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MATRIX PAVINGS IN B(H)

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§1. INTRODUCTION

Let $\mathcal{A} \subset \mathcal{B}$ be two C^*-algebras in B(H). The operator $x \in \mathcal{B}$ is said to have the relative Dixmier property (**RDP**) if $\overline{\mathbf{co}}\{uxu^* \mid u \in \mathcal{U}(\mathcal{A})\} \cap \mathcal{A}' \neq \emptyset$, where $\mathcal{U}(\mathcal{A})$ denotes the unitary group of $\mathcal{A}$ and $\overline{\mathbf{co}}$ denotes the uniformly closed convex hull [5, 7]. Then $N = \{x \in \mathcal{B} \mid x$ has the **RDP**$\}$ is uniformly closed, and is a subspace of $\mathcal{B}$ provided $\mathcal{A}$ is abelian (see [5]).

Dixmier proved that if $\mathcal{A} = \mathcal{B}$ is a von Neumann algebra then $N = \mathcal{B}$ [4]. Determining N is of interest and generally difficult, particularly in the case of the diagonal algebra $\mathcal{D} \subset B(H)$. J. Anderson [1, 2] proved that N = B(H) if and only if the extension property for pure states holds, that is every complex homomorphism on $\mathcal{D}$ extends uniquely to B(H). Define $\alpha_N(x) = \mathbf{inf}\{\|\sum_1^N p_k(x - E(x))p_k\| \mid \langle p_k\rangle_1^N$ is an orthogonal N-sequence of diagonal projections with $\sum_1^N p_k = I\}$, where E(x) denotes the diagonal of x. Define $\alpha(x) = \lim_{N\to\infty} \alpha_N(x)$, $\quad \alpha_N = \mathbf{sup}\{\alpha_N(x) \mid \|x\| = 1,\ E(x) = 0\}$, $\alpha = \mathbf{sup}\{\alpha(x) \mid \|x\| = 1,$ $E(x) = 0\}$. Clearly $\alpha_N \downarrow \alpha$. We say x is paveable if $\alpha(x) = 0$. Then the following are equivalent: N = = B(H) (open question), $\alpha = 0$, $\alpha < 1$, $\alpha(x) = 0$ for all $x \in B(H)$, $\alpha_N(x) \downarrow 0$ as $N \to \infty$ for all $x \in B(H)$, $\alpha_N < 1$ for some N. In view of this, computing α_N becomes important. Setting

$$U = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

we get $\alpha_2(U) = 1$ hence $\alpha_2 = 1$. Let V denote the $(2n + 2) \times (2n + 2)$ matrix with entries all 1 except on the lower diagonal and upper right-hand corner where the entries are all $-n$, then $x = (1/(n + 1))V$ is unitary, and $\alpha_3(x) \geq (2/3) - 1/(n + 1)$ for $n \geq 2$, hence $\alpha_3 \geq 2/3$.

OPEN PROBLEM. Is $\alpha_3 = 2/3$ or is there a finite matrix x with $\|x\| = 1$ and

$E(x) = 0$ for which $\alpha_3(x) > 2/3$?

If $\alpha_3 = 2/3$ then the extension property would hold. If on the other hand we could find a finite matrix x with $E(x) = 0$, $\|x\| = 1$ for which $\alpha_3(x) > 2/3$, we believe one could extract the properties of x to construct y's for which $\alpha_N(y)$ is arbitrarily close to 1, which would settle the problem in the negative.

Determining which Laurent operators L_ϕ (with symbol $\phi \in L^\infty(0,2\pi)$) are paveable is an open question in its own right. We know that the class of Laurent operators L is contained in N if and only if $L_{\chi_E} \in N$ for every measurable $E \subset (0,2\pi)$ if and only if $L_{\chi_O} \in N$ for every open set $O \subset (0,2\pi)$. The latter condition follows from the *Squeeze Principle*: $x \in N$ if for every $\epsilon > 0$ there exists $a, b \in N$ with $a \leq x \leq b$ and $\|E(b) - E(a)\| < \epsilon$. Another consequence of the squeeze principle is that $\alpha(L_{\chi_E}) \leq (1/2\pi) \cdot \mathbf{max}(m(\bar{E} \setminus E), m(\bar{E} \setminus E^\circ))$. Hence a necessary condition that $L_{\chi_O} \notin N$ is that $m(\bar{O} \setminus O) > 0$.

We know that if ϕ is a step function then $L_\phi \in N$. Hence the C^*-algebra $\tilde{L}$ generated by these L_ϕ, namely the L_ϕ where ϕ is partially continuous, are all paveable. However more is true. We call a diagonal projection a "uniform projection" if its 1's along the diagonal are equally spaced. Define $\tilde{N}$, $\tilde{\alpha}_N(x)$, $\tilde{\alpha}(x)$, $\tilde{\alpha}_N$, $\tilde{\alpha}$ as before but using only uniform projections, so $\tilde{N}$ denotes the "uniformly paveable" operators and is a uniformly closed subspace of N. Then the squeeze principle and what follows it in the preceding paragraph hold true replacing N by $\tilde{N}$ and α by $\tilde{\alpha}$, and in additon $\tilde{L} \subset \tilde{N}$.

Our study of Laurent operators and these aforementioned results will be presented in [5] and [6]. Here we shall prove the following results for the diagonal basis e_n.

THEOREM 4.2. *For* $O = \bigcup_{n=2}^{\infty} \bigcup_{k=0}^{n-1} ((0,2\pi/n^3) + 2\pi k/n)$, $L_{\chi_O} \notin \tilde{N}$. *Indeed if* p *is the projection onto* $\mathbf{span}\langle e_{a+jd} \rangle_{j=-\infty}^{\infty}$, *then* $\|p(L_{\chi_O} - E(L_{\chi_O}))p\| \geq 1 - m(O)/2\pi$.

QUESTION. Is L_{χ_O} paveable ?

If we associate p with its range basis and so with the canonical subset of $\mathbb{Z}$, then an orthogonal diagonal projection N-decomposition of I can be thought of as an N-partition of $\mathbb{Z}$, which leads us to the field of combinatorics. Van der Waerden proved that at least one subset of each such partition contains arbitrarily long arithmetic progressions.

THEOREM 4.3. *If* (1) p *contains an infinite arithmetic progression or* (2) *for some* $D \in \mathbf{Z}^+$, p *contains arbitrarily long finite arithmetic progressions with difference* $d \leq D$, *then* $\| p(L_{\chi_O} - E(L_{\chi_O}))p \| \geq 1 - m(O)/2\pi$.

We wish to thank D. Larson for his information and encouragement in this subject.

§2. MATRIX EQUIVALENCES TO THE EXTENSION PROPERTY

THEOREM 2.1. *The following are equivalent:*

1) $N = B(H)$ *(open question)*

2) $\alpha = 0$

3) $\alpha < 1$

4) $\alpha(x) = 0$ *for all* $x \in B(H)$

5) $\alpha_N(x) \downarrow 0$ *as* $N \to \infty$ *for all* $x \in B(H)$

6) $\alpha_N < 1$ *for some* N.

PROOF. 1)$\Longleftrightarrow$2) was proved by J. Anderson in [2, Theorem 3.6 and Corollary 3.7]. Also 2)$\Longleftrightarrow$4)$\Longleftrightarrow$5) and 2)$\Longrightarrow$3)$\Longrightarrow$6) are obvious. Hence it suffices to show 6)$\Longrightarrow$2). For each x with $\|x\| = 1$, $E(x) = 0$, we have $\alpha_{N^2}(x) \leq \alpha_N^2(x) \leq \alpha_N^2$, where the first inequality holds by applying $\alpha_N(\cdot)$ to each of the N compressions of x appearing in the computation of $\alpha_N(x)$. Therefore, $\alpha_{N^2} \leq \alpha_N^2$. Iterating we obtain $\alpha_{N^k} \leq \alpha_N^k$. But $\alpha \leq \alpha_i$ for each i. Hence $\alpha = 0$.

FINITE MATRIX EQUIVALENT

Let q_m be the projection onto $\mathbf{span}\langle e_j \rangle_{j=1}^m$ and canonically imbed $B(q_m H)$ into B(H). Then every $m \times m$ matrix is identified naturally with an element of B(H). Define $\beta_N = \mathbf{sup}\{\alpha_N(x) \mid \|x\| = 1, E(x) = 0, x \text{ is a finite matrix}\}$.

We shall prove that $\beta_N = \alpha_N$ and hence $N = B(H)$ if ond only if $\beta_N < 1$ for some positive integer N. This reduces the paving problem to a problem of finite matrices.

LEMMA 2.2. $\alpha_N(x) = \mathbf{sup}_m\, \alpha_N(q_m x q_m)$, *for every* $x \in B(H)$ *and every positive integer* N.

PROOF. Without loss of generality we may assume $E(x) = 0$. Given $x \in B(H)$ and $N \in \mathbf{Z}^+$ it is clear that $\alpha_N(q_m x q_m)$ is increasing in m and bounded above by $\alpha_N(x)$. Set $a = \mathbf{sup}_m\, \alpha_N(q_m x q_m)$. Then $a \leq \alpha_N(x)$ and in order to prove the equality it suffices to construct an N-partition of I, $\langle p_k\rangle_1^N$ such that

$$\|\sum_1^N p_k x p_k\| < a + \varepsilon .$$

Let $\hat{P}_m$ denote the set of ordered N-partitions $\langle p_k\rangle_1^N$ of q_m such that $\|\sum_1^N p_k x p_k\| < a + \varepsilon$. At least one such partition exists by the definition of a, so $\hat{P}_m$ is not empty. Obviously $\hat{P}_m$ is finite. Set $\hat{P} = \bigcup_1^\infty \hat{P}_m$, so $\hat{P}$ is infinite. Partial order $\hat{P}$ by setting $\langle p_k\rangle_1^N \ll \langle p'_k\rangle_1^N$ whenever $p_k \leq p'_k$ for each $1 \leq k \leq N$. By using a pidgeon-hole argument common in combinatorics, we shall construct an infinite increasing chain in $\hat{P}$.

Notice first that if $\langle p_k\rangle_1^N \in \hat{P}_m$ and if $i \leq m$ then $\langle p_k q_i\rangle_1^N \in \hat{P}_i$ and hence $\langle p_k q_i\rangle_1^N \ll \langle p_k\rangle_1^N$. In particular $\langle p_k q_1\rangle_1^N \in \hat{P}_1$ for all $\langle p_k\rangle_1^N \in \hat{P}$. Since $\hat{P}$ is infinite but there are only a finite number of possible N-partitions of q_1 (at most N), for one of these, say $\langle r_k^{(1)}\rangle_1^N \in \hat{P}_1$, the set $\hat{R}_1 = \{\langle p_k\rangle_1^N \in \hat{P} \mid \langle r_k^{(1)}\rangle_1^N \ll \langle p_k\rangle_1^N\}$ is infinite. Since $\langle p_k q_2\rangle_1^N \in \hat{P}_2$ for each $\langle p_k\rangle_1^N \in \hat{R}_1 \setminus \hat{P}_1$, we can again find a partition $\langle r_k^{(2)}\rangle_1^N \in \hat{P}_2$ such that $\hat{R}_2 = \{\langle p_k\rangle_1^N \in \hat{R}_1 \mid \langle r_k^{(2)}\rangle_1^N \ll \langle p_k\rangle_1^N\}$ is infinite. Of course then $\langle r_k^{(1)}\rangle_1^N \ll \langle r_k^{(2)}\rangle_1^N$. Proceeding inductively, we find an increasing chain of N-partitions $\langle r_k^{(m)}\rangle_1^N \in \hat{P}_m$. Set $p_k = \sup_m r_k^{(m)}$. Then $\langle p_k\rangle_1^N$ is an N-partition of I. Moreover for $1 \leq k \leq N$, $\|p_k x p_k\| \leq$ $\leq \liminf \|r_k^{(m)} x r_k^{(m)}\|$ by the lower semicontinuity of $\|\ \|$ in the strong operator topology. Hence $\|\sum_1^N p_k x p_k\| < a + \varepsilon$. Q.E.D.

THEOREM 2.3. $\beta_N = \alpha_N$ *for every* N.

PROOF. Clearly $\beta_N \leq \alpha_N$. On the other hand, if $\|x\| \leq 1$, $E(x) = 0$ then $q_m x q_m \in B(q_m H) \subset B(H)$, $\|q_m x q_m\| \leq 1$, $E(q_m x q_m) = 0$ and so $\alpha_N(q_m x q_m) \leq \beta_N$. But by Lemma 2.2, $\alpha_N(x) = \sup_m\{\alpha_N(q_m x q_m)\}$. Hence $\alpha_N(x) \leq \beta_N$ for each $\|x\| = 1$, $E(x) = 0$, so $\alpha_N \leq \beta_N$. Q.E.D.

§ 3. SOME RESULTS ON THE COMPUTATION OF α_N

NOTE. If $T = (t_{ij})$ and p is a diagonal projection which we view as a subset of $\mathbf{Z}^+$, then $(pTp)_{ij} = t_{ij}$ if $i, j \in p$ and 0 otherwise.

THE COMPUTATION OF α_2

PROPOSITION 3.1. *If* $T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, *or more generally if* T *is any permutation matrix with odd dimension, then* $\alpha_2(T) = 1$, *and so* $\alpha_2 = 1$.

PROOF. If T is an $n \times n$ permutation matrix, then regarding it as a 1 - 1 map of $\{1,\ldots,n\}$ onto itself, we can express it as a disjoint product of cycles. This yields T as a direct sum of "cycles", each of which is unitarily equivalent to

$$U = \begin{bmatrix} 0 & 0 & \ldots & 1 \\ 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & . \\ . & . & \ldots & . \\ 0 & . & \ldots 1 & 0 \end{bmatrix}$$

via a permutation matrix which leaves the class of diagonals invariant. Since n is odd, at least one cycle has odd dimension (i.e. size) and without loss of generality assume it is U. Now if $p_1 + p_2 = I$, p_1, p_2 diagonal orthogonal projections, then either $\|p_i U p_i\| = 1$ or 0. If $p_i U p_i = 0$, $i = 1, 2$, then no consecutive $j, j+1 \in p_i$ for $1 \le j \le \dim U$. Without loss of generality $p_1 = \{1, 3, 5, \ldots, n\}$ and $p_2 = \{2, 4, 6, \ldots, n-1\}$. But then $1, n \in p_1$ hence $(p_1 U p_1)_{1n} = 1$, a contradiction. Q.E.D.

PROPOSITION 3.2. *Let* F *denote the* $n \times n$ *matrix with entries all* 1, *so* $(1/n)F$ *is a rank one projection. Then for* $n \ge 3$, $\alpha_2((1/n)F) = \frac{1}{2} - 1/n$ *or* $\frac{1}{2} - 1/2n$ *depending respectively on whether or not n is even. More generally, for* $n > N$, $n = pN + r$, $p \ge 0$, $0 \le r < N$, *we have* $\alpha_N((1/n)F) = 1/N - 1/n$ *or* $(N + r/p)^{-1}$ *depending respectively on whether or not* N *divides* n.

PROOF. If $\sum_1^N P_k = I$ then for the projection $P = P_k$ with largest dimension, $\dim P \ge [n/N] = p$ or $[n/N] + 1 = p + 1$ depending respectively on whether or not N divides n. Therefore $\|P((1/n)F)P\| \ge p/n = 1/N$ or $(p+1)/n$ if $N | n$ or $N \nmid n$, respectively. Since $P((1/n)F)P$ is a rank one positive operator of norm not less than $1/N$, then we remove the diagonal $(1/n)I$ and we get $\alpha_N((1/N)F) \ge 1/N - 1/n$ or $p/n = (N + r/p)^{-1}$ if $N | n$ or $N \nmid n$, respectively. To obtain the equality simply choose all P_k's of the same dimension if $N | n$, otherwise choose r projections of dimension $p + 1$ and $N - r$ projections of dimension p. Such a decomposition of I yields the equality. Q.E.D.

THE COMPUTATION OF α_3

COROLLARY 3.3. $\alpha_3 \geq 1/3$.

PROOF. This follows directly from Proposition 3.2, noting also that $\|(1/n)F - E((1/n)F)\| = 1 - 1/n$.

PROPOSITION 3.4. *If* T *is a matrix (finite or infinite) with entries* 0, 1 *only and each row and column contains at most one* 1, *then* $\alpha_3(T) = 0$.

PROOF. Without loss of generality, assume $E(T) = 0$. Since for each n, $Te_n = e_m$ for some m, or $Te_n = 0$, we may regard T as a map of $\mathbf{Z}^+$ into $\mathbf{Z}^+$. Define p_1, p_2, p_3 according to the following procedure. Put $e_1 \in p_1$ and assume $e_1, \ldots, e_{n-1}$ has been placed inside p_1, p_2 or p_3. If $e_j = Te_n$ with $j < n$ and/or if $e_n = Te_k$ with $k < n$, place e_n in a p_i which does not contain either j or k. Otherwise, place n arbitrarily. Then it is easy to verify that $p_i T p_i = 0$ for i = 1, 2, 3. Q.E.D.

THEOREM 3.5. (Blending 3.1 & 3.2). *Let* T *denote the unitary* $(2n+2)\times(2n+2)$ *matrix* $(1/(n+1))F - U$ *with all entries* $1/(n+1)$ *except the lower diagonal and the upper right-hand corner where those entries are all* $-n/(n+1)$. *For* $3 \leq N < 2n+2 = pN + r$ *with* $p \geq 1$, $0 \leq r < N$ *we have for* $n \geq 2$ *that* $\alpha_N(T) = 2/N - 1/(n+1)$ *or* $2/N - r/N(n+1)$ *if* $N \mid 2n+2$ *or* $N \nmid 2n+2$, *respectively. Hence* $\alpha_N \geq 2/N$ *and in particular* $\alpha_3 \geq 2/3$.

PROOF. If $\sum_1^N P_k = I$, then for the projection of largest dimension $P = P_k$, $\dim P \geq [(2n+2)/N] = p$ or $\dim P \geq p+1$ if $N \mid 2n+2$ or $N \nmid 2n+2$, respectively. Either all the non-zero entries of PTP are $1/(n+1)$ or at least one entry is $-n/(n+1)$. In the latter case, $\|PTP\| \geq n/(n+1)$. In the first case PTP is a rank one positive operator with $\|PTP\| \geq (2n+2)/N(n+1) = 2/N$ or $\|PTP\| \geq (p+1)/(n+1)$ depending respectively on whether or not $N \mid 2n+2$. Since PTP is a rank one positive operator of norm larger than 2/N, then removing the diagonal $(1/(n+1))I$, we get $\alpha_N(T) \geq 2/N - 1/(n+1)$ or $\alpha_N(T) \geq (p+1)/(n+1) - 1/(n+1) = (2n+2-r)/N(n+1) = 2/N - r/N(n+1)$ if $N \mid 2n+2$ or $N \nmid 2n+2$, respectively.

To obtain equality one needs to divide $[1, 2n+2] \cap \mathbf{Z}$ into N disjoint subsets $P_1, \ldots, P_N$ of size $(2n+2)/N$ or $[(2n+2)/N]+1$ so that no subset contains consecutive pairs of integers nor the pair $\{1, 2n+2\}$. Indeed, if $r > 1$, set $P_1 = \{qN \mid 1 \leq q \leq p\}$, set $P_j = \{qN + j - 1 \mid 0 \leq q \leq p\}$ for $1 \leq j-1 \leq r$, i.e. $2 \leq j \leq r+1$, and set $P_j = \{qN + j - 1 \mid 0 \leq q \leq p-1\}$ for $r < j-1 < N$, i.e. $r+2 \leq j \leq N$. If $r = 1$, set $P_1 = \{qN \mid 1 \leq q \leq p\} \cup \{2n+2\}$, set $P_2 = \{qN+1 \mid 0 \leq q \leq p-1\}$, $P_j = \{qN + j - 1 \mid 0 \leq q \leq p\}$ for $3 \leq j \leq r+1$, and set $P_j = \{qN + j - 1 \mid$

$|0 \leq q \leq p-1\}$ for $r+2 \leq j \leq N$. This is the required type of decomposition, and so $\alpha_N(T) = 2/N - 1/(n+1)$ or $2/N - r/N(n+1)$ if $N \mid 2n+2$ or not, respectively.

Since $1 - 1/(n+1) \leq \|T - E(T)\| \leq 1 + 1/(n+1)$ we have $\alpha_N \geq 2/N$. In particular, if $3 \mid 2n+2$ then $\alpha_3(T) = 2/3 - 1/(n+1)$ and so $\alpha_3 \geq 2/3$. Q.E.D.

PROBLEM 3.6. Is $\alpha_3 < 1$ or $\alpha_3 = 1$? At least is $\alpha_3 > 2/3$?

REMARK. If $\alpha_3 < 1$, this would be a remarkable result and would immediately yield $N = B(H)$.

If $\alpha_3 = 1$ we hope, indeed expect the proof to generalize to $\alpha_N = 1$ for all N, hence $N \neq B(H)$, that is $D \subset B(H)$ would not have the extension property for pure states. To obtain $\alpha_3 = 1$ one must either produce a matrix T, $\|T\| = 1$, $E(T) = 0$ such that $\alpha_3(T) = 1$ or, probably more realistically, an asymptotic example $\langle T_n \rangle$ with $\|T_n\| \to 1$, $\|E(T_n)\| \to 0$, and $\alpha_3(T_n) \to 1$.

Computing $\alpha_N(T)$ has not been easy. Hence we asked a more modest question: is $\alpha_3 > 2/3$, that is, is there a matrix T with $\|T\| = 1$, $E(T) = 0$ such that $\alpha_3(T) > 2/3$?

§ 4. LAURENT OPERATORS

In [1] Anderson also asked whether or not all Toeplitz operators T_ϕ with symbol $\phi \in L^\infty(0, 2\pi)$ were paveable.

In [5], [6] we study Laurent operators. A few of the results are mentioned in the introduction to this article. For example all Laurent operators are paveable if and only if L_{χ_O} is paveable for every open set $O \subset (0, 2\pi)$. This was obtained from the "squeeze principle" and the result that N is a uniformly closed subspace of B(H). Also from the squeeze principle we obtain $\alpha(L_{\chi_E}) \leq (1/2\pi)\mathbf{max}(m(\overline{E} \setminus E), m(E \setminus E^\circ))$, a link between paveability, the α-measure, measure theory and the topological structure of E.

Our study of Laurent operators (equivalently, of their Toeplitz compressions) can be motivated by considering the Hilbert matrix $H = (1/(i-j))_{i \neq j}$ (0 along the diagonal, so $E(H) = 0$), and asking about $\alpha_2(H)$.

PROPOSITION 4.1. $\alpha_2(H) \leq \frac{1}{2}\|H\|$ *and hence* $\alpha(H) = 0$.

PROOF. Choose $P_1 = \{1, 3, 5, 7, \ldots,\}$ and $P_2 = \{2, 4, 6, \ldots,\}$, so $P_1 \cup P_2 = \mathbf{Z}^+$. Then $(P_1HP_1)_{ij} = 1/(i-j)$ if $i \neq j$ are both odd and 0 otherwise, and $(P_2HP_2)_{ij} = 1/(i-j)$ if $i \neq j$ are both even and 0 otherwise. Hence

$$P_1HP_1 = \begin{bmatrix} 0 & 0 & -\frac{1}{2} & 0 & -\frac{1}{4} & 0 & -1/6 & 0 & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdot \\ \frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} & 0 & -\frac{1}{4} & 0 & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdot \\ \frac{1}{4} & 0 & \frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} & 0 & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdot \\ 1/6 & 0 & 1/8 & 0 & \frac{1}{2} & 0 & 0 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \simeq \tfrac{1}{2}H \oplus 0$$

$$P_2HP_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdot \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & -\frac{1}{4} & 0 & -1/6 & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdot \\ 0 & \frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} & 0 & -\frac{1}{4} & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdot \\ 0 & \frac{1}{4} & 0 & \frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdot \\ 0 & 1/6 & 0 & \frac{1}{4} & 0 & \frac{1}{2} & 0 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \simeq \tfrac{1}{2}H \oplus 0 .$$

Hence $\alpha_2(H) \leq \frac{1}{2} \| H \|$ and thus by iterating, $\alpha_{2^k}(H) \leq \frac{1}{2}\alpha_{2^{k-1}}(H) \leq \ldots \leq (1/2^k) \| H \|$. Hence $\alpha(H) \leq \lim_{k\to\infty} \alpha_{2^k}(H) = 0$.

Another proof that $\alpha(H) = 0$ can be given using the machinery developed in [5] where it is proved that all L_ϕ with piecewise continuous ϕ are paveable, and noting the well-known fact that $H = L_\phi$ where $\phi(x) = i(x - \pi) \in L^\infty(0,2\pi)$. Q.E.D.

NOTE. This does not compute $\alpha_2(H)$ precisely and we in fact do not know if equality holds. Nevertheless the key motivating property for us here is that both $P_iHP_i \simeq \frac{1}{2}H \oplus 0$. The uniformity of P_1, P_2 suggests defining $\tilde{\alpha}$, $\tilde{N}$ as we did in the introduction. This together with the above construction yields $\tilde{\alpha}(H) = 0$. Here we present our central example of an open set $O \subset (0, 2\pi)$ for which $\tilde{\alpha}(L_{\chi_O}) > 0$ (so L_{χ_O} is not uniformly paveable) and which has some bearing on α and general paveability of L_{χ_O}.

THEOREM 4.2. *Let* $O = \bigcup_{n=2}^{\infty} \bigcup_{k=2}^{n-1} ((0, 2\pi/n^3) + 2\pi k/n)$ *(so* $m(O) < 2\pi$*). If* p *is any infinite arithmetic progression (i.e.* **Range** $p =$ **span** $\langle e_{a+jd} \rangle_{j=-\infty}^{\infty}$ *), then*

$$\| p(L_{\chi_O} - m(O)/2\pi)p \| \geq 1 - m(O)/2\pi .$$

PROOF. Set $O_n = \bigcup_{k=0}^{n-1} ((0, 2\pi/n^3) + 2\pi k/n)$ so $O = \bigcup_{2}^{\infty} O_n$. Hence $\chi_{O_d} \leq \chi_O$, which implies $L_{\chi_{O_d}} \leq L_{\chi_O}$. Therefore $pL_{\chi_{O_d}} p \leq pL_{\chi_O} p$. From this it follows that

$$p(L_{\chi_O} - m(O)/2\pi)p \geq pL_{\chi_O} p - m(O)/2\pi \geq pL_{\chi_{O_d}} p - m(O)/2\pi .$$

From the Note after Proposition 4.1 and the fact that $\phi \in L^\infty(0, 2\pi)$ is $2\pi/d$ periodic if and only if the Fourier series $\sum_{j=-\infty}^{\infty} \hat{\phi}(j)e^{2\pi ijt} = \sum_{j=-\infty}^{\infty} \hat{\phi}(dj)e^{2\pi idjt}$ (i.e., $\hat{\phi}(k) = 0$ for $d \nmid k$) we have that $pL_{\chi_{O_d}} p$ is unitarily equivalent to $L_{\chi_O}(t/d) \oplus 0$ and hence is a projection q. Therefore $p(L_{\chi_O} - m(O)/2\pi)p \geq (1 - m(O)/2\pi)q - (m(O)/2\pi)q^\perp$, and hence

$$\| p(L_{\chi_O} - m(O)/2\pi)p \| \geq 1 - m(O)/2\pi . \qquad \text{Q.E.D.}$$

COROLLARY 4.3. $\tilde{\alpha}(L_{\chi_O}) \geq 1 - m(O)/2\pi > 0$, *hence* $L_{\chi_O} \notin \tilde{N}$.

THEOREM 4.4. *Let* O *be the open set in* Theorem 4.2. *If* p *is any diagonal projection whose range contains an infinite arithmetic progression, or more generally, if there exists a positive integer* D *for which* **Range** p *contains arbitrarily long finite arithmetic progressions with* $d \leq D$ *(*d *denotes the difference of the arithmetic progression* $\langle a + jd \rangle_{j=k}^{m}$*), then*

$$\| p(L_{\chi_O} - m(O)/2\pi)p \| \geq 1 - m(O)/2\pi .$$

PROOF. Suppose first $p \geq q \quad p, q \in D$ where **Range** q = **span** $\langle e_{a+jd} \rangle_{-\infty}^{\infty}$. Then by Theorem 4.2,

$$\| p(L_{\chi_O} - m(O)/2\pi)p \| \geq \| q(L_{\chi_O} - m(O)/2\pi)q \| \geq 1 - m(O)/2\pi .$$

Now only assume that for some D, p contains arbitrarily long finite arithmetic progressions with $d \leq D$. By a pigeon hole argument there is one fixed d so that for every length ℓ one prescribes, there exists $a \in \mathbf{Z}$ such that $\langle e_{a+jd} \rangle_{j=1}^{\ell} \subset$ **Range** p. Let q_ℓ denote the diagonal projection with range **span**$\langle e_{a+jd} \rangle_{j=1}^{\ell}$, assume without loss of generality that $\ell = 2k + 1$ and define r_k to be the diagonal projection with range

$\mathrm{span}\langle e_{jd}\rangle_{j=-k}^{k}$. Then since $q_\ell L_{\chi_O} q_\ell \simeq r_k L_{\chi_O} r_k$, because Laurent operators commute with the bilateral shift, we have

$$\| p(L_{\chi_O} - m(O)/2\pi)p\| \geq \| q_\ell(L_{\chi_O} - m(O)/2\pi)q_\ell\| = \| r_k(L_{\chi_O} - m(O)/2\pi)r_k\| .$$

But $r_k \uparrow r$ (SOT) where $\mathbf{span}\langle e_{jd}\rangle_{j=-\infty}^{\infty} = \mathbf{Range}\, r$, hence as is well-known $\lim_{k\to\infty} \| r_k L_{\chi_O} r_k\| = \| r L_{\chi_O} r\|$. Therefore

$$\| p(L_{\chi_O} - m(O)/2\pi)p\| \geq \mathbf{lim}\, \| r_k(L_{\chi_O} - m(O)/2\pi)r_k\| =$$

$$= \| r(L_{\chi_O} - m(O)/2\pi)r\| \geq 1 - m(O)/2\pi .$$ Q.E.D.

PROBLEM 4.5. Is $\alpha(L_{\chi_O}) = 0$?

If $\alpha(L_{\chi_O}) \neq 0$ we would have a counterexample to $N = B(H)$.

REMARK. Theorem 4.4 can be strengthened. There exists a function $D(\ell) \uparrow \infty$ such that the conclusion of Theorem 4.4 essentially holds under the milder condition $d \leq D(\ell)$. This function and its relation to the van der Waerden theorem are discussed in [6].

§5. RELATED TECHNIQUES FROM COMPACT DERIVATIONS

One can obtain some information on general compressions PTP (and pavings) using techniques related to the well-known work of Johnson and Parrott on compact derivations [8]. We believe this information is well-known to some specialists in the area of compact derivations but may be useful to those interested in matrix pavings, so we present it here.

NOTATIONS. For each $N|n$, let $P(N,n)$ denote the collection of all N-orthogonal diagonal rank n/N projection decompositions of I (the $n\times n$ identity matrix), that is, the set of all N-sequences $\langle P_k\rangle_1^N$ of $n\times n$ orthogonal diagonal projections where each **rank** $P_k = n/N$ and $I = \sum_1^N P_k$. Let $\tilde{P}_n$ denote the set of all $n\times n$ diagonal projections and when $N|n$, let $\tilde{P}_n^N$ denote all $n\times n$ diagonal projections of rank n/N. Let **card**· denote cardinality.

PROPOSITION 5.1. *If an* $n \times n$ *matrix* T *has diagonal* 0 *(i.e.* $E(T) = 0$*) then*

I. $$\sum_{P \in \tilde{P}_n} PTP = \sum_{P \in \tilde{P}_n} PTP^{\perp} = \tfrac{1}{4}(\mathbf{card}\, \tilde{P}_n)T.$$

In addition, if $N \mid n$ *then*

II. $$\sum_{\langle P_k \rangle_1^N \in P(N,n)} \sum_{k=1}^{N} P_k T P_k = (N/n)(\mathbf{card}\, P(N,n))T\,,$$

and

III. $$\sum_{P \in \tilde{P}_n^N} PTP = (N/n)^2(\mathbf{card}\, \tilde{P}_n^N)T\,.$$

PROOF OF I. We give two proofs of **I.** The first depends on the conditional expectation formula known by some specialists:

$$E(T) = (1/\mathbf{card}\, \tilde{P}_n) \sum_{P \in \tilde{P}_n} (P - P^{\perp})T(P - P^{\perp}).$$

The other proceeds from scratch.

By expanding this identity we get $E(T) = (2/\mathbf{card}\, \tilde{P}_n)(\sum_{\tilde{P}_n} PTP - \sum_{\tilde{P}_n} PTP^{\perp})$ hence $E(T) = 0$ iff $\sum_{\tilde{P}_n} PTP = \sum_{\tilde{P}_n} PTP^{\perp}$. Now summing over $\tilde{P}_n$, the identity $T = (P + P^{\perp})T(P + P^{\perp}) = PTP + P^{\perp}TP^{\perp} + PTP^{\perp} + P^{\perp}TP$ yields equations **I** immediately.

The proof from scratch is the following.

Note that the (i,j)-entry of $PTP^{\perp}$ is zero except when $e_i \in \mathbf{Ran}\, P$ and $e_j \in \mathbf{Ran}\, P^{\perp}$ in which case it is the same as the (i,j)-entry of T itself, and $PTP(i,j) = T(i,j)$ when $e_i, e_j \in \mathbf{Ran}\, P$, and 0 otherwise. For each (i,j) set

$$D_1 = \{P \in \tilde{P}_n : e_i \notin \mathbf{Ran}\, P,\ e_j \notin \mathbf{Ran}\, P^{\perp}\},\ D_2 = \{P \in \tilde{P}_n : e_i \notin \mathbf{Ran}\, P,\ e_j \in \mathbf{Ran}\, P^{\perp}\},$$

$$D_3 = \{P \in \tilde{P}_n : e_i \in \mathbf{Ran}\, P,\ e_j \notin \mathbf{Ran}\, P^{\perp}\} \text{ and } D_4 = \{P \in \tilde{P}_n : e_i \in \mathbf{Ran}\, P \text{ and } e_j \in \mathbf{Ran}\, P^{\perp}\}.$$

Then $\langle D_k \rangle_{k=1}^4$ are pairwise disjoint and $\tilde{P}_n = \bigcup_{k=1}^{4} D_k$. It is straightforward to verify that the cardinalities of all the D_k's are the same, and so $\mathbf{card}\, D_k = \tfrac{1}{4}\mathbf{card}\, \tilde{P}_n$. Hence for each (i,j),

$$\Big(\sum_{P\in\tilde{P}_n} PTP^{\perp}\Big)(i,j) = \sum_{P\in D_4} (PTP^{\perp})(i,j) = (\mathbf{card}\, D_4)T(i,j) = \tfrac{1}{4}(\mathbf{card}\, \tilde{P}_n)T(i,j)\,.$$

The other equality follows similary by

$$\sum_{P\in D_3} (PTP)(i,j) = (\mathbf{card}\, D_3)T(i,j) = \tfrac{1}{4}(\mathbf{card}\, \tilde{P}_n)T(i,j).$$

INDICATIONS OF PROOFS FOR **II**, **III**. The precise proofs of **II** and **III** are rather involved so we simply present a heuristic simplification. Each P_kTP_k has "non--zero size" $(n/N)\times(n/N)$ so the probability that $P_kTP_k(i,j) = T(i,j)$ and not 0 is $(N/n)^2$ if $i \neq j$ and the probability that $\sum_{k=1}^{N} P_kTP_k(i,j) = T(i,j)$ rather than 0 is $(N/n)^2 \cdot (n/N) = = (N/n)$ if $i \neq j$, which suggests **II**. Also for each $P \in \tilde{P}_n^N$, PTP has "non-zero size" $(n/N)\times(n/N)$ so the probability that $PTP(i,j) = T(i,j)$, not 0, is $(N/n)^2$ if $i \neq j$ which suggests **III**. To make these precise one must show that "probability" here is truly the exact degree of occurrence of $T(i,j)$.

PROPOSITION 5.2. *Let* T *be any* $n\times n$ *matrix. Then*

(1) $$\sum_{P\in\tilde{P}_n} PTP = \tfrac{1}{4}(\mathbf{card}\, \tilde{P}_n)(T + E(T))$$

and

(2) $$\sum_{P\in\tilde{P}_n} PTP^{\perp} = \tfrac{1}{4}(\mathbf{card}\, \tilde{P}_n)(T - E(T))\,.$$

Furthermore if $N\,|\,n$ *then*

(3) $$\sum_{P\in\tilde{P}_n} PTP = (\mathbf{card}\, \tilde{P}_n^N)((N/n)^2 T + (N/n)(1 - N/n)E(T)).$$

PROOF. (1) and (2) follow easily from Proposition 5.1. Indeed for any diagonal operator, and in particular $E(T)$, we have $PE(T)P^{\perp} = 0$ for every diagonal projection. Applying Proposition 5.1-I to $T - E(T)$ yields (2). To obtain (1) note that $E(T) = (P + P^{\perp}) = = E(T)(P + P^{\perp}) = PE(T)P + P^{\perp}E(T)P^{\perp}$. By Proposition 5.1-I,

$$\tfrac{1}{4}(\mathbf{card}\, \tilde{P}_n)(T - E(T)) = \sum_{P\in\tilde{P}_n} P(T - E(T))P = \sum_{P\in\tilde{P}_n} PTP - \sum_{P\in\tilde{P}_n} PE(T)P =$$

$$= \sum_{P\in\tilde{P}_n} PTP - \tfrac{1}{2}(\mathbf{card}\, \tilde{P}_n)E(T),$$

and adding we obtain

$$\sum_{\tilde{P}_n} PTP = \tfrac{1}{2}(\mathbf{card}\,\tilde{P}_n)(T + E(T)).$$

To prove (3) use Proposition 5.1-III to obtain

$$\sum_{P\epsilon\tilde{P}_n^N} P(T - E(T))P = (N/n)^2(\mathbf{card}\,\tilde{P}_n^N)(T - E(T)).$$

But $\sum_{P\epsilon\tilde{P}_n^N} PE(T)P = (N/n)(\mathbf{card}\,\tilde{P}_n^N)E(T)$. To see this requires an involved argument as before, so as before, we appeal to the heuristic simplification described at the end of the proof for Proposition 5.1. Hence

$$\sum_{P\epsilon\tilde{P}_n^N} PTP = ((N/n)^2(T - E(T)) + (N/n)E(T))\mathbf{card}\,\tilde{P}_n^N = ((N/n)^2T + (N/n)(1 - N/n)E(T))\mathbf{card}\,\tilde{P}_n^N.$$

Q.E.D.

LEMMA 5.3. *If* T *has a zero diagonal then there exists a diagonal projection* $P \epsilon \tilde{P}_n$ *such that* $\|PTP^{\perp}\| \geq \tfrac{1}{4}\|T\|$.

PROOF. By Proposition 5.1

$$\tfrac{1}{4}(\mathbf{card}\,\tilde{P}_n)\|T\| = \|\sum_{P\epsilon\tilde{P}_n} PTP^{\perp}\| \leq \tfrac{1}{4}(\mathbf{card}\,\tilde{P}_n)\max_{P\epsilon\tilde{P}_n} \|PTP^{\perp} + P^{\perp}TP\| =$$

$$= \tfrac{1}{4}(\mathbf{card}\,\tilde{P}_n)\max_{P\epsilon\tilde{P}_n} (\|PTP^{\perp}\|, \|P^{\perp}TP\|).$$

Therefore $\tfrac{1}{4}\|T\| \leq \max_{P\epsilon\tilde{P}_n} \|PTP^{\perp}\|$.

REMARK. The same identity yields many $\|PTP^{\perp}\| \geq \varepsilon\|T\|$ for ε small. A similar identity together with similar lower bound estimates also holds for $\sum_{P\epsilon\tilde{P}_n} PTP$.

THEOREM 5.4. (Johnson and Parrott, the atomic case [8]). *If* T *commutes with the diagonals D modulo the compacts* K(H), *then* $T - E(T) \epsilon K(H)$.

PROOF. Since $E(T) \epsilon D$, $T - E(T)$ commutes with D modulo K(H), so we may asume without loss of generality that T has a 0 diagonal.

Assume $T \notin K(H)$. Then $\|\pi(T)\| = \alpha > 0$ where π is the projection of B(H) onto the Calkin algebra B(H)/K(H). It is easy to construct an infinite sequence of orthogonal diagonal finite projections $\langle F_k \rangle_1^\infty$ such that $\| F_k T F_k \| > \alpha/2$. By Lemma 5.3 there exists a diagonal projection $P_k \le F_k$ with

$$\| P_k T(F_k - P_k) \| = \| P_k (F_k T F_k) P_k^\perp \| \ge \tfrac{1}{2} \| F_k T F_k \| \ge \alpha/4.$$

Set $P = \sum P_k \in D$. We claim that this implies that $TP - PT \notin K(H)$, which contradicts the hypothesis. To see this, since $\langle F_k \rangle_1^\infty$ are orthogonal projections, it suffices to show that $\| F_k(TP - PT)F_k \| \not\to 0$. Indeed

$$\| F_k(TP - PT)F_k \| = \| F_k(P^\perp TP - PTP^\perp)F_k \| =$$

$$\max(\| F_k P^\perp T P F_k \|, \| F_k P T P^\perp F_k \|) \ge \| F_k P T P^\perp F_k \| = \| P_k T(F_k - P_k) \| \ge \alpha/4.$$

Q.E.D.

REMARK. To achieve the full Johnson and Parrott theorem for the atomic case: that every compact derivation on D is inner and can be implemented by a compact, in view of Theorem 5.4 it suffices to merely prove that each derivation from D into B(H) is inner. But $T \in B(H)$, $E(T) = 0$ can be produced which implements the derivation δ by first restricting δ to the finite case (acting on $q_n H$), using 5.1-I replacing $PTP^\perp$ by $\delta(P)$ to produce $T_n = \delta \mid q_n D$, then using the weak compactness of B(H), and the well-known weak continuity of derivations to obtain T.

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