Integral Equations
and Operator Theory
Vol. 11 (1988)

0378-620X/88/010028-21$1.50+0.20/0

MATRIX NORM INEQUALITIES AND THE RELATIVE DIXMIER PROPERTY

Kenneth Berman, Herbert Halpern, Victor Kaftal (*) and Gary Weiss (**)

If x is a selfadjoint matrix with zero diagonal and non-negative entries, then there exists a decomposition of the identity into k diagonal orthogonal projections $\{p_m\}$ for which

$$\|\sum p_m x p_m\| \leq (1/k)\|x\| .$$

From this follows that all bounded matrices with non-negative entries satisfy the relative Dixmier property or, equivalently, the Kadison Singer extension property. This inequality fails for large Hadamard matrices. However a similar inequality holds for all matrices with respect to the Hilbert-Schmidt norm with constant $k^{-1/2}$ and for Hadamard matrices with respect to the Schatten 4-norm with constant $2^{1/4}k^{-1/2}$.

§1 INTRODUCTION

A long standing open problem, first discussed by Kadison and Singer in 1959, is whether every pure state on the atomic masa D of diagonal operators on a separable Hilbert space H has a unique extension to a pure state of B(H) (extension property for the embedding of D into B(H)).

An equivalent problem is whether every element x of B(H) has the Dixmier property relative to D , i.e., whether the norm closed convex hull

$$K(x) = co^{-}\{ uxu^* \mid u \in D, u \text{ unitary} \}$$

has non-empty intersection with D , in which case $K(x) \cap D = \{E(x)\}$, where E(x) denotes the diagonal of x (relative Dixmier property for the embedding of D into B(H)).

The distance $\alpha(x)$ from E(x) to K(x) or equivalently, the distance from 0 to K(x - E(x)) , can be measured using decompositions of the identity into a sum of finitely many mutually orthogonal diagonal projections (diagonal decompositions or d.d. for short), i.e. we have:

$$\alpha(x) = \inf \{ \|\sum p_m(x - E(x))p_m\| \mid \{p_m\} \text{ is a d.d.} \} .$$

Partially supported by grants (*) of the Taft Foundation, (**) NSF .

Clearly, if we denote

$$\alpha_k(x) = \inf \{ \|\sum p_m(x - E(x))p_m\| \mid \{p_m\}_{m=1,2,\ldots k} \text{ is a d.d.} \},$$

then $\alpha(x) = \lim_k \alpha_k(x)$.

While it is obvious that $\alpha(x) = 0$ for all finite matrices x , it is in general highly non-trivial to compute, or at least estimate, the values of $\alpha_k(x)$. This task involves analyzing some deep norm inequalities for matrices which have interest in their own right and which also have a direct bearing on the Dixmier property discussed above. For instance, proving that for some positive integer k ,

$$\beta_k = \sup\{ \alpha_k(x) \mid x \text{ finite matrix}, \|x\| \leq 1, E(x) = 0 \} < 1,$$

is equivalent to proving that the embedding of D into B(H) has the relative Dixmier property (Proposition 2.8).

It is known that the set N of elements having the relative Dixmier property is a selfadjoint uniformly closed subspace of B(H) and that it trivially contains D and the ideal of compact operators. Already in **[10]** Kadison and Singer showed that N contains the Banach space generated by the permutation matrices (by showing that $\alpha_3(x) = 0$ for each permutation matrix x). Apart from the recent result that N contains all Toeplitz operators with Riemann integrable symbol **[8]**, not much more was known about its elements.

We prove here that *every bounded matrix* x *with non-negative entries is in* N , by showing that $\alpha_k(x)/\|x - E(x)\| \leq 2/k$ (Corollary 3.5). Our proof uses techniques from graph theory combined with the Perron - Frobenius - Schur theory of finite matrices with non-negative entries. In the selfadjoint case (Theorem 3.4) we obtain the same inequality, but with constant 1/k , and we prove that this result is sharp.

In Proposition 2.11 we exhibit selfadjoint operators (direct sums of Hadamard matrices) for which $\alpha_k(x)/\|x - E(x)\| \geq k^{-1/2}$, which provide, so far, the largest known value for this ratio (for $k > 3$).

Related to this are our results on Schatten p-norms. If in the definition of $\alpha_k(x)$ we use the Schatten p-norm $\| \; \|_p$ instead of the operator norm $\| \; \|$, we obtain a new parameter $\alpha_{k,p}(x)$. Then in Theorem 4.4 we prove that $\alpha_{k,2}(x)/\|x - E(x)\|_2 \leq k^{-1/2}$ or every finite matrix x (or more generally, for every Hilbert-Schmidt operator x) and this result is sharp. Moreover, we show that for $p = 2$ and $p = 4$ the ratio $\alpha_{k,p}(x)/\|x - E(x)\|_p$ for Hadamard matrices behaves like $k^{-1/2}$ (Propositions 4.2 and 4.6).

We were recently informed that in his 1986 doctoral dissertation at the University of Aberdeen, K. Gregson has independently obtained that bounded matrices with non-negative entries have the relative Dixmier property.

We would like to thank D. Larson for information and encouragement throughout the preparation of this work.

§2 THE RELATIVE DIXMIER PROPERTY AND NORM INEQUALITIES FOR MATRICES.

Let H denote a separable infinite-dimensional Hilbert space with an orthonormal basis $\{\eta_j\}_{j=1,2,\ldots}$ and let q_n be the projection on $\mathrm{span}\{\eta_j \mid j = 1,\ldots, n\}$. Let D be the maximal abelian subalgebra (masa) of B(H) generated by $\{q_n\}_{n=1,2,\ldots}$; let $E : B(H) \to D$ be the canonical normal conditional expectation onto D and let U(D) be the group of unitary operators of D . Henceforth we shall identify B(H) (resp. D) with the algebra of bounded (resp. bounded diagonal) matrices and E with the operation of taking the main diagonal of a matrix. Whenever we shall talk of direct sums, we shall always mean relative to the given basis $\{\eta_j\}$. For every $x \in B(H)$, let

(1) $$K(x) = \overline{co}\{ uxu^* \mid u \in U(D) \}$$

denote the norm closure of the convex hull of the diagonal-unitary orbit of x .

We say that x has the relative Dixmier property , (RDP for short), if K(x) has non-empty intersection with D . We refer the reader to [**7**, **8**] for a discussion of the RDP in the more general setting of the embedding of a von Neumann algebra into its discrete crossed product.

A special reason to investigate the case $D \subset B(H)$ is that already in [**10**] and more explicitly in [**2**, Theorem 3.6 and **4**, Theorem 3.4], it was shown that the embedding $D \subset B(H)$ has the extension property , i.e., every pure state on D has a unique extension to a (necessarily pure) state on B(H) , if and only if the embedding has the relative Dixmier property (i.e. every $x \in B(H)$ has the RDP).

It is thus natural to consider the set N of the elements of B(H) that have the RDP. We know that N is a selfadjoint uniformly closed subspace of B(H) and a two sided D - module [**7**, Proposition 2.4 and Corollary 2.5]. Moreover

(2) $$N = D \oplus \overline{[D, B(H)]} = D \oplus \overline{[D^+, B(H)]},$$

where $\oplus$ denotes the algebraic direct sum, D^+ is the positive part of D , [,] denotes the linear span of the set of commutators and the bar denotes the closure in the uniform topology [**4**, Theorem 2.4, **2**, Corollary 3.6].

Clearly, N contains the ideal of compact operators. By [**7**, Theorems 3.5 and 4.1], N contains the C^*-algebra generated by the normalizer group of D :

(3) $$U(B(H), D) = \{ u \in U(B(H)) \mid uDu^* = D \}$$
$$= \{ u \in U(B(H)) \mid E \circ \mathrm{ad}\, u = \mathrm{ad}\, u \circ E \}.$$

In particular, N contains all operators with matrix representation (in the given basis $\{\eta_j\}$) with at most one 1 in each row and column and zeros elsewhere [cfr. **10**, Theorem 3]. In [**8**, Corollary 4.2] we have seen that N also contains all Laurent (or, equivalently, all Toeplitz) operators with Riemann integrable symbol. In the next section we shall prove that N contains all bounded matrices with non-negative entries.

It is well known (e.g. see [**4**, Theorem 2.4]) and easy to verify directly that if $K(x) \cap D \neq \emptyset$, then $E(x) \in K(x)$, and actually $K(x) \cap D = \{E(x)\}$. Therefore, in order to measure ' how far ' x is from having the RDP we consider the distance $\alpha(x)$ from $E(x)$ to $K(x)$ (or equivalently, the distance from 0 to $K(x - E(x))$). By using the parameter α, we rewrite:

(4) $$N = \{ x \in B(H) \mid \alpha(x) = 0 \}.$$

Due to the commutativity of D, we can compute $\alpha(x)$ by using diagonal projections instead of diagonal unitaries. More precisely, let a diagonal decomposition (or d.d. for short) be a partition of the identity into a sum of finitely many mutually orthogonal diagonal projections. Then by [**3**, Theorem 3.6 and **7**, Lemma 2.2 and proof of Proposition 2.3] we have:

(5) $$\begin{aligned} \alpha(x) &= \inf \{ \|\textstyle\sum p_m(x - E(x))p_m\| \mid \{p_m\} \text{ is a d.d. } \} \\ &= \inf \{ \max_m \|p_m(x - E(x))p_m\| \mid \{p_m\} \text{ is a d.d. } \}. \end{aligned}$$

For completeness' sake we also recall from [**4**, proof of Theorem 2.4 and **8**, Proposition 2.3] that

(6) $$\alpha(x) = \sup\{ | f(x - E(x)) | \mid f \in (B(H)^*)_1 ,\ f \text{ is D-invariant } \}.$$

Let us define

(7) $$\beta = \sup \{ \alpha(x)/\|x - E(x)\| \mid x \in B(H)\backslash D \}.$$

We start with some simple properties of the parameters $\alpha(\cdot)$ and the constant β that we shall use throughout this paper. It is obvious that $\alpha(\cdot)$ and hence β are invariant under embedding-preserving isomorphisms. Explicitly, let $D_1 \subset B(H_1)$ be an embedding of an atomic masa into $B(H_1)$, let $E_1 : B(H_1) \to D_1$ be the (unique) conditional expectation onto D_1 and let $\Phi : B(H) \to B(H_1)$ be an isomorphism such that $\Phi(D) = D_1$. Then $\Phi \circ E = E_1 \circ \Phi$ and hence we have for all $x \in B(H)$ that

$$\begin{aligned} \alpha(\Phi(x)) &= \inf\{ \|\textstyle\sum p_m(\Phi(x) - E_1(\Phi(x))p_m\| \mid \{p_m\} \text{ is a d.d. in } B(H_1) \} \\ &= \inf\{ \|\textstyle\sum q_m(x - E(x))q_m\| \mid \{q_m\} \text{ is a d.d. in } B(H) \} \\ &= \alpha(x). \end{aligned}$$

Consider now the case of a direct sum of Hilbert spaces $H = \sum \oplus H_n$ and let $D = \sum \oplus D_n$ be the direct sum of atomic masas in $B(H_n)$ (this is equivalent to choosing a basis in H by 'combining' the bases of H_n). Identify the identity $I_n \in B(H_n)$ with the projection onto $H_n \subset H$, then $I_n \in D$ and $E = \sum \oplus E_n(I_n \cdot I_n)$ where E_n is the canonical conditional expectation from $B(H_n)$ onto D_n. Let $x = \sum \oplus x_n \in B(H)$, then we have:

LEMMA 2.1. $\alpha(\sum \oplus x_n) \geq \sup_n \alpha(x_n)$.

PROOF: Let $\varepsilon > 0$ and let $\{p_m\}$ be a d.d. in $B(H)$ such that $\|\sum_m p_m(x - E(x))p_m\| \leq \alpha(x) + \varepsilon$. Then for every n we have

$$\begin{aligned} \|\sum_m (p_m I_n)(x_n - E_n(x_n))(p_m I_n)\| &= \|I_n(\sum_m p_m(x - E(x))p_m)I_n\| \\ &\leq \|\sum_m p_m(x - E(x))p_m\| \\ &\leq \alpha(x) + \varepsilon . \end{aligned}$$

Since $\{p_m I_n\}$ is a d.d. in $B(H_n)$, we have by definition that $\alpha(x_n) \leq \alpha(x) + \varepsilon$, whence the conclusion follows.

Q.E.D.

Clearly, if the RDP holds for $D \subset B(H)$, i.e., $\alpha(x) = 0$ for all $x \in B(H)$, then $\alpha(x) = \sup_n \alpha(x_n)$ holds trivially for all direct sums. The converse is also true (see Remark 2.7 iii)).

PROPOSITION 2.2. *There is an* $x \in B(H)$ *with* $\|x\| = 1$ *such that* $E(x) = 0$ *and* $\alpha(x) = \beta$.

PROOF : It is clear from (7) and (5) that

(8) $$\beta = \sup \{ \alpha(x) \mid x \in B(H)_1 ,\ E(x) = 0 \}.$$

Take a sequence $x_n \in H$ with $\|x_n\| = 1$ and $E(x_n) = 0$ such that $\alpha(x_n) > \beta - 1/n$. Let $y = \sum_{n=1}^{\infty} \oplus x_n \in \sum_{n=1}^{\infty} \oplus H$ and let $x \in B(H)$ be the image of y under the isomorphism of $B(\sum_{n=1}^{\infty} \oplus H)$ onto $B(H)$; then $\|x\| = 1$, $E(x) = 0$, and from Lemma 2.1 and preceding remarks, we have $\alpha(x) = \alpha(y) \geq \sup_n \alpha(x_n) \geq \beta$. But $\alpha(x) \leq \beta$ by (8), hence $\alpha(x) = \beta$.

Q.E.D.

REMARK 2.3. The same proof actually shows that the supremum of $\alpha(x)$ over any norm bounded class which is closed under direct summation (e.g. $(B(H)_{sa})_1$ or $(B(H)^+)_1$) is always attained.

We can now reformulate the relative Dixmier property for the embedding $D \subset B(H)$ in terms of the constant β :

PROPOSITION 2.4. *The following conditions are equivalent:*

i) *The embedding* $D \subset B(H)$ *has the RDP, i.e.,* $\alpha(x) = 0$ *for every* $x \in B(H)$,

ii) $\beta = 0$,

iii) $\beta < 1$,

iv) $\alpha(x) < \|x - E(x)\|$ *for every* $x \in B(H)\setminus D$.

PROOF : The implications i) $\Leftrightarrow$ ii) $\Rightarrow$ iii) $\Rightarrow$ iv) are obvious and iv) $\Rightarrow$ iii) is an immediate consequence of Proposition 2.2. Assume now that $\beta < 1$. If $\beta > 0$, we can find a γ such that $0 < \gamma^2 < \beta < \gamma$. By Proposition 2.2 there is an operator x with $\|x\| = 1$ and $E(x) = 0$ such that $\alpha(x) = \beta$ and hence there is a d.d. $\{p_m\}$ such that $\|\sum p_m x p_m\| < \gamma$. Let $y = \gamma^{-1}\sum p_m x p_m$, then $\|y\| < 1$ and $E(y) = 0$. Hence by (8), $\alpha(y) \leq \beta$ so that there is a second d.d. $\{q_i\}$ such that $\|\sum q_i y q_i\| < \gamma$. Hence if $\{r_j\}$ is the d.d. refinement of the two, i.e., for every j there is an i and an m such that $r_j = q_i p_m$, then $\|\sum r_j x r_j\| < \gamma^2 < \beta$, against the assumption that $\alpha(x) = \beta$. Therefore $\beta = 0$ and hence iii) implies ii).

Q.E.D.

For a finer analysis of $\alpha(x)$, and in order to be able to reduce the discussion to finite matrices, we consider the set of diagonal decompositions of the identity into k projections (k-d.d. for short) and define:

(9) $$\alpha_k(x) = \inf\{ \|\textstyle\sum p_m(x - E(x))p_m\| \mid \{p_m\} \text{ is a } k\text{-d.d.} \}.$$

Clearly, for every $x \in B(H)$ the sequence $\alpha_k(x)$ is monotone non-increasing and

(10) $$\alpha(x) = \lim_k \alpha_k(x).$$

Let us collect some properties of the parameters $\alpha(x)$ and $\alpha_k(x)$. In order to simplify the formulations, we set $\alpha_o = \alpha$.

LEMMA 2.5. *For all* $x, y \in B(H)$, $\gamma \in \mathbb{C}$ and $k, h = 0,1,2, \ldots$ *we have* :

i) $\alpha_k(x) \leq \|x - E(x)\|$,

ii) $\alpha_k(\gamma x) = |\gamma|\, \alpha_k(x)$,

iii) $\alpha_k(dxd') \leq \alpha_k(x)$ *for all* $d, d' \in (D)_1$, *with equality holding when* d, d' *are in* $U(D)$,

iv) $\alpha_k(uxu^*) = \alpha_k(x)$ *for all* $u \in U(B(H), D)$ (*see* (3)),

v) $\alpha_k(\sum \oplus x_n) = \sup_n \alpha_k(x_n)$ *for all* $k \geq 1$,

vi) $\alpha_{kh}(x + y) \leq \alpha_k(x) + \alpha_h(y)$,

vii) $\alpha(x + z) = \alpha(x)$ *for all* $z \in N$,

viii) $\alpha_{kh}(y) \leq \max\{ \alpha_k(x_1) + \|E(x_1 - y)\|, \alpha_h(x_2) + \|E(x_2 - y)\| \}$, *for all selfadjoint operators* x_1, x_2 and y such that $x_1 \leq y \leq x_2$,

ix) $|\alpha_k(x) - \alpha_k(y)| \leq \|x - y\| + \|E(x - y)\|$.

PROOF : i)-iii) are obvious and iv) follows easily from the definitions (9) and (3).

v) Let $x = \sum \oplus x_n$ and assume without loss of generality that $E_n(x_n) = 0$ for all n, and hence that $E(x) = 0$. Reasoning as in the proof of Lemma 2.1 we obtain

$$\alpha_k(x) \geq \sup_n \alpha_k(x_n).$$

Now let $\varepsilon > 0$ and choose for each n a k-d.d. $\{p_{n,m}\}$ in $B(H_n)$ such that

$$\|\sum p_{n,m} x_n p_{n,m}\| \leq \alpha_k(x_n) + \varepsilon .$$

Then, if we let $p_m = \sum \oplus p_{n,m}$, we see that $\{p_m\}$ is a k-d.d. in $B(H)$ and

$$\|\sum p_m x p_m\| = \sup_n \|\sum p_{n,m} x_n p_{n,m}\| \leq \sup_n \alpha_k(x_n) + \varepsilon .$$

Therefore $\alpha_k(x) \leq \sup_n \alpha_k(x_n) + \varepsilon$, and since ε is arbitrary, we obtain the equality.

vi) is obtained by taking refinements of partitions as in the proof of Proposition 2.4 ; vii) is an immediate consequence of (10), vi) and (4); viii) is proven as in [**8**, Lemma 2.8].

ix). Let $\varepsilon > 0$ and let $\{p_m\}$ be a k-d.d. such that $\|\sum p_m(x - E(x))p_m\| < \alpha_k(x) + \varepsilon$. Then

$$\sum p_m(y - E(y))p_m = \sum p_m(x - E(x))p_m + \sum p_m(y - x)p_m + E(x) - E(y),$$

hence

$$\begin{aligned} \alpha_k(y) &\leq \|\sum p_m(y - E(y))p_m\| \\ &< \alpha_k(x) + \varepsilon + \|\sum p_m(y - x)p_m\| + \|E(x) - E(y)\| \\ &< \alpha_k(x) + \|y - x\| + \|E(y - x)\| + \varepsilon . \end{aligned}$$

Since ε is arbitrary, and by switching x with y, we obtain the required inequality.

Q.E.D.

Notice that ix) implies the norm continuity of α_k for all $k \geq 0$. The situation is different for the strong operator topology. The lower semicontinuity of α in the strong operator topology is trivially equivalent to the RDP (since the compact operators are in N) and hence is equivalent to continuity. The continuity in the strong operator topology of α_k for $k \geq 1$ is false (e.g. use v) to construct a counterexample), while the lower semicontinuity of α_k in the S.O.T. follows from Proposition 2.6 below and from its continuity on finite dimensional subspaces.

Let us henceforth embed into $B(H)$ the algebra M_n of $n \times n$ complex matrices by identifying it with $B(q_n H)$. When there is no risk of confusion, we shall not distinguish between $I \in M_n$, $I \in B(q_n H)$ and $I \in B(H)$ and thus we shall talk of k-d.d. without specifying if taken in M_n, in $B(q_n H)$ or in $B(H)$.

To simplify notation, let us also identify diagonal projections with subsets of the integers: henceforth say for a projection $p \in D$ and an integer j, that $j \in p$ when $\eta_j \in pH$.

Thus each d.d. corresponds to a coloring of the integers, and its restriction to a d.d. of $B(q_nH)$ corresponds to the induced coloring of $\{1, \dots, n\}$. As in similar problems in combinatorics, we can apply the Rado compactness (or selection) principle :

PROPOSITION 2.6. $\alpha_k(x) = \lim_n \alpha_k(q_nxq_n)$ *for every* $x \in B(H)$ *and* $k = 1,2,\dots$

PROOF : Fix $x \in B(H)$ and $k \geq 1$; by Lemma 2.5 iii) , $\alpha_k(q_nxq_n)$ increases monotonically with n and is bounded by $\alpha_k(x)$. Let $\gamma = \lim_n \alpha_k(q_nxq_n)$, then $\gamma \leq \alpha_k(x)$. Let Φ be the class of all k-d.d. and let $\Phi_n \subset \Phi$ be the class of the k-d.d. $\{p_m\}$ such that

$$\|\sum p_mq_n(x - E(x))q_np_m\| \leq \gamma . \tag{11}$$

Since the number of k-d.d. in $B(q_nH)$ is finite, the infimum in the definition of $\alpha_k(q_nxq_n)$ is attained and hence there is a k-d.d. $\{p_m\}$ such that

$$\|\sum p_mq_n(x - E(x))q_np_m\| = \alpha_k(q_nxq_n) \leq \gamma .$$

Thus Φ_n is never empty. Moreover, as the left hand side in (11) is monotone non-decreasing in n , we see that $\Phi_h \subset \Phi_n$ for all $h \geq n$. Thus $\{\Phi_n\}$ has the finite intersection property, and since Φ is compact in its Tychonoff topology, by reasoning as in [**5**, Proof 2 of Theorem 1.4 (Compactness Principle) or **3**, proof of Theorem 2], we see that $\{\Phi_n\}$ has the infinite intersection property, so there exists a k-d.d. $\{p_m\} \in \bigcap_1^\infty \Phi_n$, i.e., $\{p_m\}$ satisfies (11) for all n. Thus, by passing to the limit as $n \to \infty$, and by the lower semicontinuity of the norm in the S.O.T. we obtain that $\|\sum p_m(x - E(x))p_m\| \leq \gamma$. Since $\alpha_k(x) \leq \|\sum p_m(x - E(x))p_m\|$, and since $\gamma \leq \alpha_k(x)$, we conclude that $\|\sum p_m(x - E(x))p_m\| = \gamma = \alpha_k(x)$.

Q.E.D.

REMARK 2.7 i) We have actually proven more, namely that the infimum in the definition of $\alpha_k(x)$ is always attained, even in the infinite dimensional case. This last fact is trivially false for α (take any compact operator with strictly non-zero off diagonal entries). The statement in Proposition 2.6 holds true for α if and only if $D \subset B(H)$ has the RDP (since $\alpha(q_nxq_n) = 0$ for all $x \in B(H)$ and all positive integers n).

ii) Notice that the only properties of the operator norm $\| \ \|$ that we have used here are that $\|pxp\| \leq \|x\|$ for every $x \in B(H)$ and every projection p , and that the norm is lower semicontinuous in the strong operator topology. Thus Proposition 2.6 holds also for the parameters α_k when defined using any other norm with the above properties. We shall use this fact for the Schatten p-norms in §4 .

iii) If there is an operator x such that $\alpha(x) > 0$ then we can find a direct sum of finite rank operators y such that $\alpha(y) \geq \alpha(x) > 0$ (and hence for y the inequality in Lemma 2.1 is strict). Indeed, $\alpha_k(x) \geq \alpha(x)$ for all k , and hence, by Proposition 2.6, we can find an index n(k) such that $\alpha_k(q_{n(k)}xq_{n(k)}) > \alpha(x) - 1/k$. Thus $y = \sum_{k=1}^{\infty} \oplus q_{n(k)}xq_{n(k)}$ satisfies the condition $\alpha(y) \geq \alpha(x)$ because, by Lemma 2.5 v), we have

$$\alpha_k(y) \geq \alpha_k(q_{n(k)}xq_{n(k)}) > \alpha(x) - 1/k$$

and hence, by (10), $\alpha(y) \geq \alpha(x)$.

Proposition 2.6 thus reduces the problem of computing (or estimating) the parameters α_k (for $k \neq 0$) to the finite dimensional case. More precisely, define

(12) $$\beta_k = \sup \{ \alpha_k(x) \mid x \in (\bigcup_1^{\infty} M_n)_1 ,\ E(x) = 0 \} ,$$

then we have:

PROPOSITION 2.8. i) $\beta_k = \max\{ \alpha_k(x) \mid x \in B(H)_1 ,\ E(x) = 0 \}$ *for* $k \geq 1$,

i) β_k *decreases monotonically to* β ,

iii) $D \subset B(H)$ *has the RDP iff there is a* k *such that* $\beta_k < 1$

PROOF : i) From Proposition 2.6 we easily obtain that

$$\beta_k = \sup\{ \alpha_k(x) \mid x \in B(H)_1 ,\ E(x) = 0 \}.$$

By Lemma 2.5 v), and reasoning as in Proposition 2.2 (cfr. Remark 2.3), we see that the supremum in this formula is attained, and so we obtain i).

ii) Clearly, β_k decreases monotonically, and as $\alpha(x) \leq \alpha_k(x)$ for all x, we see that $\beta \leq \lim_k \beta_k$. From Proposition 2.2 we know that there is an $x \in B(H)_1$, with $E(x) = 0$ such that $\beta = \alpha(x) = \lim_k \alpha_k(x)$. Thus if we had $\beta < \lim_k \beta_k$, there would be a j with $\alpha_j(x) < \lim_k \beta_k \leq \beta_j$, against i).

iii) This follows immediately from ii) and Proposition 2.4.

Q.E.D.

Thus the extension problem is reduced to the study of the β_k, (i.e. to the problem of finding estimates for the parameters $\alpha_k(x)$ that are independent of the size of the matrix x), which seems to be an interesting area of study in its own right.

The only cases for which the values of β_k are known to the authors (and equal to 1), are the cases $k = 1$, where $\alpha_1(x) = \|x - E(x)\|$ for all x, and $k = 2$, where the Example 2.9 i) below provides an operator for which $\alpha_2(x) = \|x - E(x)\|$. It is therefore important to find upper and lower bounds for the ratio $\alpha_k(x)/\|x - E(x)\|$, at least for some particular classes of matrices. In the case of matrices with non-negative entries, we are able to show in the next section that $2/k$ is an upper bound for this ratio (Theorem 3.4).

Here we shall consider matrices that are either ' very sparse ' or ' completely diffuse '. If x is a matrix with only one non-zero entry, then obviously $\alpha_2(x) = 0$. If x contains only a ' small ' number of non-zero entries which are ' sparse ' (e.g. at most one non-zero entry per row and column, as in [**10**, Theorem 3]), then $\alpha_3(x) = 0$ (Example 2.9 ii)).

Thus it is natural to consider, at the other extreme, matrices with entries all of the same magnitude, e.g. $n \times n$ matrices x with $|x_{ij}| = n^{-1/2}$. Then, while the Hilbert-Schmidt norm $\|x\|_2$ of x is $n^{1/2}$, the operator norm $\|x\|$ can vary between $n^{1/2}$ and 1.

In the case that $\|x\| = n^{1/2}$, it is elementary to show that x is rank one, so there exist two diagonal unitary matrices u and v such that $x = n^{1/2} u f_n v$, where f_n is the rank one projection defined below in (13) (whence $\alpha_k(x) = n^{1/2}\alpha_k(f_n)$ by Lemma 2.5 iii)). We shall analyze f_n in Example 2.9 iii) and repeatedly use it in Proposition 3.1, Corollary 3.3, Theorem 3.4 and Theorem 4.4 for proving sharpness results.

In the case that $\|x\| = 1$, it follows that x is unitary and thus, if it has only real entries (as it is natural to consider here), x is a Hadamard matrix (a unitary matrix with entries $\pm n^{-1/2}$). Lemma 2.10 and Proposition 2.11 study the ratio $\alpha_k(x)/\|x - E(x)\|$ for Hadamard matrices and yield $k^{-1/2}$ as the largest lower bound that we know for β_k (for $k > 3$). We shall then return to investigate both f_n and Hadamard matrices in the context of the Schatten p-norms in §4.

Let us define the following two $n \times n$ matrices:

(13) $$u_n = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ . & . & \dots & . & . \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix} \qquad f_n = (1/n)\begin{bmatrix} 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & \dots & 1 & 1 \\ . & . & \dots & . & . \\ 1 & 1 & \dots & 1 & 1 \end{bmatrix}$$

EXAMPLE 2.9. i) $\alpha_2(u_n) = 1$ if n is odd, and $\alpha_2(u_n) = 0$ if n is even. If u is the infinite unilateral or bilateral shift, then $\alpha_2(u) = 0$. This is easy to verify directly (see also [**9**, Proposition 3.1]).

ii) $\alpha_3(u_n) = 0$ and $\alpha_3(u) = 0$. Moreover, the same result holds for all permutation matrices and more generally, for all bounded matrices with at most one non-zero entry per row and column [**9**, Proposition 3.4]. Notice that this is essentially the content of [**10**, Theorem 3].

iii) $\alpha_k(f_n)/\|f_n - E(f_n)\| = 1/k - \varepsilon(n,k)$ where

$$\varepsilon(n,k) = \begin{cases} (1-1/k)/(n-1) & \text{if } k \text{ divides } n, \\ (r/k - 1/k)/(n-1) & \text{if } n = sk + r,\ 1 \le r < k. \end{cases}$$

This can easily be seen by noticing that $\|f_n - E(f_n)\| = 1 - 1/n$ and that for every diagonal projection p, pf_np is unitarily equivalent to $n^{-1} \dim p\, f_{\dim p}$ and hence

$$\|p(f_n - E(f_n))p\| = n^{-1}(\dim p - 1).$$

Now every k-d.d. contains at least one projection with $\dim p \ge n/k$, whence in either case, we obtain that

$$\alpha_k(f_n)/\|f_n - E(f_n)\| \geq 1/k - \varepsilon(n,k) .$$

Conversely, if we take a k-d.d. with projections having dimensions either $[n/k]$ or $[n/k] + 1$, we achieve the equality (here $[\gamma]$ denotes the integer part of γ).

iv) Let $x = 2f_n - u_n$, then x is a unitary operator, $\alpha_k(x) \geq 2/k - 2/n$ and

$$\| x - E(x) \| = \sqrt{1 + 4/n^2 + (4/n)\cos 2\pi/n}$$

Moreover, $\alpha_k(x) = 2/k - 2/n$ if k divides n ([9, Theorem 3.5]).

We now begin our examination of Hadamard matrices with the following slightly more general lemma.

LEMMA 2.10. *Let* x *be an* $n \times n$ *matrix with* $|x_{ij}| \geq \delta n^{-1/2}$ *for all* i *and* j; *then* $\alpha_k(x) \geq \delta k^{-1/2} - \|E(x)\|$.

PROOF : By Hölder's inequality, $\|x\| \geq n^{-1/2}\|x\|_2$, implying $\|x\| \geq \delta$. Thus if p is a diagonal projection of dimension h, pxp is an $h \times h$ matrix with

$$|(pxp)_{ij}| \geq \delta(h/n)^{1/2}h^{-1/2} \quad \text{for all } i,j \in p ,$$

which implies $\| pxp \| \geq \delta(h/n)^{1/2}$. As every k-d.d. $\{p_m\}$ contains at least one projection p with $\dim p \geq n/k$, we have that $\|\sum p_m x p_m\| \geq \delta k^{-1/2}$, and hence

$$\|\sum p_m(x-E(x))p_m\| \geq \delta k^{-1/2} - \| E(x) \| .$$

Q.E.D.

PROPOSITION 2.11 i) $\alpha_k(x)/\|x - E(x)\| \geq (k^{-1/2} - n^{-1/2})/(1 + n^{-1/2})$ *for every* $n \times n$ *Hadamard matrix* x. ii) $\sup\{ \alpha_k(x)/\|x - E(x)\| \mid x = x^* \in B(H)\setminus D \} \geq k^{-1/2}$. *In particular*, $\beta_k \geq k^{-1/2}$.

PROOF : i) Lemma 2.10 applied to an $n \times n$ Hadamard matrix x yields: $\alpha_k(x) \geq k^{-1/2} - \| E(x) \| = k^{-1/2} - n^{-1/2}$. Since $\| x - E(x) \| \leq \| x \| + \| E(x) \| = 1 + n^{-1/2}$, we obtain the inequality in i).

ii) Since selfadjoint Hadamard matrices of arbitrarily large size exist , e.g. tensor products of the selfadjoint Hadamard matrix $2^{-1/2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, we obtain the inequality in ii).

Q.E.D.

By Remark 2.3 we see that the supremum in ii) is attained, for instance by direct sums of Hadamard matrices of increasing size. Notice also that, in contrast to the results in § 3, which yield different upper bounds for $\alpha_k(x)/\|x - E(x)\|$ for the selfadjoint and non-selfadjoint cases, we obtain the same lower bound for this ratio for both selfadjoint and non-selfadjoint Hadamard matrices.

For $k > 3$, $k^{-1/2}$ is the largest lower bound for β_k that is known to the authors. The value $k^{-1/2}$ is of particular interest due to the fact that it is also an upper bound for the analogous ratio relative to the C_2-norm for all operators (see Corollary 4.5), and $2^{1/4}k^{-1/2}$ is (asymptotically) an upper bound for the ratio relative to the C_4-norm for all Hadamard matrices (see Proposition 4.6).

The case $k = 3$ is special, since $3^{-1/2} < 2/3$, and thus a larger lower bound can be obtained using a different, albeit non-selfadjoint, unitary (see Example 2.9 iv)). The constant $2/3$ has appeared in the literature for the case of matrices with non-negative entries [**3**, Theorem 2 and following remarks, cfr.**1**].

§ 3. MATRICES WITH NON-NEGATIVE ENTRIES.

For every operator $x \in B(H)$ and every integer k we know that $\alpha_k(x)/\|x-E(x)\| \leq 1$; in this section we shall show that if x, identified with the matrix (x_{ij}) in the basis $\{\eta_j\}$, has non-negative entries, then the above ratio is bounded by $2/k$ (or by $1/k$ if $x = x^*$).

Our tools consist of the reduction to finite matrices obtained in Proposition 2.6, the Perron-Frobenius-Schur theory for finite matrices with non-negative entries [**11**], and techniques motivated by graph theory. We start with a property on colorings of graphs, which for the reader's convenience, we shall translate into the language of matrices.

Recall that under the identification of diagonal projections with subsets of the integers, we say that $j \in p$ when $\eta_j \in pH$.

PROPOSITION 3.1. *Let* $x = x^*$ *be an* $n \times n$ *matrix with non-negative entries and with zero diagonal. Then for every integer* k *there is a* k-d.d. $\{p_m\}$ *such that*

$$\sum_{j=1}^{n}\left(\sum p_m x p_m\right)_{ij} \leq (1/k)\sum_{j=1}^{n} x_{ij} \quad \textit{for every } i.$$

Furthermore, the result is sharp in the sense that the constant $1/k$ *is the best possible.*

PROOF : Let Q be the function defined on the collection of k-d.d. by :

$$Q(\{p_m\}) = \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\sum p_m x p_m\right)_{ij} = \sum\sum\{\, x_{ij} \mid i,j \in p_m \,\}. \tag{14}$$

Since there are only a finite number of k-d.d. in M_n (the algebra of all $n \times n$ complex matrices), we can find a k-d.d. $\{p_m\}$ which minimizes the function Q. We claim that this k-d.d. satisfies the condition of the proposition. Indeed, otherwise there would be an $1 \leq h \leq k$ and an $i \in p_h$ such that

$$\sum_{j=1}^{n}\left(\sum p_m x p_m\right)_{ij} = \sum\{\, x_{ij} \mid j \in p_h \,\} > (1/k)\sum_{j=1}^{n} x_{ij}.$$

Since $\sum_{j=1}^{n} x_{ij} = \sum\sum\{\, x_{ij} \mid j \in p_m \,\}$, by the averaging principle there has to be at least one t (necessarily $t \neq h$) such that

(15) $$\sum\{ x_{ij} \mid j \in p_t \} \leq (1/k)\sum_{j=1}^{n} x_{ij} < \sum\{ x_{ij} \mid j \in p_h \}.$$

Change the k-d.d. $\{p_m\}$ into the k-d.d. $\{\tilde{p}_m\}$ by moving i out of p_h and into p_t. A straightforward computation shows that

$$\begin{aligned} Q(\{\tilde{p}_m\}) - Q(\{p_m\}) &= \sum\{ x_{ij} \mid j \in p_t \} - \sum\{ x_{ij} \mid j \in p_h \} \\ &\quad + \sum\{ x_{ji} \mid j \in p_t \} - \sum\{ x_{ji} \mid j \in p_h \} \\ &= 2(\sum\{ x_{ij} \mid j \in p_t \} - \sum\{ x_{ij} \mid j \in p_h \} \\ &< 0 . \end{aligned}$$

Here we used (15) and the fact that x is selfadjoint. But this contradicts the minimality of $Q(\{p_m\})$ and thus proves our claim.

In order to see that the result is sharp, it is enough to consider $x = f_n - E(f_n)$.

Q.E.D.

REMARK 3.2 i) A similar result, but with a bound of 2/3, has been proven for $k = 3$ without the condition of selfadjointness in [**3**, Theorem 2]. It remains an open question whether the constant 2/3 is sharp in the non-selfadjoint case.

ii) Proposition 3.1 can easily be extended to selfadjoint operators using Rado's selection principle (cfr. Proposition 2.6).

iii) By summing the inequality in Proposition 3.1 over all i, we see that the minimal value of the function Q satisfies:

(16) $$Q(\{p_m\}) \leq (1/k)\sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij}.$$

The same inequality is obtained in the next section (Proof of Theorem 4.4) by using different averaging methods. Moreover, there we obtain (16) also for non-selfadjoint operators, and in addition we show that when k divides n, (16) holds true even for the minimum of Q taken over the class of equidimensional k- d.d. (i.e. k-d.d. into projections of equal dimensions).

As an application of Proposition 3.1 and Remark 3.2 ii), we obtain the following inequality for vector norms.

COROLLARY 3.3 *For any selfadjoint operator* x *and every integer* k *there is a* k-d.d.*such that* $\|\sum p_m(x - E(x))p_m\eta_i\| \leq k^{-1/2}\|x\eta_i\|$ *for all* i. *The constant* $k^{-1/2}$ *is sharp.*

PROOF : Without loss of generality we can assume that x has zero diagonal. Consider the selfadjoint operator y with entries $y_{ij} = |x_{ji}|^2$; then by Proposition 3.1 along with Remark 3.2.ii), we can find a k-d.d. $\{p_m\}$ such that

$$\sum_{j=1}^{n}(\sum p_m y p_m)_{ij} \leq (1/k)\sum_{j=1}^{n} y_{ij} \quad \text{for every } i .$$

But then

$$\begin{aligned}\|\textstyle\sum p_m x p_m \eta_i\|^2 &= \sum_{j=1}^{n} |(\textstyle\sum p_m x p_m)_{ji}|^2 \\ &= \sum_{j=1}^{n} (\textstyle\sum p_m y p_m)_{ij} \\ &\le (1/k)\sum_{j=1}^{n} y_{ij} \\ &= (1/k)\sum_{j=1}^{n} |x_{ji}|^2 \\ &= (1/k)\|x\eta_i\|^2 .\end{aligned}$$

As usual, the sharpness is obtained by considering $x = f_n - E(f_n)$.

Q.E.D.

Notice that by the Perron-Frobenius theorem [**11**, Theorem 2.1], if x and y are matrices with non-negative entries, then $\|x\| \le \|x + y\|$. In particular, $\|x - E(x)\| \le \|x\|$. We can now prove the main theorem of this section.

THEOREM 3.4. *Let* x *be an* $n \times n$ *matrix with non-negative entries and let* k *be any positive integer. Then* $\alpha_k(x) \le (2/k)\|x\|$. *If* $x = x^*$, *then* $\alpha_k(x) \le (1/k)\|x\|$. *The constant* $1/k$ *is sharp.*

PROOF : By the remark preceding this theorem, we can assume without loss of generality that $E(x) = 0$. Clearly, we can also assume that x has no zero direct summands (relative to the given basis $\{\eta_j\}$).

Consider first the case of $x = x^*$. By the Perron-Frobenius theorem [**11**,Theorem 2.1], x has an eigenvector ζ corresponding to the eigenvalue $\|x\|$, with entries $\zeta_i \ge 0$. Let $pH = \text{span}\{\eta_i \mid \zeta_i > 0\}$; then from $\sum_{j=1}^{n} x_{ij}\zeta_j = \|x\|\zeta_i$ and the fact that $x_{ij}\zeta_j \ge 0$, we see that $x_{ij} = 0$ for all $j \in p$, $i \notin p$, i.e. $p^\perp x p = 0$, and hence p reduces x. Thus pxp has the eigenvector $p\zeta$ with $(p\zeta)_i > 0$ for all $i \in p$. Moreover, $p^\perp x p^\perp$ is again a matrix with zero diagonal and non-negative entries, and since we are assuming that x has no zero direct summand, we have that $0 < \|p^\perp x p^\perp\| \le \|x\|$. Thus, by iterating, we can decompose x into a direct sum (relative to the basis $\{\eta_j\}$) $x = \Sigma \oplus x(h)$ of matrices $x(h)$ that have some eigenvector $\xi(h)$ with strictly positive entries (in the h-th block). Let $\xi = \Sigma \oplus \xi(h)$; then

(17) $\qquad (x\xi)_i \le \|x\|\,\xi_i \quad$ and $\quad \xi_i > 0 \quad$ for all $\quad 1 \le i \le n$.

Define the matrix $y = (\xi_i \xi_j x_{ij})$, then y satisfies the conditions of Proposition 3.1 and thus there is a k-d.d. $\{p_m\}$, such that for every i we have $\sum_{j=1}^{n} (\sum p_m y p_m)_{ij} \le (1/k)\sum_{j=1}^{n} y_{ij}$. Then, for every $1 \le s \le k$ and every $i \in p_s$, we have

$$\begin{aligned}\xi_i (p_s x p_s \xi)_i &= \xi_i \sum_{j=1}^{n} (p_s x p_s)_{ij}\, \xi_j \\ &= \sum_{j=1}^{n} (p_s y p_s)_{ij}\end{aligned}$$

$$= \sum_{j=1}^{n} (\sum p_m y p_m)_{ij}$$
$$\leq (1/k) \sum_{j=1}^{n} y_{ij}$$
$$= (1/k)\, \xi_i \sum_{j=1}^{n} x_{ij}\, \xi_j$$
$$\leq (1/k) \|x\|\, \xi_i^{\,2}.$$

This inequality also holds for all $i \notin p_s$, since in this case the left-hand side is 0. Dividing by ξ_i, which by (17) is non-zero, we obtain for all s that $(p_s x p_s \xi)_i \leq (1/k)\|x\|\, \xi_i$ for all i. Therefore by the Perron-Schur test [**11**, Theorem 2.2 or **6**, Problem 45] we have that $\|p_s x p_s\| \leq (1/k)\|x\|$ for all $1 \leq s \leq k$, and hence

(18) $$\|\sum p_m x p_m\| \leq (1/k)\|x\|.$$

Thus we conclude that $\alpha_k(x) \leq (1/k)\|x\|$. Example 2.9 iii) shows that for the selfadjoint case the constant $1/k$ is sharp.

In the non-selfadjoint case take a k-d.d. $\{p_m\}$ that satisfies (18) for the operator $x + x^*$. Then by the remarks preceding this theorem, we have that

(19) $$\begin{aligned} \|\sum p_m x p_m\| &\leq \|\sum p_m (x + x^*) p_m\| \\ &\leq (1/k)\|x + x^*\| \\ &\leq (2/k)\|x\|, \end{aligned}$$

so $\alpha_k(x) \leq (2/k)\|x\|$.

Q.E.D.

Clearly, the obstruction to extending this result to general matrices with real entries, by splitting them into a difference of two matrices with non-negative entries, is that their norms may be much larger than the norm of their difference.

By using Proposition 2.6 and (10) we easily pass to the infinite case:

COROLLARY 3.5 i) *If* $x \in B(H)$ *and* x *has non-negative entries, then* $\alpha_k(x) \leq (2/k)\|x\|$ *and if* $x = x^*$, *then* $\alpha_k(x) \leq (1/k)\|x\|$. *The constant* $1/k$ *is sharp.*
ii) *Every bounded matrix with non-negative entries has the RDP.*

Since the set N of elements having the RDP is a uniformly closed subspace of B(H), N contains all bounded matrices with non-negative entries together with the C^*-algebra that they generate (which coincides with the Banach space that they generate).

As an application of this method, consider the following example.

EXAMPLE 3.6. Let $x = T_\varphi$ be a Toeplitz operator with symbol $\varphi \in L^\infty(0,2\pi)$, having Fourier coefficients $\hat{\varphi}(n) \geq 0$ for all $n \in \mathbb{Z}$. Then it is well known that $E(x) = \hat{\varphi}(0)I$ and $\|x\| = \|\varphi\|_\infty$. Also, one sees from standard techniques in harmonic analysis that $\{\hat{\varphi}(n)\} \in \ell^1$ and $\|\varphi\|_\infty = \sum_{n=-\infty}^{\infty} \hat{\varphi}(n)$. Thus φ is continuous, hence $x \in N$, i.e., $\alpha(x) = 0$ [8, Corollary 4.2]. But by Corollary 3.5 and Remark 2.7 i) we can say more, namely that for every k, there is a k-d.d. $\{p_m\}$ such that

(20) $$\|\sum p_m(x - E(x))p_m\| \leq (2/k)\|\varphi - \hat{\varphi}(0)\|_\infty .$$

§ 4 SCHATTEN P-NORM INEQUALITIES.

We have seen in the previous sections that the relative Dixmier property (and hence the extension property) can be reduced to the finite dimensional case, and the latter can be analyzed in terms of the parameters α_k. The following generalization in terms of the Schatten p-norms (for $p \geq 1$) is of independent interest and may also further our understanding of the inequalities involving α_k.

Let C_p be the Schatten p-ideal and let $\| \ \|_p$ be the p-norm ($p \geq 1$); then define for all $x \in C_p$

(21) $$\alpha_{k,p}(x) = \inf \{ \|\sum p_m(x - E(x))p_m\|_p \mid \{p_m\} \text{ is a k-d.d. } \}$$

The infimum in (21) is always attained (see Remark 2.7 i) and ii)). By Remark 2.7 ii),

(22) $$\alpha_{k,p}(x) = \lim_n \alpha_{k,p}(q_n x q_n) \quad \text{for all } x \in C_p ,\ k \geq 1 \text{ and } p \geq 1 .$$

Thus it shall again suffice to consider finite matrices.

Notice that if x is a finite matrix, or more generally, if $x \in C_q$ for some q, then $\{\|x\|_p\}_{p \geq q}$ decreases monotonically to $\|x\|$. As a consequence, $\alpha_{k,p}(x)$ decreases monotonically to $\alpha_k(x)$ and thus an upper bound for β_k is given by

(23) $$\beta_k \leq \underline{\lim}_p \sup\{ \alpha_{k,p}(x)/\|x - E(x)\|_p \mid x \in (\bigcup_1^\infty M_n)\setminus D \} .$$

Thus analyzing the upper bounds of the ratio $\alpha_{k,p}(x)/\|x - E(x)\|_p$, for all p (or at least for all p of the form 2^m, $m = 1,2,\ldots$), has a bearing on the relative Dixmier property for $D \subset B(H)$. A first step in this direction is given by Theorem 4.4 (for $p = 2$) and by Proposition 4.6 (for $p = 4$ and the class of Hadamard matrices).

We start our analysis with generalizations of Lemma 2.10 and Proposition 2.11.

LEMMA 4.1. *Let* x *be an* $n \times n$ *matrix with* $|x_{ij}| \geq \delta n^{-1/2}$ *for all* i, j *and let* $p \geq 2$*; then* $\alpha_{k,p}(x) \geq \delta\, n^{1/p} k^{-1/2} - \|E(x)\|_p$.

PROOF : Let $1/p + 1/q = 1/2$. Then from Hölder's inequality we have

$$\|x\|_p \geq n^{-1/q} \|x\|_2 \geq \delta\, n^{1/p}.$$

Likewise, for every diagonal projection r we have

$$|(rxr)_{ij}| \geq \delta\, n^{-1/2} = \delta(n^{-1} \dim r)^{1/2}(\dim r)^{-1/2} \quad \text{for all } i, j \in r ,$$

and hence

$$\|rxr\|_p \geq \delta(n^{-1}\dim r)^{1/2}(\dim r)^{1/p} = \delta\, n^{-1/2}(\dim r)^{1/2+1/p}.$$

Thus if $\{p_m\}$ is a k-d.d. , then

$$\|\sum p_m x p_m\|_p^p = \sum \| p_m x p_m \|_p^p$$
$$\geq \delta^p n^{-p/2} \sum (\dim p_m)^{1+p/2}$$
$$\geq \delta^p n^{-p/2} n^{1+p/2} k^{-p/2},$$

where the last inequality is obtained from an application of Hölder's inequality for $(1+p/2)^{-1} + (1+2/p)^{-1} = 1$ to the sum $\sum \dim p_m = n$. Thus

$$\|\sum p_m x p_m\|_p \geq \delta\, n^{1/p} k^{-1/2}$$

and hence

$$\|\sum p_m(x - E(x))p_m\|_p \geq \delta\, n^{1/p} k^{-1/2} - \|E(x)\|_p .$$

Q.E.D.

PROPOSITION 4.2 i) *If* x *is an* $n \times n$ *Hadamard matrix and* $p \geq 2$, *then* $\alpha_{k,p}(x)/\|x - E(x)\|_p \geq (k^{-1/2} - n^{-1/2})/(1 + n^{-1/2})$.

ii) $\sup\{ \alpha_{k,p}(x)/\|x - E(x)\|_p \mid x = x^* \in C_p\backslash D \} \geq k^{-1/2}$.

PROOF : Let x be an $n \times n$ Hadamard matrix; then $\|x\|_p = n^{1/p}$, $\|E(x)\|_p = n^{1/p - 1/2}$ and by Lemma 4.1 we have

$$\alpha_{k,p}(x)/\|x - E(x)\|_p \geq (k^{-1/2} - n^{-1/2})n^{1/p}/\|x - E(x)\|_p$$
$$\geq (k^{-1/2} - n^{-1/2})/(1 + n^{-1/2}).$$

As we have noticed in the proof of Proposition 2.11, there are selfadjoint Hadamard matrices of arbitrarily large size, hence ii) follows immediately from i).

Q.E.D.

Let us also compute the parameters $\alpha_{k,p}(f_n)$ for the projection f_n, which we have used so often to prove the sharpness of our results. To simplify our task, we can assume without much loss that k divides n.

EXAMPLE 4.3 $\alpha_{k,p}(f_n)/\|f_n - E(f_n)\|_p = k^{-1/q} - \varepsilon(n, k, p)$ where $1/p + 1/q = 1$ (if $p = 1$ we take $1/q = 0$). This relation defines $\varepsilon(n, k, p)$ and we have $\lim_n \varepsilon(n, k, p) = 0$ for all k and p. Actually,

(24) $$\alpha_{k,p}(f_n)/\|f_n - E(f_n)\|_p =$$
$$= k^{-1/q}\{ (1-k/n)((1-k/n)^{p-1} + (k/n)^{p-1}) / (1-1/n)((1-1/n)^{p-1} + (1/n)^{p-1}) \}^{1/p},$$

Indeed, if r is any diagonal projection, we have seen in Example 2.9 iii) that rf_nr is unitarily equivalent to $(1/n \dim r)f_{\dim r}$ and hence

$$r(f_n - E(f_n))r \simeq (1/n)\{ (\dim r - 1)f_{\dim r} - (I_r - f_{\dim r}) \},$$

where I_r is the $r \times r$ identity matrix. Then

$$\|r(f_n - E(f_n))r\|_p = (1/n)\{ (\dim r - 1)^p + \dim r - 1 \}^{1/p} .$$

In particular, taking $r = I_n$, we have that

(25) $$\|f_n - E(f_n)\|_p = \{ (1 - 1/n)((1 - 1/n)^{p-1} + (1/n)^{p-1}) \}^{1/p} .$$

Let $\{r_m\}$ be any k-d.d. ; then

(26) $$\begin{aligned}\|\sum r_m(f_n - E(f_n))r_m\|_p^p &= (1/n^p)\sum \{ (\dim r_m - 1)^p + (\dim r_m - 1) \} \\ &\geq (1/n^p) \{ (n-k)^p k^{1-p} + n - k \} \\ &= k^{-p/q}(1 - k/n)((1 - k/n)^{p-1} + (k/n)^{p-1}) ,\end{aligned}$$

using Hölder's inequality for $1/p + 1/q = 1$ applied to $\sum(\dim r_m - 1) = n - k$. On the other hand, since we assume that k divides n, we can choose, as in Example 2.9 iii), a k-d.d. with projections r_m having all dimension n/k. Substituting $\dim r_m = n/k$ in the first equality in (26), we have

$$\begin{aligned}\|\sum r_m(f_n - E(f_n))r_m\|_p &= (k^{1/p}/n)((n/k - 1)^p + n/k - 1)^{1/p} \\ &= k^{-1/q}\{ (1 - k/n)((1 - k/n)^{p-1} + (k/n)^{p-1}) \}^{1/p} .\end{aligned}$$

Thus combining this with the inequality in (26), we obtain (24) .

Notice that (at least for $n \geq 2k$) we have:

(27) $$\alpha_{k,p}(f_n)/\|f_n - E(f_n)\|_p \leq k^{-1/q} ,$$

which follows from (24) and the fact that the function $(1 - x)^p + (1 - x)x^{p-1}$ is decreasing on $(0, 1/2)$. For $p = 2$, (27) holds for all Hilbert-Schmidt operators. Indeed Corollary 3.3 immediately yields the fact that for every finite selfadjoint matrix x and every k there is a k-d.d. $\{p_m\}$ such that

(28) $$\|\sum p_m(x - E(x))p_m\|_2 \leq k^{-1/2}\|x - E(x)\|_2 .$$

We are going to prove (28) also in the non-selfadjoint case by using a different averaging method.

THEOREM 4.4. i) *Let* x *be an* $n \times n$ *matrix and assume that* k *divides* n. *Then there is a* k-d.d. $\{p_m\}$ *with* $\dim p_m = n/k$ *for all* m, *such that*

$$\|\sum p_m(x - E(x))p_m\|_2 \leq k^{-1/2}\|x - E(x)\|_2 .$$

ii) *For every Hilbert-Schmidt operator* x *we have* $\alpha_{k,2}(x) \leq k^{-1/2}\|x\|_2$. *This result is sharp. Thus* $\sup\{ \alpha_{k,2}(x) \mid x \in (C_2)_1 \} = k^{-1/2}$.

PROOF : i) Without loss of generality we can assume that $E(x) = 0$. Consider the set $\mathcal{P}$ of ordered equidimensional k-d.d. $\{p_m\}$ i.e., k-d.d. where $\dim p_m = n/k$ for all $1 \leq m \leq k$, and let $|\mathcal{P}|$ be its (finite) cardinality. Then for each $\{p_m\} \in \mathcal{P}$ we have

(29) $$\|\sum p_m x p_m\|_2^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} |(p_m x p_m)_{ij}|^2 .$$

Summing (29) over all $\{p_m\} \in \mathcal{P}$ we have

(30) $$\sum \{ \|\sum p_m x p_m\|_2^2 \mid \{p_m\} \in \mathcal{P} \} = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{m=1}^{n} \sum \{ |(p_m x p_m)_{ij}|^2 \mid \{p_m\} \in \mathcal{P} \} .$$

For any given $1 \le i, j \le n$ and $1 \le s \le k$ we have that $(p_s x p_s)_{ij} = x_{ij}$ if $i, j \in p_s$ and 0 otherwise, hence

(31) $$\sum \{ |(p_s x p_s)_{ij}|^2 \mid \{p_m\} \in \mathcal{P} \} = c\,|x_{ij}|^2$$

where c is the number of k-d.d. $\{p_m\} \in \mathcal{P}$ such that both i and j are in projection p_s at the s place in the decomposition. It is clear that c does not depend on x, nor on i, j and s; we can compute c by using a purely probabilistic argument. Indeed, for any given projection p_s, the probability that $i \in p_s$ is equal to the ratio of the dimensions of p_s to the dimension of I; thus it is $(n/k)/n = 1/k$. Similarly, the probability that for $j \neq i$ both i and j are in p_s is $(1/k)((n/k)-1)/n-1)$. Therefore

(32) $$c = |\mathcal{P}|\,(1/k)(1/k - 1/n)/(1 - 1/n) .$$

Substituting (32) in (31) and then in (30) and summing over m, we have

(33) $$\sum \{ \|\sum p_m x p_m\|_2^2 \mid \{p_m\} \in \mathcal{P} \} = |\mathcal{P}|\,(1/k - 1/n)/(1 - 1/n)\,\|x\|_2^2 .$$

By the averaging principle, there has to be at least one k-d.d. $\{p_m\}$ of $\mathcal{P}$ such that

$$\|\sum p_m x p_m\|_2^2 \le (1/k - 1/n)/(1 - 1/n)\|x\|_2^2 \le (1/k)\|x\|_2^2 .$$

Now ii) follows easily from i), since $\alpha_{k,2}(x) = \lim_n \alpha_{k,2}(q_{nk} x q_{nk})$ by (22). The sharpness follows from Proposition 4.2 or Example 4.3.

Q.E.D.

As a consequence of Proposition 4.2 i) and Theorem 4.4 i) we obtain

COROLLARY 4.5 *Let* x *be an* $n \times n$ *Hadamard matrix and let* k *divide* n; *then* $(k^{-1/2} - n^{-1/2})/(1 + n^{-1/2}) \le \alpha_{k,2}(x)/\|x - E(x)\|_2 \le k^{-1/2}$.

For Hadamard matrices, we can extend the proof of Theorem 4.4 to the case $p = 4$.

PROPOSITION 4.6. *Let* x *be an* $n \times n$ *Hadamard matrix and let* k *divide* n; *then* $(k^{-1/2} - n^{-1/2})/(1 + n^{-1/2}) \le \alpha_{k,4}(x)/\|x - E(x)\|_4 \le 2^{1/4}k^{-1/2} + \varepsilon(n,k)$ *where the function* $\varepsilon(n, k)$ *does not depend on* x and $\lim_n \varepsilon(n, k) = 0$ *for each* k.

PROOF : For every $n \times n$ matrix y with zero diagonal, we can decompose $\|y\|_4^4 = \mathrm{tr}\,((A^*A)^2)$ into

(34) $$\|y\|_4^4 = \gamma_1(y) + \gamma_2(y) + \gamma_3(y)$$

where

$$\gamma_1(y) = \sum \{ \bar{y}_{is} y_{it} \bar{y}_{jt} y_{js} \mid 1 \le i,j,s,t, \le n ;\ i,j,s,t \text{ all distinct} \}$$

$$\gamma_2(y) = \Sigma\{ |y_{sj}|^2|y_{tj}|^2 + |y_{js}|^2|y_{jt}|^2 \mid 1 \le j,s,t \le n ;\ j,s,t \text{ all distinct} \},$$

$$\gamma_3(y) = \Sigma\{ |y_{st}|^4 \mid 1 \le s,t \le n ;\ s \ne t \}.$$

In the notation used in the proof of Theorem 4.4, we have:

$$\Sigma\{ \|\Sigma p_m y p_m\|_4^4 \mid \{p_m\} \in \mathcal{P} \} = \delta_1(y) + \delta_2(y) + \delta_3(y) \qquad (35)$$

where

$$\delta_3(y) = \Sigma\{ \Sigma \gamma_3(p_m y p_m) \mid \{p_m\} \in \mathcal{P} \}$$
$$= \Sigma\Sigma\{ \Sigma\{ |(p_m y p_m)_{st}|^4 \mid \{p_m\} \in \mathcal{P} \} \mid 1 \le s,t \le n ;\ s \ne t \}$$

and similar expressions hold for $\delta_1(y)$ and $\delta_2(y)$. As in the proof of Theorem 4.4 (cfr. (32)) we can use a probabilistic argument to obtain for all $s \ne t$ that

$$\Sigma\{ |(p_m y p_m)_{st}|^4 \mid \{p_m\} \in \mathcal{P} \} = |\mathcal{P}|(1/k)(1/k - 1/n)/(1 - 1/n)\, |y_{st}|^4$$

and thus

$$|\mathcal{P}|^{-1}\delta_3(y) = ((1/k - 1/n) / (1 - 1/n))\, \delta_3(y).$$

Similarly,

$$|\mathcal{P}|^{-1}\delta_2(y) = ((1/k-1/n)(1/k-2/n) / (1-1/n)(1-2/n))\, \gamma_2(y),$$

$$|\mathcal{P}|^{-1}\delta_1(y) = ((1/k-1/n)(1/k-2/n)(1/k-3/n) / (1-1/n)(1-2/n)(1-3/n))\, \gamma_1(y).$$

For a general y, we cannot proceed further for lack of a good estimate of $\gamma_i(y)$.

Now take $y = x - E(x)$ where x is the given Hadamard matrix. Since $\|x\|_4 = n^{1/4}$ and $\|E(x)\|_4 = n^{-1/4}$ we have $n(1 - n^{-1/2})^4 \le \|y\|_4^4 \le n(1 + n^{-1/2})^4$ and hence

$$\gamma_3(y) = 1 - 1/n,$$

$$\gamma_2(y) = \sum_{j=1}^{n} (\|y\eta_j\|^4 + \|y^*\eta_j\|^4) - 2\gamma_3(y),$$
$$\le \sum_{j=1}^{n} ((\|x\eta_j\| + \|E(x)\eta_j\|)^4 + (\|x^*\eta_j\| + \|E(x)^*\eta_j\|)^4) - 2\gamma_3(y)$$
$$= 2n(1 + n^{-1/2})^4 - 2 + 2/n,$$

and similarly,

$$\gamma_2(y) \ge 2n(1 - n^{-1/2})^4 - 2 + 2/n.$$

Thus, from (34) we have

$$\gamma_1(y) = \|y\|_4^4 - \gamma_2(y) - \gamma_3(y)$$
$$\le n(1 + n^{-1/2})^4 - 2n(1 - n^{-1/2})^4 + 1 - 1/n.$$

Therefore

$$|\mathcal{P}|^{-1}\delta_1(y) \le -n/k^3 + \varepsilon_1(n,k),$$
$$|\mathcal{P}|^{-1}\delta_2(y) \le 2n/k^2 + \varepsilon_2(n,k),$$
$$|\mathcal{P}|^{-1}\delta_3(y) = \varepsilon_3(n,k),$$

$$|\mathcal{P}|^{-1}(\delta_1(y) + \delta_2(y) + \delta_3(y)) \le 2n/k^2 + \varepsilon_4(n,k)$$
$$= (2/k^2 + \varepsilon_5(n,k))\, n\, (1 - n^{-1/2})^4$$
$$\le (2^{1/4}k^{-1/2} + \varepsilon_6(n,k))^4 \|y\|_4^4 ,$$

where $\lim_n \varepsilon_i(n,k)/n = 0$ for $i = 1,2,3,4$ and $\lim_n \varepsilon_i(n,k) = 0$ for $i = 5,6$.

Therefore, by the averaging principle applied to (35), we can find a k-d.d. in $\mathcal{P}$ such that

$$\|\sum p_m y p_m\|_4 / \|y\|_4 \le 2^{1/4}k^{-1/2} + \varepsilon_6(n,k) .$$

This, together with Proposition 4.2, completes the proof.

Q.E.D.

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Herbert Halpern, Victor Kaftal, Gary Weiss
Department of Mathematical Sciences
University of Cincinnati
Cincinnati, Ohio 45221, USA

Kennneth Berman
Department of Computer Science
University of Cincinnati
Cincinnati, Ohio 45221, USA

Submitted: June 24, 1987