = \begin{cases} 1, & \phi(\sigma) \cap \sigma \neq \emptyset \\ 0, & \phi(\sigma) \cap \sigma = \emptyset \end{cases}.$$

Proof. Let $j \in \phi(\sigma) \cap \sigma$. Then $j \in \sigma$ and there exists an $i \in \sigma$ such that $\phi(i) = j$. Thus

$$(U_\phi)_\sigma \vec{e}_i = P_\sigma U_\phi P_\sigma \vec{e}_i = P_\sigma U_\phi \vec{e}_i = P_\sigma \vec{e}_{\phi(i)} = P_\sigma \vec{e}_j = \vec{e}_j,$$

which implies that

$$1 \leq \|(U_\phi)_\sigma\| \leq \|U_\phi\| = 1.$$

Now suppose $\phi(\sigma) \cap \sigma = \emptyset$. If $i \in \sigma$, then $\phi(i) \notin \sigma$, which implies that

$$(U_\phi)_\sigma \vec{e}_i = P_\sigma U_\phi P_\sigma \vec{e}_i = P_\sigma U_\phi \vec{e}_i = P_\sigma \vec{e}_{\phi(i)} = \vec{0}.$$

If, on the other hand, $i \notin \sigma$, then

$$(U_\phi)_\sigma \vec{e}_i = P_\sigma U_\phi P_\sigma \vec{e}_i = \vec{0}.$$

Thus, $(U_\phi)_\sigma = 0$. □

The following result refines [7, Proposition 3.1].

Theorem 5.2.4. *Let $\phi : \{1, 2, ..., n\} \to \{1, 2, ..., n\}$ be a permutation and $U_\phi \in \mathbb{M}_n(\mathbb{R}_+)$ be the corresponding permutation matrix. If the cycle decomposition of ϕ contains an odd cycle, then $\beta_2(U_\phi) = 1$. Otherwise, if the cycle decomposition of ϕ contains only even cycles, then $\beta_2(U_\phi) = 0$.*

Proof. Assume that the odd cycle $(i_1 i_2 ... i_{2k+1})$ is in the cycle decomposition of ϕ. Let $\pi = \{\sigma_1, \sigma_2\} \in \Pi_2^n$ and define $\sigma_1' = \sigma_1 \cap \{i_1, i_2, ..., i_{2k+1}\}$, $\sigma_2' = \sigma_2 \cap \{i_1, i_2, ..., i_{2k+1}\}$. Then $\{\sigma_1', \sigma_2'\}$ is a 2-partition of $\{i_1, i_2, ..., i_{2k+1}\}$. Without loss of generality, $\text{card}(\sigma_1') \geq k+1$. Since $\phi(\sigma_1') \subseteq \{i_1, i_2, ..., i_{2k+1}\}$, it must be that $\phi(\sigma_1') \cap \sigma_1' \neq \emptyset$, which implies $\phi(\sigma_1) \cap \sigma_1 \neq \emptyset$. By Lemma 5.2.3, $\|(U_\phi)_{\sigma_1}\| = 1$, which implies $\|(U_\phi)^\pi\| = 1$. Since the choice of π was arbitrary, $\alpha_2(U_\phi) = 1$. Since $\|U_\phi\| = 1$, $\beta_2(U_\phi) = 1$. Now assume that the cycle decomposition of ϕ contains only even cycles. Let $\pi = \{\sigma_1, \sigma_2\} \in \Pi_2^n$ be such that for every even cycle $(i_1 i_2 ... i_{2k})$ in the cycle decomposition of ϕ, $\{i_1, i_3, ..., i_{2k-1}\} \subseteq \sigma_1$ and $\{i_2, i_4, ..., i_{2k}\} \subseteq \sigma_2$. Then $\phi(\sigma_1) \cap \sigma_1 = \emptyset$ and $\phi(\sigma_2) \cap \sigma_2 = \emptyset$. By Lemma 5.2.3, $\|(U_\phi)_{\sigma_1}\| = 0$ and $\|(U_\phi)_{\sigma_2}\| = 0$, which implies $\|(U_\phi)^\pi\| = 0$. Then $\alpha_2(U_\phi) = 0$, which implies $\beta_2(U_\phi) = 0$. □

Remark 5.2.5. *Although it is not important to our development, we take the opportunity to point out that* $\beta_3(U_\phi)$ *equals* 1 *if* ϕ *has any fixed points, and* 0 *otherwise.*

5.3. **Computation of 2-Paving Parameters.** The following proposition establishes all but the second entry of the second column of Table 1.4.3:

Proposition 5.3.1. *Let* $n \geq 3$, *with* $n \neq 2^k$. *Then*

$$\beta_2(\mathbb{M}_n^0(\mathbb{C})_\circlearrowleft) = \beta_2(\mathbb{M}_n^0(\mathbb{R})_\circlearrowleft) = \beta_2(\mathbb{M}_n^0(\mathbb{R}_+)_\circlearrowleft) = 1.$$

Proof. Let $S_n \in \mathbb{M}_n^0(\mathbb{R}_+)_\circlearrowleft$ be the $n \times n$ shift matrix. Then in the notation of Theorem 5.2.4, $S_n = U_\phi$, where $\phi = (12...n)$. Since $n \geq 3$ and $n \neq 2^k$, $n = rs$, where $s \geq 3$ is odd. Since $1 \leq r < n$, $S_n^r \in \mathbb{M}_n^0(\mathbb{R}_+)_\circlearrowleft$. It is easy to see that $S_n^r = U_{\phi^r}$, where ϕ^r denotes ϕ composed with itself $r-1$ times. Since the cycle decomposition of ϕ^r consists of r cycles of length s, $\beta_2(S_n^r) = 1$ by Theorem 5.2.4. It follows that

$$1 \leq \beta_2(\mathbb{M}_n^0(\mathbb{R}_+)_\circlearrowleft) \leq \beta_2(\mathbb{M}_n^0(\mathbb{R})_\circlearrowleft) \leq \beta_2(\mathbb{M}_n^0(\mathbb{C})_\circlearrowleft) \leq 1.$$

□

The following proposition establishes the second entry of the second column of Table 1.4.3. It shows that Proposition 5.3.1 cannot be extended to the case $n = 2^k$.

Proposition 5.3.2. $\beta_2(\mathbb{M}_4^0(\mathbb{C})_\circlearrowleft) \leq \frac{2}{\sqrt{11}} \approx .6030.$

Proof. Let

$$A = \begin{bmatrix} 0 & a & b & c \\ c & 0 & a & b \\ b & c & 0 & a \\ a & b & c & 0 \end{bmatrix} \in \mathbb{M}_4^0(\mathbb{C})_\circlearrowleft,$$

with $\alpha_2(A) = 1$. We aim to show that $\|A\| \geq \frac{\sqrt{11}}{2} \approx 1.6583$. By Lemma 5.2.2, $\max\{|a|, |c|\} \geq 1$ and $|b| \geq 1$. Taking the adjoint of A, if necessary, we may assume that $|a| \geq |c|$, which implies $|a| \geq 1$. To prove that $\|A\| \geq \frac{\sqrt{11}}{2}$, it suffices to show that $\|B\| \geq \frac{\sqrt{11}}{2}$, where

$$B = A_{\{1,2,3,4\},\{1,2\}} = \begin{bmatrix} 0 & a \\ c & 0 \\ b & c \\ a & b \end{bmatrix}.$$

Now let $\alpha, \beta \in \mathbb{R}$ be such that $a = |a|e^{i\alpha}$ and $b = |b|e^{i\beta}$. Since

$$\begin{bmatrix} e^{i(\beta-\alpha)} & 0 & 0 & 0 \\ 0 & e^{i(2\alpha-2\beta)} & 0 & 0 \\ 0 & 0 & e^{i(\alpha-\beta)} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & a \\ c & 0 \\ b & c \\ a & b \end{bmatrix} \begin{bmatrix} e^{-i\alpha} & 0 \\ 0 & e^{-i\beta} \end{bmatrix} = \begin{bmatrix} 0 & |a| \\ ce^{i(\alpha-2\beta)} & 0 \\ |b| & ce^{i(\alpha-2\beta)} \\ |a| & |b| \end{bmatrix},$$

we may assume that $a, b \geq 0$. If $c = x + iy$, where $x, y \in \mathbb{R}$, then by Lemma 5.2.1,

$$\begin{aligned} \|B\|^2 &= \frac{2|a|^2 + 2|b|^2 + 2|c|^2 + \sqrt{0^2 + 4|a\overline{b} + b\overline{c}|^2}}{2} \\ &= a^2 + b^2 + x^2 + y^2 + b\sqrt{(a+x)^2 + y^2} \\ &\geq 2 + x^2 + y^2 + \sqrt{(a+x)^2 + y^2} \\ &\geq 2 + x^2 + |a + x|. \end{aligned}$$

Since $|c| \leq |a|$, it must be that $a + x \geq 0$. Thus,

$$\|B\|^2 \geq 2 + x^2 + a + x \geq x^2 + x + 3 \geq \frac{11}{4}.$$

Since the choice of A was arbitrary, $\beta_2(\mathbb{M}_4^0(\mathbb{C})_\circlearrowleft) \leq \frac{2}{\sqrt{11}}$. □

We suspect that $\beta_2(\mathbb{M}_4^0(\mathbb{C})_\circlearrowleft) = \frac{3}{5}$. At least we have the following result.

Proposition 5.3.3. $\beta_2(\mathbb{M}_4^0(\mathbb{R})_\circlearrowleft) = \frac{3}{5} = .6000$.

Proof. Let

$$A = \begin{bmatrix} 0 & a & b & c \\ c & 0 & a & b \\ b & c & 0 & a \\ a & b & c & 0 \end{bmatrix} \in \mathbb{M}_4^0(\mathbb{R})_\circlearrowleft,$$

with $\alpha_2(A) = 1$. We aim to show that $\|A\| \geq \frac{5}{3} \approx 1.6667$. Arguing as in the proof of Proposition 5.3.2, we may assume that $|a|, |b| \geq 1$ and $|a| \geq |c|$. Replacing A by $-A$, if necessary, we may assume that $a \geq 0$. Then by Equation (1),

$$\begin{aligned} \|A\| &= \max\{|az^2 + bz + c| : z = \pm 1, \pm i\} \\ &= \max\{|a + c| + |b|, \sqrt{(a-c)^2 + b^2}\} \\ &\geq \max\{a + c + 1, \sqrt{(a-c)^2 + 1}\}. \end{aligned}$$

Examining the graphs of $f(c) := a + c + 1$ and $g(c) := \sqrt{(a-c)^2 + 1}$, we see that the minimum value of $\max\{f(c), g(c)\}$ on the interval $-a \leq c \leq a$ equals $\frac{2a^2+2a+1}{2a+1}$ (at $c = -\frac{a}{2a+1}$). Using Calculus, the minimum value of $h(a) := \frac{2a^2+2a+1}{2a+1}$ on the interval $a \geq 1$ is $\frac{5}{3}$ (at $a = 1$). Thus, $\|A\| \geq \frac{5}{3}$. Since the choice of A was arbitrary, $\beta_2(\mathbb{M}_4^0(\mathbb{R})_\circlearrowleft) \leq \frac{3}{5}$. Since

$$\beta_2\left(\begin{bmatrix} 0 & 1 & 1 & -1/3 \\ -1/3 & 0 & 1 & 1 \\ 1 & -1/3 & 0 & 1 \\ 1 & 1 & -1/3 & 0 \end{bmatrix}\right) = \frac{3}{5},$$

we have $\beta_2(\mathbb{M}_4^0(\mathbb{R})_\circlearrowleft) = \frac{3}{5}$. □

Corollary 5.3.4. $\beta_2(\mathbb{M}_4^0(\mathbb{C})_\circlearrowleft) \in \left[\frac{3}{5}, \frac{2}{\sqrt{11}}\right] \approx [.6000, .6030]$.

Remark 5.3.5. *The second column of Table 1.4.3 suggests that* $\lim_{k\to\infty} \beta_2(\mathbb{M}^0_{2^k}(\mathbb{C})_\circlearrowleft) = 1$. *Is that really the case? If so, does* $\beta_2(\mathbb{M}^0_{2^k}(\mathbb{C})_\circlearrowleft) = 1$ *for* k *sufficiently large?*

5.4. **Computation of 3-Paving Parameters.** This proposition establishes the first entry of the last column of Table 1.4.1:

Proposition 5.4.1. $\frac{3}{5} = \beta_3(\mathbb{M}^0_4(\mathbb{R})_\circlearrowleft) \leq \beta_3(\mathbb{M}^0_4(\mathbb{C})_\circlearrowleft) \leq \frac{2}{\sqrt{11}}$.

Proof. Proposition 5.3.3, Proposition 5.3.2, and Lemma 5.2.2. □

This proposition establishes the third entry of the last column of Table 1.4.1 (the second entry is a consequence of Proposition 2.2.2):

Proposition 5.4.2. $\beta_3(\mathbb{M}^0_6(\mathbb{C})_\circlearrowleft) \leq \sqrt{\frac{2}{5}} \approx .6325$.

Proof. Let

$$A = \begin{bmatrix} 0 & a & b & c & d & e \\ e & 0 & a & b & c & d \\ d & e & 0 & a & b & c \\ c & d & e & 0 & a & b \\ b & c & d & e & 0 & a \\ a & b & c & d & e & 0 \end{bmatrix} \in \mathbb{M}^0_6(\mathbb{C})_\circlearrowleft,$$

with $\alpha_3(A) = 1$. We aim to show that $\|A\| \geq \sqrt{\frac{5}{2}} \approx 1.5811$. Since the $\{\{1,2\},\{3,4\},\{5,6\}\}$-paving of A has norm greater than or equal to 1, either $|a| \geq 1$ or $|e| \geq 1$. Likewise, since the $\{\{1,4\},\{2,5\},\{3,6\}\}$-paving of A has norm greater than or equal to 1, $|c| \geq 1$. Finally, since the $\{\{1,3,5\},\{2,4\},\{6\}\}$-paving of A has norm greater than or equal to 1,

$$\|A_{\{1,3,5\}}\| = \left\| \begin{bmatrix} 0 & b & d \\ d & 0 & b \\ b & d & 0 \end{bmatrix} \right\| \geq 1.$$

By Equation (1) applied to $A_{\{1,3,5\}}$, which is circulant, $|b| + |d| \geq 1$. Thus,

$$|b|^2 + |d|^2 \geq \frac{(|b| + |d|)^2}{2} \geq \frac{1}{2}.$$

Hence,

$$\|A\| \geq \sqrt{|a|^2 + |b|^2 + |c|^2 + |d|^2 + |e|^2} \geq \sqrt{\frac{5}{2}}.$$

Since the choice of A was arbitrary, $\beta_3(\mathbb{M}^0_6(\mathbb{C})_\circlearrowleft) \leq \sqrt{\frac{2}{5}}$. □

Remark 5.4.3. *Almost certainly a more painstaking analysis would lead to improvements in Proposition 5.4.2.*

6. Paving Self-Adjoint Matrices

This section establishes the first column of Table 1.4.2 and the third column of Table 1.4.3.

6.1. **Tools.**

Lemma 6.1.1. *For* $A = \begin{bmatrix} a_{ij} \end{bmatrix} \in \mathbb{M}_n(\mathbb{C})$ *and* $|A| = \begin{bmatrix} |a_{ij}| \end{bmatrix} \in \mathbb{M}_n(\mathbb{R}_+)$, $\|A\| \leq \||A|\|$.

Proof. Let $\vec{x} \in \mathbb{C}^n$. Then

$$\begin{aligned} \|A\vec{x}\|^2 &= \sum_{i=1}^n |(A\vec{x})_i|^2 = \sum_{i=1}^n \left| \sum_{j=1}^n a_{ij} x_j \right|^2 \leq \sum_{i=1}^n \left(\sum_{j=1}^n |a_{ij}||x_j| \right)^2 \\ &= \sum_{i=1}^n (|A||\vec{x}|)_i^2 = \||A||\vec{x}|\|^2 \leq \||A|\|^2 \||\vec{x}|\|^2 = \||A|\|^2 \|\vec{x}\|^2. \end{aligned}$$

Since the choice of $\vec{x}$ was arbitrary, $\|A\| \leq \||A|\|$. □

Theorem 6.1.2. *Let* $A \in \mathbb{M}_n(\mathbb{C})_{sa}$. *Then* $\mathrm{Tr}(A) = 0$ *if and only if there exists a unitary* $U \in \mathbb{M}_n(\mathbb{C})$ *such that* $U^*AU \in \mathbb{M}_n^0(\mathbb{C})_{sa}$.

Proof. ($\Rightarrow$) Suppose $\mathrm{Tr}(A) = 0$. We aim to produce an orthonormal basis $\vec{u}_1, \vec{u}_2, ..., \vec{u}_n$ of $\mathbb{C}^n$ such that $\langle A\vec{u}_i, \vec{u}_i \rangle = 0$ for all $1 \leq i \leq n$. Then $U := \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \dots & \vec{u}_n \end{bmatrix} \in \mathbb{M}_n(\mathbb{C})$ will be a unitary such that $U^*AU \in \mathbb{M}_n^0(\mathbb{C})_{sa}$. Assume that $\vec{u}_1, \vec{u}_2, ..., \vec{u}_k$ have already been constructed. Let $V = \{\vec{u}_1, \vec{u}_2, ..., \vec{u}_k\}^\perp$ and $P_V \in \mathbb{M}_n(\mathbb{C})$ be the orthogonal projection onto V. Then $P_V A|_V : V \to V$ is self-adjoint and so there exists an orthonormal basis $\vec{v}_{k+1}, \vec{v}_{k+2}, ..., \vec{v}_n$ of V consisting of eigenvectors of $P_V A|_V$. It follows that for all $k+1 \leq i, j \leq n$,

$$\langle A\vec{v}_i, \vec{v}_j \rangle = \langle P_V A|_V \vec{v}_i, \vec{v}_j \rangle = \langle \lambda_i \vec{v}_i, \vec{v}_j \rangle = \begin{cases} \lambda_i, & i = j \\ 0, & i \neq j \end{cases},$$

where λ_i is the eigenvalue of $P_V A|_V$ corresponding to $\vec{v}_i$. Define

$$\vec{u}_{k+1} = \frac{1}{\sqrt{n-k}} \sum_{i=k+1}^n \vec{v}_i.$$

Then $\vec{u}_{k+1}$ is a unit vector in V and

$$\langle A\vec{u}_{k+1}, \vec{u}_{k+1} \rangle = \frac{1}{n-k} \sum_{i,j=k+1}^n \langle A\vec{v}_i, \vec{v}_j \rangle = \frac{1}{n-k} \sum_{i=k+1}^n \lambda_i.$$

Since $\vec{u}_1, \vec{u}_2, ..., \vec{u}_k, \vec{v}_{k+1}, \vec{v}_{k+2}, ..., \vec{v}_n$ form an orthonormal basis of $\mathbb{C}^n$,

$$0 = \mathrm{Tr}(A) = \sum_{i=1}^k \langle A\vec{u}_i, \vec{u}_i \rangle + \sum_{i=k+1}^n \langle A\vec{v}_i, \vec{v}_i \rangle = 0 + \sum_{i=k+1}^n \lambda_i.$$

Thus, $\langle A\vec{u}_{k+1}, \vec{u}_{k+1} \rangle = 0$. ($\Leftarrow$) Conversely, suppose there exists a unitary $U \in \mathbb{M}_n(\mathbb{C})$ such that $B = U^*AU \in \mathbb{M}_n^0(\mathbb{C})_{sa}$. Then

$$\mathrm{Tr}(A) = \mathrm{Tr}(UBU^*) = \mathrm{Tr}(B) = 0.$$

□

Lemma 6.1.3. *For* $A \in \mathbb{M}_n^0(\mathbb{C})_{sa}$,

$$\|A\|^2 \leq \frac{n-1}{n} \|A\|_{HS}^2.$$

This inequality is sharp.

Proof. Let $-\|A\| \le \lambda_1 \le \lambda_2 \le ... \le \lambda_n \le \|A\|$ be the eigenvalues of A. Since $\mathrm{Tr}(A) = 0$, $\lambda_1 + \lambda_2 + ... + \lambda_n = 0$. Replacing A by $-A$, if necessary, we may assume that $\lambda_n = \|A\|$. By the Cauchy-Schwarz inequality,

$$\|A\| = \lambda_n = -\lambda_1 - \lambda_2 - ... - \lambda_{n-1} \le \sqrt{n-1}(\lambda_1^2 + \lambda_2^2 + ... + \lambda_{n-1}^2)^{1/2}.$$

Thus,

$$\|A\|^2 \le (n-1)(\|A\|_{HS}^2 - \|A\|^2),$$

and the inequality follows. Now let

$$D = \mathrm{diag}(n-1, \overbrace{-1, -1, ..., -1}^{n-1}) \in \mathbb{M}_n(\mathbb{C})_{sa}.$$

By Theorem 6.1.2, there exists a unitary $U \in \mathbb{M}_n(\mathbb{C})$ such that $A := U^*DU \in \mathbb{M}_n^0(\mathbb{C})_{sa}$. Since

$$\|A\|^2 = \|D\|^2 = (n-1)^2$$

and

$$\|A\|_{HS}^2 = \|D\|_{HS}^2 = (n-1)^2 + \overbrace{(-1)^2 + (-1)^2 + ... + (-1)^2}^{n-1} = n(n-1),$$

we have

$$\|A\|^2 = \frac{n-1}{n}\|A\|_{HS}^2,$$

which proves that the inequality is sharp. □

Lemma 6.1.4. *For $A \in \mathbb{M}_n^0(\mathbb{C})_{sa}$ and n odd,*

$$\|A\|_{HS}^2 \le (n-1)\|A\|^2.$$

This inequality is sharp.

Proof. We may assume that $\|A\| = 1$. Let $0 < \lambda_1 \le \lambda_2 \le ... \le \lambda_{n_+} \le 1$ be the strictly positive eigenvalues of A and $-1 \le \sigma_1 \le \sigma_2 \le ... \le \sigma_{n_-} < 0$ be the strictly negative eigenvalues of A. Replacing A by $-A$, if necessary, we may assume that $n_+ \le n_-$. Since n is odd, $n_+ \le \frac{n-1}{2}$. Thus,

$$\|A\|_{HS}^2 = \sum_{i=1}^{n_-} \sigma_i^2 + \sum_{j=1}^{n_+} \lambda_j^2 \le \sum_{i=1}^{n_-} -\sigma_i + \sum_{j=1}^{n_+} \lambda_j = 2\sum_{j=1}^{n_+} \lambda_j \le 2n_+ \le n-1.$$

Now let

$$D = \mathrm{diag}(\overbrace{1, 1, ..., 1}^{\frac{n-1}{2}}, 0, \overbrace{-1, -1, ..., -1}^{\frac{n-1}{2}}) \in \mathbb{M}_n(\mathbb{C})_{sa}.$$

By Theorem 6.1.2, there exists a unitary $U \in \mathbb{M}_n(\mathbb{C})$ such that $A := U^*DU \in \mathbb{M}_n^0(\mathbb{C})_{sa}$. Since

$$\|A\|^2 = \|D\|^2 = 1^2 = 1$$

and

$$\|A\|_{HS}^2 = \|D\|_{HS}^2 = \overbrace{1^2 + 1^2 + ... + 1^2}^{\frac{n-1}{2}} + 0^2 + \overbrace{(-1)^2 + (-1)^2 + ... + (-1)^2}^{\frac{n-1}{2}} = n-1,$$

we have

$$\|A\|_{HS}^2 = (n-1)\|A\|^2.$$

Thus the inequality is sharp. □

Corollary 6.1.5. *For* $A = \left[a_{ij}\right] \in \mathbb{M}_n^0(\mathbb{C})_{sa}$ *and* $|A| = \left[|a_{ij}|\right] \in \mathbb{M}_n^0(\mathbb{R}_+)_{sa}$*,*

$$|||A||| \leq \begin{cases} \sqrt{n-1}\|A\|, & n \text{ even} \\ \frac{n-1}{\sqrt{n}}\|A\|, & n \text{ odd} \end{cases}.$$

Proof. By Lemmas 6.1.3 and 6.1.4,

$$|||A|||^2 \leq \frac{n-1}{n}|||A|||_{HS}^2 = \frac{n-1}{n}\|A\|_{HS}^2 \leq \begin{cases} (n-1)\|A\|^2, & n \text{ even} \\ \frac{(n-1)^2}{n}\|A\|^2, & n \text{ odd} \end{cases}.$$

□

6.2. **Computation of 2-Paving Parameters.** The following proposition establishes the first entry of the third column of Table 1.4.3:

Proposition 6.2.1. $\beta_2(\mathbb{M}_3^0(\mathbb{C})_{sa}) = \frac{1}{\sqrt{3}} \approx .5774$.

Proof. Let $A \in \mathbb{M}_3^0(\mathbb{C})_{sa}$. Then by Lemma 6.1.1, Theorem 1.3.4, and Corollary 6.1.5,

$$\alpha_2(A) \leq \alpha_2(|A|) \leq \frac{1}{2}|||A||| \leq \frac{1}{2}\left(\frac{2}{\sqrt{3}}\|A\|\right) = \frac{1}{\sqrt{3}}\|A\|.$$

Since the choice of A was arbitrary, $\beta_2(\mathbb{M}_3^0(\mathbb{C})_{sa}) \leq \frac{1}{\sqrt{3}}$. Since

$$\beta_2\left(\begin{bmatrix} 0 & 1 & i \\ 1 & 0 & 1 \\ -i & 1 & 0 \end{bmatrix}\right) = \frac{1}{\sqrt{3}},$$

we have $\beta_2(\mathbb{M}_3^0(\mathbb{C})_{sa}) = \frac{1}{\sqrt{3}}$. □

The following proposition establishes the second entry of the third column of Table 1.4.3:

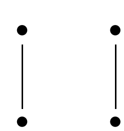
Proposition 6.2.2. $\beta_2(\mathbb{M}_4^0(\mathbb{C})_{sa}) = \frac{1}{\sqrt{3}} \approx .5774$.

Proof. By Proposition 6.2.1,

$$\frac{1}{\sqrt{3}} = \beta_2(\mathbb{M}_3^0(\mathbb{C})_{sa}) \leq \beta_2(\mathbb{M}_4^0(\mathbb{C})_{sa}).$$

Now let $A \in \mathbb{M}_4^0(\mathbb{C})_{sa}$, with $\alpha_2(A) = 1$. We aim to show that $\|A\| \geq \sqrt{3}$. Associate a graph $G = (V, E)$ with A as follows: $V = \{1, 2, 3, 4\}$ and $(i, j) \in E$ if and only if $\|A_{\{i,j\}}\| < 1$. We may assume that G has the following properties:

I. The graph



is not a subgraph of G. Otherwise, A has a $(2, 2)$-paving of norm less than 1, contradicting $\alpha_2(A) = 1$.

II. G has no isolated vertices. Otherwise, if vertex i is isolated, then the ith row (and column) of A has three entries of modulus greater than or equal to 1, which implies that $\|A\| \geq \sqrt{3}$ (and we are done).

III. By removing a vertex (and all associated edges) from G one cannot arrive at the "edgeless" graph $(\{i,j,k\},\emptyset)$. Otherwise $\alpha_2(A_{\{i,j,k\}}) \geq 1$ (since every $(1,2)$-paving has norm greater than or equal to 1), which implies $\|A_{\{i,j,k\}}\| \geq \sqrt{3}$ (by Proposition 6.2.1). Then $\|A\| \geq \|A_{\{i,j,k\}}\| \geq \sqrt{3}$ (and we are done).

Checking [11, p. 8], there are no such graphs G. Thus $\|A\| \geq \sqrt{3}$. Since the choice of A was arbitrary, $\beta_2(\mathbb{M}_4^0(\mathbb{C})_{sa}) \leq \frac{1}{\sqrt{3}}$, and so $\beta_2(\mathbb{M}_4^0(\mathbb{C})_{sa}) = \frac{1}{\sqrt{3}}$. □

The following proposition establishes the third entry of the third column of Table 1.4.3:

Proposition 6.2.3. $\beta_2(\mathbb{M}_5^0(\mathbb{C})_{sa}) = \beta_2(\mathbb{M}_5^0(\mathbb{R})_{sa}) = \frac{2}{\sqrt{5}} \approx .8944.$

Proof. Let $A \in \mathbb{M}_5^0(\mathbb{C})_{sa}$. Then by Lemma 6.1.1, Theorem 1.3.4, and Corollary 6.1.5,

$$\alpha_2(A) \leq \alpha_2(|A|) \leq \frac{1}{2}\||A|\| \leq \frac{1}{2}\left(\frac{4}{\sqrt{5}}\|A\|\right) = \frac{2}{\sqrt{5}}\|A\|.$$

Since the choice of A was arbitrary, $\beta_2(\mathbb{M}_5^0(\mathbb{C})_{sa}) \leq \frac{2}{\sqrt{5}}$. Since

$$\beta_2\left(\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & -1 \\ 1 & 1 & 0 & -1 & 1 \\ 1 & 1 & -1 & 0 & -1 \\ 1 & -1 & 1 & -1 & 0 \end{bmatrix}\right) = \frac{2}{\sqrt{5}},$$

we have $\frac{2}{\sqrt{5}} \leq \beta_2(\mathbb{M}_5^0(\mathbb{R})_{sa}) \leq \beta_2(\mathbb{M}_5^0(\mathbb{C})_{sa}) \leq \frac{2}{\sqrt{5}}$. □

The following proposition establishes the fourth entry of the third column of Table 1.4.3:

Proposition 6.2.4. $\beta_2(\mathbb{M}_6^0(\mathbb{C})_{sa}) = \beta_2(\mathbb{M}_6^0(\mathbb{R})_{sa}) = \frac{2}{\sqrt{5}} \approx .8944.$

Proof. By Proposition 6.2.3,

$$\frac{2}{\sqrt{5}} = \beta_2(\mathbb{M}_5^0(\mathbb{R})_{sa}) \leq \beta_2(\mathbb{M}_6^0(\mathbb{R})_{sa}) \leq \beta_2(\mathbb{M}_6^0(\mathbb{C})_{sa}).$$

Now let $A \in \mathbb{M}_6^0(\mathbb{C})_{sa}$, with $\alpha_2(A) = 1$. We aim to show that $\|A\| \geq \frac{\sqrt{5}}{2}$. By assumption, every $(3,3)$-paving of A has norm greater than or equal to 1. Thus for all $\{\{i,j,k\},\{l,m,n\}\} \in \Pi_2^6$, either $\|A_{\{i,j,k\}}\| \geq 1$ or $\|A_{\{l,m,n\}}\| \geq 1$. Since there are 10 such partitions, we can break the proof into $2^{10} = 1024$ cases. Here are four such cases:

Case 1: Suppose $\|A_\sigma\| \geq 1$ for all

$$\begin{aligned}\sigma \in \Sigma \ := \ & \{\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,2,6\},\{1,3,4\}, \\ & \{1,3,5\},\{1,3,6\},\{1,4,5\},\{1,4,6\},\{1,5,6\}\}.\end{aligned}$$

Then by Lemmas 6.1.4 and 6.1.3,

$$\begin{aligned}
12\|A\|^2 &= 4\|A\|^2 + 8\|A\|^2 \\
&\geq 4\|A_{\{2,3,4,5,6\}}\|^2 + 8\|\vec{A}_1\|^2 \\
&\geq \|A_{\{2,3,4,5,6\}}\|_{HS}^2 + 8\|\vec{A}_1\|_{HS}^2 \\
&= \sum_{\sigma\in\Sigma} \|A_\sigma\|_{HS}^2 \geq \frac{3}{2}\sum_{\sigma\in\Sigma}\|A_\sigma\|^2 \geq \frac{3}{2}(10) = 15.
\end{aligned}$$

Here $\vec{A}_1$ is the first column of A.

Case 256: Suppose $\|A_\sigma\| \geq 1$ for all

$$\begin{aligned}
\sigma \in \Sigma \;:=\; &\{\{1,2,3\},\{1,2,4\},\{3,4,6\},\{3,4,5\},\{2,5,6\}, \\
&\{2,4,6\},\{2,4,5\},\{2,3,6\},\{2,3,5\},\{2,3,4\}\}.
\end{aligned}$$

Then

$$\begin{aligned}
12\|A\|^2 &= 4\|A\|^2 + 4\|A\|^2 + 2\|A\|^2 + 2\|A\|^2 \\
&\geq 4\|A_{\{2,3,4,5,6\}}\|^2 + 4\|\vec{A}_2\|^2 + 2\|\vec{A}_3\|^2 + 2\|\vec{A}_4\|^2 \\
&\geq \|A_{\{2,3,4,5,6\}}\|_{HS}^2 + 4\|\vec{A}_2\|_{HS}^2 + 2\|\vec{A}_3\|_{HS}^2 + 2\|\vec{A}_4\|_{HS}^2 \\
&= \sum_{\sigma\in\Sigma} \|A_\sigma\|_{HS}^2 \geq \frac{3}{2}\sum_{\sigma\in\Sigma}\|A_\sigma\|^2 \geq \frac{3}{2}(10) = 15.
\end{aligned}$$

Case 257: Suppose $\|A_\sigma\| \geq 1$ for all

$$\begin{aligned}
\sigma \in \Sigma \;:=\; &\{\{1,2,3\},\{3,5,6\},\{1,2,5\},\{1,2,6\},\{1,3,4\}, \\
&\{1,3,5\},\{1,3,6\},\{1,4,5\},\{1,4,6\},\{1,5,6\}\}.
\end{aligned}$$

Then

$$\begin{aligned}
12\|A\|^2 &= 4\|A\|^2 + 4\|A\|^2 + 4\|A\|^2 \\
&\geq 4\|A_{\{1,2,3,5,6\}}\|^2 + 4\|A_{\{1,3,4,5,6\}}\|^2 + 4\|\vec{A}_1\|^2 \\
&\geq \|A_{\{1,2,3,5,6\}}\|_{HS}^2 + \|A_{\{1,3,4,5,6\}}\|_{HS}^2 + 4\|\vec{A}_1\|_{HS}^2 \\
&= \sum_{\sigma\in\Sigma} \|A_\sigma\|_{HS}^2 \geq \frac{3}{2}\sum_{\sigma\in\Sigma}\|A_\sigma\|^2 \geq \frac{3}{2}(10) = 15.
\end{aligned}$$

Case 683: Suppose $\|A_\sigma\| \geq 1$ for all

$$\begin{aligned}
\sigma \in \Sigma \;:=\; &\{\{4,5,6\},\{1,2,4\},\{3,4,6\},\{1,2,6\},\{2,5,6\}, \\
&\{1,3,5\},\{2,4,5\},\{1,4,5\},\{2,3,5\},\{1,5,6\}\}.
\end{aligned}$$

Then

$$\begin{aligned}
12\|A\|^2 &= 6\|A\|^2 + 4\|A\|^2 + 2\|A\|^2 \\
&\geq 6\|A\|^2 + 4\|A_{\{1,2,4,5,6\}}\|^2 + 2\|\vec{A}_5\|^2 \\
&\geq \|A\|_{HS}^2 + \|A_{\{1,2,4,5,6\}}\|_{HS}^2 + 2\|\vec{A}_5\|_{HS}^2 \\
&= \sum_{\sigma\in\Sigma} \|A_\sigma\|_{HS}^2 \geq \frac{3}{2}\sum_{\sigma\in\Sigma}\|A_\sigma\|^2 \geq \frac{3}{2}(10) = 15.
\end{aligned}$$

The remaining 1020 cases work similarly (they are checked by computer). Thus, $\|A\| \geq \frac{\sqrt{5}}{2}$. Since the choice of A was arbitrary, $\beta_2(\mathbb{M}_6^0(\mathbb{C})_{sa}) \leq \frac{2}{\sqrt{5}}$, and so $\beta_2(\mathbb{M}_6^0(\mathbb{C})_{sa}) = \beta_2(\mathbb{M}_6^0(\mathbb{R})_{sa}) = \frac{2}{\sqrt{5}}$. □

Remark 6.2.5. *The proof of Proposition 6.2.4 only uses the fact that every* $(3,3)$*-paving of* A *has norm greater than or equal to* 1 *rather than the full set of implications arising from* $\alpha_2(A) = 1$.

6.3. **Computation of 3-Paving Parameters.** The following proposition establishes the first entry in the first column of Table 1.4.2:

Proposition 6.3.1. $\beta_3(\mathbb{M}_4^0(\mathbb{C})_{sa}) = \frac{1}{\sqrt{3}} \approx .5774$.

Proof. Let $A \in \mathbb{M}_4^0(\mathbb{C})_{sa}$ with $\alpha_3(A) = 1$. We aim to show that $\|A\| \geq \sqrt{3}$. By assumption, every $(1,1,2)$-paving of A has norm greater than or equal to 1. Thus, every 2-compression of A has norm greater than or equal to 1, i.e., $|a_{ij}| \geq 1$ for all $i \neq j$. But then every row and column of A has norm greater than or equal to $\sqrt{3}$, which implies $\|A\| \geq \sqrt{3}$. Since the choice of A was arbitrary, $\beta_3(\mathbb{M}_4^0(\mathbb{C})_{sa}) \leq \frac{1}{\sqrt{3}}$. Since

$$\beta_3\left(\begin{bmatrix} 0 & i & 1 & 1 \\ -i & 0 & 1 & -1 \\ 1 & 1 & 0 & i \\ 1 & -1 & -i & 0 \end{bmatrix}\right) = \frac{1}{\sqrt{3}},$$

we have $\beta_3(\mathbb{M}_4^0(\mathbb{C})_{sa}) = \frac{1}{\sqrt{3}}$. □

The following proposition establishes the second entry in the first column of Table 1.4.2:

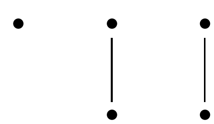
Proposition 6.3.2. $\beta_3(\mathbb{M}_5^0(\mathbb{C})_{sa}) = \frac{1}{\sqrt{3}} \approx .5774$.

Proof. By Proposition 6.3.1,

$$\frac{1}{\sqrt{3}} = \beta_3(\mathbb{M}_4^0(\mathbb{C})_{sa}) \leq \beta_3(\mathbb{M}_5^0(\mathbb{C})_{sa}).$$

Now let $A \in \mathbb{M}_5^0(\mathbb{C})_{sa}$ with $\alpha_3(A) = 1$. We aim to show that $\|A\| \geq \sqrt{3}$. Associate a graph $G = (V, E)$ with A as follows: $V = \{1, 2, ..., 5\}$ and $(i, j) \in E$ if and only if $\|A_{\{i,j\}}\| < 1$. We may assume that G has the following properties:

I. The graph



is not a subgraph of G. Otherwise, A has a $(1,2,2)$-paving of norm less than 1, contradicting $\alpha_3(A) = 1$.

II. For every i, the degree of vertex i is greater than or equal to 2. Otherwise the ith row (and column) of A has at least three entries of modulus greater than or equal to 1, which implies that $\|A\| \geq \sqrt{3}$ (and we are done).

Checking [11, p. 8], there are no such graphs G. Thus $\|A\| \geq \sqrt{3}$. Since the choice of A was arbitrary, $\beta_3(\mathbb{M}_5^0(\mathbb{C})_{sa}) \leq \frac{1}{\sqrt{3}}$, and so $\beta_3(\mathbb{M}_5^0(\mathbb{C})_{sa}) = \frac{1}{\sqrt{3}}$. □

The following proposition establishes the third entry in the first column of Table 1.4.2:

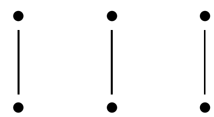
Proposition 6.3.3. $\beta_3(\mathbb{M}_6^0(\mathbb{C})_{sa}) = \frac{1}{\sqrt{3}} \approx .5774$.

Proof. By Proposition 6.3.1,

$$\frac{1}{\sqrt{3}} = \beta_3(\mathbb{M}_4^0(\mathbb{C})_{sa}) \leq \beta_3(\mathbb{M}_6^0(\mathbb{C})_{sa}).$$

Now let $A \in \mathbb{M}_6^0(\mathbb{C})_{sa}$ with $\alpha_3(A) = 1$. We aim to show that $\|A\| \geq \sqrt{3}$. Associate a graph $G = (V, E)$ with A as follows: $V = \{1, 2, ..., 6\}$ and $(i, j) \in E$ if and only if $\|A_{\{i,j\}}\| < 1$. We may assume that G has the following properties:

I. The graph



is not a subgraph of G. Otherwise, A has a $(2, 2, 2)$-paving of norm less than 1, contradicting $\alpha_3(A) = 1$.

II. For every i, the degree of vertex i is greater than or equal to 3. Otherwise the ith row (and column) of A has at least three entries of modulus greater than or equal to 1, which implies that $\|A\| \geq \sqrt{3}$ (and we are done).

Checking [11, p. 9-11], there are no such graphs G. Thus $\|A\| \geq \sqrt{3}$. Since the choice of A was arbitrary, $\beta_3(\mathbb{M}_6^0(\mathbb{C})_{sa}) \leq \frac{1}{\sqrt{3}}$, and so $\beta_3(\mathbb{M}_6^0(\mathbb{C})) = \frac{1}{\sqrt{3}}$. □

The following proposition establishes the fourth entry in the first column of Table 1.4.2:

Proposition 6.3.4. $\beta_3(\mathbb{M}_7^0(\mathbb{C})_{sa}) \leq \frac{2}{\sqrt{7}} \approx .7559$.

Proof. Let $A \in \mathbb{M}_7^0(\mathbb{C})_{sa}$. Then by Lemma 6.1.1, Theorem 1.3.4, and Corollary 6.1.5,

$$\alpha_3(A) \leq \alpha_3(|A|) \leq \frac{1}{3}|||A||| \leq \frac{1}{3}\left(\frac{6}{\sqrt{7}}\|A\|\right) = \frac{2}{\sqrt{7}}\|A\|.$$

□

The following proposition establishes the fifth entry in the first column of Table 1.4.2:

Proposition 6.3.5. $\beta_3(\mathbb{M}_8^0(\mathbb{C})_{sa}) \leq \frac{\sqrt{7}}{3} \approx .8819$.

Proof. Let $A \in \mathbb{M}_8^0(\mathbb{C})_{sa}$. Then by Lemma 6.1.1, Theorem 1.3.4, and Corollary 6.1.5,

$$\alpha_3(A) \leq \alpha_3(|A|) \leq \frac{1}{3}|||A||| \leq \frac{1}{3}(\sqrt{7}\|A\|) = \frac{\sqrt{7}}{3}\|A\|.$$

□

The following proposition establishes the sixth entry in the first column of Table 1.4.2:

Proposition 6.3.6. $\beta_3(\mathbb{M}_9^0(\mathbb{C})_{sa}) \leq \frac{8}{9} \approx .8889$.

Proof. Let $A \in \mathbb{M}_9^0(\mathbb{C})_{sa}$. Then by Lemma 6.1.1, Theorem 1.3.4, and Corollary 6.1.5,

$$\alpha_3(A) \leq \alpha_3(|A|) \leq \frac{1}{3}|||A||| \leq \frac{1}{3}\left(\frac{8}{3}\|A\|\right) = \frac{8}{9}\|A\|.$$

□

7. Paving Real Symmetric Matrices

In this section we establish the second column of Table 1.4.2 and the fourth column of Table 1.4.3.

7.1. **Tools.**

Lemma 7.1.1. *For*

$$0 \neq A = \begin{bmatrix} 0 & a & b \\ \overline{a} & 0 & c \\ \overline{b} & \overline{c} & 0 \end{bmatrix} \in \mathbb{M}_3^0(\mathbb{C})_{sa},$$

we have

$$\frac{|a|^2 + |b|^2 + |c|^2}{\|A\|^2} + \frac{2|Re(a\overline{b}c)|}{\|A\|^3} = 1.$$

In fact, $\lambda = \|A\|$ *is the largest solution of the equation*

$$\frac{|a|^2 + |b|^2 + |c|^2}{\lambda^2} + \frac{2|Re(a\overline{b}c)|}{\lambda^3} = 1.$$

Proof. An easy calculation shows that the characteristic polynomial of A equals

$$P(\lambda) = \lambda^3 - (|a|^2 + |b|^2 + |c|^2)\lambda - 2Re(a\overline{b}c).$$

Let $-\|A\| \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \|A\|$ be the eigenvalues of A.

Case 1: Suppose $\lambda_3 = \|A\|$. Since $\lambda_1 + \lambda_2 + \lambda_3 = 0$, it must be that $\lambda_1, \lambda_2 \leq 0$. Thus, $2Re(a\overline{b}c) = \lambda_1\lambda_2\lambda_3 \geq 0$, which implies that

$$P(\lambda) = \lambda^3 - (|a|^2 + |b|^2 + |c|^2)\lambda - 2|Re(a\overline{b}c)|.$$

Since $\lambda = \|A\|$ is the largest solution of the equation $P(\lambda) = 0$, it is also the largest solution of the equation

$$\frac{|a|^2 + |b|^2 + |c|^2}{\lambda^2} + \frac{2|Re(a\overline{b}c)|}{\lambda^3} = 1.$$

Case 2: Suppose $\lambda_1 = -\|A\|$. Then $\|-A\| = \|A\|$ is an eigenvalue of $-A$. By Case 1, $\lambda = \|-A\|$ is the largest solution of the equation

$$\frac{|-a|^2 + |-b|^2 + |-c|^2}{\lambda^2} + \frac{|Re((-a)\overline{(-b)}(-c))|}{\lambda^3} = 1.$$

Thus, $\lambda = \|A\|$ is the largest solution of the equation

$$\frac{|a|^2 + |b|^2 + |c|^2}{\lambda^2} + \frac{2|Re(a\overline{b}c)|}{\lambda^3} = 1.$$

□

Corollary 7.1.2. *For*

$$A = \begin{bmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{bmatrix} \in \mathbb{M}_3^0(\mathbb{R})_{sa},$$

we have $|||A||| = \|A\|$.

Proof. By Lemma 7.1.1, both $\lambda = |||A|||$ and $\lambda = \|A\|$ are the largest solution of the equation

$$\frac{a^2 + b^2 + c^2}{\lambda^2} + \frac{2|abc|}{\lambda^3} = 1.$$

□

Remark 7.1.3. *For $A \in \mathbb{M}_{k\times m}(\mathbb{C})$, $B \in \mathbb{M}_{k\times n}(\mathbb{C})$, $C \in \mathbb{M}_{l\times m}(\mathbb{C})$, and $D \in \mathbb{M}_{l\times n}(\mathbb{C})$,*

$$\left\| \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right\| = \left\| \begin{bmatrix} -A & B \\ C & -D \end{bmatrix} \right\|.$$

This follows from the identity

$$\begin{bmatrix} -A & B \\ C & -D \end{bmatrix} = \begin{bmatrix} -I_k & 0 \\ 0 & I_l \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I_m & 0 \\ 0 & -I_n \end{bmatrix}.$$

Lemma 7.1.4. *Let $X \in \mathbb{M}_3^0(\mathbb{R})_{sa}$ and $Y \in \mathbb{M}_3(\mathbb{R})$. If $\|X\| \geq 1$ and $|y_{ij}| \geq 1$ for all $1 \leq i, j \leq 3$, then $\left\| \begin{bmatrix} X & Y \end{bmatrix} \right\| \geq \frac{\sqrt{17}}{2} \approx 2.0616$. This inequality is sharp.*

Proof. Let

$$X = \begin{bmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{bmatrix} \text{ and } Y = \begin{bmatrix} r & s & t \\ u & v & w \\ x & y & z \end{bmatrix}.$$

Since

$$\left\| \begin{bmatrix} \frac{1}{\|X\|}X & Y \end{bmatrix} \right\| = \left\| \begin{bmatrix} X & Y \end{bmatrix} \begin{bmatrix} \frac{1}{\|X\|}I_3 & 0 \\ 0 & I_3 \end{bmatrix} \right\| \leq \left\| \begin{bmatrix} X & Y \end{bmatrix} \right\|,$$

we may assume that $\|X\| = 1$. Permuting the indices 1, 2, and 3, we may assume that $|a| \geq |b| \geq |c|$. Pre- and post-multiplying $\begin{bmatrix} X & Y \end{bmatrix}$ by diagonal orthogonal matrices, we may assume that $a, b, r, s, t \geq 0$. By Remark 7.1.3, we may assume that $c \geq 0$. Finally, permuting the indices 4, 5, and 6, we may assume that u and v have the same sign. Applying Lemma 6.1.3 to X, we conclude that $a^2 + b^2 + c^2 \geq 3/4$. It follows that $a^2 \geq 1/4$. We claim that $bc \leq 1/4$. Indeed, if $bc > 1/4$, then $2abc > 1/4$. But then, applying Lemma 7.1.1 to X, we have that

$$1 = (a^2 + b^2 + c^2) + 2abc > 3/4 + 1/4 = 1,$$

a contradiction. Therefore (by Lemma 5.2.1)

$$\begin{aligned} \left\| \begin{bmatrix} X & Y \end{bmatrix} \right\|^2 &\geq \left\| \begin{bmatrix} 0 & a & b & r & s \\ a & 0 & c & u & v \end{bmatrix} \right\|^2 \\ &\geq \frac{2a^2 + b^2 + c^2 + 4 + \sqrt{0^2 + 4(bc + ru + sv)^2}}{2} \\ &\geq \frac{1 + 4 + \sqrt{4(1.75)^2}}{2} = \frac{17}{4}. \end{aligned}$$

The matrices

$$X = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

show that the inequality is sharp. □

Corollary 7.1.5. *Let*

$$A = \begin{bmatrix} X & Y \\ Y^* & Z \end{bmatrix} \in \mathbb{M}_6^0(\mathbb{R})_{sa},$$

where $X, Z \in \mathbb{M}_3^0(\mathbb{R})_{sa}$ and $Y \in \mathbb{M}_3(\mathbb{R})$. If $\|X\|, \|Z\| \geq 1$ and $|y_{ij}| \geq 1$ for all $1 \leq i, j \leq 3$, then $\|A\| \geq \frac{\sqrt{17}}{2} \approx 2.0616$. This inequality is sharp.

Proof. By Lemma 7.1.4,

$$\|A\| \geq \left\| \begin{bmatrix} X & Y \end{bmatrix} \right\| \geq \frac{\sqrt{17}}{2}.$$

The matrices

$$X = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}, \ Y = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ -1 & -1 & 1 \end{bmatrix}, \text{ and } Z = \begin{bmatrix} 0 & 1/2 & -1/2 \\ 1/2 & 0 & 1/2 \\ -1/2 & 1/2 & 0 \end{bmatrix}$$

show that the inequality is sharp. □

Lemma 7.1.6. *Let*

$$A = \begin{bmatrix} 0 & a & b & c \\ a & 0 & d & e \\ b & d & 0 & f \\ c & e & f & 0 \end{bmatrix} \in \mathbb{M}_4^0(\mathbb{R})_{sa}.$$

If $|a|, |b|, |c| \geq 1$ *and* $\|A_{\{2,3,4\}}\| \geq 1$*, then* $\|A\| \geq \mu := 1.79333220781535$.

Proof. Conjugating by a diagonal orthogonal matrix, we may assume that $a, b, c \geq 0$. Permuting the indices 2, 3, and 4, we may assume that d and e have the same sign. Finally, by Remark 7.1.3, we may assume that $d, e \geq 0$.

Case 1: Suppose $d^2 + e^2 \geq r^2$, where $r := -1 + \sqrt{2\mu^2 - 3} \approx .8526$. Then $(d+e)^2 \geq d^2 + e^2 \geq r^2$. Thus (by Lemma 5.2.1)

$$\begin{aligned} \|A\|^2 &\geq \|A_{\{1,2\},\{1,2,3,4\}}\|^2 = \left\| \begin{bmatrix} 0 & a & b & c \\ a & 0 & d & e \end{bmatrix} \right\|^2 \\ &= \frac{2a^2 + b^2 + c^2 + d^2 + e^2 + \sqrt{(b^2 + c^2 - d^2 - e^2)^2 + 4(bd + ce)^2}}{2} \\ &\geq \frac{4 + r^2 + \sqrt{0^2 + 4r^2}}{2} = \frac{r^2 + 2r + 4}{2} = \mu^2. \end{aligned}$$

Case 2: Suppose $s^2 < d^2 + e^2 \leq r^2$, where $s := \sqrt{\frac{\mu^4 - 4\mu^2 + 3}{\mu^2 - 2}} \approx .6275$. Define a sequence $\{s_n : n \in \mathbb{N}\}$ recursively as follows: $s_1 := r$ and

$$s_{n+1} := \sqrt{2(\mu^2 - 1) - \sqrt{4(2\mu^2 - 3) + (2 - s_n^2)^2}}, \ n \in \mathbb{N}.$$

Since $s_n \searrow s$, it suffices to consider the subcase $s_{n+1}^2 \leq d^2 + e^2 \leq s_n^2$. Then

$$(d+e)^2 \geq d^2 + e^2 \geq s_{n+1}^2.$$

Thus (by Lemma 5.2.1)

$$\begin{aligned} \|A\|^2 &\geq \|A_{\{1,2\},\{1,2,3,4\}}\|^2 = \left\| \begin{bmatrix} 0 & a & b & c \\ a & 0 & d & e \end{bmatrix} \right\|^2 \\ &= \frac{2a^2 + b^2 + c^2 + d^2 + e^2 + \sqrt{(b^2 + c^2 - d^2 - e^2)^2 + 4(bd + ce)^2}}{2} \\ &\geq \frac{4 + s_{n+1}^2 + \sqrt{(2 - s_n^2)^2 + 4s_{n+1}^2}}{2} = \mu^2. \end{aligned}$$

Case 3: Suppose $|f| \geq t$, where $t := \frac{\mu^2-2}{\mu} \approx .6781$. Then (by Lemmas 7.1.2 and 2.1.1)

$$\begin{aligned} \|A\| &\geq \|A_{\{1,3,4\}}\| = \left\| \begin{bmatrix} 0 & b & c \\ b & 0 & f \\ c & f & 0 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 0 & b & c \\ b & 0 & |f| \\ c & |f| & 0 \end{bmatrix} \right\| \\ &\geq \left\| \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & t \\ 1 & t & 0 \end{bmatrix} \right\| = \frac{t + \sqrt{t^2+8}}{2} = \mu. \end{aligned}$$

Case 4: Suppose $d^2 + e^2 \leq s^2$ and $|f| \leq t$. Applying Lemma 7.1.1 to $A_{\{2,3,4\}}$, we have that

$$1 = \frac{d^2+e^2+f^2}{\|A_{\{2,3,4\}}\|^2} + \frac{2|def|}{\|A_{\{2,3,4\}}\|^3} \leq d^2+e^2+f^2+2|def| \leq s^2+t^2+2det.$$

Thus,

$$de \geq \frac{1-s^2-t^2}{2t}.$$

Hence (by Lemma 5.2.1)

$$\begin{aligned} \|A\|^2 &\geq \|A_{\{1,2\},\{1,2,3,4\}}\|^2 = \left\| \begin{bmatrix} 0 & a & b & c \\ a & 0 & d & e \end{bmatrix} \right\|^2 \\ &= \frac{2a^2+b^2+c^2+d^2+e^2+\sqrt{(b^2+c^2-d^2-e^2)^2+4(bd+ce)^2}}{2} \\ &\geq \frac{4+d^2+e^2+\sqrt{(2-d^2-e^2)^2+4(d+e)^2}}{2} \\ &= \frac{4+d^2+e^2+\sqrt{(d^2+e^2)^2+8de+4}}{2} \\ &\geq \frac{4+2de+\sqrt{(2de)^2+8de+4}}{2} = 3+2de \\ &\geq 3 + \frac{1-s^2-t^2}{t} > \mu^2. \end{aligned}$$

□

7.2. **Computation of 2-Paving Parameters.** The following proposition establishes the first entry in the fourth column of Table 1.4.3:

Proposition 7.2.1. $\beta_2(\mathbb{M}_3^0(\mathbb{R})_{sa}) = \beta_2(\mathbb{M}_3^0(\mathbb{R}_+)_{sa}) = \frac{1}{2} = .5000$.

Proof. Let $A \in \mathbb{M}_3^0(\mathbb{R})_{sa}$. Then by Theorem 1.3.4 and Corollary 7.1.2,

$$\alpha_2(A) = \alpha_2(|A|) \leq \frac{1}{2}\||A|\| = \frac{1}{2}\|A\|.$$

Since the choice of A was arbitrary, $\beta_2(\mathbb{M}_3^0(\mathbb{R})_{sa}) \leq \frac{1}{2}$. Since

$$\beta_2\left(\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\right) = \frac{1}{2},$$

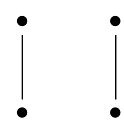
we have $\frac{1}{2} \leq \beta_2(\mathbb{M}_3^0(\mathbb{R}_+)_{sa}) \leq \beta_2(\mathbb{M}_3^0(\mathbb{R})_{sa}) \leq \frac{1}{2}$. □

The following proposition establishes the second entry in the fourth column of Table 1.4.3:

Proposition 7.2.2. *$\beta_2(\mathbb{M}_4^0(\mathbb{R})_{sa}) \leq .5577$.*

Proof. Let $A \in \mathbb{M}_4^0(\mathbb{R})_{sa}$, with $\alpha_2(A) = 1$. We aim to show that $\|A\| \geq \mu = 1.79333220781535$. Associate a graph $G = (V, E)$ with A as follows: $V = \{1, 2, 3, 4\}$ and $(i, j) \in E$ if and only if $\|A_{\{i,j\}}\| < 1$. We may assume that G has the following properties:

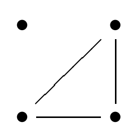
I. The graph



is not a subgraph of G. Otherwise, A has a $(2, 2)$-paving of norm less than 1, contradicting $\alpha_2(A) = 1$.

II. By removing a vertex (and all associated edges) from G one cannot arrive at the "edgeless" graph $(\{i, j, k\}, \emptyset)$. Otherwise, $\alpha_2(A_{\{i,j,k\}}) \geq 1$ (since every $(1, 2)$-paving has norm greater than or equal to 1), which implies $\|A_{\{i,j,k\}}\| \geq 2$ (Proposition 7.2.1). Then $\|A\| \geq \|A_{\{i,j,k\}}\| \geq 2 > \mu$ (and we are done).

Checking [11, p. 8], we see that G must be



Thus (after a permutation of the indices)

$$A = \begin{bmatrix} 0 & \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & 0 & \bullet & \bullet \\ \blacksquare & \bullet & 0 & \bullet \\ \blacksquare & \bullet & \bullet & 0 \end{bmatrix},$$

where "$\bullet$" indicates an entry of modulus less than 1 and "$\blacksquare$" indicates an entry of modulus greater than or equal to 1. By assumption, the $\{\{1\}, \{2, 3, 4\}\}$-paving of A has norm greater than or equal to 1. Thus, $\|A_{\{2,3,4\}}\| \geq 1$. By Lemma 7.1.6, $\|A\| \geq \mu$. Since the choice of A was arbitrary, $\beta_2(\mathbb{M}_4^0(\mathbb{R})_{sa}) \leq \frac{1}{\mu} \leq 0.5577$. $\square$

Remark 7.2.3. *The third and fourth entries in the fourth column of Table 1.4.3 were already established by Propositions 6.2.3 and 6.2.4, respectively.*

7.3. **Computation of 3-Paving Parameters.** The following proposition establishes the first entry of the second column of Table 1.4.2:

Proposition 7.3.1. *$\beta_3(\mathbb{M}_4^0(\mathbb{R})_{sa}) = \frac{1}{\sqrt{5}} \approx .4472$.*

Proof. Let

$$A = \begin{bmatrix} 0 & a & b & c \\ a & 0 & d & e \\ b & d & 0 & f \\ c & e & f & 0 \end{bmatrix} \in \mathbb{M}_4^0(\mathbb{R})_{sa},$$

with $\alpha_3(A) = 1$. We aim to show that $\|A\| \geq \sqrt{5}$. By assumption, every $(1,1,2)$-paving of A has norm greater than or equal to 1. Thus, every 2-compression of A has norm greater than or equal to 1, i.e. $|a|, |b|, |c|, |d|, |e|, |f| \geq 1$. Conjugating by a diagonal orthogonal matrix, if necessary, we may assume $a, b, c \geq 0$. Permuting the indices 2, 3, and 4, if necessary, we may assume d and e have the same sign. Finally, applying Remark 7.1.3, if necessary, we may assume $d, e \geq 0$. Thus, by Lemma 2.1.1,

$$\|A\| \geq \|A_{\{1,2\},\{1,2,3,4\}}\| = \left\| \begin{bmatrix} 0 & a & b & c \\ a & 0 & d & e \end{bmatrix} \right\| \geq \left\| \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \right\| = \sqrt{5}.$$

Since the choice of A was arbitrary, $\beta_3(\mathbb{M}_4^0(\mathbb{R})_{sa}) \leq \frac{1}{\sqrt{5}}$. Since

$$\beta_3\left(\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & -1 \\ 1 & 1 & -1 & 0 \end{bmatrix} \right) = \frac{1}{\sqrt{5}},$$

we see that $\beta_3(\mathbb{M}_4^0(\mathbb{R})_{sa}) = \frac{1}{\sqrt{5}}$. □

The following proposition establishes the second entry of the second column of Table 1.4.2:

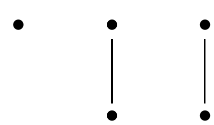
Proposition 7.3.2. $\beta_3(\mathbb{M}_5^0(\mathbb{R})_{sa}) = \frac{1}{\sqrt{5}} \approx .4472$.

Proof. By Proposition 7.3.1,

$$\frac{1}{\sqrt{5}} = \beta_3(\mathbb{M}_4^0(\mathbb{R})_{sa}) \leq \beta_3(\mathbb{M}_5^0(\mathbb{R})_{sa}).$$

Now let $A \in \mathbb{M}_5^0(\mathbb{R})_{sa}$ with $\alpha_3(A) = 1$. We aim to show that $\|A\| \geq \sqrt{5}$. Associate a graph $G = (V, E)$ with A as follows: $V = \{1, 2, ..., 5\}$ and $(i, j) \in E$ if and only if $\|A_{\{i,j\}}\| < 1$. We may assume that G has the following properties:

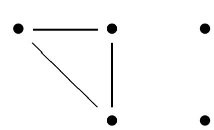
I. The graph



is not a subgraph of G. Otherwise, A has a $(1,2,2)$-paving of norm less than 1, contradicting $\alpha_3(A) = 1$.

II. By removing a vertex (and all associated edges) from G one cannot arrive at the "edgeless" graph $(\{i,j,k,l\}, \emptyset)$. Otherwise, $\alpha_3(A_{\{i,j,k,l\}}) \geq 1$ (since every $(1,1,2)$-paving has norm greater than or equal to 1) which implies that $\|A_{\{i,j,k,l\}}\| \geq \sqrt{5}$ (Proposition 7.3.1). Then $\|A\| \geq \|A_{\{i,j,k,l\}}\| \geq \sqrt{5}$ (and we are done).

Checking [11, p. 8], we see that G must be



Thus (after a permutation of the indices)

$$A = \begin{bmatrix} 0 & \bullet & \bullet & a & b \\ \bullet & 0 & \bullet & c & d \\ \bullet & \bullet & 0 & e & f \\ a & c & e & 0 & g \\ b & d & f & g & 0 \end{bmatrix},$$

where "$\bullet$" indicates an entry of modulus less than 1 and $|a|, |b|, |c|, |d|, |e|, |f|, |g| \geq 1$. Conjugating by a diagonal orthogonal matrix, if necessary, we may assume $b, d, f, g \geq 0$. Permuting the indices 1, 2, and 3, if necessary, we may assume a and c have the same sign. Finally, by Remark 7.1.3, we may assume $a, c \geq 0$. Thus (by Lemma 2.1.1)

$$\|A\| \geq \|A_{\{4,5\},\{1,2,4,5\}}\| = \left\| \begin{bmatrix} a & c & 0 & g \\ b & d & g & 0 \end{bmatrix} \right\| \geq \left\| \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \right\| = \sqrt{5}.$$

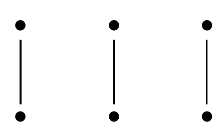
Since the choice of A was arbitrary, $\beta_3(\mathbb{M}_5^0(\mathbb{R})_{sa}) \leq \frac{1}{\sqrt{5}}$, and so $\beta_3(\mathbb{M}_5^0(\mathbb{R})_{sa}) = \frac{1}{\sqrt{5}}$. □

The following proposition establishes the third entry of the second column of Table 1.4.2:

Proposition 7.3.3. $\beta_3(\mathbb{M}_6^0(\mathbb{R})_{sa}) = \frac{2}{\sqrt{17}} \approx .4851$.

Proof. Let $A \in \mathbb{M}_6^0(\mathbb{R})_{sa}$, with $\alpha_3(A) = 1$. We aim to show that $\|A\| \geq \frac{\sqrt{17}}{2} \approx 2.0616$. Associate a graph $G = (V, E)$ with A as follows: $V = \{1, 2, ..., 6\}$ and $(i, j) \in E$ if and only if $\|A_{\{i,j\}}\| < 1$. We may assume that G has the following properties:

I. The graph

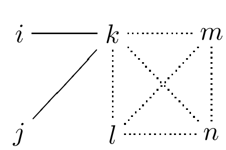


is not a subgraph of G. Otherwise, A has a $(2, 2, 2)$-paving of norm less than 1, contradicting $\alpha_3(A) = 1$.

II. G has no isolated vertices. Otherwise, if vertex i is isolated, then the ith row (and column) of A has five entries of modulus greater than or equal to 1, which implies $\|A\| \geq \sqrt{5} > \frac{\sqrt{17}}{2}$ (and we are done).

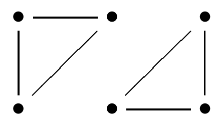
III. By removing two vertices (and all associated edges) from G, one cannot arrive at the "edgeless" graph $(\{i, j, k, l\}, \emptyset)$. Otherwise, $\alpha_3(A_{\{i,j,k,l\}}) \geq 1$ (since every $(1, 1, 2)$-paving has norm greater than or equal to 1) which implies $\|A_{\{i,j,k,l\}}\| \geq \sqrt{5}$ (Proposition 7.3.1). Then $\|A\| \geq \|A_{\{i,j,k,l\}}\| \geq \sqrt{5} > \frac{\sqrt{17}}{2}$ (and we are done).

IV. G is not the graph



where a dotted line indicates an edge which may or may not be there. Otherwise, $\alpha_3(A_{\{i,j,l,m,n\}}) \geq 1$ (see the proof of Proposition 3.3.3), which implies $\|A_{\{i,j,k,l,m\}}\| \geq \sqrt{5}$ (Proposition 7.3.2). Then $\|A\| \geq \|A_{\{i,j,k,l,m\}}\| \geq \sqrt{5} > \frac{\sqrt{17}}{2}$ (and we are done).

Checking [11, p. 9-11], we see that G must be



Thus (up to a permutation of the indices)

$$A = \begin{bmatrix} 0 & \bullet & \bullet & \blacksquare & \blacksquare & \blacksquare \\ \bullet & 0 & \bullet & \blacksquare & \blacksquare & \blacksquare \\ \bullet & \bullet & 0 & \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare & 0 & \bullet & \bullet \\ \blacksquare & \blacksquare & \blacksquare & \bullet & 0 & \bullet \\ \blacksquare & \blacksquare & \blacksquare & \bullet & \bullet & 0 \end{bmatrix},$$

where "$\bullet$" indicates an entry of modulus less than 1 and "$\blacksquare$" indicates an entry of modulus greater than or equal to 1. Since the $(\{1,2,3\},\{4\},\{5,6\})$-paving of A has norm greater than or equal to 1 and $\|A_{\{5,6\}}\| < 1$, it must be that $\|A_{\{1,2,3\}}\| \geq 1$. Likewise, $\|A_{\{4,5,6\}}\| \geq 1$. By Corollary 7.1.5, $\|A\| \geq \frac{\sqrt{17}}{2}$. Since the choice of A was arbitrary, $\beta_3(\mathbb{M}_6^0(\mathbb{R})_{sa}) \leq \frac{2}{\sqrt{17}}$. Since

$$\beta_3\left(\begin{bmatrix} 0 & 1/2 & 1/2 & 1 & 1 & 1 \\ 1/2 & 0 & 1/2 & 1 & -1 & -1 \\ 1/2 & 1/2 & 0 & -1 & -1 & 1 \\ 1 & 1 & -1 & 0 & 1/2 & -1/2 \\ 1 & -1 & -1 & 1/2 & 0 & 1/2 \\ 1 & -1 & 1 & -1/2 & 1/2 & 0 \end{bmatrix}\right) = \frac{2}{\sqrt{17}},$$

we have that $\beta_3(\mathbb{M}_6^0(\mathbb{R})_{sa}) = \frac{2}{\sqrt{17}}$. □

Remark 7.3.4. *The fourth, fifth, and sixth entries of the second column of Table 1.4.2 follow from Propositions 6.3.4, 6.3.5, and 6.3.6, respectively.*

8. Paving Self-Adjoint Circulants

In this section we establish the third (and last) column of Table 1.4.2 and the fifth (and last) column of Table 1.4.3.

8.1. **Computation of 2-Paving Parameters.** The following proposition establishes the first entry of the last column of Table 1.4.3:

Proposition 8.1.1. $\beta_2(\mathbb{M}_3^0(\mathbb{C})_{\circlearrowleft,sa} = \frac{1}{\sqrt{3}} \approx .5774.$

Proof. Let

$$A = \begin{bmatrix} 0 & a & \overline{a} \\ \overline{a} & 0 & a \\ a & \overline{a} & 0 \end{bmatrix} \in \mathbb{M}_3^0(\mathbb{C})_{\circlearrowleft,sa},$$

with $\alpha_2(A) = 1$. We aim to show that $\|A\| \geq \sqrt{3} \approx 1.7321$. Since the $\{\{1,2\},\{3\}\}$-paving of A has norm greater than or equal to 1, $|a| \geq 1$. Thus (by Lemma 6.1.4)

$$6 \leq \|A\|_{HS}^2 \leq 2\|A\|^2,$$

which implies $\|A\| \geq \sqrt{3}$. Since the choice of A was arbitrary, $\beta_2(\mathbb{M}_3^0(\mathbb{C})_{\circlearrowleft,sa}) \leq \frac{1}{\sqrt{3}}$. Since

$$\beta_2\left(\begin{bmatrix} 0 & i & -i \\ -i & 0 & i \\ i & -i & 0 \end{bmatrix}\right) = \frac{1}{\sqrt{3}},$$

we see that $\beta_2(\mathbb{M}_3^0(\mathbb{C})_{\circlearrowleft,sa}) = \frac{1}{\sqrt{3}}$. □

The following proposition establishes the second entry of the last column of Table 1.4.3:

Proposition 8.1.2. $\beta_2(\mathbb{M}_4^0(\mathbb{C})_{\circlearrowleft,sa}) = \frac{1}{1+\sqrt{2}} \approx .4142$.

Proof. Let

$$A = \begin{bmatrix} 0 & a & b & \overline{a} \\ \overline{a} & 0 & a & b \\ b & \overline{a} & 0 & a \\ a & b & \overline{a} & 0 \end{bmatrix} \in \mathbb{M}_4^0(\mathbb{C})_{\circlearrowleft,sa},$$

with $\alpha_3(A) = 1$. We aim to show that $\|A\| \geq 1 + \sqrt{2} \approx 2.4142$. By Lemma 5.2.2, $|a|, |b| \geq 1$. Let $a = re^{i\theta}$, where $r \geq 1$ and $\theta \in \mathbb{R}$. By Equation (1),

$$\begin{aligned} \|A\| &= \max\{|az^3 + bz^2 + \overline{a}z| : z = \pm 1, \pm i\} \\ &= \max\{2r|\cos(\theta)| + |b|, 2r|\sin(\theta)| + |b|\} \\ &\geq \max\{2|\cos(\theta)| + 1, 2|\sin(\theta)| + 1\} \geq 1 + \sqrt{2}. \end{aligned}$$

Since the choice of A was arbitrary, $\beta_3(\mathbb{M}_4^0(\mathbb{C})_{\circlearrowleft,sa}) \leq \frac{1}{1+\sqrt{2}}$. Since

$$\beta_3\left(\begin{bmatrix} 0 & e^{\pi i/4} & 1 & e^{-\pi i/4} \\ e^{-\pi i/4} & 0 & e^{\pi i/4} & 1 \\ 1 & e^{-\pi i/4} & 0 & e^{\pi i/4} \\ e^{\pi i/4} & 1 & e^{-\pi i/4} & 0 \end{bmatrix}\right) = \frac{1}{1+\sqrt{2}},$$

we have that $\beta_3(\mathbb{M}_4^0(\mathbb{C})_{\circlearrowleft,sa}) = \frac{1}{1+\sqrt{2}}$. □

The following proposition establishes the third entry of the last column of Table 1.4.3:

Proposition 8.1.3. $\beta_2(\mathbb{M}_5^0(\mathbb{C})_{\circlearrowleft,sa}) = \frac{2}{\sqrt{5}} \approx .8944$.

Proof. By Proposition 6.2.3, $\beta_2(\mathbb{M}_5^0(\mathbb{C})_{\circlearrowleft,sa}) \leq \beta_2(\mathbb{M}_5^0(\mathbb{C})_{sa}) \leq \frac{2}{\sqrt{5}}$. Since

$$\beta_2\left(\begin{bmatrix} 0 & e^{2\pi i/5} & e^{-\pi i/5} & e^{\pi i/5} & e^{-2\pi i/5} \\ e^{-2\pi i/5} & 0 & e^{2\pi i/5} & e^{-\pi i/5} & e^{\pi i/5} \\ e^{\pi i/5} & e^{-2\pi i/5} & 0 & e^{2\pi i/5} & e^{-\pi i/5} \\ e^{-\pi i/5} & e^{\pi i/5} & e^{-2\pi i/5} & 0 & e^{2\pi i/5} \\ e^{2\pi i/5} & e^{-\pi i/5} & e^{\pi i/5} & e^{-2\pi i/5} & 0 \end{bmatrix}\right) = \frac{2}{\sqrt{5}},$$

we have that $\beta_2(\mathbb{M}_5^0(\mathbb{C})_{\circlearrowleft,sa}) = \frac{2}{\sqrt{5}}$. □

Remark 8.1.4. *The sixth entry of the last column of Table 1.4.3 follows from Proposition 6.2.4.*

8.2. **Computation of 3-Paving Parameters.** The following proposition establishes the first entry of the last column of Table 1.4.2:

Proposition 8.2.1. $\beta_3(\mathbb{M}_4^0(\mathbb{C})_{\circlearrowleft,sa}) = \frac{1}{1+\sqrt{2}} \approx .4142.$

Proof. Proposition 8.1.2 and Lemma 5.2.2. □

The following proposition establishes the second entry of the last column of Table 1.4.2:

Proposition 8.2.2. $\beta_3(\mathbb{M}_5^0(\mathbb{C})_{\circlearrowleft,sa}) = \frac{1}{\sqrt{5}} \approx .4472.$

Proof. Let

$$A = \begin{bmatrix} 0 & a & b & \overline{b} & \overline{a} \\ \overline{a} & 0 & a & b & \overline{b} \\ \overline{b} & \overline{a} & 0 & a & b \\ b & \overline{b} & \overline{a} & 0 & a \\ a & b & \overline{b} & \overline{a} & 0 \end{bmatrix} \in \mathbb{M}_5^0(\mathbb{C})_{\circlearrowleft,sa},$$

with $\alpha_3(A) = 1$. We aim to show that $\|A\| \geq \sqrt{5} \approx 2.2361$. Since the $\{\{1,2\},\{3,4\},\{5\}\}$-paving of A has norm greater than or equal to 1, $|a| \geq 1$. Likewise, since the $\{\{1,3\},\{2,4\},\{5\}\}$-paving of A has norm greater than or equal to 1, $|b| \geq 1$. By Lemma 6.1.4,

$$20 \leq \|A\|_{HS}^2 \leq 4\|A\|^2,$$

which implies $\|A\| \geq \sqrt{5}$. Since the choice of A was arbitrary, $\beta_3(\mathbb{M}_5^0(\mathbb{C})_{\circlearrowleft,sa}) \leq \frac{1}{\sqrt{5}}$. Since

$$\beta_3\left(\begin{bmatrix} 0 & e^{2\pi i/5} & e^{-\pi i/5} & e^{\pi i/5} & e^{-2\pi i/5} \\ e^{-2\pi i/5} & 0 & e^{2\pi i/5} & e^{-\pi i/5} & e^{\pi i/5} \\ e^{\pi i/5} & e^{-2\pi i/5} & 0 & e^{2\pi i/5} & e^{-\pi i/5} \\ e^{-\pi i/5} & e^{\pi i/5} & e^{-2\pi i/5} & 0 & e^{2\pi i/5} \\ e^{2\pi i/5} & e^{-\pi i/5} & e^{\pi i/5} & e^{-2\pi i/5} & 0 \end{bmatrix}\right) = \frac{1}{\sqrt{5}},$$

we have that $\beta_3(\mathbb{M}_5^0(\mathbb{C})_{\circlearrowleft,sa}) = \frac{1}{\sqrt{5}}$. □

The following proposition establishes the third entry of the last column of Table 1.4.2:

Proposition 8.2.3. $\beta_3(\mathbb{M}_6^0(\mathbb{C})_{\circlearrowleft,sa}) \leq \frac{1}{\delta} \approx 0.4495$, *where* $\delta := 2\cos(5\pi/12) + 1 + \cos(\pi/4) \approx 2.2247$.

Proof. Let

$$A = \begin{bmatrix} 0 & a & b & c & \overline{b} & \overline{a} \\ \overline{a} & 0 & a & b & c & \overline{b} \\ \overline{b} & \overline{a} & 0 & a & b & c \\ c & \overline{b} & \overline{a} & 0 & a & b \\ b & c & \overline{b} & \overline{a} & 0 & a \\ a & b & c & \overline{b} & \overline{a} & 0 \end{bmatrix} \in \mathbb{M}_6^0(\mathbb{C})_{\circlearrowleft,sa},$$

with $\alpha_3(A) = 1$. We aim to show that $\|A\| \geq \cos(\pi/4) + 2\cos(5\pi/12) + 1 \approx 2.2247$. Arguing as in the proof of Proposition 5.4.2, we see that $|a|, |c| \geq 1$ and $|b| \geq \frac{1}{2}$.

Replacing A by $-A$, if necessary, we may assume that $c \geq 0$. Now by Equation (1),

$$\begin{aligned}\|A\| &= \max\{|az^5 + bz^4 + cz^3 + \overline{b}z^2 + \overline{a}z| : z^6 = 1\} \\ &= \max\{|2Re(az^5) + 2Re(bz^4) + cz^3| : z^6 = 1\} \\ &= \max\{|2Re(az^2) + 2Re(bz) + c| : z^6 = 1\} \\ &= \max\left\{|2Re(az^2) + c| + 2|Re(bz)| : z = 1, e^{2\pi i/3}, e^{4\pi i/3}\right\}.\end{aligned}$$

For $k \in \mathbb{Z}/24\mathbb{Z}$, define $W_k = \left\{re^{i\theta} : r \geq 0,\ \theta \in \left[\frac{k\pi}{12}, \frac{(k+1)\pi}{12}\right]\right\}$. Clearly $a \in W_k$ implies $ae^{2\pi i/3} \in W_{k+8}$ and $ae^{4\pi i/3} \in W_{k+16}$.

Case 1: Suppose $a \notin W_k$ for any $k = 3, 4, 11, 12, 19, 20$. Then there exists a $z \in \{1, e^{2\pi i/3}, e^{4\pi i/3}\}$ such that $az^2 \in W_j$ for some $j = 0, 1, 2, 21, 22, 23$. Then

$$\|A\| \geq |2Re(az^2) + c| \geq 2\cos(\pi/4) + 1 \approx 2.4142 > \delta.$$

Case 2: Suppose $a \in W_k$ for some $k = 3, 11, 19$. Then there exists a $z \in \{1, e^{2\pi i/3}, e^{4\pi i/3}\}$ such that $az^2 \in W_3$.

Case 2.1: Suppose $bz \notin W_j$ for any $j = 5, 6, 17, 18$. Then

$$\begin{aligned}\|A\| &\geq |2Re(az^2) + c| + 2|Re(bz)| \\ &\geq 2\cos(\pi/3) + 1 + \cos(5\pi/12) \approx 2.2588 > \delta.\end{aligned}$$

Case 2.2: Suppose $bz \in W_j$ for some $j = 5, 6, 17, 18$. Set $w = ze^{2\pi i/3}$. Then $w \in \{1, e^{2\pi i/3}, e^{4\pi i/3}\}$, $aw^2 \in W_{19}$, and $bw \in W_i$ for some $i = 1, 2, 13, 14$. Thus

$$\begin{aligned}\|A\| &\geq |2Re(aw^2) + c| + 2|Re(bw)| \\ &\geq 2\cos(5\pi/12) + 1 + \cos(\pi/4) = \delta.\end{aligned}$$

Case 3: Suppose $a \in W_k$ for some $k = 4, 12, 20$. Then arguing as in Case 2, we can show that $\|A\| \geq \delta$.

□

Remark 8.2.4. *The fourth, fifth, and sixth entries of the last column of Table 1.4.2 follow from Propositions 6.3.4, 6.3.5, and 6.3.6, respectively.*

Appendix A. Worst-known 3-Paver

The worst-known 3-paver A is a 13×13 complex circulant obtained by computer experimentation. The first column of A is (approximately) equal to

$$\vec{A}_1 = \begin{bmatrix} 0 \\ -0.055522930135728 + 0.149717916185917i \\ -0.085982594349687 - 0.167559358391542i \\ 0.012524801908532 - 0.005174683700118i \\ 0.211884289354117 - 0.450037958090483i \\ 0.181822822115818 + 0.190955159891972i \\ 0.351168610117535 - 0.052615522797929i \\ 0.003304818602041 + 0.071138805339765i \\ -0.242643523991422 + 0.113229168904351i \\ 0.147040327638516 + 0.000763498011691i \\ 0.306857154117503 - 0.502250996138940i \\ -0.333648956442746 - 0.012814790427734i \\ -0.255497016354932 - 0.470756522956261i \end{bmatrix}$$

We have that $A^*A \approx 1.3474 I_{13}$, i.e. A is (approximately) a scalar multiple of a unitary. The eigenvalues of A are (somewhat) uniformly distributed on the circle of radius $\sqrt{1.3474}$, except for a (nearly) multiplicity-two eigenvalue near the negative real axis.

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