A reprint from the

JOURNAL OF OPERATOR THEORY

Published by the
Department of Mathematics
of the
NATIONAL INSTITUTE
FOR SCIENTIFIC
AND TECHNICAL CREATION
Bucharest

INCREST

J. OPERATOR THEORY
16(1986), 355–374

MATRIX PAVINGS AND LAURENT OPERATORS

H. HALPERN, V. KAFTAL, G. WEISS

1. INTRODUCTION

Let $D \subset B(H)$ be the algebra of diagonal operators relative to a fixed basis of a Hilbert space H, then a long standing open question is whether the following equivalent properties hold [1, 2, 3, 9, 10, 11].

The extension property: every pure state of D has a unique extension to a pure state of $B(H)$.

The relative Dixmier property: for every $x \in B(H)$ the set $K(x) = \overline{\mathrm{co}}\{wxw^* \mid w \in D, w \text{ unitary}\}$ has nonempty intersection with D.

The paving property: for every $x \in B(H)$ and $\varepsilon > 0$, there is a diagonal decomposition of the identity (a decomposition of the identity into finitely many mutually orthogonal diagonal projections $p(j)$) such that $\|p(j)(x - E(x))p(j)\| < \varepsilon$ for every j, where $E(x)$ denotes the diagonal of x.

The relative Dixmier property can be formulated in the more general setting of the embedding of two C^*-algebras $A \subset B$ and was first proven to hold true by J. Dixmier for the case that $A = B$ is a von Neumann algebra [5]. We study this property further in [8, 9, 10].

In this paper we investigate a related problem: if $\{u_t, t \in [0, 1)\}$ is the one parameter group of diagonal unitaries that implements the dual action in the decomposition of $B(H)$ as the crossed product of D by $\mathbf{Z}$, [4, 10, 13] and if $K_0(x) = \overline{\mathrm{co}}\{u_t x u_t^* \mid t \in \mathbf{Q} \cap [0, 1)\}$, then for which $x \in B(H)$ does $K_0(x)$ have a nonempty intersection with D? We call such an x *uniformly paveable* because of its characterization obtained in terms of "uniform decompositions of the identity", i.e. decompositions of the identity into diagonal projections which correspond to infinite arithmetic progressions under the natural identification of diagonal projections with subsets of the integers (Definition 2.1).

It is well known that $K(x) \cap D \neq \emptyset$ iff $E(x) \in K(x)$, so the distance $\alpha(x)$ (resp. $\tilde{\alpha}(x)$) from $E(x)$ to $K(x)$ (resp. to $K_0(x)$) can be seen as a modulus of paveability (resp. uniform paveability). We express α and $\tilde{\alpha}$ in terms of decompositions of the identity (resp. uniform decompositions of the identity) and in terms of $U(D)$-invariant (resp. $\{u_t \mid t \in Q \cap [0, 1)\}$-invariant) functionals (Proposition 2.3).

It was shown in [11, Theorem 2] that the class $\mathscr{L}$ of Laurent operators, i.e. the operators L_φ of multiplication by functions φ in $L^\infty(0, 1)$ acting on $L^2(0, 1)$, does not have the extension property with respect to $B(H)$. This was achieved by considering a type of "uniform" decompositions of the identity into Laurent projections and by constructing a suitable diagonal operator for a counterexample. Thus in the context of uniform paveability (with respect to D) it is of particular interest to study $\mathscr{L}$.

In §3 we reduce uniform paveability for $\mathscr{L}$ to an ergodic property in $L^\infty(0, 1)$ (Proposition 3.5), and we prove the existence of Laurent operators that are not uniformly paveable (Theorem 3.6).

In §4 we find an upper bound for the modulus of uniform paveability $\tilde{\alpha}(L_\varphi)$ of a Laurent operator L_φ in terms of the upper and lower Riemann integrals of φ. As a consequence, we obtain that all Laurent operators with symbol in the C^*-subalgebra of $L^\infty(0,1)$ of Riemann integrable functions are uniformly paveable and hence are paveable.

Conversely, in Theorem 4.5 we show that $\tilde{\alpha}(L_\varphi) = 1 - m(V)$ if φ is the characteristic function of an open dense subset V of $(0,1)$, hence L_φ is not uniformly paveable when $1 - m(V) > 0$.

In Theorem 5.4 we strengthen this result by showing that if V is an open dense set with $m(V) \leqslant 1 - m(V)$, then for every diagonal projection p "containing" an infinite arithmetic progression, we have that $\|p(L_\varphi - E(L_\varphi))p\| = 1 - m(V)$.

Moreover, for every such V and $0 < \beta < 1$, there is an integer valued function $f(n, \beta)$ such that $\|p(L_\varphi - E(L_\varphi))p\| > \beta(1 - m(V))$ for every diagonal projection p which "contains" a β-sufficiently long arithmetic progression, (i.e. an arithmetic progression of difference n and length at least $f(n, \beta)$).

The function $f(n, \beta)$ is related to the rate of ℓ^2-convergence of the Fourier coefficients of $L^\infty(0, 1)$ functions. In Propositions 5.5, 5.6 we compute an upper and a lower bound for $f(n, \beta)$ (for certain values of β) in terms of the geometry of the set V.

These results relate also to the paving problem. The fact that $\|p(L_\varphi - E(L_\varphi))p\| > \beta(1 - m(V))$ for the large class of diagonal projections p containing β-sufficiently long arithmetic progressions, makes L_φ a natural candidate for a nonpaveable operator.

An analysis of the decompositions of the identity containing projections of this class pertains to combinatorics. The van der Waerden theorem [6, Theorem 2.1] guarantees that for every finite coloring of the integers (or equivalently, for every diagonal decomposition of the identity), there are arbitrarily long monochromatic arithmetic progressions. However, little is known about the connection between the length t of each such arithmetic progressions and its difference n.

Thus our results lead to an interplay between combinatorics, operator theory and Fourier analysis that might shed some light on the paving problem.

2. PRELIMINARY RESULTS

Let H be a separable Hilbert space with basis $\{\eta_k \mid k \in \mathbf{Z}\}$, let $D \subset B(H)$ be the algebra of diagonal operators relative to this basis and let $E: B(H) \to D$ be the canonical faithful normal conditional expectation onto D ("taking the main diagonal"). Denote by $U(D)$ the group of the unitary diagonal operators. For every $x \in B(H)$, let $K(x)$ be the norm closure of the convex hull of the diagonal unitary orbit of x. In symbols,

$$K(x) = \overline{\mathrm{co}}\{wxw^* \mid w \in U(D)\}.$$

We say that a finite collection $\{p(j) \mid j = 1, 2, \ldots, n\}$ of mutually orthogonal projections $p(j) \in D$ for which $\sum_{j=1}^{n} p(j) = 1$ is a *diagonal decomposition of the identity*.

If furthermore $p(j)$ is the projection on the closure of $\mathrm{span}\{\eta_k \mid k = j + mn, m \in \mathbf{Z}\}$, we call $\{p(j)\}$ a *uniform decomposition of the identity* or, more precisely, the n-uniform decomposition of the identity.

Under the natural identification of diagonal projections and subsets of $\mathbf{Z}$, an n-uniform decomposition of the identity corresponds to a disjoint partition of $\mathbf{Z}$ (a coloring, in the language of combinatorics) in n infinite arithmetic progressions with the same difference n. By Remark 2.2 we shall see that there is no gain in generality in considering all the partitions of $\mathbf{Z}$ into infinite arithmetic progressions with arbitrary differences.

DEFINITION 2.1. Let $x \in B(H)$, then we define

$$\alpha(x) = \inf\left\{\left\|\sum_{j=1}^{n} p(j)(x - E(x))p(j)\right\| \,\Big|\, \{p(j)\} \text{ is a diagonal decomposition of the identity}\right\},$$

$$\tilde{\alpha}(x) = \inf\left\{\left\|\sum_{j=1}^{n} p(j)(x - E(x))p(j)\right\| \,\Big|\, \{p(j)\} \text{ is a uniform decomposition of the identity}\right\}.$$

If $\alpha(x) = 0$ (resp. $\tilde{\alpha}(x) = 0$) we say that x is *paveable* (resp. *uniformly paveable*).

REMARK 2.2. It is easy to verify that $\tilde{\alpha}(x)$ coincides with the infimum of $\left\|\sum_{j=1}^{n} p(j)(x - E(x))p(j)\right\|$ over the collection of all the diagonal decompositions of the identity $\{p(j)\}$ such that

$$p(j)H = \overline{\text{span}}\{\eta_k \mid k = a_j + mn_j, m \in \mathbf{Z}\} \quad \text{for some } a_j, n_j \in \mathbf{N}.$$

In other words, to consider all decompositions of the identity into diagonal projections that "are infinite arithmetic progressions" or to consider only uniform decompositions of the identity is equivalent for computing $\tilde{\alpha}$.

By the norm continuity of E, it is clear that $K(x) \cap D \neq \emptyset$ iff $K(x) \cap D = \{E(x)\}$ iff $E(x) \in K(x)$. From [10, Lemma 2.2 and proof of Proposition 2.3] we see that $\alpha(x)$ is the distance from $E(x)$ (or, equivalently, from D) to the set $K(x)$.

By the same reasoning and by Lemma 3.1, $\tilde{\alpha}(x)$ is the distance from $E(x)$ to the norm closed convex set $K_0(x) = \overline{\text{co}}\{u_t x u_t^* \mid t \in \mathbf{Q} \cap [0, 1)\}$, where $\{u_t \mid t \in \mathbf{Q} \cap [0, 1)\}$ is a rational subgroup of $U(D)$ which will be discussed in the next section.

More generally, for every subgroup G of $U(D)$, define for all $x \in B(H)$

$$K_G(x) = \overline{\text{co}}\{uxu^* \mid u \in G\},$$

$$\alpha_G(x) = \inf\{\|y - E(x)\| \mid y \in K_G(x)\}.$$

The distance $\alpha_G(x)$ can be characterized in terms of G-invariant functionals, where a functional f is said to be G-invariant if $f(uxu^*) = f(x)$ for all $u \in G$ and $x \in B(H)$ cf. [1, 3].

PROPOSITION 2.3. *For every $x \in B(H)$ we have*

$$\alpha_G(x) = \sup\{|f(x - E(x))| \mid f \in (B(H)^*)_1 \text{ and } f \text{ is } G\text{-invariant}\}.$$

Proof. Without loss of generality, we can assume that $E(x) = 0$. Since continuous G-invariant functionals are constant on each set $K_G(x)$, we have

$$\sup\{|f(x)| \mid f \in (B(H)^*)_1, f \text{ is } G\text{-invariant}\} =$$

$$= \inf\{\sup\{|f(y)| \mid f \in (B(H)^*)_1, f \text{ is } G\text{-invariant}\} \mid y \in K_G(x)\} \leqslant$$

$$\leqslant \inf\{\|y\| \mid y \in K_G(x)\} = \alpha_G(x).$$

Thus if $\alpha_G(x) = 0$ we are done. Assume not and let $0 < \varepsilon < \alpha_G(x)$. Let B be the open ball of $B(H)$ with radius $\alpha_G(x) - \varepsilon$, then by the Hahn-Banach theorem there is a $c \in \mathbf{R}$ and a $\varphi \in B(H)^*$ such that for all $z \in B$ and $y \in K_G(x)$, we have $\text{Re}(\varphi(z)) < c \leqslant \text{Re}(\varphi(y)) \leqslant |\varphi(y)|$. Since $\sup\{\text{Re}(\varphi(z)) \mid z \in B\} = \|\varphi\|(\alpha_G(x) - \varepsilon) \leqslant c$, we have that $\psi = \varphi/\|\varphi\| \in (B(H)^*)_1$ and $|\psi(y)| \geqslant \alpha_G(x) - \varepsilon$ for every $y \in K_G(x)$. Let $F = \overline{\text{co}}\{\psi \circ \text{ad}\, u \mid u \in G\}$ where the closure is in the weak*-topology of $B(H)^*$; then for all $f \in F$ we have $|f(x)| \geqslant \alpha_G(x) - \varepsilon$. By the Alaoglu-Bourbaki theorem, F is a weak*-compact convex subset of $(B(H)^*)_1$. For every $u \in G$, the map T_u: $B(H)^* \to B(H)^*$, given by $T_u(f) = f \circ \text{ad}\, u$ is a weak*-continuous linear contraction that maps F onto F. Since G is abelian, all the maps T_u commute and thus, by the Markov-Kakutani fixed point theorem, there is an $f \in F$ which is fixed under all T_u or, in other words, is G-invariant. Thus $f \in (B(H)^*)_1$ is a G-invariant functional for which $|f(x)| \geqslant \alpha_G(x) - \varepsilon$. Since ε was arbitrary, we obtain the thesis. Q.E.D.

REMARK 2.4. a) By using the same ideas as in [3, Proof of Theorem 2.4, (i) $\Rightarrow$ (iii)] we can show that

$$\alpha_G(x) \leqslant 4 \sup\{|f(x - E(x))| \mid f \text{ is a } G\text{-invariant state of } B(H)\}.$$

If $x = x^*$ (resp. $x \geqslant 0$), instead of 4 we can substitute 2 (resp. 1) in the above inequality.

b) As a consequence of a), $E(x) \in K_G(x)$ iff $f(x) = f(E(x))$ for all G-invariant states f of $B(H)$ cf. [3, Theorem 2.4].

Let us denote by A the abelian C^*-subalgebra of D generated by G. It is easy to see that a continuous functional f of $B(H)$ (and in particular a state) is G-invariant iff for all $x \in B(H)$ we have $f(ax) = f(xa)$ for all a in the span of G and hence for all $a \in A$, iff $f(uxu^*) = f(x)$ for all unitary $u \in U(A)$, i.e. iff f is $U(A)$-invariant cf. [3, § 2]. Thus we have:

COROLLARY 2.5. *Let G be a subgroup of $U(D)$ and let A be the abelian C^*-subalgebra of D generated by G, then $\alpha_{U(A)}(x) = \alpha_G(x)$ for every $x \in B(H)$.*

In particular, $K_{U(A)}(x) \cap D \neq \emptyset$ iff $K_G(x) \cap D \neq \emptyset$. Notice that unless $A = D$, i.e. A is maximal abelian, an application of the Stone-Weierstrass theorem shows that A cannot have the extension property relative to $B(H)$ [11, § 1]. This is equivalent to the existence of an operator $x \in B(H)$ for which $K_G(x) \cap D = \emptyset$.

Let us return to the case when G is $U(D)$ or $\{u_t \mid t \in \mathbf{Q} \cap [0, 1)\}$. Clearly, $\alpha(x) \leqslant \tilde{\alpha}(x)$ for all $x \in B(H)$. We have studied in [10, Proposition 2.4, Corollary 2.5], the class N of the paveable operators and we have seen that N is a selfadjoint Banach space and a two sided D-module cf. [11, Remark 6].

The same proof, with only minor modifications, shows that *the class $\tilde{N} \subset N$ of the uniformly paveable operators* is also a selfadjoint Banach space and a two sided D-module. Thus we have:

PROPOSITION 2.6. *Let $u \in B(H)$ be the bilateral shift, then the C^*-algebra generated by D and u is contained in $\tilde{N}$. In particular, every compact operator is uniformly paveable.*

Proof. Let $n \in \mathbf{N}$ and let $\{p(j) \mid j = 1, \ldots, n+1\}$ be the $(n+1)$-uniform decomposition of the identity. It is easy to verify that $p(j)u^n p(j) = 0$ for all j and hence $\tilde{\alpha}(u^n) = 0$. Thus $u^n \in \tilde{N}$ and hence $a_n u^n \in \tilde{N}$ for all $a_n \in D$ and $n \in \mathbf{Z}$ as $\tilde{N}$ is a selfadjoint D-module. But linear combinations of $a_n u^n$ are dense in the C^*-algebra generated by D and u and $\tilde{N}$ is a Banach space. Q.E.D.

In [10, Proposition 5.6] we have seen that the C^*-algebra generated by D and u is a proper subalgebra of the C^*-algebra generated by D and the permutation matrices. By [10, Theorem 4.1] (cf. [11, Theorem 5]) the elements of this latter C^*-algebra are all paveable. Moreover all the matrices having only 0 and 1 entries, but with at most one 1 per row and column, are also paveable, i.e. are in N [9, Proposition 3.4]. It is natural to ask whether they are also in $\tilde{N}$.

The following is a counterexample, which shows at the same time that $N \neq \tilde{N}$.

EXAMPLE 2.7. Let $x \in B(H)$ be the matrix with 1 in the $(i, 2i)$ position for all $i \in \mathbf{Z}$ and 0 elsewhere. Let $\{p(j) \mid j = 1, \ldots, n\}$ be the uniform n-decomposition of the identity, then we have $p(j)xp(j) = 0$ for all $j \neq n$ but $\|p(n)xp(n)\| = 1$. Thus $\left\| \sum_{j=1}^{n} p(j)(x - E(x))p(j) \right\| = 1$ for all n and hence $\tilde{\alpha}(x) = 1$.

It is however quite non-trivial to show that there are Laurent (or equivalently, Toeplitz) operators that are not uniformly paveable. This will be the task of Theorem 3.6 and of Theorem 4.5.

The following lemma will be used in § 4.

LEMMA 2.8 (The Squeeze Principle). *Let x_1, x_2 and y be selfadjoint operators such that $x_1 \leqslant y \leqslant x_2$. Then*

a) $\alpha(y) \leqslant \max\{\alpha(x_1) + \|E(x_1 - y)\|, \alpha(x_2) + \|E(x_2 - y)\|\}$,

b) $\tilde{\alpha}(y) \leqslant \max\{\tilde{\alpha}(x_1) + \|E(x_1 - y)\|, \tilde{\alpha}(x_2) + \|E(x_2 - y)\|\}$.

Proof. We shall prove b). The proof of a) is essentially identical. Let $\varepsilon > 0$, let $\{p_i(j) \mid j = 1, 2, \ldots, n_i\}$, $i = 1, 2$ be two uniform decompositions of the identity such that

$$\left\| \sum_{j=1}^{n_i} p_i(j)(x_i - E(x_i))p_i(j) \right\| < \tilde{\alpha}(x_i) + \varepsilon \quad \text{for } i = 1, 2.$$

Let $\{p(j) \mid j = 1, 2, \ldots, n\}$ be the n-uniform decomposition of the identity which is the refinement of the two partitions, i.e. $p(j) = p_1(k)p_2(m)$ for some k, m and where n is the least common multiple of n_1 and n_2. Then it is easy to verify that

$$\left\| \sum_{j=1}^{n} p(j)(x_i - E(x_i))p(j) \right\| < \tilde{\alpha}(x_i) + \varepsilon \quad \text{for } i = 1, 2.$$

Let

$$a = \sum_{j=1}^{n} p(j)(y - E(y))p(j).$$

then $E(x_1) \leqslant E(y) \leqslant E(x_2)$ and

$$a \leqslant \sum_{j=1}^{n} p(j)(x_2 - E(y))p(j) = \sum_{j=1}^{n} p(j)(x_2 - E(x_2))p(j) + E(x_2 - y) \leqslant$$

$$\leqslant \left(\left\| \sum_{j=1}^{n} p(j)(x_2 - E(x_2))p(j) \right\| + \|E(x_2 - y)\| \right) I \leqslant (\tilde{\alpha}(x_2) + \varepsilon + \|E(x_2 - y)\|)I.$$

Similarly

$$a \geqslant \sum_{j=1}^{n} p(j)(x_1 - E(x_1))p(j) + E(x_1 - y) \geqslant -(\tilde{\alpha}(x_1) + \varepsilon + \|E(x_1 - y)\|)I.$$

These two inequalities give

$$\|a\| \leqslant \max\{\tilde{\alpha}(x_1) + \|E(x_1 - y)\|, \tilde{\alpha}(x_2) + \|E(x_2 - y)\|\} + \varepsilon.$$

Since $\tilde{\alpha}(y) \leqslant \|a\|$ and ε is arbitrary, we obtain the thesis. Q.E.D.

We leave the proof of the following lemma to the reader. An analogous version holds for α.

LEMMA 2.9. a) $\tilde{\alpha}(x + y) \leqslant \tilde{\alpha}(x) + \tilde{\alpha}(y)$ *for every* $x, y \in B(H)$.

b) $\tilde{\alpha}(x + y) = \tilde{\alpha}(x)$ *for every* $x \in B(H)$ *and* $y \in \tilde{N}$.

In particular $\tilde{\alpha}(x) = \tilde{\alpha}(1 - x)$.

3. UNIFORM PAVEABILITY AND LAURENT OPERATORS

Let us identify H with $L^2(0, 1)$ and $\{\eta_k\}$ with $\{\exp 2\pi ikt\}$. Then a Laurent operator $L_\varphi \in B(H)$ is the operator of multiplication by the function $\varphi \in L^\infty(0, 1)$ acting on $L^2(0, 1)$.

The algebra $\mathscr{L}$ of the Laurent operators is the von Neumann algebra (masa) generated by the bilateral shift u on H. The map $L: \varphi \to L_\varphi$ from $L^\infty(0, 1)$ onto $\mathscr{L}$ is an isomorphism of von Neumann algebras and hence an isometry, i.e. $\|L_\varphi\| = \|\varphi\|_\infty$ for all $\varphi \in L^\infty(0, 1)$.

The matricial representation of L_φ relative to the basis $\{\eta_k\}$ is given by $(L_\varphi)_{ij} = \hat{\varphi}(i - j)$, where $\hat{\varphi}(m) = \int_0^1 \varphi(t)\exp(-2\pi imt)\,dt$ [7, Problem 241]. Thus $E(L_\varphi) = \hat{\varphi}(0)I$ and $\hat{\varphi}(m)I = E(L_\varphi u^{-m})$. Equivalently, $L_\varphi = \sum_{m=-\infty}^{\infty} \hat{\varphi}(m)u^m$ where the series converges in the Bures topology [12]. Recall that $B(H)$ can be identified with the crossed product of D by the action $\theta = \mathrm{ad}\, u$ of $\mathbf{Z}$. Then the dual action $\hat{\theta}$ of the dual of $\mathbf{Z}$, which we identify here with the interval $[0, 1)$, is given by $\hat{\theta}_t = \mathrm{ad}\, u_t$, where $u_t = d^{2\pi i t}$ has for its generator the (unbounded) diagonal operator d defined by $d\eta_k = \exp(-k)\eta_k$. In other words, $u_t\eta_k = \exp(-2\pi i k t)\eta_k$ for all $k \in \mathbf{Z}$ [4, Part I, § 4]. In particular, $\hat{\theta}_t(a) = a$ for all $a \in D$, $\hat{\theta}_t(u) = \exp(-2\pi i t)u$ and $E(x) = \int_0^1 \hat{\theta}_t(x)\, dt$ for all $x \in B(H)$ (where the convergence of this integral is in the σ-weak topology). The action $\hat{\theta}$ has been further studied in this context in a general setting in [10].

The connection between the dual action $\hat{\theta}_t$ (or equivalently between the unitary operators u_t) for t rational and the uniform decompositions of the identity is given by the following lemma, which strengthens [10, Lemma 2]; see also [3, remarks after Corollary 2.3].

Lemma 3.1. *Let $\{p(j) \mid j = 1, \ldots, n\}$ be the n-uniform decomposition of the identity, then* $(1/n)\sum_{k=1}^{n} u_{k/n} x u_{k/n}^* = \sum_{j=1}^{n} p(j)xp(j)$ *for every* $x \in B(H)$.

Proof. It is easy to verify that $u_{1/n} = \sum_{j=1}^{n} \exp(-2\pi i j/n)p(j)$. Thus

$$(1/n)\sum_{k=1}^{n} u_{k/n} x u_{k/n}^* = (1/n)\sum_{k=1}^{n}\sum_{m=1}^{n}\sum_{j=1}^{n} \exp(-2\pi i(m-j)k/n)p(m)xp(j) =$$

$$= \sum_{m=1}^{n}\sum_{j=1}^{n} (1/n)\sum_{k=1}^{n} \exp(-2\pi i(m-j)k/n)p(m)xp(j) = \sum_{j=1}^{n} p(j)xp(j),$$

since $\sum_{k=1}^{n} \exp(-2\pi i(m-j)k/n) = 0$ unless $(m-j)/n \in \mathbf{Z}$, i.e. unless $m = j$. Q.E.D.

The C^*-algebra $B(H)_c$ of the continuous elements relative to $\hat{\theta}$ [14, 7.5] (i.e. the elements x for which $t \to \hat{\theta}_t(x)$ is norm continuous) coincides with the C^*-algebra generated by D and by the bilateral shift u [10, 13]. The algebra $B(H)_c \cap \mathscr{L}$ is then precisely the algebra of Laurent operators with continuous symbol [10, Section 5]. Thus $B(H)_c \cap \mathscr{L} \subset \tilde{N}$ by Proposition 2.6.

The action $\hat{\theta}$ operates on L_φ as a translation on the symbol φ. Indeed extend by periodicity all the functions on $(0, 1)$ to functions on $\mathbf{R}$ with period 1 and thus embed $L^\infty(0, 1)$ in $L^\infty(\mathbf{R})$. Denote by $\lambda(s)$ the translation by s on $L^\infty(\mathbf{R})$ (and hence on $L^\infty(0, 1)$), i.e. $(\lambda(s)\varphi)(t) = \varphi(t - s)$ for almost all $t \in \mathbf{R}$. Then we have:

Lemma 3.2. $\hat{\theta}_s(L_\varphi) = L_{\lambda(s)\varphi}$ *for all* $\varphi \in L^\infty(0, 1)$ *and* $s \in (0, 1)$.

Proof. Reasoning in part as in the proof of [10, Lemma 4.4] we have for all $m \in \mathbf{Z}$:

$$E(\hat{\theta}_s(L_\varphi)u^{-m}) = E(\hat{\theta}_s(L_\varphi \exp(-2\pi i m s)u^{-m})) = \exp(-2\pi i m s)E(L_\varphi u^{-m}) =$$

$$= \exp(-2\pi i m s)\hat{\varphi}(m)I = (\lambda(s)\varphi)^\wedge(m)I = E(L_{\lambda(s)\varphi}u^{-m}),$$

where we have used the fact that $E = E \circ \hat{\theta}_s$ for all s. Q.E.D.

Let $Q_n = (1/n)\sum_{j=1}^{n} \lambda(j/n)$, then it is easy to verify that Q_n is a faithful normal conditional expectation of $L^\infty(0, 1)$ onto the algebra of $(1/n)$-periodic functions of $L^\infty(0, 1)$.

The following proposition shows that the paving of a Laurent operator induced by a uniform decomposition of the identity corresponds to the Q_n averaging process for the symbol.

Proposition 3.3. *Let $\{p(j) \mid j = 1, \ldots, n\}$ be the n-uniform decomposition of the identity and let $\varphi \in L^\infty(0, 1)$, then* $\sum_{j=1}^{n} p(j)L_\varphi p(j) = L_{Q_n\varphi}$.

Proof.

$$\sum_{j=1}^{n} p(j)L_\varphi p(j) = (1/n)\sum_{j=1}^{n} u_{j/n}L_\varphi u_{j/n}^* = \qquad \text{(Lemma 3.1)}$$

$$= (1/n)\sum_{j=1}^{n} \hat{\theta}_{j/n}(L_\varphi) = (1/n)\sum_{j=1}^{n} L_{\lambda(j/n)\varphi} = L_{Q_n\varphi}. \qquad \text{(Lemma 3.2)}$$

Q.E.D.

We shall need the following characterization of Q_n. We leave the proof to the reader.

Lemma 3.4. *For every $\varphi \in L^\infty(0, 1)$ and $n \in \mathbf{N}$ we have that*

$$(Q_n\varphi)^\wedge(k) = \begin{cases} \hat{\varphi}(k) & \text{if } n \text{ divides } k \\ 0 & \text{if } n \text{ does not divide } k. \end{cases}$$

It is easy to verify that $K(L_\varphi) = \overline{\mathrm{co}}\{wL_\varphi w^* \mid w \in U(D)\} \subset \mathscr{L}$ iff φ is constant a.e.. However, by Lemma 3.2, for every $\varphi \in L^\infty(0, 1)$ we have that $\overline{\mathrm{co}}\{u_t L_\varphi u_t^* \mid t \in (0, 1)\} \subset \mathscr{L}$. In particular

$$K_0(L_\varphi) = \overline{\mathrm{co}}\{u_t L_\varphi u_t^* \mid t \in \mathbf{Q} \cap [0, 1)\} \subset \mathscr{L}.$$

Let us define for all $\varphi \in L^\infty(0, 1)$

$$K_0(\varphi) = \overline{\mathrm{co}}\{\lambda(t)\varphi \mid t \in \mathbf{Q} \cap [0, 1)\},$$

where the closure is in the L^∞-norm.

The operator $L\colon \varphi \to L_\varphi$ maps $K_0(\varphi)$ onto $K_0(L_\varphi)$ and hence $K_0(\varphi) \cap \mathbf{C}I$ onto $K_0(L_\varphi) \cap \mathbf{C}I = K_0(L_\varphi) \cap D$. Thus $\hat{\varphi}(0)I \in K_0(\varphi)$ iff $E(L_\varphi) \in K_0(L_\varphi)$. From Proposition 3.3 we have

$$\left\| \sum_{j=1}^{n} p(j)(L_\varphi - E(L_\varphi))p(j) \right\| = \| L_{Q_n\varphi} - \hat{\varphi}(0)I \| = \| Q_n\varphi - \hat{\varphi}(0)I \|_\infty .$$

Thus we obtain:

PROPOSITION 3.5. *For every* $\varphi \in L^\infty(0, 1)$ *we have*
a) $\tilde{\alpha}(L_\varphi) = 0$ *iff* $K_0(\varphi) \cap \mathbf{C}I \neq \emptyset$;
b) $\tilde{\alpha}(L_\varphi) = \inf_n \| Q_n\varphi - \hat{\varphi}(0) \|_\infty$.

This reduces uniform paveability of Laurent operators to an ergodic property of $L^\infty(0, 1)$. We deal with it in the following theorem.

THEOREM 3.6. *There is a* $\varphi \in L^\infty(0, 1)$ *such that* $\tilde{\alpha}(L_\varphi) \neq 0$.

Proof. Consider the group $U_{\mathbf{R}} = \{\lambda(t) \mid t \in [0, 1)\}$ and its subgroup $U_{\mathbf{Q}} = \{\lambda(t) \mid t \in [0, 1) \cap \mathbf{Q}\}$. By [15, Corollary 22.4], there are two distinct $U_{\mathbf{R}}$-invariant means on $L^\infty(0, 1)$, i.e. two states μ_1 and μ_2 on $L^\infty(0, 1)$ which are invariant under all the elements of $U_{\mathbf{R}}$ and hence under all the elements of $U_{\mathbf{Q}}$. Therefore there is an element $\varphi \in L^\infty(0, 1)$ such that $\mu_1(\varphi) \neq \mu_2(\varphi)$. Then for every $\psi \in K_0(\varphi)$ we have $\mu_i(\psi) = \mu_i(\varphi)$, $i = 1, 2$. Thus $\tilde{\alpha}(L_\varphi) \neq 0$ since otherwise, by Proposition 3.5 a), $\hat{\varphi}(0)I \in K_0(\varphi)$, which yields the contradiction:

$$\mu_1(\varphi) = \mu_1(\hat{\varphi}(0)I) = \hat{\varphi}(0) = \mu_2[\hat{\varphi}(0)I] = \mu_2(\varphi).$$

Q.E.D.

REMARK 3.7. a) We have actually proven that there is a $\varphi \in L^\infty(0, 1)$ such that

$$\overline{\mathrm{co}}\{\lambda(t)\varphi \mid t \in [0, 1)\} \cap \mathbf{C}I = \emptyset.$$

Thus by Corollary 2.5, $E(L_\varphi) \notin \overline{\mathrm{co}}\{uL_\varphi u^* \mid u \in A\}$ where A is the C^*-subalgebra of D generated by $\{u_t \mid t \in [0, 1)\}$.

b) Note that this argument is nonconstructive and it does not yield an evaluation of $\tilde{\alpha}(L_\varphi)$.

4. THE MODULUS OF UNIFORM PAVEABILITY OF LAURENT OPERATORS

In the previous section we have seen in Theorem 3.6 (see also Remark 3.7) that there exist Laurent operators that are not uniformly paveable. In this section we shall provide concrete examples of Laurent projections that are not uniformly paveable and evaluate their modulus of uniform paveability $\tilde{\alpha}$.

Firstly, we can obtain an upper bound for $\tilde{\alpha}(L_\varphi)$ for a selfadjoint Laurent operator L_φ in terms of the upper and lower Riemann integrals of its symbol φ.

Let $M^\infty(0, 1)$ be the algebra of real-valued everywhere bounded Lebesgue measurable functions on $(0, 1)$ and let $f \in M^\infty(0, 1)$, then we denote by $\overline{R}f$, $\underline{R}f$, $\int_0^1 f\,\mathrm{d}t$ the upper and lower Riemann integrals and the Lebesgue integral of f over $(0, 1)$ respectively.

PROPOSITION 4.1. *Let* L_φ *be a selfadjoint Laurent operator, then*

$$\tilde{\alpha}(L_\varphi) \leqslant \inf\left\{\max\left(\overline{R}f - \int_0^1 \varphi\,\mathrm{d}t, \int_0^1 \varphi\,\mathrm{d}t - \underline{R}f\right) \,\Big|\, f \in M^\infty(0, 1) \text{ and } f = \varphi \text{ a.c.}\right\}.$$

Proof. Let $f \in M^\infty(0, 1)$, $f = \varphi$ a.e. and let $\varepsilon > 0$, then there are two real-valued, continuous functions g_i such that $g_1 \leqslant f \leqslant g_2$ (and hence $L_{g_1} \leqslant L_\varphi \leqslant L_{g_2}$) and $\int_0^1 g_2\,\mathrm{d}t \leqslant \overline{R}f + \varepsilon$, $\int_0^1 g_1\,\mathrm{d}t \geqslant \underline{R}f - \varepsilon$. We have already noticed in § 3 that by Proposition 2.6 all Laurent operators with continuous symbol are uniformly paveable. Thus $\tilde{\alpha}(L_{g_i}) = 0$ and by the squeeze principle (Lemma 2.8)

$$\tilde{\alpha}(L_\varphi) \leqslant \max\{\|E(L_{g_2} - L_\varphi)\|, \|E(L_\varphi - L_{g_1})\|\} =$$
$$= \max\left\{\int_0^1 (g_2 - \varphi)\,\mathrm{d}t, \int_0^1 (\varphi - g_1)\,\mathrm{d}t\right\} \leqslant \max\left\{\overline{R}f - \int_0^1 \varphi\,\mathrm{d}t, \int_0^1 \varphi\,\mathrm{d}t - \underline{R}f\right\} + \varepsilon.$$

But $\varepsilon > 0$ and $f = \varphi$ a.e. are arbitrary. Q.E.D.

Let $\mathscr{R}$ be the C^*-subalgebra of $\mathscr{L}$ of the Laurent operators with Riemann integrable symbol (or more precisely, with symbol almost everywhere equal to a Riemann integrable function). Then as an immediate consequence of Proposition 4.1 and Lemma 2.9 (for the nonselfadjoint case) we have:

COROLLARY 4.2. *Every element of $\mathscr{R}$ is uniformly paveable.*

We shall henceforth consider Laurent projections, i.e. operators L_φ with symbol $\varphi = \chi_E$, the characteristic function of a measurable set $E \subset (0, 1)$. Let us denote by $\bar{E}$, E^0, $E^\sim$, and $\partial E = \bar{E} \setminus E^0$ the closure, the interior, the complement in $(0, 1)$ and the boundary of the set E respectively and let m be the Lebesgue measure on $(0, 1)$. Then $\bar{R}\varphi = m(\bar{E})$, $\underline{R}\varphi = m(E^0)$ and thus Proposition 4.1 can be reformulated as follows:

COROLLARY 4.3. *Let $\varphi = \chi_E$ for a measurable set $E \subset (0, 1)$. Then*

$$\tilde{\alpha}(L_\varphi) \leqslant \inf\{\max(m(\bar{F} - F), m(F - F^0)) \mid \chi_F = \chi_E \text{ a.e.}\}.$$

We can exhibit cases where the upper bound is attained (Theorem 4.5). The following lemma is a routine measure theory result.

LEMMA 4.4. *If $E \subset (0, 1)$ is either a closed or an open set, then*

$$m(\partial E) = \inf\{\max(m(\bar{F} - F), m(F - F^0)) \mid \chi_F = \chi_E \text{ a.e.}\}.$$

THEOREM 4.5. *If $V \subset (0, 1)$ is an open dense set and $\varphi = \chi_V$, then $\tilde{\alpha}(L_\varphi) = m(\partial V) > 0$. If furthermore $m(V) \leqslant 1/2$, then for every uniform decomposition of the identity $\{p(j) \mid j = 1, \ldots, n\}$ we have*

$$\left\| \sum_{j=1}^{n} p(j)(L_\varphi - E(L_\varphi))p(j) \right\| = m(\partial V).$$

Proof. Because of Corollary 4.3 and Lemma 4.4 we know that

$$\tilde{\alpha}(L_\varphi) \leqslant \inf\{\max(m(\bar{F} - F), m(F - F^0)) \mid \chi_F = \chi_V \text{ a.e.}\} = m(\partial V).$$

In order to prove the reverse inequality, by Proposition 3.5 b), it suffices to show that for every $n \in \mathbf{N}$ we have $\|Q_n\varphi - \hat{\varphi}(0)I\|_\infty \geqslant m(\partial V)$. Fix $n \in \mathbf{N}$, and let $J_1 = V \cap (0, 1/n)$, then $J_1 \neq \emptyset$ because V is dense in $(0, 1)$. Let $\tilde{J}_2 = V \cap (J_1 + 1/n)$, then we also have $\tilde{J}_2 \neq \emptyset$. Let $J_2 = \tilde{J}_2 - 1/n$, then $J_2 \subset J_1$. Iterating we find a nested sequence $J_n \subset J_{n-1} \subset \ldots \subset J_1$ of open nonempty sets such that $J_k + (k - 1)n \subset V$. In particular $J_n + k/n \subset V$ for $k = 0, 1, \ldots, n - 1$. Thus

$$V_n = \bigcup\{(J_n + k/n) \mid k = 0, 1, \ldots, n - 1\} \subset V.$$

Let $\varphi_n = \chi_{V_n}$, then φ_n is $(1/n)$-periodic and $\varphi \geqslant \varphi_n$. But then $Q_n\varphi \geqslant Q_n\varphi_n = \varphi_n$ and thus we have:

$$Q_n\varphi - m(V)I \geqslant \varphi_n - m(V)I = (1 - m(V))\varphi_n - m(V)(1 - \varphi_n).$$

Therefore

$$(Q_n\varphi - m(V)I)\varphi_n \geqslant (1 - m(V))\varphi_n \geqslant 0$$

and hence, since $\hat{\varphi}(0) = m(V)$, we have

$$\|Q_n\varphi - \hat{\varphi}(0)I\|_\infty \geqslant 1 - m(V) = m(\partial V).$$

Assume furthermore that $m(V) \leqslant 1/2$, hence $m(V) \leqslant m(\partial V)$. Since Q_n is a contraction, $Q_nI = I$ and φ is a characteristic function, we have

$$\|Q_n\varphi - m(V)I\|_\infty = \|Q_n(\varphi - m(V)I)\|_\infty \leqslant$$

$$\leqslant \|\varphi - m(V)I\|_\infty = \|(1 - m(V))\varphi - m(V)(1 - \varphi)\|_\infty = m(\partial V).$$

Thus, by Proposition 3.3 and remarks before Proposition 3.5, we have for all n that $\left\| \sum_{j=1}^{n} p(j)(L_\varphi - E(L_\varphi))p(j) \right\| = m(\partial V)$. Q.E.D.

We shall see in Corollary 5.4 that we actually have more:

$$\|p(j)(L_\varphi - E(L_\varphi))p(j)\| = m(\partial V) \quad \text{for each } j.$$

We wish to point out for later use (Propositions 5.5, 5.6) a special set that possesses the essential properties used in the proof of Theorem 4.5.

EXAMPLE 4.6. Let $a_n > 0$ be a sequence of real numbers such that $\sum_{n=1}^{\infty} na_n \leqslant 1/2$, let

$$V_n = \bigcup\{(0, a_n) + j/n \mid j = 0, 1, \ldots, n - 1\}$$

and let $V = \bigcup_1^{\infty} V_n$. Then V is an open dense set and

$$m(V) < \sum_{n=1}^{\infty} m(V_n) = \sum_{n=1}^{\infty} na_n \leqslant 1/2.$$

Then the sets V_n coincide with the "periodic" subsets constructed in the proof of Theorem 4.5.

The following example will show that if an open set V is not dense in $(0, 1)$ the inequality in Corollary 4.3 could be strict. We thank A. Sourour for the idea that led to this construction.

EXAMPLE 4.7. Let $1/6<\beta<1/2$ and let $K \subset (0, 1/2)$ be the Cantor set of measure β, so that its complement $K^{\sim}$ in $(0, 1)$ is the union of disjoint open intervals J_k with length $3^{-n}\beta$ for some n. Split each interval J_k into the union of two disjoint nonempty open intervals J_k^1 and J_k^2 and the division point. Let $W_i = \bigcup_1^{\infty} J_k^i$, then it is easy to verify that $\overline{W}_1 = [0, 1/2] \setminus W_2$ and, similarly, that $\overline{W}_2 = [0, 1/2] \setminus W_1$. Let $V_1 = W_1 \subset (0, 1/2)$, $V_2 = W_2 + 1/2 \subset (1/2, 1)$ and let $V = V_1 \cup V_2$, $\tilde{V} = (V_1 + 1/2) \cup (V_2 - 1/2)$. Then V and $\tilde{V}$ are disjoint and

$$m(V) = m(\tilde{V}) = m(W_1 \cup W_2) = m(K^{\sim}) = 1/2 - \beta.$$

V is not dense in $(0, 1)$, actually $\overline{V} = [0, 1] \setminus \tilde{V}$, so that $m(\overline{V}) = 1 - m(V) = 1/2 + \beta$. Let $\varphi = \chi_V$, then $\hat{\varphi}(0) = 1/2 - \beta$ and $\lambda(1/2)\varphi = \chi_{\tilde{V}}$. Hence

$$Q_2\varphi = (1/2)(\varphi + \lambda(1/2)\varphi) = (1/2)(\chi_{V \cup \tilde{V}})$$

and since $\beta > 1/6$, we have

$$\|Q_2\varphi - \hat{\varphi}(0)I\|_\infty = \|(1/2)(\chi_{V\cup\tilde{V}}) - (1/2 - \beta)I\|_\infty =$$

$$= \|\beta(\chi_{V\cup\tilde{V}}) - (1/2 - \beta)(1 - \chi_{V\cup\tilde{V}})\|_\infty = \max(\beta, 1/2 - \beta) < 2\beta = m(\partial V).$$

Therefore $\tilde{\alpha}(L_\varphi) < m(\partial V)$.

5. UNIFORM PAVINGS AND FINITE ARITHMETIC PROGRESSIONS

The techniques developed for dealing with uniform pavings can be used to shed some light on a larger class of pavings. Let us introduce some further notations.

Define an operator $\tilde{Q}_n$ on $L^\infty(0, 1)$ by setting $(\tilde{Q}_n\varphi)(t) = (Q_n\varphi)(t/n)$ for all $\varphi \in L^\infty(0, 1)$ and for almost all $t \in (0, 1)$. An equivalent characterization is given by property a) in the following lemma.

LEMMA 5.1. a) $(\tilde{Q}_n\varphi)^\wedge(k) = \hat{\varphi}(kn)$ *for all* $\varphi \in L^\infty(0, 1)$ *and* $k \in \mathbf{Z}$,

b) $\|\tilde{Q}_n\varphi\|_\infty = \|Q_n\varphi\|_\infty$ *for all* $\varphi \in L^\infty(0, 1)$,

c) $\tilde{Q}_n$ *is a positive identity-preserving linear contraction on* $L^\infty(0, 1)$,

d) $\tilde{Q}_n Q_n = \tilde{Q}_n$,

e) *if* $0 < n\alpha < 1$, *then* $n\tilde{Q}_n\chi_{(0,\alpha)} = \chi_{(0,n\alpha)}$.

Proof. a). From the $1/n$-periodicity of $Q_n\varphi$ we have:

$$(\tilde{Q}_n\varphi)^\wedge(k) = \int_0^{\frac{1}{n}} (Q_n\varphi)(t/n)\exp(-2\pi ikt)dt =$$

$$= \int_0^1 (Q_n\varphi)(t)\exp(-2\pi iknt)dt = (Q_n\varphi)^\wedge(kn) = \hat{\varphi}(kn) \quad \text{(Lemma 3.4)}.$$

b) and c) are obvious and d) follows from the fact that Q_n is idempotent. e). From a) we have for every $n \in \mathbf{Z}$,

$$(n\tilde{Q}_n\chi_{(0,\alpha)})^\wedge(k) = (n\chi_{(0,\alpha)})^\wedge(nk) = (\chi_{(0,n\alpha)})^\wedge(k). \qquad \text{Q.E.D.}$$

Let $p(n, t, a)$, r and $r(t)$ be the (diagonal) projections on $\overline{\text{span}}\{\eta_k \mid k = a + jn, j = 1, \ldots, t\}$, $\overline{\text{span}}\{\eta_k \mid k \in \mathbf{N}\}$ and $\text{span}\{\eta_k \mid k = 1, \ldots, t\}$ respectively. Then $r = \sup r(t)$.

The next proposition enables us to compute the norm of certain compressions of Laurent operators.

PROPOSITION 5.2. *Let* $\varphi \in L^\infty(0, 1)$, *then for all* $n, t \in \mathbf{N}$ *and* $a \in \mathbf{Z}$ *the operators* $p(n, t, a)L_\varphi p(n, t, a)$ *and* $r(t)L_{\tilde{Q}_n\varphi}r(t)$ *are unitarily equivalent. In particular we have that*

$$\|p(n, t, a)(L_\varphi - E(L_\varphi))p(n, t, a)\| = \|r(t)(L_{\tilde{Q}_n\varphi} - E(L_\varphi))r(t)\|.$$

Proof. Let $x = p(n, t, a)L_\varphi p(n, t, a)$, and let $[x_{ij}]$ be the matricial representation of x relative to the basis $\{\eta_k\}$. If $i = a + i_0n$ and $j = a + j_0n$ for some $1 \leqslant i_0, j_0 \leqslant t$, then we have

$$x_{ij} = p(n, t, a)_{ii}(L_\varphi)_{ij}p(n, t, a)_{jj} =$$

$$= \hat{\varphi}(i - j) = (\tilde{Q}_n\varphi)^\wedge(i_0 - j_0) = \qquad \text{by Lemma 5.1 a)}$$

$$= (L_{\tilde{Q}_n\varphi})_{i_0j_0} = (r(t)(L_{\tilde{Q}_n\varphi})r(t))_{i_0j_0}$$

and $x_{ij} = 0$ elsewhere. Therefore x is unitarily equivalent to $r(t)L_{\tilde{Q}_n\varphi}r(t)$. Since $\tilde{Q}_nI = I$ and $E(L_\varphi) = \hat{\varphi}(0)I$, by applying the result obtained above to $\varphi - \hat{\varphi}(0)I$ we see that $p(n, t, a)(L_\varphi - E(L_\varphi))p(n, t, a)$ and $r(t)(L_{\tilde{Q}_n\varphi} - E(L_\varphi))r(t)$ are unitarily equivalent and hence have the same norm. Q.E.D.

By the continuity of the norm in the S.O.T. we can define:

DEFINITION 5.3. Assume that $x \in B(H)$ and $rxr \neq 0$. For each $0 < \beta < 1$ we define $\nu(x, \beta)$ to be the minimal integer k such that $\|r(k)xr(k)\| > \beta\|rxr\|$.

Thus the integer valued function $\nu(x, \beta)$ measures the rate of norm convergence of the finite blocks $r(k)xr(k)$ to the infinite matrix rxr. As we shall see in Propositions 5.5 and 5.6, if x is a Laurent operator with symbol φ, this rate is related to the rate of convergence of the series of the Fourier coefficients of φ, which is easier to estimate.

THEOREM 5.4. *Let V be an open dense set in $(0, 1)$ with $m(V) \leqslant m(\partial V)$, let φ be the characteristic function of V and let p be a diagonal projection. Let $f(n, \beta) = \nu(L_{\tilde{Q}_n\varphi} - E(L_\varphi), \beta)$ for $0 < \beta < 1$ and $n \in \mathbf{N}$; then*

a) *If p contains in its range $\{\eta_k \mid k = a + jn,\ j \in \mathbf{N}\}$ for some $n \in \mathbf{N}$ and $a \in \mathbf{Z}$, then $\|p(L_\varphi - E(L_\varphi))p\| = m(\partial V)$.*

b) *If $p \geqslant p(n, t, a)$ for some $t \geqslant f(n, \beta)$ and some $0 < \beta < 1$, $n \in \mathbf{N}$ and $a \in \mathbf{Z}$, then $\|p(L_\varphi - E(L_\varphi))p\| > \beta m(\partial V)$.*

Proof. a) Since $p \geqslant p(n, t, a)$ for every t, we have from Proposition 5.2 that

$$\|p(L_\varphi - E(L_\varphi))p\| \geqslant \|p(n, t, a)(L_\varphi - E(L_\varphi))p(n, t, a)\| =$$
$$= \|r(t)(L_{\tilde{Q}_n\varphi} - E(L_\varphi))r(t)\|.$$

Since $r(t)(L_{\tilde{Q}_n\varphi} - E(L_\varphi))r(t)$ converges in the S.O.T. to the Toeplitz operator $r(L_{\tilde{Q}_n\varphi} - E(L_\varphi))r$ with symbol $\tilde{Q}_n(\varphi - \hat{\varphi}0)I$ (which has the same norm as the Laurent operator with the same symbol), we have that

$$\|r(t)(L_{\tilde{Q}_n\varphi} - E(L_\varphi))r(t)\| \uparrow \|r(L_{\tilde{Q}_n\varphi} - E(L_\varphi))r\| =$$
$$= \|L_{\tilde{Q}_n\varphi} - E(L_\varphi)\| = \|\tilde{Q}_n\varphi - \hat{\varphi}(0)I\|_\infty = \qquad \text{from Lemma 5.1 b),}$$
$$= \|Q_n\varphi - \hat{\varphi}(0)I\|_\infty = m(\partial V) \qquad \text{from the proof of Theorem 4.4.}$$

Thus $\|p(L_\varphi - E(L_\varphi))p\| \geqslant m(\partial V)$. On the other hand,

$$\|L_\varphi - E(L_\varphi)\| = \|\varphi - \hat{\varphi}(0)I\|_\infty = m(\partial V)$$

which proves the equality.

b) By the definition of $f(n, \beta) = \nu(L_{\tilde{Q}_n\varphi} - E(L_\varphi), \beta)$ and by the above computation, for all $t \geqslant f(n, \beta)$ we have

$$\|r(t)(L_{\tilde{Q}_n\varphi} - E(L_\varphi))r(t)\| > \beta\|r(L_{\tilde{Q}_n\varphi} - E(L_\varphi))r\| = \beta m(\partial V).$$

Therefore if $p \geqslant p(n, t, a)$ for some $t \geqslant f(n, \beta)$, by Proposition 5.2 we obtain the inequality. Q.E.D.

In particular for all projections p belonging to any uniform decomposition of the identity we have that

$$\|p(L_\varphi - E(L_\varphi))p\| = m(\partial V).$$

Under the identification of diagonal projections with subsets of $\mathbf{Z}$, we say that a projection p contains a β-sufficiently long arithmetic progression (for L_φ) if $p \geqslant p(n, t, a)$ for some $t \geqslant f(n, \beta)$. Thus, by Theorem 5.4 b), we have that

$$\left\| \sum_{j=1}^{k} p(j)(L_\varphi - E(L_\varphi))p(j) \right\| > \beta m(\partial V)$$

for the large class of diagonal decompositions of the identity for which one of the projections contains a β-sufficiently long arithmetic progression.

This fact makes L_φ a natural candidate for a counterexample to the paving problem.

A characterization of the decompositions of the identity or, equivalently, of the colorings of $\mathbf{Z}$ that contain β-sufficiently long monochromatic arithmetic progressions, pertains to combinatorics and is dependent on an estimate of the function $f(n, \beta)$.

As an example of the techniques involved in such an estimate, for the remainder of this section we shall consider the characteristic function φ of the open dense subset $V = \bigcup V_n \subset (0, 1)$ given in Example 4.6 and exhibit an upper and lower bound for $f(n, \beta) = \nu(L_{\tilde{Q}_n\varphi} - E(L_\varphi), \beta)$.

Let φ_n be the characteristic function of V_n and let $\delta = m(V)$, so that $E(L_\varphi) = \delta I$. Choose a_n so that $na_n \leqslant 1/4$ for all $n \in \mathbf{N}$. Define $g(t) = 2(t - (\pi/3)^2 t^3)$ for $t \in (0, \sqrt{3}/\pi)$; then g is monotone increasing and has maximum $4/\sqrt{3}\pi$. Choose β small enough to insure that $\gamma = \beta(1 - \delta) + \delta \leqslant g(\sqrt{3}/\pi - 1/4)$. Then we have:

PROPOSITION 5.5. *$f(n, \beta) < 2 + 2g^{-1}(\gamma)/na_n$ for all $n \in \mathbf{N}$.*

Proof. By definition, $\varphi_n = \sum_{j=1}^{n} \lambda(j/n)\chi_{(0,a_n)} = nQ_n\chi_{(0,a_n)}$. Then by Lemma 5.1 d) and e), we have that

$$\tilde{Q}_n\varphi_n = n\tilde{Q}_nQ_n\chi_{(0,a_n)} = n\tilde{Q}_n\chi_{(0,a_n)} = \chi_{(0,na_n)}.$$

Thus $L_{\tilde{Q}_n\varphi_n}$ is a Laurent projection and $r(L_{\tilde{Q}_n\varphi_n})r$ is the corresponding Toeplitz operator, so $\|r(L_{\tilde{Q}_n\varphi_n})r\| = \|L_{\tilde{Q}_n\varphi_n}\| = 1$ and

$$\|r(L_{\tilde{Q}_n\varphi} - \delta I)r\| = \|L_{\tilde{Q}_n\varphi} - \delta I\| = 1 - \delta.$$

Since $0 \le \varphi_n \le \varphi$ and hence $0 \le L_{\tilde Q_n\varphi_n} \le L_{\tilde Q_n\varphi}$, we have for all t that $\|r(t)L_{\tilde Q_n\varphi_n}r(t)\| \le \|r(t)L_{\tilde Q_n\varphi}r(t)\|$. Let $t = \nu(L_{\tilde Q_n\varphi_n}, \gamma)$ then

$$\|r(t)(L_{\tilde Q_n\varphi} - \delta I)r(t)\| \geq \|r(t)L_{\tilde Q_n\varphi}r(t)\| - \delta \geq$$

$$\geq \|r(t)L_{\tilde Q_n\varphi_n}r(t)\| - \delta > \gamma\|r(L_{\tilde Q_n\varphi_n})r\| - \delta =$$

$$= \beta(1-\delta) = \beta\|r(L_{\tilde Q_n\varphi} - \delta I)r\|.$$

Therefore by Definition 5.3, $f(n,\beta) = \nu(L_{\tilde Q_n\varphi} - \delta I, \beta) \le t$, i.e.,

$$f(n,\beta) \le \nu(L_{\tilde Q_n\varphi_n}, \beta(1-\delta)+\delta).$$

This inequality holds true under the milder condition $\gamma = \beta(1-\delta)+\delta < 1$. Set $s = na_n$ and $\xi = \chi_{(0,s)} \in L^2(0,1)$, then ξ is in the range of $L_{\tilde Q_n\varphi_n}$. $\|\xi\|^2 = (\xi, \eta_0) = s$ and for $j \ne 0$ we have

$$|(\xi,\eta_j)|^2 = (1/2)\pi^{-2}j^{-2}(1-\cos 2\pi js).$$

Denote by $q(t)$ the projection on $\overline{\operatorname{span}}\{\eta_k \mid -t \le k \le t\}$. Then it is easy to see (cf. proof of Proposition 5.2), that for all Laurent operators L_ψ

$$\|r(2t+1)L_\psi r(2t+1)\| = \|q(t)L_\psi q(t)\|.$$

Thus we have for all $k \ge 0$

$$\|r(2k+1)L_{\tilde Q_n\varphi_n}r(2k+1)\| = \|q(k)L_{\tilde Q_n\varphi_n}q(k)\| = \|L_{\tilde Q_n\varphi_n}q(k)L_{\tilde Q_n\varphi_n}\| \geq$$

$$\geq \|\xi\|^{-2}(L_{\tilde Q_n\varphi_n}q(k)L_{\tilde Q_n\varphi_n}\xi,\xi) = \|\xi\|^{-2}\|q(k)\xi\|^2 =$$

$$= \|\xi\|^{-2}\left\{|(\xi,\eta_0)|^2 + 2\sum_{j=1}^k |(\xi,\eta_j)|^2\right\} = s^{-1}\left\{s^2 + \pi^{-2}\sum_{j=1}^k j^{-2}(1-\cos 2\pi js)\right\} >$$

$$> s^{-1}\left\{s^2 + \pi^{-2}\sum_{j=1}^k j^{-2}((1/2!)(2\pi js)^2 - (1/4!)(2\pi js)^4)\right\} =$$

$$= s + 2s\sum_{j=1}^k (1-(1/3)(\pi s)^2 j^2) = (2k+1)s - (2/3)s(\pi s)^2\sum_{j=1}^k j^2 =$$

$$= (2k+1)s - (2/9)s(\pi s)^2 k(k+1)(k+1/2) > 2(k+1/2)s - 2(\pi/3)^2(k+1/2)^3 s^3 =$$

$$= g((k+1/2)s).$$

Therefore, if k satisfies the inequality $g^{-1}(\gamma) \le (k+1/2)s \le \sqrt{3}/\pi$, then by the monotonicity of the function $g(t)$ we have $g((k+1/2)s) \ge \gamma$ and hence

$$\|r(2k+1)L_{\tilde Q_n\varphi_n}r(2k+1)\| > \gamma = \gamma\|r(L_{\tilde Q_n\varphi_n})r\|.$$

In particular, the inequality holds for the smallest integer k such that $2g^{-1}(\gamma)/s \le 2k+1$. Therefore, by Definition 5.3 we have $\nu(L_{\tilde Q_n\varphi_n},\gamma) \le 2k+1$, hence we obtain that $\nu(L_{\tilde Q_n\varphi_n},\gamma) \le 2 + 2g^{-1}(\gamma)/s$. Thus we conclude that

$$f(n,\beta) \le \nu(L_{\tilde Q_n\varphi_n},\gamma) < 2 + 2g^{-1}(\gamma)/na_n. \qquad \text{Q.E.D.}$$

Thus we have an upper bound on the growth of $f(n,\beta)$ with n. The following proposition shows that indeed $f(n,\beta)\to\infty$ for $n\to\infty$. Here φ is seen also as a vector in $H = L^2(0,1)$ and thus

$$\left\{\sum_{j=n+1}^\infty |\hat\varphi(j)|^2\right\}^{1/2} = \|(r - r(n))\varphi\| \to 0.$$

PROPOSITION 5.6.

$f(n,\beta) > (1/2)(\beta(1-\delta))^2\|(r - r(n-1))\varphi\|^{-2}$ *for all* $0<\beta<1$ *and* $n>1$.

Proof. Let $t = f(n,\beta)$ and let $\|\ \|_2$ denote the Hilbert-Schmidt norm, then we have:

$$(\beta(1-\delta))^2 = \beta^2\|L_{\tilde Q_n\varphi} - \delta I\|^2 < \|r(t)(L_{\tilde Q_n\varphi} - \delta I)r(t)\|^2 \le$$

$$\le (\|r(t)(L_{\tilde Q_n\varphi}-\delta I)r(t)\|_2)^2 = \sum\{|(L_{\tilde Q_n\varphi}-\delta I)_{ij}|^2 \mid 1\le i,j\le t\} =$$

$$= \sum\{|(\tilde Q_n\varphi - \delta I)^\wedge(i-j)|^2 \mid 1\le i,j\le t\} =$$

$$= \sum\{(t-|k|)|(\tilde Q_n\varphi-\delta I)^\wedge(k)|^2 \mid 0\le|k|\le t-1\} \le$$

$$\le t\sum\{|(\tilde Q_n\varphi)^\wedge(k)|^2 \mid 1\le|k|\le t-1\} = \qquad \text{as } (Q_n\varphi)^\wedge(0)=\delta$$

$$= 2t\sum\{|(\tilde Q_n\varphi)^\wedge(k)|^2 \mid 1\le k\le t-1\}\le \qquad \text{as } \tilde Q_n\varphi \text{ is real valued}$$

$$\le 2t\sum\{|\hat\varphi(nk)|^2 \mid 1\le k\le t-1\}\le \qquad \text{by Lemma 5.1 a)}$$

$$\le 2t\sum\{|\hat\varphi(j)|^2 \mid j\ge n\} = 2t\|(r-r(n-1))\varphi\|^2. \qquad \text{Q.E.D.}$$

Part of this research was done while the second author was a member of MSRI at Berkeley.
Research of the third author was partially supported by NSF grant DMS 8503390.

REFERENCES

1. Anderson, J., Extensions, restrictions and representations of states on C^*-algebras, *Trans. Amer. Math. Soc.*, **249**(1979), 303–329.
2. Anderson, J., A conjecture concerning the pure states of $B(H)$ and a related theorem, in *Topics in modern operator theory*, Birkhäuser, 1981, pp. 27–43.
3. Archbold, R.; Bunce, J.; Gregson, K., Extensions of states of C^*-algebras. II. *Proc. Roy. Soc. Edinburgh*, **92A**(1982), 113–122.
4. van Daele, A., *Continuous crossed products and type* III *von Neumann algebras*, London Math. Soc. Lecture Notes Series, **31**, Cambridge University Press, Cambridge, 1978.
5. Dixmier, J., Applications dans les anneaux d'opérateurs, *Compositio Math.*, **10**(1952), 1–55.
6. Graham, R.; Rotschild, B.; Spencer, J., *Ramsey theory*, Wiley, New York, 1980.
7. Halmos, P., *A Hilbert space problem book*, 2nd Ed., Springer-Verlag, New York–Heidelberg–Berlin, 1982.
8. Halpern, H.; Kaftal, V., Compact operators in type III_λ and type III_0 factors. II, *Tôhoku Math. J.*, to appear.
9. Halpern, H.; Kaftal, V.; Weiss, G., Matrix pavings in $B(H)$, in *Operators in indefinite metric spaces, scattering theory, and other topics*, Birkhäuser, 1987, pp. 201–214.
10. Halpern, H.; Kaftal, V.; Weiss, G., The relative Dixmier property in discrete crossed products, *J. Funct. Anal.*, to appear.
11. Kadison, R.; Singer, I., Extensions of pure states, *Amer. J. Math.*, **81**(1959), 547–564.
12. Mercer, R., Convergence of Fourier series in discrete cross products of von Neumann algebras, *Proc. Amer. Math. Soc.*, **94**(1985), 254–258.
13. Olesen, D.; Pedersen, G., On a certain C^*-crossed product inside a W^*-crossed product, *Proc. Amer. Math. Soc.*, **79**(1980), 587–590.
14. Pedersen, G., *C^*-algebras and their automorphism groups*, Academic Press, London, 1979.
15. Pier, J., *Amenable locally compact groups*, Wiley-Interscience, New York, 1984.

H. HALPERN, V. KAFTAL, G. WEISS
Department of Mathematical Sciences,
University of Cincinnati,
Cincinnati, Ohio 45221,
U.S.A.

Received February 14, 1986.