PAVING SMALL MATRICES AND THE KADISON-SINGER EXTENSION PROBLEM II - COMPUTATIONAL RESULTS

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Abstract. This article is part exposition of a recent rather technical paper of the last two authors on matrix pavings related to the 1959 Kadison-Singer Extension Problem and part a report on further computational results providing new bounds for the paving parameters for classes of small matrices investigated there and subsequently.

(Website address with file to be created) provides to all interested the matrices experimentally discovered that yield these bounds along with the propietary Matlab software with simple operational directions to load them, pave them, and to perform paving searches.

The convergence to 1 or not of the infinite sequence of these parameters in most cases is equivalent to the Kadison-Singer Extension Problem and in all cases, convergence to 1 negates the problem. The last two sections describe the search process and an interpretation of the data integrated with the results of the precursor to this paper.

Contents

1. Introduction

The Kadison-Singer Extension Problem has a number of matrix equivalences, one of which is on the behavior of so-called paving parameters. Here we recall the definitions and report our recent experimental data on these parameters, extending data in [12] to larger matrices up to sizes over 20. There we provided sharp results for very small matrices whose proofs depended on blending this kind of experimental data with rigorous mathematics. This data expands our scope and is the next natural step in this program, which is to investigate paving small matrices towards

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generating ideas for the larger problem. The proprietary software created for Matlab for this project with simple operational directions and all concrete matrices produced to support our data we make available on request and encourage all who are interested to join the search.

One alternate source for learning about this subject is the website for the 2006 workshop: http://www.aimath.org/pastworkshops/kadisonsinger.html

2. The Kadison-Singer Extension Problem

Let ℓ^2 be the Hilbert space of absolutely square-summable complex sequences and $B(\ell^2)$ denote the von Neumann algebra of all bounded linear operators on ℓ^2. Every $T \in B(\ell^2)$ has an infinite matrix representation with respect to the standard orthonormal basis $\mathcal{E} = \{e_n : n \in \mathbb{N}\}$ of ℓ^2, namely $\big[T\big]_{\mathcal{E}} = \big[\langle Te_j, e_i\rangle\big]_{i,j\in\mathbb{N}}$. Identify ℓ^∞ with the set of all $T \in B(\ell^2)$ for which $\big[T\big]_{\mathcal{E}}$ is diagonal. Then ℓ^∞ is a maximal abelian von Neumann subalgebra (or MASA) of $B(\ell^2)$. A fundamental open problem in the theory of operator algebras is the *Kadison-Singer Extension Problem* (hereafter KS) [9]:

Does every pure state on ℓ^∞ extend uniquely to a pure state on $B(\ell^2)$?

A state on a C^*-algebra is a norm one positive linear functional and a pure state is an extreme point of this convex class. Existence of an extension of every pure state on ℓ^∞ is straightforward, the issue is uniqueness. Indeed, any Hahn-Banach extension of a state on ℓ^∞ is a state on $B(\ell^2)$. If the original state is pure, then the Krein-Milman Theorem implies the existence of a pure state extension. Alternatively, an explicit construction is available—the composition of a pure state on ℓ^∞ with the usual canonical conditional expectation of $B(\ell^2)$ onto ℓ^∞ is a pure state on $B(\ell^2)$ [3]. An affirmative answer to KS would entail a complete description of those pure states on $B(\ell^2)$ which restrict to pure states on ℓ^∞. They would be precisely the states of the form $\Phi_{\mathcal{U}}(T) = \lim_{\mathcal{U}}\langle Te_n, e_n\rangle$, where $\mathcal{U}$ is an ultrafilter (or Banach limit) on $\mathbb{N}$. While this would not cover all pure states on $B(\ell^2)$ [1], it would be a substantial step in that direction. Kadison and Singer doubted the truth of KS [9], and that is also the prevailing opinion among experts today. Although we share this sentiment, the data presented herein, in a weak experimental sense, appears to suggest that otherwise might be the case.

3. Anderson's Paving Problem

A major advance in the study of KS was made by Anderson, who reformulated the problem in terms of finite matrices [2]. (See also [11] in which their Proposition 2.2 combined with Theorems 2.7-2.8 provide a simplified transparent proof. That "KS is like Riemann integration" was an observation by Hadwin to Paulsen and Raghupathi which led to their Proposition 2.2 (communication by Paulsen).)

We state Anderson's result in terms of certain *paving parameters*. To define these we need the notion of a *paving*, which in turn relies on the idea of a *compression*.

Definition 3.1 (compression). *For $A \in \mathbb{M}_n(\mathbb{C})$ ($n \times n$ complex matrices) and $\sigma \subseteq \{1, 2, ..., n\}$, the σ-compression of A is $A_\sigma := P_\sigma A P_\sigma$, where $P_\sigma \in \mathbb{M}_n(\mathbb{C})$ is* *the orthogonal projection onto span*$\{\vec{e}_i : i \in \sigma\}$*. By a p-compression of A we mean a compression* A_σ *with* $\operatorname{card}(\sigma) = p$.

Note that $\|A_\sigma\| \leq \|P_\sigma\|\|A\|\|P_\sigma\| = \|A\|$, where $\|\cdot\|$ is the operator norm.

Definition 3.2 (paving)**.** *For* $A \in \mathbb{M}_n(\mathbb{C})$ *and* $\pi \in \Pi_k^n$ *(the set of all k-partitions of* $\{1, 2, ..., n\}$*), the* π*-paving of A is* $A^\pi := \sum_{\sigma\in\pi} A_\sigma$*. By a k-paving of A we mean a paving* A^π *with* $\operatorname{card}(\pi) = k$*. By an* $(n_1, n_2, ..., n_k)$*-paving of A we mean a paving* A^π *with* $\pi = \{\sigma_1, \sigma_2, ..., \sigma_k\}$*, where* $\operatorname{card}(\sigma_i) = n_i$ *for all* $1 \leq i \leq k$*.*

Note that
$$\|A^\pi\| = \left\| \sum_{\sigma\in\pi} A_\sigma \right\| = \max\{\|A_\sigma\| : \sigma \in \pi\} \leq \|A\|.$$

Definition 3.3 (paving parameters for a matrix)**.** *For* $0 \neq A \in \mathbb{M}_n(\mathbb{C})$*, the k-paving parameter of A is*
$$\alpha_k(A) := \min\{\|A^\pi\| : \pi \in \Pi_k^n\} \in [0, \|A\|].$$
The normalized k-paving parameter of A is
$$\beta_k(A) := \frac{\alpha_k(A)}{\|A\|} \in [0, 1].$$

There are two other simpler parameters with consequent faster computing:

Packing parameters (motivated by paving results of Bourgain and Tzafriri [5, Corollary 1.2]): where instead of measuring compression norms using finite diagonal projection decompositions of the identity, we use only single diagonal projections of size $\lceil \frac{n}{k} \rceil$, which resulting parameters, it is straightforward to see, provide lower bounds for paving parameters. Formally, the packing parameters are defined the same as the paving parameters but replacing $\beta_k(A)$ with the normalized k-packing parameter of A:
$$\beta'_k(A) := \frac{\alpha'_k(A)}{\|A\|} \in [0, 1]$$
where
$$\alpha'_k(A) := \min\{\|A_\sigma\| \mid card(\sigma) = \lceil \frac{n}{k} \rceil\} \in [0, \|A\|].$$

Equi-paving parameters: where we restrict the projections to "equal dimension." For example, to decompose the identity for paving a 13×13 matrix we use diagonal projections only of dimensions $4, 4, 5$ rather than all possible triples summing to 13 as used for defining paving parameters. Equi-paving parameters are upper bounds for paving parameters and bounded by 1.

Definition 3.4 (paving parameters for matrix classes)**.** *For* $\emptyset \neq \mathcal{S} \subseteq \mathbb{M}_n(\mathbb{C})$*, the (normalized) k-paving parameter of* $\mathcal{S}$ *is*
$$\beta_k(\mathcal{S}) := \sup\{\beta_k(A) : A \in \mathcal{S}\} \in [0, 1].$$

In this paper, $\mathcal{S}$ above will be one of the following classes:

i. $\mathbb{M}_n^0(\mathbb{C})$, the set of all $n \times n$ zero-diagonal complex matrices.
ii. $\mathbb{M}_n^0(\mathbb{R})$, the set of all $n \times n$ zero-diagonal real matrices.
iii. $\mathbb{M}_n^0(\mathbb{R}_+)$, the set of all $n \times n$ zero-diagonal non-negative (entried) matrices.
iv. $\mathbb{M}_n^0(\mathbb{C})_\triangle$, the set of all $n \times n$ strictly upper-triangular complex matrices.
v. $\mathbb{M}_n^0(\mathbb{C})_T$, the set of all $n \times n$ zero-diagonal complex Toeplitz matrices (See http://en.wikipedia.org/wiki/Toeplitz_matrix).

v'. $\mathbb{M}_n^0(\mathbb{C})_\circlearrowleft$, the set of all $n \times n$ zero-diagonal complex circulants (a class of Toeplitz matrices: See http://en.wikipedia.org/wiki/Circulant_matrix).

vi. $\mathbb{M}_n^0(\mathbb{C})_{sa} = \{A \in \mathbb{M}_n^0(\mathbb{C}) : A^* = A\}$ (A^* denotes the adjoint of A, i.e., the conjugate-transpose).

vii. $\mathbb{M}_n^0(\mathbb{R})_{sa} = \{A \in \mathbb{M}_n^0(\mathbb{R}) : A^* = A\}$.

viii. $\mathbb{M}_n^0(\mathbb{C})_{\circlearrowleft,sa} = \{A \in \mathbb{M}_n^0(\mathbb{C})_\circlearrowleft : A^* = A\}$.

Using the fact that $\beta_k(A \oplus 0) = \beta_k(A)$ [4], we deduce that $\beta_k(\mathbb{M}_n^0(\mathbb{C})) \le \beta_k(\mathbb{M}_{n+1}^0(\mathbb{C}))$, and so

$$\lim_{n\to\infty} \beta_k(\mathbb{M}_n^0(\mathbb{C})) = \sup_n \beta_k(\mathbb{M}_n^0(\mathbb{C})) \in [0,1].$$

This increasing limit formula is true for all matrix classes considered above, with the exception of $\mathbb{M}_n^0(\mathbb{C})_\circlearrowleft$ and $\mathbb{M}_n^0(\mathbb{C})_{\circlearrowleft,sa}$ and $\mathbb{M}_n^0(\mathbb{C})_T$. Indeed, monotonicity follows when a class is closed under direct summing with zero, but the direct sum of a nonzero circulant with zero is never circulant and the direct sum of a nonzero Toeplitz matrix with zero is never a Toeplitz matrix.

We can now state Anderson's theorem [2] on the equivalence of KS and the so-called *Paving Problem*:

Theorem 3.5. *The following are equivalent:*

1. *Every pure state on ℓ^∞ extends uniquely to a pure state on $B(\ell^2)$, i.e., KS is true.*
2. *There exists a $k \in \mathbb{N}$ such that $\lim_{n\to\infty} \beta_k(\mathbb{M}_n^0(\mathbb{C})) < 1$.*
3. *For every $0 < \epsilon < 1$, there exists a $k \in \mathbb{N}$ such that $\lim_{n\to\infty} \beta_k(\mathbb{M}_n^0(\mathbb{C})) < \epsilon$.*

Using the formula $\alpha_{k^2}(A+B) \le \alpha_k(A) + \alpha_k(B)$ [4], one can show that $\mathbb{M}_n^0(\mathbb{C})$ in Theorem 3.5 may be replaced by $\mathbb{M}_n^0(\mathbb{R})$, $\mathbb{M}_n^0(\mathbb{C})_{sa}$, or $\mathbb{M}_n^0(\mathbb{R})_{sa}$. Owing to recent work of Paulsen and Raghupathi on logmodular algebras, $\mathbb{M}_n^0(\mathbb{C})_\triangle$ also works [11]. So solving the Paving Problem for any of these classes would settle KS.

Not relevant here but curious and an easy consequence is an infinite matrix KS equivalent free from requiring a universal $\epsilon < 1$ and free from requiring a universal fixed k is:

Corollary 3.6. *KS is equivalent to the condition: Every norm one zero diagonal $B(\ell^2)$-matrix A admits a finite commuting diagonal decomposition of the identity, $I = P_1 + \cdots + P_k$ with k depending on A, for which $\|\sum_{i=1}^k P_i A P_i\| < 1$.*

Laurent, Toeplitz and upper-triangular operators.
The paveability of the special class of Laurent operators (L^∞ multiplication operators on L^2 of the torus-so in the standard basis is a bi-infinite matrix with constant diagonals) has been studied in depth by the paving community as well, making central to this subject the behavior of their paving parameters (tending to 1 or not) and the paving parameters of their Toeplitz matrices (compressions of Laurent matrices using only finite consecutive integer projections). The Paulsen and Raghupathi work on logmodular algebras also yields the equivalence of the paveability of the Toeplitz matrices $\mathbb{M}_n^0(\mathbb{C})_T$ to the paveability of the upper triangular Toeplitz matrices $\mathbb{M}_n^0(\mathbb{C})_\triangle \cap \mathbb{M}_n^0(\mathbb{C})_T$. So we include both these categories in our experiments. A counterexample sequence (i.e., tending to 1) to the latter provides a counterexample to KS but also not tending to 1 is equivalent to the paveability

of the Laurents.

4. Paving Results

Because of Theorem 3.5, it is of substantial interest to compute $\lim_{n\to\infty} \beta_k(\mathbb{M}_n^0(\mathbb{C}))$ for $k \in \mathbb{N}$, $k \geq 2$ (as well as the corresponding limits for other matrix classes). Nonetheless, heretofore this has only been accomplished for $k = 2$. Since

$$\beta_2\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\right) = 1,$$

we have (trivially) that

$$\lim_{n\to\infty} \beta_2(\mathbb{M}_n^0(\mathbb{C})) = \lim_{n\to\infty} \beta_2(\mathbb{M}_n^0(\mathbb{R})) = \lim_{n\to\infty} \beta_2(\mathbb{M}_n^0(\mathbb{R}_+)) = 1.$$

The self-adjoint case, which is much more delicate, was recently settled by Casazza, Edidin, Kalra, and Paulsen [6]:

Theorem 4.1. $\lim_{n\to\infty} \beta_2(\mathbb{M}_n^0(\mathbb{C})_{sa}) = 1$.

Remark 4.2. *The question of attainment in Theorem 4.1, i.e., whether or not there exists an $A \in \mathbb{M}_n^0(\mathbb{C})_{sa}$ with $\beta_2(A) = 1$, is still open and of considerable interest. For this we have some new data (Theorem 5.3).*

Turning to $k = 3$, prior to [12] the only result in the literature is due to Halpern, Kaftal and Weiss [8]:

Theorem 4.3. $\lim_{n\to\infty} \beta_3(\mathbb{M}_n^0(\mathbb{C})) \geq \lim_{n\to\infty} \beta_3(\mathbb{M}_n^0(\mathbb{R})) \geq \frac{2}{3}$.

On the other hand, the Paving Problem for non-negative matrices is known to have a positive answer, thanks to work of Berman, Halpern, Kaftal and Weiss [4]:

Theorem 4.4. *For $k \in \mathbb{N}$,*

$$\lim_{n\to\infty} \beta_k(\mathbb{M}_n^0(\mathbb{R}_+)_{sa}) = \frac{1}{k} \text{ and } \lim_{n\to\infty} \beta_k(\mathbb{M}_n^0(\mathbb{R}_+)) \leq \frac{2}{k}.$$

Unfortunately, KS seems not equivalent to the Paving Problem for non-negative matrices.

5. Summary of the Papers

The impetus for [12] was a question of Halpern, Kaftal and Weiss concerning Theorem 4.3 [8]:

Is $\lim_{n\to\infty} \beta_3(\mathbb{M}_n^0(\mathbb{C})) < 1$? or Is $\lim_{n\to\infty} \beta_3(\mathbb{M}_n^0(\mathbb{C})) = 1$?
At least is $\lim_{n\to\infty} \beta_3(\mathbb{M}_n^0(\mathbb{C})) > \frac{2}{3}$?

By computing $\beta_3(\mathbb{M}_n^0(\mathbb{C}))$ for small values of n, we are able to answer the second question affirmatively. Using paving parameters, packing parameters and equi-paving parameters, we obtain the following 3-paving tables, which are thus far the main results of this investigation. For detailed explanations of how to read these tables see the remarks following them (Remarks 5.5 - 5.6).

Theorem 5.1 (3-Paving Tables-Nonselfadjoints-monotone/nonmonotone β_3)**.**

n	$\beta_3(\mathbb{M}_n^0(\mathbb{C}))$	$\beta_3(\mathbb{M}_n^0(\mathbb{R}))$	$\beta_3(\mathbb{M}_n^0(\mathbb{R}_+))$	$\beta_3(\mathbb{M}_n^0(\mathbb{C})_\triangle)$
4	.6180	.6180	.5550	.5412
5	.6180	.6180	.5550	[.5609, .5774]
6	.7071	.7071	[.5550, .5774]	[.5725, .5774]
7	[.8239, 1]	[.8029, 1]	[.5550, .6667]	[.6503, .9258]
8	″	″	″	[.6599, 1]
9	″	″	″	[.6630, 1]
10	[.8540, 1]	[.8387, 1]	″	[.6703, 1]
11	″	″	″	″
12	″	″	″	″
13	[.8615, 1]	″	″	[.6852, 1]
14	″	″	″	″
15	″	″	″	″
16	″	″	″	[.6963, 1]
$17-22$	″	″	″	″

n	$\beta_3(\mathbb{M}_n^0(\mathbb{C})_\circlearrowleft)$	$\beta_3(\mathbb{M}_n^0(\mathbb{C})_T)$	$\beta_3(\mathbb{M}_n^0(\mathbb{C})_T \cap \mathbb{M}_n^0(\mathbb{C})_\triangle)$
4	[.6000, .6030]	[,]	[,]
5	[.6120, .6180]	[,]	[,]
6	[.5726, .6325]	[,]	[,]
7	[.8239, 1]	[,]	[,]
8	[.7651, 1]	[,]	[,]
9	[.7132, 1]	[,]	[,]
10	[.8540, 1]	[,]	[,]
11		[,]	[,]
12		[,]	[,]
13	[.8615, 1]	[,]	[,]
14	[.8119, 1]	[,]	[,]
15	[.7802, 1]	[,]	[,]
16	[.8523, 1]	[,]	[,]
17	[.8125, 1]	[,]	[,]
18	[.7617, 1]	[,]	[,]
19	[.8424, 1]	[,]	[,]
22	[.8230, 1]	[,]	[,]

Theorem 5.2 (3-Paving Table for Selfadjoint Matrices)**.**

n	$\beta_3(\mathbb{M}_n^0(\mathbb{C})_{sa})$	$\beta_3(\mathbb{M}_n^0(\mathbb{R})_{sa})$	$\beta_3(\mathbb{M}_n^0(\mathbb{C})_{\circlearrowleft,sa})$
4	.5774	.4472	.4142
5	.5774	.4472	.4472
6	.5774	.4851	[.4069, .4495]
7	[.6872, .7559]	[.6667, .7559]	[.6544, .7559]
8	[.6872, .8819]	[.6667, .8819]	[.5797, .8819]
9	[.6872, .8889]	[.6667, .8889]	[.5539, .8889]
10	[.7536, 1]	[.7454, 1]	[.6686, 1]
11	″	″	
12	[″	″	
13	″	″	[.6983, 1]
14	″	″	
15	″	″	
16	[.7574, 1]	[.7454, 1]	[.7019, 1]

To support bootstrapping arguments for the 3-paving tables, as well as because of intrinsic interest, we also compute the following 2-paving table:

Theorem 5.3 (2-Paving Table)**.**

n	$\beta_2(\mathbb{M}_n^0(\mathbb{C})_{\triangle})$	$\beta_2(\mathbb{M}_n^0(\mathbb{C})_{\circlearrowleft})$	$\beta_2(\mathbb{M}_n^0(\mathbb{C})_{sa})$	$\beta_2(\mathbb{M}_n^0(\mathbb{R})_{sa})$	$\beta_2(\mathbb{M}_n^0(\mathbb{C})_{\circlearrowleft,sa})$
3	.6180	1	.5774	.5000	.5774
4	.7071	[.6000, .6030]	.5774	[.5493, .5577]	.4142
5	[.7715, 1]	1	.8944	.8944	.8944
6	[.8337, 1]	1	.8944	.8944	[.7454, .8944]
7	[.8500, 1]	1	[.9225, 1]	[.8944, 1]	[.9073, 1]
8	[.8866, 1]	[.9623, 1]	[.9225, 1]	″	[.7689, 1]
9	[.8965, 1]	1	[.9414, 1]	″	[.8920, 1]
10	[.9149, 1]	1	[.9414, 1]	″	
11	[.9207, 1]	1	[.9477, 1]	″	
12	″	1	[.9477, 1]	″	
13	″	1	[.9547, 1]	″	
14	″	1	[.9547, 1]	″	
15	″	1	[.9625, 1]	″	
16	″	[.9846, 1]	[.9625, 1]	″	
17	″	1	[.9692, 1]	″	
18	″	1	[.9692, 1]	″	
19	″	1	[.9742, 1]	″	
20	″	1	[.9742, 1]	″	
21	″	1	[.9762, 1]	″	
22	″	1	[.9762, 1]	″	
23	″	1	[.9780, 1]	″	

Remark 5.4 ($\beta_2(\mathbb{M}_n^0(\mathbb{C})_{sa})$ implication)**.** *The shrinking gaps for* $\beta_2(\mathbb{M}_n^0(\mathbb{C})_{sa})$ *suggest a reasonable experimental conjecture to Remark 4.2: that* 1 *is not attainable. To give a sense of the computational effort involved, the latter six bounds took manpower months of parallel independent experiments using over 40 modern computers.*

The remainder of the precursor to this paper [12] is divided into seven sections consisting of a myriad of propositions, each of which establishes particular entries in our tables. Each section corresponds to a certain class of matrices. Although the arguments are structurally similar, the details vary from class to class. Each section begins with a subsection (or two) which gathers together the needed tools. We assumed there and here that the reader is familiar with basic operator theory and basic graph theory.

Remark 5.5 (exact vs. approximate, ditto and blank boxes)**.** *The numbers in our paving tables are decimal approximants. The corresponding exact expressions (i.e., closed forms when available) appear in the proposition statements of* [12]. *It is natural to expect that when a single number is listed in the tables, rather than an interval, a proof is available for it that is likely to generate a precise value rather than a decimal approximant, for instance, the reciprocal of Fibonacci's Golden Ratio:* $\beta_3(\mathbb{M}_4^0(\mathbb{C})) = \frac{2}{1+\sqrt{5}}$.

A blank data box means no experiments or not enough of them have been made to yield credible values. A ditto means the interval for a previous matrix size remains theoretically (provably) valid, usually because the respective column of paving parameters is one which is easily seen to be theoretically increasing.

Remark 5.6 (computer-generated examples)**.** *For those table entries which consist of an interval (e.g. the* $n = 7$ *entry of the first column of Table 5.1), the lower bound is (almost always) the result of a computer-generated example. To our knowledge, these examples do not have closed-form expressions, and (with one exception) we do not include them in the paper. In Section 6 we do show the worst-known 3-paver, a* 13×13 *complex circulant* A *such that* $\beta_3(A) \approx .8615$.

Remark 5.7 (open questions)**.** *This paper invites many questions. In particular, can any of the non-sharp table entries be improved? Here are some other interesting questions:*

1. *Is* $\lim_{n\to\infty} \beta_3(\mathbb{M}_n^0(\mathbb{C})) < 1$ *or is* $\lim_{n\to\infty} \beta_3(\mathbb{M}_n^0(\mathbb{C})) = 1$*? At least is* $\lim_{n\to\infty} \beta_3(\mathbb{M}_n^0(\mathbb{C})) > .8615$*? This is the aforementioned question of Halpern, Kaftal and Weiss, amended to reflect the information in Table 5.1.*
2. *Does there exist* $n \in \mathbb{N}$ *and* $A \in \mathbb{M}_n^0(\mathbb{C})_{sa}$ *such that* $\beta_2(A) = 1$.
3. *Early on Table 5.3 suggested an affirmative answer with* $n \approx 30$ *but recent data as mentioned in Remark 5.4 has changed our view. Remember that it is known that* $\lim_{n\to\infty} \beta_2(\mathbb{M}_n^0(\mathbb{C})_{sa}) = 1$.
4. *Is* $\lim_{n\to\infty} \beta_2(\mathbb{M}_n^0(\mathbb{C})_\triangle) = 1$*? Table 5.3 suggests an affirmative answer. In that case, is there an* $n \in \mathbb{N}$ *and* $A \in \mathbb{M}_n^0(\mathbb{C})_\triangle$ *such that* $\beta_2(A) = 1$*?*
5. *Is* $\lim_{n\to\infty} \beta_3(\mathbb{M}_n^0(\mathbb{R}_+)) < \frac{2}{3}$ *or is* $\lim_{n\to\infty} \beta_3(\mathbb{M}_n^0(\mathbb{R}_+)) = \frac{2}{3}$*?*
   Table 5.1 along with substantial computer experimentation suggest that $\beta_3(\mathbb{M}_n^0(\mathbb{R}_+)) = \frac{1}{\kappa} \approx .5550$ *for all* $n \geq 4$ *where*
   $$\kappa := \left\| \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \right\| = \sqrt{\frac{5 + 2\sqrt{7}\cos(\tan^{-1}(3\sqrt{3})/3)}{3}} \approx 1.8019.$$
   Trivially $\beta_3(\mathbb{M}_3^0(\mathbb{R}_+)) = 0$.
6. *Is KS equivalent to the Paving Problem for circulants?*

6. Worst-known 3-Paver

The worst-known 3-paver A is a 13×13 complex circulant obtained by computer experimentation. The first column of A is (approximately) equal to

$$\vec{A}_1 = \begin{bmatrix} 0 \\ -0.055522930135728 + 0.149717916185917i \\ -0.085982594349687 - 0.167559358391542i \\ 0.012524801908532 - 0.005174683700118i \\ 0.211884289354117 - 0.450037958090483i \\ 0.181822822115818 + 0.190955159891972i \\ 0.351168610117535 - 0.052615522797929i \\ 0.003304818602041 + 0.071138805339765i \\ -0.242643523991422 + 0.113229168904351i \\ 0.147040327638516 + 0.000763498011691i \\ 0.306857154117503 - 0.502250996138940i \\ -0.333648956442746 - 0.012814790427734i \\ -0.255497016354932 - 0.470756522956261i \end{bmatrix}$$

We have that $A^*A \approx 1.3474 I_{13}$, i.e., A is (approximately) a scalar multiple of a unitary. The eigenvalues of A distributed on the circle of radius $\sqrt{1.3474}$ are approximately

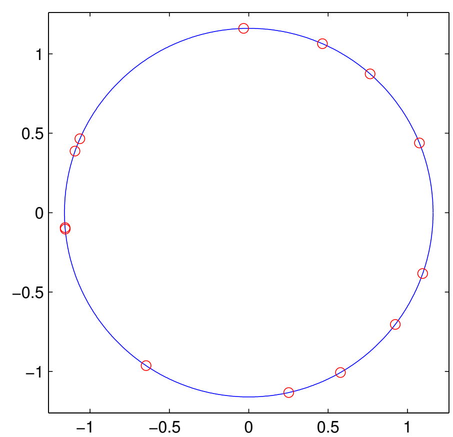

Figure 1. Eigenvalues of the Circulant13 near radius 1.1608

7. Informal summary description of search process

In the beginning, the fact that

$$\lim_{n\to\infty} \beta_2(\mathbb{M}_n^0(\mathbb{C})) = \lim_{n\to\infty} \beta_2(\mathbb{M}_n^0(\mathbb{R})) = \lim_{n\to\infty} \beta_2(\mathbb{M}_n^0(\mathbb{R}_+)) = 1.$$

follows simply from

$$\beta_2(\mathbb{M}_3^0(\mathbb{C})) = \beta_2\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\right) = 1$$

prompted us to try to theoretically compute $\beta_3(\mathbb{M}_4^0(\mathbb{C}))$. Computer experiments led us to the matrix

$$\beta_3\left(\begin{bmatrix} 0 & 1 & 1 & -\frac{2}{1+\sqrt{5}} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}\right) = \frac{2}{1+\sqrt{5}} \approx .6180,$$

and theoretical arguments involving graphs and basic matrix facts provided a proof that this bound was sharp and this matrix was an optimal "bad paver," that is, $\frac{2}{1+\sqrt{5}} \leq \beta_3(\mathbb{M}_4^0(\mathbb{R})) \leq \beta_3(\mathbb{M}_4^0(\mathbb{C})) \leq \frac{2}{1+\sqrt{5}}$. We proved the same bound holds for $n = 5$ with optimal bad paver the direct sum of this matrix with 0. And from a graph theory perspective, optimal bad pavers were rare. So also for $n = 6$ where

$$\beta_3\left(\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ -\frac{1}{2} & 1 & \frac{1}{2} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & -\frac{1}{2} & 0 & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}\right) = \frac{1}{\sqrt{2}} \approx .7071,$$

and from expanded theoretical arguments, that this was an optimal bad paver, that is, $\frac{1}{\sqrt{2}} \leq \beta_3(\mathbb{M}_6^0(\mathbb{R})) \leq \beta_3(\mathbb{M}_6^0(\mathbb{C})) \leq \frac{1}{\sqrt{2}}$.

After working on theoretical arguments for $n = 7$ (the selfadjoint case) combined with computer searches, all described in the prequel, and because the paving community bias was against KS, so against paveability, we concentrated more recently mainly on computer searches and proceeded as follows.

Choose a class among those listed above (for instance, circulants or selfadjoint matrices), choose a matrix size n, say $n = 7$ or $n = 23$, and perform a random search. Do this repeatedly. Such searches consist of alternating between the off the shelf Matlab simplex search program and the proprietary search program based on von Neumann's alternating projection method for finding points of intersection between closed convex sets in a Hilbert space. When the searcher detects little improvement, that is, when one finds an apparent local peak (for the paving parameter), the search is finished. Another search by perturbing this matrix occasionally produces a bounce, that is, knocks the matrix off its local peak and a new ascension begins. Repeat this many times.

We soon found that the most successful searches began with a prior bad paver initial matrix, then searching and perturbing sometimes mixing search types to worsen it as measured by increasing the paving parameter. There is another variable constraint we can control called the maxradius (reciprocal of the norm of admissible sample matrices) which restricts the domain, and hence the complexity of the search. It turns out experimentally that that can be crucial in the more successful searches, though we understand little why. Setting the maxradius button closer to the target paving parameter often yields faster searches and larger paving parameter results. In particular, the largest maxradius theoretically is 1, but this almost never results in a good search result.

By combining algebraically, or through direct summing and weighted combinations, we created many initial matrices of increasing sizes that again occasionally achieved significantly larger paving parameters. After time and little search progress on a matrix class and size, we then accept the matrix and its paving parameter as an experimental approximant of a possible theoretical optimal bad paver.

The next steps in this project might be to analyze such matrices for their properties and how optimal they really are, as we did for $n = 4 - 6$ and the selfadjoint case for $n = 7$. Indeed, one hope of this project is to gain insight into which classes or special matrix structures and their properties achieve bad optimal status and why. Such information might lead to constructing an infinite sequence of increasing size matrices with paving parameters tending to 1, or otherwise, reveal structures that constrain paving parameters from approaching 1. At the very least this project generated in the prequel a myriad of questions on small matrices that may be of interest.

8. Interpretation of results and limitations

Recognizing that experimental work can be off the mark, but at the same time respecting it for its center stage role in the theoretical results obtained so far, we provide the following data analysis and observations on its weaknesses.

1. $\beta_3(\mathbb{M}_n^0(\mathbb{C}))$ may stay bounded by approximately .8615. Despite our searches for several years now throughout all these classes and for matrices up to size 22, we have been unable to surpass this paving parameter bound. Likewise, for month we have been unable to surpass the paving parameter bound of .8387 for real-entried matrices up to size 14. Likewise for all the other monotone paving parameters, they appear bounded and if not, very slow growing to 1.

Weakness of the evidence. Comparing the proofs of the $n = 4, 6$ cases, we see that the rarity of the bad paver classes in a graph theory class sense jumps dramatically. In the best precise calculations of paving parameters in the prequel [12, Propositions 2.2.1-2.2.3], we use digraph data [13] that demonstrates this. For the 4 case, we showed that optimal bad pavers lie in one of 218 classes defined by their properties related to digraphs on 4 vertices. For the 6 case, that number is much larger. So if bad pavers are very rare with rarity increasing with matrix size, then a random matrix search approach has its obvious weaknesses. Nevertheless, bootstraping by using improved initial matrices has shown alot of promise in this effort.

2. *Experimental evidence* for positive-entried zero-diagonal matrices:

$$\beta_3(\mathbb{M}_n^0(\mathbb{R}_+)) \approx .5550 \text{ for all } n \geq 4.$$

Work on $\beta_3(\mathbb{M}_6^0(\mathbb{R}_+)) \in [.5550, .5774]$ yielded for this line of investigation some theoretical results and several questions [12, Section 3, esp. Proposition 3.3.3].

3. Improvement in computer speed and in our software since the project began contributed significantly to improving our theoretical and experimental results. We hope that with others joining the search and with newer technology, better evidence and more proofs evolve.

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