

B(H) LATTICES, DENSITY AND ARITHMETIC MEAN IDEALS

VICTOR KAFTAL AND GARY WEISS

Communicated by Vern I. Paulsen

ABSTRACT. This paper is part of an eight paper project [14]-[20] studying the arithmetic mean operator ideals in $B(H)$ introduced by Dykema, Figiel, Weiss and Wodzicki in [10]. Every ideal I is generated by diagonal operators with positive decreasing sequences, and its arithmetic mean ideal I_a is generated by diagonal operators with the arithmetic means of those sequences. In this paper we focus on lattice properties: we prove that the lattices of all principal ideals, of principal ideals with $\Delta_{1/2}$ -generators, of arithmetic mean stable principal ideals ($I = I_a$), and of arithmetic mean at infinity stable principal ideals ($I = I_{a_\infty}$) are all both upper and lower dense in the lattice of general ideals, i.e., between any ideal and an ideal in one of these sublattices lies another ideal in that sublattice. Among the applications: a principal ideal is am-stable (similarly for am- ∞ stable) if and only if any (hence all) of its first order arithmetic mean ideals are am-stable, e.g., its am-interior, am-closure and others. A principal ideal I is am-stable (similarly for am- ∞ stable) if and only if it satisfies any (hence all) of the first order equality cancelation properties, e.g., $J_a = I_a \Rightarrow J = I$. Cancelations can fail even for am-stable countably generated ideals. We prove that while the inclusion cancelation $J_a \supset I_a \Rightarrow J \supset I$ does not hold in general, even for I am-stable and principal, there is always a largest ideal \hat{I} for which $J_a \supset I_a \Rightarrow J \supset \hat{I}$. Furthermore, if I is principal and has generator $\text{diag}(\xi)$, then \hat{I} has generator $\text{diag}(\hat{\xi})$ and $\hat{\xi}$ is a sequence optimal in the majorization sense. That is, $\eta \geq \hat{\xi}$ asymptotically for all sequences η decreasing to zero for which $\sum_1^n \eta_j \geq \sum_1^n \xi_j$ for every n and $\hat{\xi}$ is asymptotically sharp. In particular, $\widehat{\omega^{1/p}} = \omega^{1/p'}$ for the harmonic sequence $\omega = \langle \frac{1}{n} \rangle$, $0 < p < 1$, and $1/p - 1/p' = 1$.

2000 *Mathematics Subject Classification*. Primary: 47B47, 47B10; Secondary: 46A45, 46B45, 47L20.

Key words and phrases. Soft ideals, operator ideals, arithmetic means.

The first named author was partially supported by The Charles Phelps Taft Research Center, the second named author was partially supported by NSF Grants DMS 95-03062 and DMS 97-06911 and by The Charles Phelps Taft Research Center.

1. INTRODUCTION

Operator ideals, the two-sided ideals of $B(H)$, have played an important role in operator theory and operator algebras since they were first studied by J. Calkin [8] in 1941.

Many of the questions in the theory have involved the structure of the lattice \mathcal{L} of all operator ideals and of its distinguished sublattices, like the lattice \mathcal{PL} of all principal ideals. For instance, one basic question due to Brown, Pearly and Salinas [7] was whether or not the ideal of compact operators $K(H)$ was the sum of two proper ideals. This was settled affirmatively in [6] for any proper ideal strictly larger than F (the ideal of finite rank operators) using the continuum hypothesis and the techniques employed led to the set-theoretic concept of groupies density that has proved useful in point-set topology and abelian group theory ([4], [5], [24]). Other work by Salinas (e.g., [25]) studied increasing or decreasing nests of special classes of symmetric norm ideals (Banach ideals).

Another central topic in the theory was the study of commutator spaces (also called commutator ideals) on which much work was done from the early years. More recently, the introduction of cyclic cohomology in the 1980's by A. Conned (e.g., see [9]) and the connection with algebraic K-theory by M. Wodzicki in the 1990's (e.g., see [31]) provided a powerful motivation for further work on operator ideals and in particular on commutator spaces. Arithmetic means of monotone sequences were first connected, albeit only implicitly, to commutator spaces in [27] (see also [28]-[29] and survey [30]) and then explicitly for the trace class in [22]. They provided the key tool for the full characterization of commutator spaces achieved in [10] in terms of arithmetic mean operations on ideals. This led to the definition of a number of arithmetic mean ideals derived from an ideal I : the arithmetic mean I_a , the pre-arithmetic mean ${}_aI$, the am-interior I^o and the am-closure I^- . After [10], the connection between arithmetic means and operator ideals were further studied in [23], [11], [12] and in an ongoing program by the authors of this paper which was announced in [14] and which includes [15]-[17].

At the beginning of this project it soon became apparent that to investigate questions such as "how many traces can an ideal support", i.e., the codimension of the commutator space of the ideal, which question formed the main focus of [15], a more systematic study of the actions of the arithmetic mean operations on operator ideals was necessary and the action of the arithmetic mean at infinity needed also to be introduced formally (see Section 2 herein for definitions). This program was carried out in part in [15] and further in [16], but many questions, and in particular those involving arithmetic mean cancelation properties, led to an

analysis of the relation between operator lattices and arithmetic mean operations. The goal of this paper is to provide this analysis and its applications.

Section 2 introduces the notations and some of the necessary preliminaries for arithmetic mean ideals.

In Section 3 we study the density properties of ideal sublattices of \mathcal{L} . Given two lattices $\mathcal{L}_1 \subset \mathcal{L}_2$, \mathcal{L}_1 is said to be upper dense in \mathcal{L}_2 if for every pair of ideals $I_1 \subsetneq I_2$ with $I_i \in \mathcal{L}_i$, there is an intermediate ideal $I_1 \subsetneq L \subsetneq I_2$ with $L \in \mathcal{L}_1$. The notion of lower density is similar. We consider the following sublattices of \mathcal{L} . (See the next section for their precise definitions.)

- \mathcal{PL} : principal ideals,
- \mathcal{L}_{\aleph_0} : countably generated ideals,
- $\Delta_{1/2}\mathcal{PL}$: principal ideals with a generator that satisfies the $\Delta_{1/2}$ -condition,
- \mathcal{SPL} : principal ideals with an am-regular generator,
- $\mathcal{S}_\infty\mathcal{PL}$ principal ideals with an am- ∞ regular generator.

A first result, which is then used throughout the paper, is that \mathcal{PL} and \mathcal{L}_{\aleph_0} are upper dense in \mathcal{L} (Corollary 3.7). These lattices are, however, not lower dense in \mathcal{L} as every nonzero principal ideal I has a gap underneath it, i.e., an ideal $M \subsetneq I$ with no other ideals in-between (Corollary 3.4).

The main technical results of this section and of the paper are that $\Delta_{1/2}\mathcal{PL}$, \mathcal{SPL} , and $\mathcal{S}_\infty\mathcal{PL}$ are both upper and lower dense in \mathcal{PL} (Theorems 3.9, 3.10 and 3.13). The proofs require us to construct between a “nice” sequence (the s -numbers of the generator) and a possibly “bad” but comparable sequence, a new “nice” intermediate sequence that is inequivalent (in the ideal sense) to either of the two given sequences. The key tool in the case of \mathcal{SPL} and $\mathcal{S}_\infty\mathcal{PL}$ are the Potter-type conditions satisfied by the regular or ∞ -regular sequences. For the am-case, these are just the properties of the Matuszewska β -index studied in [10, Theorem 3.10]. For the am- ∞ case, these are the properties of the Matuszewska α -index and were introduced in [15, Theorem 4.12].

In Section 4 we show that a Banach ideal is am-stable if and only if it is the union of am-stable principal ideals (Proposition 4.3). While it is clear that every ideal is the union of the principal ideals that it contains, it is not known which ideals are the union of an increasing chain of principal ideals. Assuming the continuum hypothesis, we show that every ideal is the union of an increasing chain of countably generated ideals, and we prove that if the ideal is the power of a Banach ideal then it is also the union of an increasing chain of principal ideals (see Proposition 4.4).

In Section 5 we use the density results from Section 3 to analyze the relation between the properties of an ideal and those of its first order arithmetic mean

ideals (resp., arithmetic mean ideals at infinity, see the next section for definitions). The first results (Theorems 5.1 and 5.2) are: for principal ideals, the am-stability (resp., am- ∞ stability) of I is equivalent to the am-stability of any of its first order arithmetic mean ideals (resp., first order arithmetic mean ideals at infinity). What is perhaps interesting is that this “rigidity” property does not extend even to countably generated ideals: Examples 5.4 and 5.5 exhibit ideals that are not am-stable but have various first-order am-ideals that are am-stable.

In Section 6 we apply density to investigate first order cancelation properties for arithmetic mean operations. (Second order cancelation properties are more complex and are studied in [14] and [17].) Equality cancelation questions ask: Under which conditions on an ideal I , does $J_a = I_a$ imply $J = I$? (and similarly for the other derived am and am- ∞ ideals). Inclusion cancelation questions ask: Under which conditions on an ideal I does $I_a \subset J_a$ imply $I \subset J$, or $J_a \subset I_a$ imply $J \subset I$? (and similarly for the other derived am and am- ∞ ideals). Beyond the elementary cancelations (e.g., for fixed I , $J_a \subset I_a \Rightarrow J \subset I$ if and only if I is am-closed (see Lemma 6.1)), we know of no natural conditions for general am-cancelation to hold. Examples 5.4 and 5.5 show that the equality am-cancelations can fail even for am-stable countably generated ideals.

For principal ideals however, the necessary and sufficient condition for equality cancelations is that the principal ideal I is am-stable (resp., am- ∞ stable) (see Theorem 6.7). One key ingredient in the proof is the fact that for countably generated ideals, and hence for principal ideals, am-stability is equivalent to am-closure, and similarly for the am- ∞ case (see [16, Theorems 2.11 and 3.5]). For other cancelations we need instead a technical result that shows that if a principal ideal is am-open but not am-stable, then it properly contains and is properly contained in two other principal ideals that share with it, respectively, the same am-open envelope and the same am-interior, and similarly for the am- ∞ case (Proposition 6.5).

For the nontrivial inclusion cancelations, am-stability is not sufficient even in the principal ideal case. For instance, for every principal ideal I , there is a principal ideal J with ${}_a J \subset {}_a I$ but $J \not\subset I$ (Proposition 6.8), i.e., no cancelation whatsoever is possible.

For the cancelation $J_a \supset I_a \Rightarrow J \supset I$, a stronger condition than am-stability is required for the ideal I . Indeed, we show that for every ideal I there is an “optimal” ideal \widehat{I} for which $J_a \supset I_a \Rightarrow J \supset \widehat{I}$. If $I = (\xi)$, then $\widehat{(\xi)} = (\widehat{\xi})$. Furthermore, the sequence $\widehat{\xi}$ is also optimal in the memorization sense, i.e., if $\eta_a \geq \xi_a$ then $\eta \geq \widehat{\xi}$ asymptotically (Theorem 6.14). Sequences for which

$(\xi) = (\widehat{\xi})$ are necessarily regular (Proposition 6.17) and they form a distinguished proper subclass of the regular sequences. Indeed, for the harmonic sequence ω , $\widehat{\omega^p} \asymp \omega^{p'}$, where $0 < p < 1$ and $\frac{1}{p} - \frac{1}{p'} = 1$ (Corollary 6.16). Thus $\omega^{1/2}$ is regular but $\omega^{1/2} \not\asymp \widehat{\omega^{1/2}} \asymp \omega$. This result is linked to the single commutator problem implicit in [10, Chapter 7] on whether containment in the class of single $(I, B(H))$ -commutators, $[I, B(H)]_1$, of a finite rank nonzero trace operator implies $\omega^{1/2} \in \Sigma(I)$.

This paper employs commutative techniques to study noncommutative objects (operator ideals). The main source of these connections derives from Calkin’s $B(H)$ -ideal characterization in terms of positive cones of sequences discussed next.

2. NOTATIONS AND PRELIMINARY RESULTS

Calkin [8] established the inclusion preserving lattice isomorphism

$$I \rightarrow \Sigma(I) := \{s(X) \mid X \in I\}$$

between two-sided ideals I of $B(H)$, the algebra of bounded linear operators on a separable infinite-dimensional complex Hilbert space H and the characteristic sets $\Sigma \subset c_o^*$, i.e., the hereditary (solid) ampliation invariant (see below) subcones of the collection c_o^* of sequences decreasing to 0. Here $s(X)$ denotes the sequence of the s -numbers of the compact operator X and the inverse of this correspondence maps a characteristic set Σ to the ideal generated by the collection of diagonal operators $\{diag \xi \mid \xi \in \Sigma\}$.

Two sequence operations, the arithmetic mean acting on c_o^* and the arithmetic mean at infinity acting on $(\ell^1)^*$ (the monotone nonincreasing summable sequences) respectively,

$$\xi_a := \left\langle \frac{1}{n} \sum_1^n \xi_j \right\rangle \quad \text{and} \quad \xi_{a_\infty} := \left\langle \frac{1}{n} \sum_{n+1}^\infty \xi_j \right\rangle$$

are essential for the study of commutator spaces (i.e., commutator ideals) and hence traces on ideals, as mentioned in the introduction (e.g., see [10] and [14]-[16]). For the readers’ convenience we list the definitions and first properties from [10, Sections 2.8 and 4.3] of the ideals derived via arithmetic mean operations (am-ideals for short).

If I is an ideal, then the arithmetic mean ideals ${}_aI$ and I_a , called respectively the *pre-arithmetic mean* and *arithmetic mean* of I , are the ideals with characteristic sets

$$\begin{aligned} \Sigma({}_aI) &:= \{\xi \in c_o^* \mid \xi_a \in \Sigma(I)\} \\ \Sigma(I_a) &:= \{\xi \in c_o^* \mid \xi = O(\eta_a) \text{ for some } \eta \in \Sigma(I)\}. \end{aligned}$$

The *arithmetic mean-closure* I^- and *arithmetic mean-interior* I^o of an ideal (am-closure and am-interior for short) are defined as

$$I^- := {}_a(I_a) \quad \text{and} \quad I^o := ({}_aI)_a.$$

For any ideal I , the following *5-chain of inclusions* holds:

$${}_aI \subset I^o \subset I \subset I^- \subset I_a$$

and the identities

$$I_a = ({}_a(I_a))_a \quad \text{and} \quad {}_aI = {}_a(({}_aI)_a).$$

Other first order arithmetic mean ideals derived from a given ideal I and introduced in [14], [16] are the largest am-closed ideal I_- contained in I and the smallest am-open ideal I^{oo} containing I .

We take this opportunity to make a shift in the terminology introduced in [14]. There we called 2nd order arithmetic mean ideals those obtained from applying twice the basic am and pre-am operations to an ideal, e.g., the am-closure ${}_a(I_a)$ and the am-interior $({}_aI)_a$. In later work we found a commonality of properties between the ideals $I_a, {}_aI, {}_a(I_a), ({}_aI)_a$ and the ideals I_- and I^{oo} mentioned above that motivated us to now call them all first order am-ideals, while ideals of the kind $I_{a^2}, {}_{a^2}(I_{a^2})$, and so on will be called 2nd order am-ideals.

As a consequence of one of the main results in [10], an ideal is am-stable (i.e., $I = {}_aI$, or equivalently, $I = I_a$) if and only if $I = [I, B(H)]$ if and only if I supports no nonzero trace. In particular, ideals contained in the trace class \mathcal{L}_1 cannot be am-stable. For them, the arithmetic mean at infinity is the relevant operation. It was often used as a sequence operation in connection with operator ideals (see for instance [24], [21], [10], [32]) and its action on operator ideals was studied in [15, Section 4]. For the readers' convenience we list the definitions and first properties.

If I is an ideal, then the arithmetic mean at infinity ideals ${}_{a_\infty}I$ and I_{a_∞} are the ideals with characteristic sets

$$\begin{aligned} \Sigma({}_{a_\infty}I) &:= \{\xi \in (\ell^1)^* \mid \xi_{a_\infty} \in \Sigma(I)\} \\ \Sigma(I_{a_\infty}) &:= \{\xi \in c_o^* \mid \xi = O(\eta_{a_\infty}) \text{ for some } \eta \in \Sigma(I \cap \mathcal{L}_1)\}, \end{aligned}$$

and the other derived am-∞ ideals are the am-∞ closure $I^{-\infty} := a_\infty(I_{a_\infty})$, the am-∞ interior $I^{o\infty} := (a_\infty I)_{a_\infty}$, the largest am-∞ closed ideal, $I_{-\infty}$, contained in I and the smallest am-∞ open ideal, $I^{oo\infty}$, containing $I \cap se(\omega)$ where $\omega = \langle \frac{1}{n} \rangle$ is the harmonic sequence and $se(\omega)$ is the ideal whose characteristic set consists of sequences $o(\omega)$.

The am-∞ analog of the 5-chain of inclusions are

$$a_\infty I \subset I^{o\infty} \subset I \cap se(\omega) \quad \text{and} \quad I \cap \mathcal{L}_1 \subset I^{-\infty} \subset I_{a_\infty} \cap \mathcal{L}_1$$

and the following identities hold

$$I_{a_\infty} = (a_\infty(I_{a_\infty}))_{a_\infty} \quad \text{and} \quad a_\infty I = a_\infty((a_\infty I)_{a_\infty}).$$

An ideal I is am-∞ stable ($I = a_\infty I$ or, equivalently, $I \subset \mathcal{L}_1$ and $I = I_{a_\infty}$) if and only if $I = F + [I, B(H)]$ if and only if I supports a nonzero trace unique up to scalar multiples (see [15, Theorem 6.6]).

For every ideal I , the lower and upper stabilizers

$$st_a(I) := \bigcap_{m=0}^\infty a^m I \subset I \subset st^a(I) := \bigcup_{m=0}^\infty I_{a^m}$$

are, respectively, the largest am-stable ideal contained in ideal I (possibly $\{0\}$) and the smallest am-stable ideal containing I .

In particular, $st^a(F) = st^a(\mathcal{L}_1) = st^a(o(\omega))$ is the smallest am-stable ideal ($K(H)$ is the largest). For the am-∞ case and $I \neq \{0\}$,

$$\{0\} \neq st_{a_\infty}(I) := \bigcap_{m=0}^\infty a_\infty^m I \subset I$$

and if $I \subset \mathcal{L}_1$, then

$$I \subset \bigcup_{m=0}^\infty I_{a_\infty^m}$$

are the largest am-∞ stable ideal contained in I and the smallest am-∞ stable ideal containing I , respectively. In particular, F is the smallest am-∞ stable ideal and the largest am-∞ stable ideal is $st_{a_\infty}(\mathcal{L}_1) = st_{a_\infty}(o(\omega))$ with characteristic set

$$\{\xi \in c_o^* \mid \sum \xi_n \log^p n < \infty \text{ for all } p > 0\}.$$

(See [15, Definition 4.16-Proposition 4.18 including proof].)

Of particular interests-and the main focus of this paper-are principal ideals. If $X \in B(H)$ is a generator of the ideal I and $\xi = s(X)$ is the s-number sequence of X , then the diagonal operator $diag \xi$ is also a generator of I , thus we will denote $I := (\xi)$, and by abuse of language, we will simply say that a sequence

$\xi \in \mathfrak{c}_o^*$ generates the ideal (ξ) . The characteristic set $\Sigma((\xi))$ of a principal ideal (ξ) consists of the sequences $\eta \in \mathfrak{c}_o^*$ for which $\eta = O(D_m(\xi))$ for some $m \in \mathbb{N}$, where the m -fold ampliation is

$$\mathfrak{c}_o^* \ni \xi \rightarrow D_m \xi := \langle \xi_1, \dots, \xi_1, \xi_2, \dots, \xi_2, \xi_3, \dots, \xi_3, \dots \rangle$$

with each entry ξ_i of ξ repeated m -times. More generally, $(D_t \xi)_n := \xi \lceil \frac{n}{t} \rceil$ for all $t > 0$, where $\lceil \frac{n}{t} \rceil$, denotes the smallest integer majorizing $\frac{n}{t}$. A sequence is said to satisfy the $\Delta_{1/2}$ -condition if $D_2 \xi \asymp \xi$, namely, $D_2 \xi = O(\xi)$ and $\xi = O(D_2 \xi)$ (equivalently, if $\xi \asymp D_m \xi$ for all $m \in \mathbb{N}$). In particular, if at least one of the generators satisfies the $\Delta_{1/2}$ -condition condition, then $(\xi) \subset (\eta)$ if and only if $\xi = O(\eta)$. Now, a principal ideal (ξ) is am-stable if and only if the sequence ξ is regular (i.e., $\xi_a = O(\xi)$) since $(\xi)_a = (\xi_a)$ and every arithmetic mean sequence satisfies the $\Delta_{1/2}$ -condition. Less trivially since ξ_{a_∞} in general does not satisfy the $\Delta_{1/2}$ -condition, it is also true that (ξ) is am- ∞ stable if and only if $\xi_{a_\infty} = O(D_m \xi)$ for some $m \in \mathbb{N}$ if and only if the sequence ξ is ∞ -regular (i.e., $\xi_{a_\infty} = O(\xi)$) (see [15, Theorem 4.12]).

Operator ideals form a lattice \mathcal{L} with inclusion as partial order and intersection (resp. sum) as meet (resp. join). To avoid tedious disclaimers herein, we assume that \mathcal{L} does not include the zero ideal $\{0\}$. Since for any $\xi, \eta \in \mathfrak{c}_o^*$, $(\xi) \cap (\eta) = (\min(\xi, \eta))$ and $(\xi) + (\eta) = (\xi + \eta)$, the collection of all principal ideals forms a sublattice which we denote by \mathcal{PL} . Since the intersections and sums of two am-stable (resp. am- ∞ stable) ideals is easily seen also to be am-stable (resp. am- ∞ stable), the collection of all am-stable (resp. am- ∞ stable) principal nonzero ideals forms a lattice that we denote by \mathcal{SPL} (resp. $\mathcal{S}_\infty \mathcal{PL}$). Similarly, the collection of all nonzero principal ideals having a generator sequence that satisfies the $\Delta_{1/2}$ -condition forms a lattice denoted by $\Delta_{1/2} \mathcal{PL}$. Finally we will consider the lattice \mathcal{L}_{\aleph_o} of countably generated nonzero ideals. Here too, we will say by abuse of language that an ideal is generated by a countable collection of sequences in \mathfrak{c}_o^* , meaning that it is generated by the corresponding diagonal operators. Notice that if I is generated by the sequences $\eta^{(k)}$, then $\eta \in \Sigma(I)$ if and only if $\eta = O(D_m(\eta^{(1)} + \eta^{(2)} + \dots + \eta^{(k)}))$ for some $m, k \in \mathbb{N}$.

3. THE GENERAL IDEAL LATTICE, SUBLATTICES, DENSITY AND GAPS

Definition 3.1.

- (i) A *gap* in a lattice of ideals is a nested pair of ideals in the lattice, $I \subsetneq J$, between which there is no ideal in the lattice. An *upper* (resp. *lower*) *gap* for an ideal I is a gap $I \subsetneq J$ (resp. $J \subsetneq I$).

- (ii) For two nested lattices $\mathcal{L}' \subset \mathcal{L}''$, \mathcal{L}' is *upper dense* (resp. *lower dense*) in \mathcal{L}'' provided that between every pair of ideals $I \in \mathcal{L}'$ and $J \in \mathcal{L}''$ with $I \subsetneq J$ (resp. $J \subsetneq I$) lies another ideal in the smaller lattice \mathcal{L}' .

Although by assumption all ideals in a lattice are nonzero, we still say that $\{0\} \subset I$ is a gap in the lattice if there is no ideal L in the lattice with $\{0\} \subsetneq L \subsetneq I$, e.g., F has a lower gap in \mathcal{L} .

First we discuss gaps in \mathcal{L} .

Lemma 3.2.

- (i) A pair of ideals $I \subsetneq J$ is a gap in \mathcal{L} if and only if $J = I + (\xi)$ for some principal ideal (ξ) and I is maximal among the ideals $I \subset L \subset J$ such that $(\xi) \not\subset L$.
- (ii) An ideal J has a lower gap in \mathcal{L} if and only if J can be decomposed into $J = N + (\xi)$ with N a (possibly zero) ideal and $(\xi) \not\subset N$.
- (iii) An ideal I has an upper gap in \mathcal{L} if and only if there is a principal ideal $(\xi) \not\subset I$ such that if $I \subset I + (\eta) \subset I + (\xi)$ for some principal ideal (η) , then either $I = I + (\eta)$ or $I + (\eta) = I + (\xi)$.

PROOF.

- (i) If $J = I + (\xi)$ for some principal ideal (ξ) and $I \subset L \subset J$, then $L \neq J$ if and only if $(\xi) \not\subset L$, thus $I \subset J = I + (\xi)$ is a gap in \mathcal{L} precisely when I is maximal among the ideals $I \subset L \subset J$ such that $(\xi) \not\subset L$. On the other hand, if $I \subset J$ is a gap in \mathcal{L} , then for every $\xi \in \Sigma(J) \setminus \Sigma(I)$ it follows that $J = I + (\xi)$.
- (ii) The condition is necessary by (i). If $J = N + (\xi)$ with N a (possibly zero) ideal and $(\xi) \not\subset N$, then the collection \mathcal{C} of all the ideals L with $N \subset L \subset J$ and L does not contain (ξ) includes at least N . It is immediate to see that the union of every chain in \mathcal{C} is also an element of \mathcal{C} . Let M be the union of a maximal chain in \mathcal{C} , which must exist by the Hausdorff Maximality Principle. Then by part (i), $M \subsetneq J$ is a gap.
- (iii) By (i) the condition is necessary. Indeed, if I has an upper gap in \mathcal{L} , the gap is of the form $I \subsetneq I + (\xi)$ for some principal ideal (ξ) . But then, any intermediate ideal $I + (\eta)$ must coincide with either I or $I + (\xi)$. The condition is also sufficient. Indeed, if $I \subset L \subset I + (\xi)$ for some ideal L , then if $I \neq L$ then there is a principal ideal $(\eta) \subset L \setminus I$ and hence $I \subsetneq I + (\eta) \subset L \subset I + (\xi)$ and from $I + (\eta) = I + (\xi)$ it follows that $L = I + (\xi)$, that is, $I \subsetneq I + (\xi)$ forms a gap in \mathcal{L} . \square

As a consequence, between any two nested ideals $I \subsetneq J$ there is always a gap. Indeed, if $\xi \in \Sigma(J) \setminus \Sigma(I)$, then by the above lemma there is gap between I and

$I + (\xi) \subset J$. Another consequence of the lemma is the following corollary that lower gaps in \mathcal{L} can never be unique.

Corollary 3.3. *If $(\xi) \not\subset M$, then there is a gap in \mathcal{L} , $N \subsetneq M + (\xi)$, with $N \not\subset M$. Consequently, if $M \subsetneq J$ is a gap in \mathcal{L} , then there is another gap $N \subsetneq J$ in \mathcal{L} with $M \neq N$.*

PROOF. Decompose the ideal (ξ) into the sum of two non-comparable principal ideals $(\xi) = (\rho) + (\eta)$, i.e. $(\eta) \not\subset (\rho)$ and $(\rho) \not\subset (\eta)$, which implies that both are properly contained in (ξ) . For this decomposition, inductively choose a sequence of indices $n_1 = 1$ and $n_k \leq kn_k < n_{k+1}$ where n_{k+1} is the smallest integer j for which $\xi_j < \frac{1}{k}\xi_{kn_k}$. Then define $\zeta_i := \frac{1}{k}\xi_{kn_k}$ for $i \in [n_k, n_{k+1})$, and $\eta_i = \zeta_i$ and $\rho_i = \xi_i$ for $i \in [n_k, n_{k+1})$ for k odd, $\rho_i = \zeta_i$ and $\eta_i = \xi_i$ for $i \in [n_k, n_{k+1})$ for k even. Then $\rho, \eta \leq \xi$, $\rho, \eta \in \mathfrak{c}_o^*$, and $(\frac{\xi}{D_k \rho})_{kn_k} = k$ for k odd and $(\frac{\xi}{D_k \eta})_{kn_k} = k$ for k even, from which it follows that $(\rho) \subsetneq (\xi)$, $(\eta) \subsetneq (\xi)$, and $(\eta) \not\subset (\rho)$, $(\rho) \not\subset (\eta)$. Moreover, $(\xi) = (\rho) + (\eta)$ since $\max(\rho, \eta) = \xi \leq \rho + \eta \leq 2\xi$.

Now let M_1 (resp. M_2) be maximal among those ideals contained in $M + (\xi)$ that contain (ρ) but do not contain (ξ) (resp. that contain (η) but do not contain (ξ)). Then each $M_i \subset M + (\xi)$ forms a gap by the maximality of the ideal M_i . But if $M \supset M_1$ and $M \supset M_2$, then $M \supset (\rho)$ and $M \supset (\eta)$ hence $M \supset (\eta) + (\rho) = (\xi)$ against the assumption, so at least one $N := M_i \not\subset M$. In particular, if $M \subsetneq J$ is a gap in \mathcal{L} , then $J = M + (\xi)$ for every principal ideal $(\xi) \not\subset M$ and for any choice of such a principal ideal, the pair $N \subsetneq J$ constructed above provides a lower gap for J distinct from $M \subsetneq J$. \square

An immediate consequence of Lemma 3.2 by choosing $N = 0$ in (ii) is:

Corollary 3.4. *Every principal ideal has lower gaps in \mathcal{L} .*

Countably generated ideals may fail to have lower gaps as the following example shows.

Example 3.5. *Let I be an ideal with generators $\eta^{(k)}$, where the sequences $\eta^{(k)}$ all satisfy the $\Delta_{1/2}$ -condition and $\eta^{(k)} = o(\eta^{(k+1)})$. Then I has no lower gaps in \mathcal{L} .*

PROOF. Assuming otherwise, by Lemma 3.2(ii), $I = N + (\xi)$ for some principal ideal $(\xi) \not\subset N$. Then $(\xi) \subset (\eta^{(k_o)})$ for some k_o and hence $\xi = O(\eta^{(k_o)})$. Without loss of generality, assume $\xi \leq \eta^{(k_o)} \leq \eta^{(k_o+1)}$. Since

$$\eta^{(k_o+1)} \leq \zeta + KD_m \xi \leq \zeta + KD_m \eta^{(k_o)} \leq \zeta + K' \eta^{(k_o)}$$

for some $\zeta \in \Sigma(N)$, $K, K' > 0$, and $m \in \mathbb{N}$, and since $\eta^{(k_o)} = o(\eta^{(k_o+1)})$, it follows that $\eta^{(k_o+1)} \leq K''\zeta$ for some $K'' > 0$. Hence $(\xi) \subset N$, contradicting the assumption $(\xi) \not\subset N$. \square

The fact that countably generated ideals and, in particular, principal ideals never have upper gaps will follow from the more general fact that *strongly soft complemented* (ssc for short) ideals never have upper gaps, which we prove in Proposition 3.6 below. An ideal I is defined in [16, Definition 4.4] to be ssc if for every sequence $\eta^{(k)} \notin \Sigma(I)$ there is a sequence of integers n_k for which if $\xi \in c_o^*$ and $\xi_i \geq \eta_i^{(k)}$ for all $1 \leq i \leq n_k$, then $\xi \notin \Sigma(I)$. Without loss of generality, n_k is increasing. In [16, Propositions 4.5, 4.6, 4.7, 4.11, 4.12, and 4.18] we show that many “classical” ideals are ssc, among them countably generated ideals, maximal symmetric norm ideals, Lorentz, Marcinkiewicz, and Orlicz ideals.

Proposition 3.6. *Let $I = \bigcap I_\gamma$ be the intersection of strongly soft-complemented ideals I_γ .*

(i) *If $I \subsetneq I + (\xi)$, then there is a principal ideal $(\eta) \subset (\xi)$ for which*

$$I \subsetneq I + (\eta) \subsetneq I + (\xi).$$

(ii) *I has no upper gaps in \mathcal{L} .*

PROOF.

(i) First we prove the statement when I itself is strongly soft-complemented. Since $\frac{1}{k}D_{\frac{1}{k}}\xi \notin \Sigma(I)$ for all k there is an increasing sequence of integers n_k such that if $\zeta \in c_o^*$ and $\zeta_i \geq \frac{1}{k}(D_{\frac{1}{k}}\xi)_i$ for $1 \leq i \leq n_k$, then $\zeta \notin \Sigma(I)$. Define $\eta_i = \frac{1}{k}(D_{\frac{1}{k}}\xi)_i$ for $i \in (n_{k-1}, n_k]$; then $\eta \in c_o^*$ and $\eta \leq \xi$. Since $\frac{1}{j}D_{\frac{1}{j}}\xi \geq \frac{1}{k}D_{\frac{1}{k}}\xi$ for every $j \leq k$, $\eta_i \geq \frac{1}{k}(D_{\frac{1}{k}}\xi)_i$ for all $i \in [1, n_k]$ and hence by the strong soft-complementedness of I , $\eta \notin \Sigma(I)$. Thus $I \subsetneq I + (\eta)$. For every $m \in \mathbb{N}$ and for every $i > mn_m$ choose that k for which $n_{k-1} < \lceil \frac{i}{m} \rceil \leq n_k$. It is easy to verify that $k \geq m + 1$, hence $k \lceil \frac{i}{m} \rceil \geq i$, and thus $(D_m\eta)_i = \eta_{\lceil \frac{i}{m} \rceil} = \frac{1}{k}\xi_{k\lceil \frac{i}{m} \rceil} \leq \frac{1}{k}\xi_i$, i.e., $D_m\eta = o(\xi)$ and hence $(\eta) \subset (\xi)$. This implies that $(\xi) \not\subset I + (\eta)$ and hence that $I + (\eta) \subsetneq I + (\xi)$. Indeed otherwise $\xi \leq \zeta + KD_m\eta$ for some $\zeta \in \Sigma(I)$, $m \in \mathbb{N}$, and $K > 0$. Since $D_m\eta = o(\xi)$, eventually $\xi_i \leq 2\zeta_i$, that is $(\xi) \subset I$, against the hypothesis.

Now we prove the general case. Since $(\xi) \not\subset I$, then $(\xi) \not\subset I_\gamma$ for some I_γ in the collection. Then by the first part of the proof there is a principal ideal $(\eta) \subset (\xi)$ such that $I_\gamma \subsetneq I_\gamma + (\eta) \subsetneq I_\gamma + (\xi)$. In particular, $(\eta) \not\subset I_\gamma$ hence $(\eta) \not\subset I$ and similarly, $(\xi) \not\subset I_\gamma + (\eta)$, hence $(\xi) \not\subset I + (\eta)$. Thus $I \subsetneq I + (\eta) \subsetneq I + (\xi)$.

(ii) Immediate from Lemma 3.2. \square

In particular, Proposition 3.6 applies to all ssc ideals (e.g., all countably generated ideals).

All the gaps proved here depend on the Hausdorff Maximality Principle or its equivalent, Zorn's Lemma. We do not know if the existence of gaps in \mathcal{L} is equivalent to these.

Next we discuss density of the lattices of principal ideals and countably generated ideals.

Corollary 3.7.

- (i) \mathcal{L}_{\aleph_0} is upper dense in \mathcal{L} .
- (ii) \mathcal{PL} is upper dense in \mathcal{L} .

PROOF.

- (i) If $I \in \mathcal{L}_{\aleph_0}$, $J \in \mathcal{L}$, and $I \subsetneq J$, then there is a $\xi \in \Sigma(J) \setminus \Sigma(I)$ and hence $I \subsetneq I + (\xi)$. By [16, Proposition 4.5], I is ssc, so by Proposition 3.6(i), there is a principal ideal (η) such that $I \subsetneq I + (\eta) \subsetneq I + (\xi) \subset J$. By adding η to each of a countable collection of generators for I , one easily sees that $I + (\eta) \in \mathcal{L}_{\aleph_0}$.
- (ii) By the same argument as in (i) and the fact that $I + (\eta) \in \mathcal{PL}$ when $I \in \mathcal{PL}$, and hence \mathcal{PL} is upper dense in \mathcal{L} . \square

In particular, \mathcal{L}_{\aleph_0} and \mathcal{PL} have no gaps. Corollary 3.7 implies:

Corollary 3.8. *Let $I \in \mathcal{L}_{\aleph_0}$ and $J \in \mathcal{PL}$.*

- (i) $\bigcap\{L \in \mathcal{L} \mid L \supsetneq I\} = \bigcap\{L \in \mathcal{L}_{\aleph_0} \mid L \supsetneq I\} = I$
- (ii) $\bigcap\{L \in \mathcal{L} \mid L \supsetneq J\} = \bigcap\{L \in \mathcal{PL} \mid L \supsetneq J\} = J$

Gaps in other lattices can be studied by similar tools as the following two simple examples illustrate.

If $I \subsetneq J$ are am-closed ideals, then $I \subsetneq I + (\xi) \subset J$ for some principal ideal $(\xi) \subset J$. But then $(\xi)^- \subset J$, hence $I \subsetneq I + (\xi)^- \subset J$. By [16, Theorem 2.5], $I + (\xi)^-$ is am-closed and by [16, Lemma 2.1(v)], the union M of a maximal chain of am-closed ideals $I \subset L \subset I + (\xi)^-$ with $(\xi) \not\subset L$ is also am-closed. If $M \subset L = L^- \subset I + (\xi)^-$ and $L \neq I + (\xi)^-$, i.e., $(\xi)^- \not\subset L$, then $(\xi) \not\subset L$ since L is am-closed. By the maximality of the chain, $I \subset M \subsetneq I + (\xi)^-$ is a gap in the lattice of am-closed ideals. Similar arguments show that pairs of am-open ideals, am- ∞ closed ideals and am- ∞ open ideals all contain gaps in their respective ideal lattices.

If $(\xi) \in \mathcal{SP}\mathcal{L}$ and $M \subset (\xi)$ is a gap in \mathcal{L} , then M must be am-stable. Indeed, if not $M \subsetneq M_a \subset (\xi)_a = (\xi)$ would imply $M_a = (\xi)$, hence $\xi = O(\mu_a)$ for some

$\mu \in \Sigma(M)$. Since also $\mu_a = O(\xi)$, $\mu_a \asymp \xi$, hence μ_a is regular. But then μ itself is regular by [10, Theorem 3.10] and thus $\mu \asymp \xi$, against the assumption that $M \neq (\xi)$.

Together with the upper density of \mathcal{PL} in \mathcal{L} , the following three theorems which are the technical core of this article, will provide the tools we need for applications to arithmetic mean ideals. Recall that a sequence $\xi \in c_o^*$ satisfies the $\Delta_{1/2}$ -condition if $\xi_n \leq M\xi_{2n}$ for some $M > 1$ and all n and that $\Delta_{1/2}\mathcal{PL}$ is the lattice of principal ideals with $\Delta_{1/2}$ generators.

Theorem 3.9. $\Delta_{1/2}\mathcal{PL}$ is upper and lower dense in \mathcal{PL} .

PROOF. First we prove lower density. Let (ξ) be a principal ideal where the generating sequence ξ satisfies the $\Delta_{1/2}$ -condition and let $(\eta) \subsetneq (\xi)$. Then $\eta = O(\xi)$, so assume without loss of generality that $\eta \leq \xi$. Construct inductively two sequences of indices $q_{k-1} < n_k < q_k$ as follows. Let $M > 1$ be a bound for which $\xi_n \leq M\xi_{2n}$ for all n , choose an integer $k_o \geq M$, set $q_{k_o} = 1$, and assume the construction up to $k - 1 \geq k_o$. Since $\xi \notin \Sigma((\eta))$ and hence $\xi \neq O(\eta)$, choose $n_k > q_{k-1}$ so that $\xi_{n_k} \geq kM^k\eta_{n_k}$ and let q_k be the largest integer i for which $\xi_i \geq \frac{1}{k}\xi_{2^k n_k}$. Since $\xi_{2^{k+1}n_k} \geq \frac{1}{M}\xi_{2^k n_k} \geq \frac{1}{k}\xi_{2^k n_k}$, it follows that $q_k \geq 2^{k+1}n_k$. Now define the sequence

$$\zeta_i := \begin{cases} \xi_i & i \in (q_{k-1}, n_k] \\ \frac{1}{j}\xi_i & i \in (2^{j-1}n_k, 2^j n_k] \text{ and } 1 \leq j \leq k \\ \frac{1}{k}\xi_{2^k n_k} & i \in (2^k n_k, q_k]. \end{cases}$$

By construction, ζ is monotone nonincreasing and $\zeta \leq \xi$. By assumption, $\eta_i \leq \xi_i = \zeta_i$ for $i \in (q_{k-1}, n_k]$ and $\zeta_i \geq \frac{1}{k}\xi_{2^k n_k} \geq \frac{1}{kM^k}\xi_{n_k} \geq \eta_{n_k} \geq \eta_i$ for $i \in (n_k, q_k]$. Thus $\eta \leq \zeta$ and hence $(\eta) \subset (\zeta) \subset (\xi)$. The following inequalities show that the sequence ζ satisfies the $\Delta_{1/2}$ -condition.

$$\frac{\zeta_i}{\zeta_{2i}} = \begin{cases} \frac{\xi_i}{\xi_{2i}} \leq M & i \in (q_{k-1}, n_k] \\ \frac{\frac{1}{j}\xi_i}{\frac{1}{j+1}\xi_{2i}} \leq 2M & i \in (2^{j-1}n_k, 2^j n_k] \text{ and } 1 \leq j \leq k - 1 \\ \frac{\frac{1}{k}\xi_i}{\frac{1}{k}\xi_{2^k n_k}} \leq \frac{\xi_{2^{k-1}n_k}}{\xi_{2^k n_k}} \leq M & i \in (2^{k-1}n_k, 2^k n_k] \\ \frac{\frac{1}{k}\xi_{2^k n_k}}{\zeta_{2i}} \leq \frac{\xi_{q_k}}{\zeta_{2q_k}} = \frac{\xi_{q_k}}{\xi_{2q_k}} \leq M & i \in (2^k n_k, q_k] \end{cases}$$

Thus $\frac{\zeta_i}{\zeta_{2i}} \leq 2M$ for all i . Finally $\zeta \neq O(\eta)$ and since $\left(\frac{\zeta}{\eta}\right)_{n_k} = \left(\frac{\xi}{\eta}\right)_{n_k} \geq kM^k$, then also $\xi \neq O(\zeta)$ and $\left(\frac{\xi}{\zeta}\right)_{2^k n_k} = k$ for all k . Since ζ satisfies the $\Delta_{1/2}$ -condition, it follows that $(\eta) \neq (\zeta) \neq (\xi)$, which concludes the proof of lower density.

Next we prove upper density. Let $(\xi) \in \Delta_{1/2}\mathcal{PL}$, let $(\xi) \subsetneq (\eta)$ and assume without loss of generality that $\xi \leq \eta$. Construct inductively two increasing sequences of integers n_k, q_k with $2^{n_{k-1}} < q_k < 2^{n_k}$ as follows. Let $M > 1$ be such that $\xi_n \leq M\xi_{2n}$ for all n , choose an integer $k_o \geq M$, set $n_{k_o} = 0$, and assume the construction up to $k-1 \geq k_o$. Since $\eta \neq O(\xi)$, we can find infinitely many integers r_k such that $\eta_{r_k} \geq kM^{k+1}\xi_{r_k}$. Let $n_k := \lceil \log_2 r_k \rceil$, then

$$\eta_{2^{n_k}} \geq \eta_{r_k} \geq kM^{k+1}\xi_{r_k} \geq kM^k \xi_{\frac{r_k}{2}} \geq kM^k \xi_{2^{n_k}}.$$

Choose r_k sufficiently large so that $\xi_{2^{n_{k-1}+1}} \geq k\xi_{2^{n_k-k}}$ and let q_k to be the largest integer i for which $\xi_i \geq k\xi_{2^{n_k-k}}$. As $k\xi_{2^{n_k-k}} \geq \frac{k}{M}\xi_{2^{n_k-k-1}} > \xi_{2^{n_k-k-1}}$, because $\xi_i > 0$ for all i as ξ satisfies the $\Delta_{1/2}$ -condition, it follows that

$$2^{n_{k-1}} < 2^{n_{k-1}+1} \leq q_k < 2^{n_k-k-1} < 2^{n_k}.$$

Now define the sequence

$$\zeta_i := \begin{cases} \xi_i & i \in [2^{n_{k-1}}, q_k] \\ k\xi_{2^{n_k-k}} & i \in (q_k, 2^{n_k-k}] \\ j\xi_i & i \in (2^{n_k-j}, 2^{n_k-j+1}] \text{ for } 1 \leq j \leq k. \end{cases}$$

Then by construction, $\zeta \in \mathfrak{c}_o^*$ and $\xi \leq \zeta$. Since $\zeta_i = \xi_i \leq \eta_i$ for $i \in [2^{n_{k-1}}, q_k]$ and

$$\zeta_i \leq k\xi_{2^{n_k-k}} \leq kM^k \xi_{2^{n_k}} \leq \eta_{2^{n_k}} \leq \eta_i \text{ for } i \in (q_k, 2^{n_k}],$$

it follows that $\zeta \leq \eta$ and hence $(\xi) \subset (\zeta) \subset (\eta)$. The following inequalities show that ζ satisfies the $\Delta_{1/2}$ -condition.

$$\frac{\zeta_i}{\zeta_{2i}} = \begin{cases} \frac{\xi_i}{\xi_{2i}} \leq M & i \in [2^{n_{k-1}}, q_k] \\ \frac{k\xi_{2^{n_k-k}}}{\xi_{2^{n_k-k+1}}} \leq M & i \in (q_k, 2^{n_k-k}] \\ \frac{j\xi_i}{(j-1)\xi_{2i}} \leq 2M & i \in (2^{n_k-j}, 2^{n_k-j+1}] \text{ for } 2 \leq j \leq k. \\ \frac{\xi_i}{\xi_{2i}} \leq M & i \in (2^{n_k-1}, 2^{n_k}] \end{cases}$$

For the last inequality, notice that $2^{n_k+1} \leq q_{k+1}$, which was proved above. Finally, from $\eta_{2^{n_k}} \geq kM^k \xi_{2^{n_k}} = kM^k \zeta_{2^{n_k}}$ and $\zeta_{2^{n_k-k}} = k\xi_{2^{n_k-k}}$ it follows that $\eta \neq O(\zeta)$ and $\zeta \neq O(\xi)$. Since both ξ and ζ satisfy the $\Delta_{1/2}$ -condition, $(\xi) \neq (\zeta) \neq (\eta)$, which concludes the proof. \square

As stated in ([10, Section 2.4 (22)] and in [14, Corollary 4.15(i)], the $\Delta_{1/2}$ -condition is equivalent to the Potter-type condition $\xi_n \geq C(\frac{m}{n})^p \xi_m$ for some $C > 0$ (necessarily $C \leq 1$), $p \in \mathbb{N}$, and all $n \geq m$. Although we did not need to employ explicitly this condition in the above proof, similar conditions characterizing regular and ∞ -regular sequences will be essential in the proofs of the next two theorems.

Theorem 3.10. *$\mathcal{SP}\mathcal{L}$ is upper and lower dense in $\mathcal{P}\mathcal{L}$.*

PROOF. Let (ξ) be an am-stable principal ideal, i.e., one whose generating sequence ξ is regular, namely, $\xi_a \leq M\xi$ for some $M > 1$. Equivalently, ξ satisfies the Potter-type condition

$$\xi_n \geq C(\frac{m}{n})^{p_o} \xi_m \quad \text{for some } 0 < C \leq 1, \quad 0 < p_o < 1 \quad \text{and all } n \geq m$$

[10, Theorem 3.10 and Remark 3.11]. (See also [3, Proposition 2.2.1], [1]). Choose any $p_o < p < 1$.

First we show the lower density of $\mathcal{SP}\mathcal{L}$ in $\mathcal{P}\mathcal{L}$. Let $(\eta) \subsetneq (\xi)$ and assume without loss of generality that $\eta \leq \xi$. Construct inductively two increasing sequences of indices $q_{k-1} < n_k < q_k$ as follows. Choose an integer $k_o \geq C^{-\frac{1}{p-p_o}}$, set $q_{k_o} = 1$, and assume the construction up to $k - 1 \geq k_o$. Since $\xi \neq O(\eta)$, choose $n_k > q_{k-1}$ so that $\xi_{n_k} \geq k^p \eta_{n_k}$ and choose q_k to be the largest integer i for which $\xi_i \geq k^{-p} \xi_{n_k}$. Then $\xi_{kn_k} \geq Ck^{-p_o} \xi_{n_k} > k^{-p} \xi_{n_k}$ and hence $n_k < kn_k \leq q_k$. Now define the sequence

$$\zeta_i := \begin{cases} \xi_i & \text{on } (q_{k-1}, n_k] \\ \max(k^{-p}, C(\frac{n_k}{i})^p) \xi_{n_k} & \text{on } (n_k, q_k] \end{cases}$$

As a further consequence of the Potter-type condition and of the inequality just obtained,

$$\zeta_{n_k} = \xi_{n_k} \geq \xi_{n_k+1} \geq \max(k^{-p}, C(\frac{n_k}{n_k+1})^p) \xi_{n_k} = \zeta_{n_k+1}.$$

Also, $\zeta_{q_k} \geq k^{-p} \xi_{n_k} > \xi_{q_{k+1}} = \zeta_{q_{k+1}}$. Thus ζ is monotone nonincreasing. By the definition of q_k and by the Potter-type condition, it follows that $\zeta_i \leq \xi_i$ for all i . By assumption, $\zeta_i = \xi_i \geq \eta_i$ for all $i \in (q_{k-1}, n_k]$, while $\zeta_i \geq k^{-p} \xi_{n_k} \geq \eta_{n_k} \geq \eta_i$ for all $i \in (n_k, q_k]$. Thus $\eta \leq \zeta \leq \xi$ and hence $(\eta) \subset (\zeta) \subset (\xi)$.

Next we prove that the sequence ζ is regular and hence the principal ideal (ζ) is am-stable. If $j \in (q_{k-1}, n_k]$, then $(\zeta_a)_j \leq (\xi_a)_j \leq M\xi_j = M\zeta_j$. If $j \in (n_k, q_k]$,

then

$$\begin{aligned}
j(\zeta_a)_j &= \sum_1^{n_k} \zeta_i + \sum_{n_k+1}^j \max\left(\frac{1}{k^p}, C\left(\frac{n_k}{i}\right)^p\right) \xi_{n_k} \\
&\leq \sum_1^{n_k} \xi_i + (j - n_k) \frac{1}{k^p} \xi_{n_k} + C \sum_{n_k+1}^j \left(\frac{n_k}{i}\right)^p \xi_{n_k} \\
&\leq n_k (\xi_a)_{n_k} + \frac{j}{k^p} \xi_{n_k} + C \sum_{i=2}^j \left(\frac{n_k}{i}\right)^p \xi_{n_k} \\
&\leq M n_k \xi_{n_k} + \frac{j}{k^p} \xi_{n_k} + \frac{j}{1-p} C \left(\frac{n_k}{j}\right)^p \xi_{n_k} \\
&\leq j \left(\frac{1}{k^p} \xi_{n_k} + \left(\frac{1}{1-p} + \frac{M}{C}\right) C \left(\frac{n_k}{j}\right)^p \xi_{n_k} \right) \\
&\leq j \left(1 + \frac{1}{1-p} + \frac{M}{C}\right) \zeta_j.
\end{aligned}$$

Thus $\zeta_a = O(\zeta)$, i.e., ζ is regular. Since $\zeta_{n_k} = \xi_{n_k} \geq k^p \eta_{n_k}$, it follows that $\zeta \neq O(\eta)$.

From the Potter-type inequality,

$$\xi_{kn_k} \geq C k^{-p_o} \xi_{n_k} = C k^{-(p_o-p)} k^{-p} \xi_{n_k} = C k^{p-p_o} \zeta_{kn_k}$$

and thus $\xi \neq O(\zeta)$. As ζ is regular and hence satisfies the $\Delta_{1/2}$ condition, we conclude that $(\eta) \not\subseteq (\zeta) \subseteq (\xi)$.

Now we prove the upper density of $\mathcal{SP}\mathcal{L}$ in \mathcal{PL} . Let $M > 1$, $p_o < p < 1$, and $0 < C \leq 1$ be as above and let (ξ) be am-stable with $(\xi) \not\subseteq (\eta)$ for some principal ideal (η) , and assume without loss of generality that $\xi \leq \eta$. Construct inductively two increasing sequences of indices $n_{k-1} < q_k < n_k$ as follows. Choose an integer $k_o \geq 2C^{-1/p}$, set $n_{k_o} = 1$, and assume the construction up to $k-1 \geq k_o$. Since ξ is regular and hence not summable, choose $q_k > n_{k-1}$ such that $\sum_{n_{k-1}}^{q_k} \xi_i \geq n_{k-1} \xi_1$. Since $\eta \neq O(\xi)$, choose an integer n_k such that $\eta_{n_k} \geq k^p \xi_{n_k}$ and $\xi_{n_k} \leq k^{-p} \xi_{q_k}$. By increasing if necessary q_k , assume without loss of generality that q_k is the largest integer i such that $\xi_i \geq k^p \xi_{n_k}$. Clearly, $q_k < n_k$. Next, define the sequence

$$\zeta_i = \begin{cases} \xi_i & i \in [n_{k-1}, q_k] \\ \min\left(k^p, \frac{1}{C} \left(\frac{n_k}{i}\right)^p\right) \xi_{n_k} & i \in (q_k, n_k) \end{cases}$$

By definition, $\zeta_{q_k} = \xi_{q_k} \geq k^p \xi_{n_k} \geq \zeta_{q_{k+1}}$ and since $k^p > k_o^p \geq \frac{2^p}{C} > \frac{1}{C} \left(\frac{n_k}{n_{k-1}}\right)^p$, it follows that $\zeta_{n_{k-1}} = \frac{1}{C} \left(\frac{n_k}{n_{k-1}}\right)^p \xi_{n_k} > \xi_{n_k} = \zeta_{n_k}$. Thus ζ is monotone nonincreasing. Since $\xi_i = \zeta_i$ for $i \in [n_{k-1}, q_k]$ and $\xi_i < \zeta_i$ for $i \in (q_k, n_k)$ because $\xi_i \leq \frac{1}{C} \left(\frac{n_k}{i}\right)^p \xi_{n_k} < \frac{1}{C} \left(\frac{n_k}{i}\right)^p \xi_{n_k}$ and $\xi_i < k^p \xi_{n_k}$ by the definition of q_k , it follows that $\xi \leq \zeta$. Since $\zeta_i = \xi_i \leq \eta_i$ for $i \in [n_{k-1}, q_k]$ and also $\zeta_i \leq k^p \xi_{n_k} \leq \eta_{n_k} \leq \eta_i$ for $i \in (q_k, n_k)$, it follows that $\zeta \leq \eta$. Therefore $(\xi) \subset (\zeta) \subset (\eta)$.

Next we prove that the sequence ζ is regular and hence the principal ideal (ζ) is am-stable. If $j \in (q_k, n_k)$, then recalling that by definition $\xi_1 = \zeta_1$, $\xi \leq \zeta$, and that $\sum_{n_{k-1}}^{q_k} \xi_i \geq n_{k-1} \xi_1$,

$$\begin{aligned} j(\zeta_a)_j &= \sum_1^{n_{k-1}-1} \zeta_i + \sum_{n_{k-1}}^{q_k} \xi_i + \sum_{q_{k+1}}^j \min\left(k^p, \frac{1}{C} \left(\frac{n_k}{i}\right)^p\right) \xi_{n_k} \\ &\leq n_{k-1} \xi_1 + \sum_{n_{k-1}}^{q_k} \xi_i + \sum_2^j \min\left(k^p, \frac{1}{C} \left(\frac{n_k}{i}\right)^p\right) \xi_{n_k} \\ &\leq 2 \sum_1^j \xi_i + \min\left(\sum_2^j k^p, \sum_2^j \frac{1}{C} \left(\frac{n_k}{i}\right)^p\right) \xi_{n_k} \\ &\leq 2j(\xi_a)_j + \min\left(jk^p, \frac{j}{1-p} \frac{1}{C} \left(\frac{n_k}{j}\right)^p\right) \xi_{n_k} \\ &\leq 2Mj\zeta_j + \frac{j}{1-p} \min\left(k^p, \frac{1}{C} \left(\frac{n_k}{j}\right)^p\right) \xi_{n_k} \\ &= \left(2M + \frac{1}{1-p}\right)j\zeta_j. \end{aligned}$$

If $j \in [n_k, q_{k+1}]$, then by using the above inequality and the definition of ζ , and recalling that $\zeta_{n_{k-1}} = \frac{1}{C} \left(\frac{n_k}{n_{k-1}}\right)^p \xi_{n_k}$, we have

$$\begin{aligned} j(\zeta_a)_j &= (n_k - 1)(\zeta_a)_{n_{k-1}} + \sum_{n_k}^j \xi_i \leq \left(2M + \frac{1}{1-p}\right)(n_k - 1)\zeta_{n_{k-1}} + \sum_1^j \xi_i \\ &= \left(2M + \frac{1}{1-p}\right) \frac{n_k}{C} \left(\frac{n_k}{n_{k-1}}\right)^{p-1} \xi_{n_k} + \sum_1^j \xi_i \\ &\leq \frac{1}{C} \left(\left(2M + \frac{1}{1-p}\right) n_k \xi_{n_k} + \sum_1^j \xi_i \right) \end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{C} \left(2M + \frac{1}{1-p} + 1\right) j(\xi_a)_j \\ &\leq \frac{1}{C} \left(2M + \frac{1}{1-p} + 1\right) Mj\xi_j \\ &= \frac{1}{C} \left(2M + \frac{1}{1-p} + 1\right) Mj\zeta_j. \end{aligned}$$

Thus $\zeta_a = O(\zeta)$, i.e., ζ is regular and hence the principal ideal (ζ) is am-stable. Since $\eta_{n_k} \geq k^p \xi_{n_k} = k^p \zeta_{n_k}$ it follows that $\eta \neq O(\zeta)$. Set $m_k = \lceil \frac{n_k}{kC^{1/p}} \rceil$, the smallest integer majorizing $\frac{n_k}{kC^{1/p}}$. Since $\frac{m_k}{n_k} < \frac{1}{kC^{1/p}} + \frac{1}{n_k} < 1$ because $k > k_o$, and using the inequality $\xi_{m_k} \leq \frac{1}{C} \left(\frac{n_k}{m_k}\right)^{p_o} \xi_{n_k} < \frac{1}{C} \left(\frac{n_k}{m_k}\right)^p \xi_{n_k} \leq k^p \xi_{n_k}$, it follows that $q_k < m_k < n_k$. By the same inequalities,

$$\zeta_{m_k} = \frac{1}{C} \left(\frac{n_k}{m_k}\right)^p \xi_{n_k} = \left(\frac{n_k}{m_k}\right)^{p-p_o} \frac{1}{C} \left(\frac{n_k}{m_k}\right)^{p_o} \xi_{n_k} > \left(\frac{1}{kC^{1/p}} + \frac{1}{n_k}\right)^{p_o-p} \xi_{m_k}.$$

and hence $\zeta \neq O(\xi)$. As ζ is regular and hence satisfies the $\Delta_{1/2}$ -condition, it follows that $(\xi) \subsetneq (\zeta) \subsetneq (\eta)$, which concludes the proof. \square

Now we consider am- ∞ stable principal ideals, i.e., ideals with ∞ -regular generating sequences. The main tool for proving the upper and lower density of $S_\infty PL$ in \mathcal{PL} is the Potter-type inequality for ∞ -regular sequences [15, Theorem 4.12] (see proof of Theorem 3.13). Similar to the am-case, but much less trivial, a sequence ξ is ∞ -regular, i.e., $(\xi)_{a_\infty} = (\xi)$ if and only if $\xi_{a_\infty} = O(\xi)$ (see [ibid.]). The main technical complication with respect to the arithmetic mean case is that ∞ -regular sequences in general do not satisfy the $\Delta_{1/2}$ -condition (e.g., see [15, Example 4.5(ii)]). Thus D_m considerations are unavoidable, i.e., to prove that the inclusion of two principal ideals $(\eta) \subset (\xi)$ is proper, it is necessary to show that $\xi \neq O(D_m \eta)$ for all positive integers m . We first need the following lemmas.

Lemma 3.11. *If $(\xi) \supseteq F$, then $(\xi) \supseteq (\eta) \supseteq F$ for some am- ∞ stable principal ideal (η) .*

PROOF. Define $\eta := \langle 2^{-j} \prod_j^{j^2} \xi_i \rangle$ and assume without loss of generality that $\xi_1 < 1$. Clearly, the sequences $\langle \prod_j^{j^2} \xi_i \rangle$ and hence η are monotone decreasing, $\eta \leq \xi$ and $\eta_i > 0$ for all i . Furthermore, for every m and every $n \geq m$,

$$(D_m \eta)_{mn} = \eta_n = 2^{-n} \prod_n^{n^2} \xi_i < 2^{-n} \xi_{n^2} \leq 2^{-n} \xi_{mn},$$

hence $\xi \neq O(D_m \eta)$. Thus $(\xi) \supsetneq (\eta) \supsetneq F$. Moreover,

$$(\eta_{a_\infty})_n = \frac{1}{n} \sum_{n+1}^{\infty} (2^{-j} \prod_j \xi_i) \leq \frac{1}{n} \left(\prod_n \xi_i \right) \sum_{n+1}^{\infty} 2^{-j} = \frac{1}{n} \eta_n,$$

hence η is ∞ -regular by [15, Theorem 4.12] and therefore (η) is am- ∞ stable. \square

Lemma 3.12. *If $\{0\} \neq (\xi) \subsetneq (\eta)$ and ξ is summable, then $(\xi) \subsetneq (\zeta) \subsetneq (\eta)$ for some summable ζ .*

PROOF. By Corollary 3.7(ii), it is enough to find a summable ζ for which $(\xi) \subsetneq (\zeta) \subsetneq (\eta)$. The case when ξ is finitely supported, i.e., $(\xi) = F$ is trivial: it is enough to construct a summable but not finitely supported sequence ζ with $(\zeta) \subsetneq (\eta)$, e.g., $\langle \frac{\eta_n}{2^n} \rangle$. Thus assume $\xi_i > 0$ for all i , and without loss of generality, assume that $\xi \leq \eta$. Construct inductively two increasing sequences of indices $n_{k-1} < q_k < n_k$ starting with $n_1 = 1$ and satisfying the following three conditions: $\eta_{n_k} \geq k \xi_{\lceil \frac{n_k}{k} \rceil}$, $\xi_{\lceil \frac{n_k}{k} \rceil} < \frac{1}{k} \xi_{n_{k-1}}$, and $n_k \xi_{\lceil \frac{n_k}{k} \rceil} \leq k^{-3}$. Indeed, since $\eta \neq O(D_k \xi)$ for all k , there are infinitely many indices i such that $\eta_i \geq k \xi_{\lceil \frac{i}{k} \rceil}$, and among those indices, we can choose n_k that satisfies the second condition since $\xi_i \rightarrow 0$ and the third since $i \xi_i \rightarrow 0$ by the summability of ξ . Set q_k to be the largest index i such that $\xi_i \geq k \xi_{\lceil \frac{n_k}{k} \rceil}$. Then

$$n_{k-1} \leq q_k < \lceil \frac{n_k}{k} \rceil < n_k.$$

Next, for a sequence $\zeta \in c_0^*$ with $\xi \leq \zeta \leq \eta$ define:

$$\zeta_i := \begin{cases} \xi_i & \text{on } (n_{k-1}, q_k) \\ k \xi_{\lceil \frac{n_k}{k} \rceil} & \text{on } [q_k, n_k] \end{cases}$$

Then $\zeta \neq O(D_m \xi)$ for all m since

$$\zeta_{n_k} = k \xi_{\lceil \frac{n_k}{k} \rceil} = k(D_k \xi)_{n_k} \geq k(D_m \xi)_{n_k}$$

for all $k \geq m$. Thus $\xi \subsetneq \zeta \subsetneq \eta$. Finally, ζ is summable because

$$\sum_{i=1}^{\infty} \zeta_i < \sum_{i=1}^{\infty} \xi_i + \sum_{k=1}^{\infty} \sum_{i=q_k}^{n_k} k \xi_{\lceil \frac{n_k}{k} \rceil} < \sum_{i=1}^{\infty} \xi_i + \sum_{k=1}^{\infty} k n_k \xi_{\lceil \frac{n_k}{k} \rceil} \leq \sum_{i=1}^{\infty} \xi_i + \sum_{k=1}^{\infty} k^{-2} < \infty.$$

\square

In the terminology of this paper, the above lemma states that the lattice of principal ideals contained in \mathcal{L}_1 (i.e., with trace class generators) is upper dense in $\mathcal{P}\mathcal{L}$. By the lack of gaps in $\mathcal{P}\mathcal{L}$ (Corollary 3.7(ii)), the lattice of principal ideals contained in \mathcal{L}_1 is also lower dense in $\mathcal{P}\mathcal{L}$.

Theorem 3.13. $\mathcal{S}_\infty\mathcal{PL}$ is upper and lower dense in \mathcal{PL} .

PROOF. Let (ξ) be an am- ∞ stable principal ideal, i.e., ξ is an ∞ -regular sequence, in particular, ξ is summable. By [15, Theorem 4.12], $\xi_{a_\infty} \leq M\xi$ for some $M > 0$, and by the same theorem, $\xi_n \leq C(\frac{m}{n})^{p_o} \xi_m$ for some $C \geq 1$, $p_o > 1$ and all $n \geq m$. Choose any $1 < p < p_o$.

First we prove the lower density of $\mathcal{S}_\infty\mathcal{PL}$ in \mathcal{PL} . Let $(\eta) \subsetneq (\xi)$. Then $\eta \leq KD_m\xi$ for some positive integer m and some $K > 0$ and since $(KD_m\xi) = (\xi)$, without loss of generality we can assume that $\eta \leq \xi$. The case where η is finitely supported, i.e. $(\eta) = F$ follows from Lemma 3.11 above, so assume that $\eta_i > 0$ for all i . We construct inductively two increasing sequences of indices $q_{k-1} < m_k < q_k$ as follows. Choose an integer $k_o \geq M + 1$, set $q_{k_o} = 1$ and assume the construction up to $k - 1 \geq k_o$. Using the fact that for all $m \in \mathbb{N}$, $\xi \neq O(D_m\eta)$, choose $n_k \geq 2kq_{k-1}$ such that $\xi_{n_k} \geq k^R(D_k\eta)_{n_k} = k^R\eta_{m_k}$ where $R := \frac{2pp_o}{p_o-p}$ and $m_k := \lceil \frac{n_k}{k} \rceil$ is the smallest integer majorizing $\frac{n_k}{k}$; thus $m_k \geq 2q_{k-1}$. Then define q_k to be the largest integer i for which $\xi_i \geq \eta_{m_k}$; thus $m_k < n_k \leq q_k$. Next define the sequence

$$\zeta_i := \begin{cases} \xi_i & \text{on } (q_{k-1}, m_k) \\ \min(\xi_i, (\frac{q_k}{i})^p \eta_{m_k}) & \text{on } [m_k, q_k]. \end{cases}$$

Since $\zeta_{m_{k-1}} = \xi_{m_{k-1}} \geq \xi_{m_k} \geq \zeta_{m_k}$ and $\zeta_{q_k} = \eta_{m_k} > \xi_{q_{k+1}} = \zeta_{q_{k+1}}$, it follows that ζ is monotone non-increasing. Clearly, $\zeta_i \leq \xi_i$ for all i , $\zeta_i \geq \eta_i$ for $i \in (q_{k-1}, m_k)$, and $\zeta_i \geq \zeta_{q_k} = \eta_{m_k} \geq \eta_i$ for $i \in [m_k, q_k]$. Thus $\eta \leq \zeta \leq \xi$ and hence $(\eta) \subset (\zeta) \subset (\xi)$.

Now we prove that ζ is am- ∞ regular by showing that $\zeta_{a_\infty} = O(\zeta)$ (see [15, Theorem 4.12]). If $j \in [m_k, q_k]$, then the Potter-type inequality implies that $\xi_j \geq \frac{1}{C}(\frac{q_k}{j})^{p_o} \xi_{q_k} \geq \frac{1}{C}(\frac{q_k}{j})^p \eta_{m_k}$, hence $\zeta_j \geq \frac{1}{C}(\frac{q_k}{j})^p \eta_{m_k}$ and thus we have $\zeta_j \leq (\frac{q_k}{j})^p \eta_{m_k} \leq C\zeta_j$. Thus, using the fact that $\xi_{q_{k+1}} < \eta_{m_k}$ and $q_k \geq k_o \geq M+1$, by definition of q_k and k_o ,

$$\begin{aligned} j(\zeta_{a_\infty})_j &\leq \sum_{j+1}^{q_k} \zeta_i + \sum_{q_{k+1}}^{\infty} \xi_i \leq \sum_{j+1}^{q_k} (\frac{q_k}{i})^p \eta_{m_k} + \xi_{q_{k+1}} + (q_k + 1)(\xi_{a_\infty})_{q_{k+1}} \\ &\leq \frac{j}{p-1} (\frac{q_k}{j})^p \eta_{m_k} + \xi_{q_{k+1}} + M(q_k + 1)\xi_{q_{k+1}} \\ &\leq (\frac{j}{p-1} (\frac{q_k}{j})^p + (M+1)q_k)\eta_{m_k} \\ &\leq (\frac{1}{p-1} + M+1)j(\frac{q_k}{j})^p \eta_{m_k} \leq C(\frac{1}{p-1} + M+1)j\zeta_j. \end{aligned}$$

If $j \in (q_{k-1}, m_k)$, then $(\zeta_{a_\infty})_j \leq (\xi_{a_\infty})_j \leq M\xi_j = M\zeta_j$. Thus $\zeta_{a_\infty} = O(\zeta)$, which proves the am- ∞ regularity of ζ . It remains to prove that both inclusions in $(\eta) \subset (\zeta) \subset (\xi)$ are proper. By the Potter-type condition, $\frac{q_k}{n_k} \leq (C \frac{\xi_{n_k}}{\xi_{q_k}})^{\frac{1}{p_o}}$ and hence

$$\begin{aligned} \zeta_{m_k} &\leq \left(\frac{q_k}{m_k}\right)^p \eta_{m_k} \leq k^p \left(\frac{q_k}{n_k}\right)^p \eta_{m_k} \leq C^{\frac{p}{p_o}} k^p \left(\frac{\xi_{n_k}}{\xi_{q_k}}\right)^{\frac{p}{p_o}} \eta_{m_k} \leq C^{\frac{p}{p_o}} k^p \left(\frac{\xi_{n_k}}{\eta_{m_k}}\right)^{\frac{p}{p_o}} \eta_{m_k} \\ &= C^{\frac{p}{p_o}} k^p \left(\frac{\eta_{m_k}}{\xi_{n_k}}\right)^{(1-\frac{p}{p_o})} \xi_{n_k} \leq C^{\frac{p}{p_o}} k^{p-R(1-\frac{p}{p_o})} \xi_{n_k} = C^{\frac{p}{p_o}} k^{-p} \xi_{n_k}. \end{aligned}$$

Thus $\xi_{n_k} \geq C^{-\frac{p}{p_o}} k^p (D_k \zeta)_{n_k}$ for all $k \geq k_o$, and hence $\xi \neq O(D_m \zeta)$ for any m , i.e., $(\xi) \not\subset (\zeta)$. Finally, recalling that $(\frac{q_k}{j})^p \eta_{m_k} \leq C\zeta_j$ for all $j \in [m_k, q_k]$ and that $n_k \geq \frac{k}{2} m_k$ and hence $\frac{k^2}{2} m_{k^2} \leq n_{k^2} \leq q_{k^2}$ we have

$$\zeta_{km_{k^2}} \geq \frac{1}{C} \left(\frac{q_{k^2}}{km_{k^2}}\right)^p \eta_{m_{k^2}} \geq \frac{1}{C} \left(\frac{n_{k^2}}{km_{k^2}}\right)^p \eta_{m_{k^2}} \geq \frac{1}{C} \left(\frac{k}{2}\right)^p \eta_{m_{k^2}}$$

and hence $\zeta \neq O(D_m \eta)$ for any m . Thus $(\zeta) \not\subset (\eta)$, completing the proof of lower density.

Now we prove the upper density of $S_\infty \mathcal{P}\mathcal{L}$ in $\mathcal{P}\mathcal{L}$. Let $(\eta) \supseteq (\xi)$ and, as in the first part of the proof, assume without loss of generality that $\eta \geq \xi$ and additionally that $\eta_1 = \xi_1$. By Lemma 3.11, assume that $\xi_i > 0$ for all i and by Lemma 3.12 assume also that η is summable.

We construct inductively two increasing sequences of indices $n_{k-1} < q_k < n_k$ as follows. Choose $n_1 = 1$ and assume the construction up to $k - 1$. For all positive integers n define $q(n) := \max\{i \in \mathbb{N} \mid \xi_i \geq \eta_n\}$; then clearly $q(n) \uparrow \infty$. Using the fact that $\eta \neq O(D_k \xi)$ and that η is summable, choose n_k so that $\eta_{n_k} \geq k^R (D_k \xi)_{n_k}$ and $\sum_{q(n_k)}^\infty \eta_i \leq \sum_{n_{k-1}}^\infty \xi_i$ where $R := \frac{2pp_o}{p_o-p}$. Set $q_k := q(n_k)$ and $m_k := \lceil \frac{n_k}{k} \rceil$, the smallest integer majorizing $\frac{n_k}{k}$. Because of the inequalities $\xi_{q_k} \geq \eta_{n_k} \geq k^R \xi_{m_k} > \xi_{m_k}$ and the summation condition, it follows that $n_{k-1} < q_k < m_k < n_k$. Next, define the sequence

$$\zeta_i := \begin{cases} \xi_i & \text{on } (n_{k-1}, q_k] \\ \max(\xi_i, (\frac{q_k}{i})^p \eta_{m_k}) & \text{on } [q_k, n_k] \end{cases}$$

Since $\zeta_{n_k} \geq \xi_{n_k} \geq \xi_{n_{k+1}} = \zeta_{n_{k+1}}$ we see that ζ is monotone nonincreasing. By definition, $\xi_i \leq \zeta_i$ for all i , $\zeta_i \leq \eta_i$ for all $i \in (n_{k-1}, q_k]$, and $\zeta_i \leq \eta_{m_k} \leq \eta_i$ for all $i \in [q_k, n_k]$. Thus $\xi \leq \zeta \leq \eta$ and hence $(\xi) \subset (\zeta) \subset (\eta)$. Now we prove that ζ is

am- ∞ -regular. If $j \in [q_k, n_k]$, then

$$\begin{aligned} j(\zeta_{a_\infty})_j &= \sum_{j+1}^{\infty} \zeta_i \leq \sum_{j+1}^{n_k} \xi_i + \sum_{j+1}^{n_k} \left(\frac{q_k}{i}\right)^p \eta_{n_k} + \sum_{n_k+1}^{q_{k+1}} \xi_i + \sum_{q_{k+1}}^{\infty} \eta_i \\ &\leq \sum_{j+1}^{q_{k+1}} \xi_i + \sum_{j+1}^{n_k} \left(\frac{q_k}{i}\right)^p \eta_{n_k} + \sum_{n_k}^{\infty} \xi_i \leq 2 \sum_{j+1}^{\infty} \xi_i + \frac{j}{p-1} \left(\frac{q_k}{j}\right)^p \eta_{n_k} \\ &\leq 2Mj\xi_j + \frac{j}{p-1} \left(\frac{q_k}{j}\right)^p \eta_{n_k} \leq j(2M + \frac{1}{p-1})\zeta_j. \end{aligned}$$

If $j \in (n_{k-1}, q_k)$, then by applying the inequality just obtained and the Potter-type inequality,

$$\begin{aligned} j(\zeta_{a_\infty})_j &= \sum_{j+1}^{q_k} \xi_i + \sum_{q_{k+1}}^{\infty} \zeta_i \\ &\leq \sum_{j+1}^{\infty} \xi_i + (2M + \frac{1}{p-1})q_k \xi_{q_k} \leq Mj\xi_j + (2M + \frac{1}{p-1})Cq_k \left(\frac{j}{q_k}\right)^p \xi_j \\ &\leq Mj\xi_j + (2M + \frac{1}{p-1})Cj\xi_j \leq C(3M + \frac{1}{p-1})j\zeta_j. \end{aligned}$$

Thus $\zeta_{a_\infty} = O(\zeta)$ and hence ζ is am- ∞ regular. Now we prove that the ideal inclusions are proper. By the Potter-type inequality $\frac{q_{k+1}}{m_k} \geq \left(\frac{1}{C} \frac{\xi_{m_k}}{\xi_{q_{k+1}}}\right)^{\frac{1}{p_o}}$, therefore

$$\begin{aligned} \zeta_{n_k} &\geq \left(\frac{q_k}{n_k}\right)^p \eta_{n_k} \geq (2k)^{-p} \left(\frac{q_k+1}{m_k}\right)^p \eta_{m_k} \geq (2k)^{-p} \left(\frac{1}{C} \frac{\xi_{m_k}}{\xi_{q_{k+1}}}\right)^{\frac{p}{p_o}} \eta_{n_k} \\ &\geq (2k)^{-p} C^{-\frac{p}{p_o}} \left(\frac{\xi_{m_k}}{\eta_{m_k}}\right)^{\frac{p}{p_o}} \eta_{m_k} = (2k)^{-p} C^{-\frac{p}{p_o}} \left(\frac{\eta_{n_k}}{\xi_{m_k}}\right)^{1-\frac{p}{p_o}} \xi_{m_k} \\ &\geq 2^{-p} C^{-\frac{p}{p_o}} k^{-p+R(1-\frac{p}{p_o})} \xi_{m_k} = 2^{-p} C^{-\frac{p}{p_o}} k^p \xi_{m_k}. \end{aligned}$$

Thus $\zeta \neq O(D_m \xi)$ for all m and hence $(\xi) \neq (\zeta)$. Finally, we see that $q_{k^2} < m_{k^2} < km_{k^2} < k^2 m_{k^2} \leq n_{k^2}$ and hence, again by the Potter-type condition,

$$\begin{aligned} \zeta_{km_{k^2}} &= \max \left(\xi_{km_{k^2}}, \left(\frac{q_{k^2}}{km_{k^2}}\right)^p \eta_{n_{k^2}} \right) \\ &\leq \max \left(C \left(\frac{q_{k^2}+1}{km_{k^2}}\right)^{p_o} \xi_{q_{k^2}+1}, k^{-p} \eta_{n_{k^2}} \right) \\ &\leq Ck^{-p} \eta_{n_{k^2}}. \end{aligned}$$

Thus $\eta \neq O(D_m \zeta)$ for all m and so $(\eta) \neq (\zeta)$, concluding the proof. \square

Since $\mathcal{P}\mathcal{L}$ is upper but not lower dense in \mathcal{L} by Corollaries 3.7 and 3.4, immediate consequences of Theorems 3.9, 3.10, and 3.13 are:

Corollary 3.14.

- (i) $\mathcal{P}\mathcal{L}$, $\Delta_{1/2}\mathcal{P}\mathcal{L}$, $\mathcal{SP}\mathcal{L}$, and $\mathcal{S}_\infty\mathcal{P}\mathcal{L}$ are upper dense in \mathcal{L} but they all have lower gaps in \mathcal{L} .
- (ii) $\mathcal{P}\mathcal{L}$, $\Delta_{1/2}\mathcal{P}\mathcal{L}$, $\mathcal{SP}\mathcal{L}$, and $\mathcal{S}_\infty\mathcal{P}\mathcal{L}$ have no gaps.

Other consequences of the same theorems and the Potter-type conditions already mentioned are:

Corollary 3.15.

- (i) Every principal ideal is contained in a principal ideal from $\mathcal{SP}\mathcal{L}$ and if it is strictly larger than F it contains a principal ideal in $\mathcal{S}_\infty\mathcal{P}\mathcal{L}$ that is strictly larger than F .
- (ii) A principal ideal (η) contains a principal ideal in $\Delta_{1/2}\mathcal{P}\mathcal{L}$ if and only if for some $0 < p < 1$, $\omega^p = O(\eta)$; it contains a principal ideal in $\mathcal{SP}\mathcal{L}$ if and only if $\omega^p = O(\eta)$ for some $0 < p < 1$; and it is contained in a principal ideal in $\mathcal{S}_\infty\mathcal{P}\mathcal{L}$ if and only if $\eta = O(\omega^p)$ for some $p > 1$.

PROOF.

- (i) Assume without loss of generality that the principal ideal I has generator η with $\eta_1 = 1$ and hence $\|\eta_{a^m}\| = 1$ for the c_o -norm for all $m \in \mathbb{N}$. Thus the series $\sum_{m=1}^\infty 2^{-m}\eta_{a^m} := \zeta$ converges in the c_o -norm, hence weakly, and thus $\zeta \in c_o^*$ and $(\eta) \subset (\zeta)$. By definition,

$$(\zeta_a)_n = \frac{1}{n} \sum_{j=1}^n \sum_{m=1}^\infty 2^{-m}(\eta_{a^m})_j = \sum_{m=1}^\infty 2^{-m} \frac{1}{n} \sum_{j=1}^n (\eta_{a^m})_j = \sum_{m=1}^\infty 2^{-m} (\eta_{a^{m+1}})_n \leq 2\zeta_n$$

and hence ζ is regular.

When $I \neq F$, by Lemma 3.11 there is an ideal $J \in \mathcal{S}_\infty\mathcal{P}\mathcal{L}$ with $F \subsetneq J \subsetneq I$.

- (ii) It is immediate to verify that the sequence ω^p satisfies the $\Delta_{1/2}$ -condition for every $p > 0$, is regular if and only if $0 < p < 1$ and is ∞ -regular if and only if $p > 1$. This establishes the sufficiency in (ii). The necessity is a direct consequence of the Potter-type conditions (in all cases taking $m = 1$) that we have already quoted in this section. □

By the lack of gaps in $\mathcal{P}\mathcal{L}$ and by Theorems 3.9, 3.10 and 3.13, the ideals in (i) and (ii) are of course never unique.

If L is an am-stable ideal, I is principal, and $I \subset L$, then while Corollary 3.15(i) ensures that there are principal am-stable ideals $J \supset I$, it may be

impossible to find any such J also contained in L , as the following example shows. The same holds in the $\text{am-}\infty$ case.

Recall from Section 2 that the am (resp. $\text{am-}\infty$) stabilizer $st^a(I)$ (resp. $st_{a_\infty}(I)$) is the smallest am -stable ideal containing I (resp. the largest $\text{am-}\infty$ stable ideal contained in I).

Example 3.16.

- (i) Let ξ be an irregular sequence. Then $L = st^a((\xi))$ is am -stable but there is no intermediate am -stable principal ideal $(\xi) \subset J \subset L$.
- (ii) $st_{a_\infty}((\omega))$ is $\text{am-}\infty$ stable but it contains principal ideals that are not contained in any $\text{am-}\infty$ stable principal ideal.

PROOF.

- (i) Assume by contradiction that there is an am -stable principal ideal $J = (\eta)$ with $(\xi) \subset J \subset L$. Since $J \supset (\xi)$ and J is am -stable, then $J \supset L$, hence $J = L = \bigcup_n (\xi_{a^n})$. Then $\eta \in \Sigma((\xi_{a^n}))$ for some $n \in \mathbb{N}$. On the other hand, $(\xi_{a^n}) \subset (\eta)$, hence $\xi_{a^n} \asymp \eta$ is regular, and thus ξ too is regular (see Theorem 5.1 and the paragraph preceding it), against the hypothesis.
- (ii) Choose an increasing sequence of indices n_k so that $\sum_{n_k}^{\infty} \frac{\log^k n}{n^{1+\frac{1}{k}}} \leq \frac{1}{k^2}$ and define $\xi_n := \frac{1}{n^{1+\frac{1}{k}}}$ for $n \in [n_k, n_{k+1})$. Then for every $p \in \mathbb{N}$,

$$\sum_{n=n_p}^{\infty} \xi_n \log^p n \leq \sum_{k=p}^{\infty} \sum_{n=n_k}^{n_{k+1}-1} \xi_n \log^k n \leq \sum_{k=p}^{\infty} \frac{1}{k^2} < \infty$$

and thus $\sum_{n=1}^{\infty} \xi_n \log^p n < \infty$. By [15, Proposition 4.18(ii)] $(\xi) \subset st_{a_\infty}((\omega))$. Furthermore, $\xi \neq O(\omega^p)$ for every $p > 1$, so by Corollary 3.15(ii), there is no $J \in \mathcal{S}_\infty \mathcal{P}\mathcal{L}$ contained in I . □

To complete our analysis of the lattices $\Delta_{1/2} \mathcal{P}\mathcal{L}$, $\mathcal{S}\mathcal{P}\mathcal{L}$, and $\mathcal{S}_\infty \mathcal{P}\mathcal{L}$ we consider also their complements in $\mathcal{P}\mathcal{L}$, i.e., the principal ideals that do not satisfy the $\Delta_{1/2}$ condition, (resp. are not am -stable, or are not $\text{am-}\infty$ stable.)

Proposition 3.17. *Every nested pair of nonzero principal ideals has strictly between them an ideal from $\mathcal{P}\mathcal{L} \setminus \Delta_{1/2} \mathcal{P}\mathcal{L}$, $\mathcal{P}\mathcal{L} \setminus \mathcal{S}\mathcal{P}\mathcal{L}$, and $\mathcal{P}\mathcal{L} \setminus \mathcal{S}_\infty \mathcal{P}\mathcal{L}$.*

PROOF. Let $I \subsetneq J$ be principal ideals. By the lack of gaps in $\mathcal{P}\mathcal{L}$, we can find principal ideals I' and J' such that $I \subsetneq I' \subsetneq J' \subsetneq J$, so for all three cases it is enough to find principal ideals $I \subset L \subset J$ outside the three special lattices considered without having to prove that they do not coincide with I or J . Notice

also that since the arithmetic mean of a c_o^* sequence always satisfies the $\Delta_{1/2}$ -condition, $\mathcal{SP}\mathcal{L} \subset \Delta_{1/2}\mathcal{P}\mathcal{L}$, and hence the second case follows from the first. Let $I = (\xi)$ and $J = (\eta)$ and by passing if necessary to other generators, assume without loss of generality that $\xi \leq \eta$ and $\xi_1 = \eta_1$.

First we prove that there is an $I \subset L \subset J$ with $L \in \mathcal{P}\mathcal{L} \setminus \Delta_{1/2}\mathcal{P}\mathcal{L}$. Since $\eta \neq O(\xi)$, we can find an increasing sequence of positive integers n_k for which $\eta_{n_k} \geq k\xi_{n_k}$ and $\eta_{n_k} \leq \xi_{n_{k-1}}$. Define $\zeta_j := \min(\xi_{n_k}, \eta_j)$ for $j \in (n_k, n_{k+1}]$. Then $\zeta \in c_o^*$, $\xi \leq \zeta \leq \eta$, and one has $\zeta_{n_k} = \eta_{n_k} \geq k\xi_{n_k} \geq k\zeta_{n_{k+1}} \geq k\zeta_{2n_k}$. Thus $L = (\zeta) \notin \Delta_{1/2}\mathcal{P}\mathcal{L}$ and $I \subset L \subset J$.

Now we prove that there is an $I \subset L \subset J$ with $L \in \mathcal{P}\mathcal{L} \setminus \mathcal{S}_\infty\mathcal{P}\mathcal{L}$. Since $\eta \neq O(D_m\xi)$ for any $m \in \mathbb{N}$, there is an increasing sequence of positive integers n_k , starting with $n_1 = 1$, such that $\eta_{n_k} \geq k^2(D_k\xi)_{n_k} = k^2\xi_{\lceil \frac{n_k}{k} \rceil}$, $\eta_{n_k} \leq \frac{1}{k}\eta_{n_{k-1}}$ and $n_k > kn_{k-1}$. Define $\zeta_i := \min(\frac{1}{k}\eta_{n_k}, \eta_i)$ for $i \in [\lceil \frac{n_k}{k} \rceil, \lceil \frac{n_{k+1}}{k+1} \rceil)$. Then $\zeta \in c_o^*$, $\zeta \leq \eta$, and since for all $i \in [\lceil \frac{n_k}{k} \rceil, \lceil \frac{n_{k+1}}{k+1} \rceil)$, $\frac{1}{k}\eta_{n_k} \geq k\xi_{\lceil \frac{n_k}{k} \rceil} \geq \xi_i$ and $\eta_i \geq \xi_i$, it follows that $\zeta \geq \xi$ and hence $I \subset L \subset J$. Let $q_k := \max\{i \in \mathbb{N} \mid \eta_i \geq \frac{1}{k}\eta_{n_k}\}$. Then $q_k \geq n_k > k(\lceil \frac{n_k}{k} \rceil - 1)$ and

$$(\zeta_{a_\infty})_{\lceil \frac{n_k}{k} \rceil} \geq \frac{1}{\lceil \frac{n_k}{k} \rceil} \sum_{\lceil \frac{n_k}{k} \rceil+1}^{q_k} \frac{1}{k}\eta_{n_k} = \frac{q_k - \lceil \frac{n_k}{k} \rceil}{\lceil \frac{n_k}{k} \rceil} \zeta_{\lceil \frac{n_k}{k} \rceil} \geq (k-2)\zeta_{\lceil \frac{n_k}{k} \rceil}$$

Thus $\zeta_{a_\infty} \neq O(\zeta)$ and hence $L = (\zeta) \notin \mathcal{S}_\infty\mathcal{P}\mathcal{L}$. □

From this we see that above and below every nonzero principal ideal (above only for F) lies a “bad” ideal. Indeed, by taking $I = F$ in Proposition 3.17, we see that every principal ideal $J \neq F$ contains (properly) principal ideals distinct from F not in the lattices $\Delta_{1/2}\mathcal{P}\mathcal{L}$, $\mathcal{SP}\mathcal{L}$, and $\mathcal{S}_\infty\mathcal{P}\mathcal{L}$. Similarly, every principal ideal is properly contained in another principal ideal, and hence in principal ideals not in the lattices $\Delta_{1/2}\mathcal{P}\mathcal{L}$, $\mathcal{SP}\mathcal{L}$, and $\mathcal{S}_\infty\mathcal{P}\mathcal{L}$.

4. UNIONS OF PRINCIPAL IDEALS

Basic questions on intersections and unions of ideals from a certain lattice play a natural role in the subject. In [25] Salinas investigated intersections and unions of ideals related to various classes of mainly Banach ideals. In this section, first we use Corollary 3.15 and density Theorems 3.9, 3.10 and 3.13 to determine intersections and unions of various classes of principal ideals. Then we investigate questions on representing ideals as unions of chains of principal or countably generated ideals.

An immediate consequence of Corollary 3.15(ii) is:

Corollary 4.1.

$$\begin{aligned}
 (i) \quad \bigcap_{p>0} (\omega^p) &= \bigcap_{L \in \Delta_{1/2} \mathcal{PL}} L; \\
 (ii) \quad \bigcap_{0 < p < 1} (\omega^p) &= \bigcap_{L \in \mathcal{SP}\mathcal{L}} L; \\
 (iii) \quad \bigcup_{p>1} (\omega^p) &= \bigcup_{L \in \mathcal{S}_\infty \mathcal{PL}} L.
 \end{aligned}$$

Remark 4.2.

(i) By Corollaries 3.8(ii) and 3.15(i) and Theorems 3.9, 3.10, and 3.13, every ideal in the lattices \mathcal{PL} , $\Delta_{1/2}\mathcal{PL}$, $\mathcal{SP}\mathcal{L}$, and $\mathcal{S}_\infty\mathcal{PL}$ is the intersection of all ideals in the lattice that properly contain it. (In contrast, a principal ideal is never the union of any chain of ideals properly contained in it.) As a consequence, every ideal in one of these lattices is the intersection of an infinite maximal chain of ideals that properly contain it and belong to the same lattice. By [16, Propositions 4.6 and 5.3], the chain must be uncountable because principal ideals are *ssc*, while intersections of countable chains of principal ideals are never *ssc*. More generally, the latter clause shows that no maximal Banach ideal (which are the symmetric normed ideals \mathfrak{S}_ϕ for a symmetric norming function ϕ), which is *ssc* by [16, Proposition 4.7], can be the proper intersection of a countable chain of principal ideals. This provides an additional special case answer to Salinas' question in [25, Section 7: (α_1)] asking whether $\mathfrak{S}_\phi \neq \bigcap I_k$ if $\mathfrak{S}_\phi \subsetneq I_k \subsetneq I_{k-1}$ for all k (with I_k not necessarily principal).

(ii) By [16, Lemmas 2.1 and 2.2 and Section 3], arbitrary intersections or directed unions of *am-stable* ideals (resp. *am- ∞ stable* ideals) are *am-stable* (resp. *am- ∞ stable*).

In the case of Banach ideals or their powers we can obtain more. The proof used in Corollary 3.15(i) to construct a regular sequence ζ majorizing a given sequence η depends on the completeness of the c_0^* -norm; a similar proof can be employed for constructing directly a sequence ζ that majorizes ξ and satisfies the $\Delta_{1/2}$ -condition. Both constructions can be extended to Banach ideals and the latter even to powers of Banach ideals (i.e., *e*-complete ideals as by [10, Sections 2.9-2.14 and Theorem 3.6]).

Proposition 4.3.

(i) Let $I = J^p$ for some $p > 0$ and some Banach ideal J . Then

$$I = \bigcup \{L \in \Delta_{1/2}\mathcal{PL} \mid L \subset I\}.$$

(ii) If I is a Banach ideal, then I is am-stable if and only if

$$I = \bigcup \{L \in \mathcal{SP}\mathcal{L} \mid L \subset I\}.$$

(iii) If I is a Banach ideal, then I is am- ∞ stable if and only if

$$I = \bigcup \{L \in \mathcal{S}_\infty\mathcal{PL} \mid L \subset I\}.$$

PROOF.

(i) Clearly, $I = \bigcup \{L \in \mathcal{PL} \mid L \subset I\}$, thus proving (i) is equivalent to proving that for every $\xi \in \Sigma(I)$ there is a $\zeta \geq \xi$ in $\Sigma(I)$ that satisfies the $\Delta_{1/2}$ -condition or, equivalently, such that $\zeta^{\frac{1}{p}}$ satisfies the $\Delta_{1/2}$ -condition. Thus without loss of generality we can assume that $p = 1$ and, from [10, Section 4.5], that $\|\cdot\|$ is a complete symmetric norm on I . Denote still by $\|\cdot\|$ the induced complete cone norm on $\Sigma(I)$. Then define $\zeta := \sum_{m=1}^\infty \frac{1}{m^2 \|D_m \xi\|} D_m \xi$. The series converges in norm and hence $\zeta \in \Sigma(I)$. Then $\xi \leq \zeta$. By the inequality for symmetric norms, $\|D_2 \eta\| \leq 2\|\eta\|$, and since $D_2 D_m = D_{2m}$, ζ satisfies the $\Delta_{1/2}$ -condition because

$$D_2 \zeta = \sum_{m=1}^\infty \frac{1}{m^2 \|D_m \xi\|} D_2 D_m \xi = \sum_{m=1}^\infty 4 \frac{\|D_{2m} \xi\|}{\|D_m \xi\|} \frac{1}{(2m)^2 \|D_{2m} \xi\|} D_{2m} \xi \leq 8\zeta.$$

(ii) Since the finite sum of am-stable principal ideals is am-stable and the union $\bigcup \{L \in \mathcal{SP}\mathcal{L} \mid L \subset I\}$ is directed, the condition is sufficient by Remark 4.2(ii).

The proof of necessity requires showing that for every $\xi \in \Sigma(I)$, there is a regular $\zeta \geq \xi$ in $\Sigma(I)$. As in the proof of Corollary 3.15(i), define $\zeta := \sum_{m=1}^\infty \frac{1}{2^m \|\xi_{a^m}\|} \xi_{a^m}$. By the am-stability of I , $\xi_{a^m} \in \Sigma(I)$ for all m . Then by the completeness of the norm, $\zeta \in \Sigma(I)$. By [10, Lemma 2.11], the map $\Sigma(I) \ni \eta \rightarrow \eta_a \in \Sigma(I)$ is bounded, hence ζ is regular because

$$\begin{aligned} \zeta_a &= \sum_{m=1}^\infty \frac{1}{2^m \|\xi_{a^m}\|} \xi_{a^{m+1}} \\ &\leq 2 \sup_m \left(\frac{\|\xi_{a^{m+1}}\|}{\|\xi_{a^m}\|} \right) \sum_{m=1}^\infty \frac{1}{2^{m+1} \|\xi_{a^{m+1}}\|} \xi_{a^{m+1}} \\ &\leq 2 \sup_m \left(\frac{\|\xi_{a^{m+1}}\|}{\|\xi_{a^m}\|} \right) \zeta. \end{aligned}$$

(iii) Same proof as (ii). □

Notice: if an ideal I is “too small”, i.e., if $I \not\supseteq st^a((\omega))$, then it cannot contain any am-stable ideal and, in particular, any ideal in $\mathcal{SP}\mathcal{L}$. The same conclusion holds for $I = st^a((\omega))$ by Example 3.16(i). Similarly, if $I \not\supseteq \bigcup_{p>0} (\omega)^p$, it cannot contain an ideal in $\Delta_{1/2}\mathcal{P}\mathcal{L}$ (Corollary 3.15(ii)). Likewise, if an ideal I is “too large”, i.e., $I \not\subseteq \bigcup_{p>1} (\omega)^p$, then by Corollary 4.1(iii) it is not the union of ideals in $\mathcal{S}_\infty\mathcal{P}\mathcal{L}$. Example 3.16(ii) shows that $st_{a_\infty}((\omega))$ is “too large.”

While every ideal is the union of principal ideals, a different and natural question is: which ideals are the union of an increasing chain of principal ideals. Salinas [25] and others have studied this kind of question and nested unions play an underlying role throughout the study of ideals. A partial answer is given by the following proposition.

Proposition 4.4. *Assuming the continuum hypothesis, the following hold.*

- (i) *All ideals are unions of increasing chains of countably generated ideals.*
- (ii) *An ideal I is the union of an increasing chain of principal ideals if for every countably generated ideal $J \subset I$ there is a principal ideal L with $J \subset L \subset I$.*

PROOF. Since $\Sigma(I)$ has cardinality c , by the continuum hypothesis,

$$\Sigma(I) = \{\xi_\gamma \mid \gamma \in [1, \Omega)\},$$

i.e., it can be indexed by the interval of ordinal numbers $[1, \Omega)$ where Ω is the first uncountable ordinal. Denote by \prec the well-order relation on $[1, \Omega)$.

- (i) For each $\gamma \in [1, \Omega)$, let I_γ be the ideal generated by the countable collection $\{\xi_\alpha \mid \alpha \prec \gamma\}$. Thus $I = \bigcup_{\gamma \in [1, \Omega)} I_\gamma$ is a nested (albeit uncountable) union of countably generated ideals.
- (ii) By (i) we can represent $I = \bigcup_{\gamma \in [1, \Omega)} I_\gamma$ as a nested union of countably generated ideals and define by using transfinite induction a collection

$$\{\eta_\gamma \in \Sigma(I) \mid \gamma \in [1, \Omega)\}$$

so that $I_\gamma \subset (\eta_\gamma)$ and $(\eta_\gamma) \subset (\eta_{\gamma'})$ whenever $\gamma \prec \gamma' \in [1, \Omega)$. Indeed, assume for a given $\beta \in [1, \Omega)$ that $\{\eta_\alpha \mid \alpha \preceq \beta\}$ have been chosen satisfying these two conditions. It suffices to choose an $\eta_\beta \in \Sigma(I)$ so that the collection $\{\eta_\gamma \mid \gamma \in [1, \beta)\}$ too satisfies these two conditions. Since $[1, \beta)$ is countable, $J = \bigcup_{\gamma \in [1, \beta)} (\eta_\gamma)$ is countably generated and hence $J \subset (\eta')$ and $I_\beta \subset (\eta'')$ for some $\eta', \eta'' \in \Sigma(I)$. Set $\eta_\beta := \eta' + \eta'' \in \Sigma(I)$. Then for every $\gamma \prec \beta$ it follows that

$$I_\gamma \subset (\eta_\gamma) \subset J + I_\beta \subset (\eta_\beta).$$

Thus $\{(\eta_\gamma)\}$ forms an uncountable chain and $I = \bigcup_{\gamma \in [1, \Omega)} (\eta_\gamma)$. □

The following corollary shows that Banach ideals and more generally e -complete ideals, i.e., powers of Banach ideals (see [10, Section 4.6]), satisfy the condition in part (ii) of the above proposition.

Corollary 4.5. *If I is an e -complete ideal and J is a countably generated ideal with $J \subset I$, then there is a principal ideal L with $J \subset L \subset I$. Thus, assuming the continuum hypothesis, I is the union of an increasing chain of principal ideals.*

PROOF. Since powers of countably generated (resp. principal) ideals are countably generated (resp. principal), without loss of generality assume that I is a Banach ideal. Let $\{\eta^{(k)}\}$ be a sequence of generators of J . Then

$$\zeta := \sum_{k=1}^{\infty} \frac{1}{k^2 \|\eta^{(k)}\|} \eta^{(k)} \in \Sigma(I),$$

and so setting $L := (\zeta)$ one has $(\eta^{(k)}) \subset L$ for all k , hence $J \subset L \subset I$. (This is a proof that countably generated Banach ideals and hence countably generated e -complete ideals are principal. See also [10, Corollary 2.23].) Then by Proposition 4.4, I is the union of an increasing chain of principal ideals. \square

Notice that neither the e -completeness condition in the corollary nor the condition in part (ii) of the proposition are necessary. Indeed, for the former, any principal ideal that is not e -complete (i.e., whose generator does not satisfy the $\Delta_{1/2}$ -condition [10, Corollary 2.23]) still satisfies automatically the conclusion of the corollary. And for the latter, any countably generated ideal that is not principal fails to satisfy the condition in part (ii) and yet is obviously the union of a (countable) chain of principal ideals.

We also do not know whether or not the continuum hypothesis or something stronger than the usual axioms of set theory is required for Proposition 4.4.

Remark 4.6. *By combining Propositions 4.3 and 4.4, and assuming the continuum hypothesis, if I is a Banach ideal, then I is am-stable if and only if it is a union of a chain of am-stable principal ideals. And if I is a positive power of a Banach ideal, then it is the union of a chain of $\Delta_{1/2}\mathcal{PL}$ ideals.*

5. APPLICATIONS TO FIRST ORDER ARITHMETIC MEAN IDEALS

By the main result of [10], an ideal I is am-stable, i.e., $I = I_a$, if and only if $I = [I, B(H)]$ (the latter is the commutator space of the ideal, i.e., the span of the commutators of elements of I with bounded operators) if and only if I does not support any nonzero trace (unitarily invariant linear functional on I).

We can reformulate some of the results obtained in the previous section in terms of traces. By Corollary 3.15(i), every principal ideal I is contained in a principal ideal J supporting no nonzero traces; so no nonzero trace on I can extend to a trace on J . Conversely, by Corollary 3.15(ii), if a principal ideal I is sufficiently large to contain (ω^p) for some $0 < p < 1$, then $(\omega^p) = [(\omega^p), B(H)] \subset [I, B(H)]$ and hence every trace on I must necessarily vanish on (ω^p) . More generally, repeatedly employing Theorem 3.10 and Proposition 3.17, we see that every am-stable principal ideal is the first ideal of an increasing (or decreasing) countable chain of principal ideals where every odd numbered ideal is am-stable and hence has no nonzero trace, and every even numbered ideal supports infinitely many nonzero traces (its space of traces is infinite-dimensional) ([15, Theorem 7.5], [16, Proposition 4.6(i)]).

The am- ∞ case is similar, but with the difference that an ideal I is am- ∞ stable (i.e., $I = {}_{a\infty}I$) if and only if $I = F + [I, B(H)]$ if and only if it supports a unique nonzero trace (up to scalar multiples). In this case, $I \subset \mathcal{L}_1$ and this trace is of course the restriction to I of the usual trace Tr on the trace class \mathcal{L}_1 [15, Theorem 6.6]. Equivalently, by a Hamel basis argument, an ideal contained in \mathcal{L}_1 is not am- ∞ stable if and only if it supports a nonzero singular trace, i.e., a trace that vanishes on F . And by Theorem 3.13 and Proposition 3.17, every am- ∞ stable principal ideal is the first ideal of an increasing countable chain of principal ideals (or a decreasing countable chain in the case that the ideal is not F) where every odd numbered ideal is am- ∞ stable and hence supports no nonzero singular trace, and every even numbered ideal supports infinitely many singular traces (again see [15, Theorem 7.5] and [16, Proposition 4.6(i)]).

First order arithmetic mean ideals are investigated in [10] and [14]-[16] and the density theorems of Section 3 provide further information for the principal ideal cases. Recall that am-stability of an ideal I is equivalent to $I = {}_aI$. Thus if I is am-stable, so are I_a and ${}_aI$ and hence also the first order arithmetic mean ideals

$$I^o := ({}_aI)_a, \quad I^- := {}_a(I_a),$$

I^{oo} (the smallest am-open ideal containing I), since $I \subset I^{oo} \subset I_a$, and I_- (the largest am-closed ideal contained in I), since ${}_aI \subset I_- \subset I$. The same relations hold for the am- ∞ case with the exception that we only have $I \cap \mathcal{L}_1 \subset I^{-\infty} \subset I_{a\infty}$. While in general these first order arithmetic mean ideals (resp. am- ∞ ideals) can be am-stable without the ideal I being am-stable (e.g., see Examples 5.4 and 5.5), principal ideals are more “rigid.”

Theorem 5.1. *For I a nonzero principal ideal, the following are equivalent.*

- (i) I is am-stable.
- (ii) ${}_aI$ is nonzero and am-stable.
- (ii') I° is nonzero and am-stable.
- (ii'') I_- is nonzero and am-stable.
- (iii) I^{oo} is am-stable.
- (iv) I_a is am-stable.
- (iv') I^- is am-stable.

PROOF. The equivalences of (ii) and (ii'), and of (iv) and (iv'), the 5-chain inclusions ${}_aI \subset I^\circ \subset I \subset I^- \subset I_a$ and ${}_aI \subset I_- \subset I \subset I^{oo} \subset I_a$ and, consequently, the fact that (i) implies all the other conditions, all hold for general ideals. (See [16, Section 2] for details.) For principal ideals, the identity $I_- = {}_aI$ is proven in [ibid., Theorem 2.8] and hence (ii) is equivalent to (ii''). Thus it remains to prove that each of the conditions (ii'), (iii) and (iv) imply (i). By [ibid., Lemma 2.12] I°, I^{oo} and I_a are principal when I is principal. Assume that $\{0\} \neq I^\circ \subsetneq I$ and that I° is am-stable. Then by Theorem 3.10, there is an am-stable principal ideal J with $I^\circ \subsetneq J \subsetneq I$. Since $J = J^\circ$, by the monotonicity of the am-interior operation, $J \subset I^\circ$, a contradiction. Similarly, if $I \subsetneq I^{oo}$ and I^{oo} is am-stable (resp. $I \subsetneq I_a$ and I_a is am-stable), then there is an am-stable principal ideal $I \subsetneq J \subsetneq I^{oo}$ (resp. $I \subsetneq J \subsetneq I_a$) and J being am-open, $I^{oo} \subset J$ (resp. $I_a \subset J$), a contradiction. \square

Notice that for any ideal I that does not contain (ω) (resp. \mathcal{L}_1), ${}_aI = I^\circ = \{0\}$ (resp. $I_- = \{0\}$). Since $\{0\}$ is am-stable while in both cases I is not, the conditions in the last corollary that I_a, I° and I_- are non-zero is essential.

There are two main differences in the am- ∞ case. The first is that if $I \neq \{0\}$, then unlike the am-case, ${}_{a_\infty}I$ is never zero, and in lieu of the two chains of inclusions, we have ${}_{a_\infty}I \subset I^{o\infty} \subset I, {}_{a_\infty}I \subset I_{-\infty} \subset I, I \cap \mathcal{L}_1 \subset I^{-\infty} \subset I_{a_\infty}$, and $I \cap se(\omega) \subset I^{oo\infty} \subset I_{a_\infty}$. (See [15, Proposition 4.8(i)-(i')] and remarks following both [16, Corollary 3.8 and Proposition 3.14]). The second is that if I is principal, then $I^{o\infty}, I^{oo\infty}$ and I_{a_∞} are either principal or they coincide with $se(\omega)$ [15, Lemma 4.7], [16, Lemma 3.9, Lemma 3.16]. As $se(\omega)$ is not am-stable, if any of the ideals $I^{o\infty}, I^{oo\infty}$ or I_{a_∞} is am-stable, then it is principal as well. With these minor changes and the properties of arithmetic mean ideals at infinity developed in [16, Section 3], the proof of Theorem 5.1 carries over to the am- ∞ case.

Theorem 5.2. *Let $I \neq \{0\}$ be a principal ideal. Then am- ∞ stability of the following ideals is equivalent: I , ${}_a\infty I$, $I^{o\infty}$, $I_{-\infty}$, $I^{oo\infty}$, I_{a_∞} , and $I^{-\infty}$.*

The fact that a sequence $\xi \in c_o^*$ is regular if and only if ξ_a is regular has been proved in a number of different ways (see [10, Remark 3.11] for a discussion). The analogous result for ∞ -regularity was obtained in [15]. Both facts are also an immediate consequence of Theorems 5.1 and 5.2:

Corollary 5.3. *Let I be a principal ideal and let $n \in \mathbb{N}$.*

- (i) I_{a^n} is am-stable if and only if I is am-stable.
- (ii) $I_{a_\infty^n}$ is am- ∞ stable if and only if I is am- ∞ stable.

It may be of interest to note that Theorem 5.1 does not extend beyond principal ideals. In fact, as the two examples below show, it can fail even for countably generated ideals. It is an open question whether or not Theorem 5.2 extends beyond principal ideals, indeed, even whether or not am- ∞ stability of I_{a_∞} implies am- ∞ stability of I .

Example 5.4. *We exhibit a countably generated ideal N which is not am-stable but for which the ideals ${}_aN$, N^o , and N_- are all equal to $st^a((\omega))$ and hence are am-stable.*

PROOF. Define the sequence $\zeta_i = 2^{-k^3}$ for $i \in [2^{k^3}, 2^{(k+1)^3})$. Since

$$\zeta_{2^{(k+1)^3-1}} = 2^{-k^3} \neq O((\omega \log^k)_{2^{(k+1)^3-1}})$$

and since $\omega \log^k$ is increasing in k , $\zeta \neq O(\omega \log^p)$ for every $p \in \mathbb{N}$. As $\omega \log^p$ satisfies the $\Delta_{1/2}$ -condition, it follows that $\zeta \notin \Sigma((\omega \log^p))$ and hence $\zeta \notin \Sigma(st^a((\omega)))$. On the other hand, if for some $p \in \mathbb{N}$ and some $\rho \in c_o^*$, $\rho_a \leq \zeta + \omega \log^p$, then for all $i \in [2^{k^3}, 2^{(k+1)^3})$ and for k large enough,

$$\begin{aligned} i(\rho_a)_i &\leq 2^{(k+1)^3} (\rho_a)_{2^{(k+1)^3}} \leq 2^{(k+1)^3} \left(2^{-(k+1)^3} + \frac{\log^p(2^{(k+1)^3})}{2^{(k+1)^3}} \right) \\ &= 1 + \log^p(2^{(k+1)^3}) \leq 2 \log^p(2^{k^3}) \leq 2 \log^p i = 2i(\omega \log^p)_i. \end{aligned}$$

Thus $\rho_a = O(\omega \log^p)$, which proves that $(\zeta + \omega \log^p)^o = (\omega \log^p)$ since $(\omega \log^p) = (\omega)_{a^p}$ which is am-open. Let $N := (\zeta) + st^a((\omega)) = \bigcup_p (\zeta + \omega \log^p)$. Then N is countably generated and $N \neq st^a((\omega))$. By [16, Lemma 2.1],

$$N^o = \bigcup_p (\zeta + \omega \log^p)^o = \bigcup_p (\omega \log^p) = st^a((\omega)).$$

Thus $N^o = {}_a(N^o) = {}_aN \neq N$. By [ibid, Theorem 2.8], $N_- = {}_aN$, which concludes the proof. \square

Example 5.5. *We exhibit a countably generated ideal L which is not am-stable but for which the ideals L_a , L^- , and L^{oo} are all equal to $st^a((\omega))$ and hence are am-stable.*

PROOF. We construct inductively a sequence of sequences $\xi^{(p)} \in c_o^*$ for which the principal ideals $(\xi^{(1)} + \dots + \xi^{(p)}) \not\preceq (\omega)$ and $(\xi^{(p)})_a = (\omega)_{a^{p+1}}$. The initial $p = 1$ case is a special case of the general induction. Assume the construction up to $p - 1 \geq 1$. Since $\omega \neq O(\xi^{(1)} + \dots + \xi^{(p-1)})$, choose an increasing sequence of integers r_i for which $(\frac{\xi^{(1)} + \dots + \xi^{(p-1)}}{\omega})_{r_i} \rightarrow 0$. For the initial case $p = 1$ simply choose $r_i = i$.

Construct inductively two strictly increasing sequences of integers $\{q_k, n_k\}$ with $q_{k-1} < n_{k-1} < q_k$ and $\frac{n_k}{\log^p n_k} > (k+1)2^p$, starting with $q_o = 1$ and with an integer n_o chosen so that $\frac{n_o}{\log^p n_o} > 2^p$ and $n_o > e^p$ (the latter condition ensures that $\frac{\log^p j}{j}$ is decreasing for $j \geq n_o$). Assuming the construction of q_j, n_j up to $k - 1 \geq 0$, choose $q_k := \max \{j \in \mathbb{N} \mid \frac{1}{kn_{k-1}} \leq \frac{\log^p j}{j}\}$; thus $kn_{k-1} < q_k < n_{k-1}^2$. Then choose $n_k > q_k + 1$ from among the sequence $\{r_i\}$ and sufficiently large to satisfy the conditions $\sum_{j=q_{k+1}}^{n_{k-1}} \frac{\log^p j}{j} \geq \frac{1}{2} \sum_{j=1}^{n_k} \frac{\log^p j}{j}$ and $\frac{n_k}{\log^p n_k} > (k+1)2^p$.

Now define the sequence $\xi_j^{(p)} := \min \{\frac{1}{kn_{k-1}}, \frac{\log^p j}{j}\}$ for $n_{k-1} \leq j < n_k$ and $k \geq 1$, setting $\xi_j^{(p)} := 1$ for $1 \leq j < n_o$. Then $\xi^{(p)} \in c_o^*$, $\xi_j^{(p)} \leq (\omega \log^p)_j$ for $j \geq n_o$, and so $(\xi^{(p)}) \subset (\omega \log^p)$. An elementary computation shows that the sequence inequalities

$$C_p \omega \log^{p+1} \leq (\omega \log^p)_a \leq \omega \log^{p+1}$$

hold for some constant $0 < C_p < 1$, hence the ideal identities

$$(\omega \log^p)_a = (\omega \log^{p+1}) = (\omega)_{a^{p+1}}$$

also hold. Thus $(\xi^{(p)})_a \subset (\omega)_{a^{p+1}}$. To obtain equality, we show that $(\omega \log^p)_a = (\xi^{(p)})_a$. Indeed, for $k \geq 2$ and $n_{k-1} \leq j \leq q_k$,

$$\begin{aligned} j(\xi_a^{(p)})_j &\geq \sum_{i=q_{k-1}+1}^{n_{k-1}-1} \xi_i^{(p)} = \sum_{i=q_{k-1}+1}^{n_{k-1}-1} \frac{\log^p i}{i} \geq \frac{1}{2} \sum_{i=1}^{n_{k-1}} \frac{\log^p i}{i} \geq \frac{C_p}{2} \log^{p+1} n_{k-1} \\ &\geq \frac{C_p}{2} \log^{p+1} \sqrt{q_k} \geq \frac{C_p}{2^{p+2}} \log^{p+1} j \geq \frac{C_p}{2^{p+2}} j((\omega \log^p)_a)_j. \end{aligned}$$

For $k \geq 2$ and $q_k < j < n_k$, by the above inequality applied at q_k we also have

$$\begin{aligned} j(\xi_a^{(p)})_j &= q_k(\xi_a^{(p)})_{q_k} + \sum_{i=q_k+1}^j \frac{\log^p i}{i} \geq \frac{C_p}{2^{p+2}} q_k((\omega \log^p)_a)_{q_k} + \sum_{i=q_k+1}^j \frac{\log^p i}{i} \\ &\geq \frac{C_p}{2^{p+2}} \left(\sum_{i=1}^{q_k} \frac{\log^p i}{i} + \sum_{i=q_k+1}^j \frac{\log^p i}{i} \right) = \frac{C_p}{2^{p+2}} j((\omega \log^p)_a)_j. \end{aligned}$$

Thus $(\omega \log^p)_a \leq \frac{2^{p+2}}{C_p} \xi_a^{(p)}$ on $[n_1, \infty)$, which proves that $(\omega)_{a^{p+1}} = (\xi^{(p)})_a$. Finally,

$$\begin{aligned} \left(\frac{\xi^{(1)} + \dots + \xi^{(p)}}{\omega} \right)_{n_k} &= \left(\frac{\xi^{(1)} + \dots + \xi^{(p-1)}}{\omega} \right)_{n_k} + \left(\frac{\xi^{(p)}}{\omega} \right)_{n_k} \\ &= \left(\frac{\xi^{(1)} + \dots + \xi^{(p-1)}}{\omega} \right)_{n_k} + \frac{1}{k+1} \rightarrow 0. \end{aligned}$$

Thus $\omega \neq O((\xi^{(1)} + \dots + \xi^{(p)}))$ and since ω satisfies the $\Delta_{1/2}$ -condition, it follows that $(\omega) \not\subset (\xi^{(1)} + \dots + \xi^{(p)})$. This completes the inductive construction of the sequence of sequences $\xi^{(p)}$.

Now let L be the ideal generated by the sequences $\xi^{(p)}$. Since L is the union of the principal ideals $(\xi^{(1)} + \dots + \xi^{(p)})$, by construction $(\omega) \not\subset L$. Since L_a is the ideal generated by $\{\xi_a^{(p)} \mid p \in \mathbb{N}\}$, and since $(\xi^{(p)})_a = (\omega_{a^{p+1}})$, L_a coincides with $st^a((\omega)) = \bigcup_{p=0}^\infty (\omega)_{a^p}$, the upper am-stabilizer of the principal ideal (ω) (see Section 2). Thus L_a is am-stable and so $L_a = {}_a(L_a) = L^-$. Also L_a contains (ω) while L does not, so $L_a \neq L$, i.e., L is not am-stable.

To obtain equality with L^{oo} , we first show that $(\xi^{(p)})^{oo} = (\omega)_{a^p}$ for every p . Because $(\xi^{(p)}) \subset (\omega \log^p) = (\omega)_{a^p}$ which is am-open, by definition $(\xi^{(p)})^{oo} \subset (\omega \log^p)$. By [16, Remark 2.19], $(\xi^{(p)})^{oo}$ is principal. Choose as a generator of it some $\rho_a \geq \xi^{(p)}$ (see [16, Lemma 2.13]). For every $k \geq 1$ and $j \in (q_k, n_k)$ one has $j(\rho_a)_j \geq j\xi_j^{(p)} = \log^p j = j(\omega \log^p)_j$ from the definition of the sequence $\xi^{(p)}$. For every $j \in [n_k, q_{k+1}]$, by using the last inequality at $n_k - 1$,

$$\begin{aligned} j(\rho_a)_j &\geq (n_k - 1)(\rho_a)_{(n_k - 1)} \geq \log^p(n_k - 1) \geq \log^p(\sqrt{q_{k+1}} - 1) \\ &\geq \log^p(\sqrt{j} - 1) \geq \log^p(j^{1/4}) = 4^{-p} j(\omega \log^p)_j. \end{aligned}$$

The last inequality holds because $j \geq n_k \geq (\frac{1+\sqrt{5}}{2})^4$. Thus $\omega \log^p = O(\rho_a)$ and hence $(\xi^{(p)})^{oo} = (\omega \log^p)$. By [16, Lemma 2.22(i) and (v)],

$$\begin{aligned} L^{oo} &= \left(\bigcup_p \left((\xi^{(1)}) + \dots + (\xi^{(p)}) \right) \right)^{oo} = \bigcup_p \left((\xi^{(1)}) + \dots + (\xi^{(p)}) \right)^{oo} \\ &= \bigcup_p \left((\xi^{(1)})^{oo} + \dots + (\xi^{(p)})^{oo} \right) = \bigcup_p (\omega \log^p) = st^a((\omega)). \end{aligned}$$

□

6. FIRST ORDER CANCELTION PROPERTIES

In this section we use sublattice density to study first order arithmetic mean ideals' cancelation properties for inclusions. Several of these results were announced in [14]. We start with the elementary equivalences.

Lemma 6.1. *Let I be an ideal.*

(C) *The following conditions are equivalent.*

- (i) *I is am-closed*
- (ii) $J_a \subset I_a \Rightarrow J \subset I$ (ii') $J^- \subset I^- \Rightarrow J \subset I$
- (iii) $J_a = I_a \Rightarrow J \subset I$ (iii') $J^- = I^- \Rightarrow J \subset I$
- (iv) $I_- \subset J_- \Rightarrow I \subset J$
- (v) $I_- = J_- \Rightarrow I \subset J$

(O) *The following conditions are equivalent.*

- (i) *I is am-open*
- (ii) ${}_a I \subset {}_a J \Rightarrow I \subset J$ (ii') $I^o \subset J^o \Rightarrow I \subset J$
- (iii) ${}_a I = {}_a J \Rightarrow I \subset J$ (iii') $I^o = J^o \Rightarrow I \subset J$
- (iv) $J^{oo} \subset I^{oo} \Rightarrow J \subset I$
- (v) $J^{oo} = I^{oo} \Rightarrow J \subset I$

(C_∞) *The following conditions are equivalent.*

- (i) *I is am-∞ closed*
- (ii) $J \subset \mathcal{L}_1$ and $J_{a_\infty} \subset I_{a_\infty} \Rightarrow J \subset I \subset \mathcal{L}_1$
- (ii') $J \subset \mathcal{L}_1$ and $J^{-\infty} \subset I^{-\infty} \Rightarrow J \subset I \subset \mathcal{L}_1$
- (iii) $J \subset \mathcal{L}_1$ and $J_{a_\infty} = I_{a_\infty} \Rightarrow J \subset I \subset \mathcal{L}_1$
- (iii') $J \subset \mathcal{L}_1$ and $J^{-\infty} = I^{-\infty} \Rightarrow J \subset I \subset \mathcal{L}_1$
- (iv) $I_{-\infty} \subset J_{-\infty} \Rightarrow I \subset J$
- (v) $I_{-\infty} = J_{-\infty} \Rightarrow I \subset J$

(O_∞) *The following conditions are equivalent.*

- (i) *I is am-∞ open*
- (ii) $a_\infty I \subset a_\infty J \Rightarrow I \subset J$ (ii') $I^{o\infty} \subset J^{o\infty} \Rightarrow I \subset J$
- (iii) $a_\infty I = a_\infty J \Rightarrow I \subset J$ (iii') $I^{o\infty} = J^{o\infty} \Rightarrow I \subset J$
- (iv) $J \subset se(\omega)$ and $J^{o\infty} \subset I^{o\infty} \Rightarrow J \subset I \subset se(\omega)$
- (v) $J \subset se(\omega)$ and $J^{o\infty} = I^{o\infty} \Rightarrow J \subset I \subset se(\omega)$

PROOF. Conditions (x) and (x') are all equivalent by the monotonicity (i.e., inclusion preserving property) of the operations $L \mapsto L_a$, $L \mapsto {}_a L$, $L \mapsto L_{a_\infty}$, and $L \mapsto {}_{a_\infty} L$ and by the identities

$$L^- = ({}_a(L_a))_a, \quad L^o = {}_a(({}_a L)_a), \quad L^{-\infty} = ({}_{a_\infty}(L_{a_\infty}))_{a_\infty}, \quad L^{o\infty} = {}_{a_\infty}(({}_{a_\infty} L)_{a_\infty})$$

(see Section 2). Also, condition (ii) always trivially implies condition (iii) and condition (iv) always trivially implies condition (v). The am-case being similar but simpler because of the 5-chains of inclusions, we prove only the am-∞ case.

(C_∞) Recall that ${}_{a_\infty} I \subset I_{-\infty} \subset I$ and $I \cap \mathcal{L}_1 \subset I^{-\infty} \subset I_{a_\infty} \cap \mathcal{L}_1$ for every ideal L (see paragraph preceding Theorem 5.2).

- (i) \Rightarrow (ii') $I = I^{-\infty} \subset \mathcal{L}_1$ and $J = J \cap \mathcal{L}_1 \subset J^{-\infty} \subset I^{-\infty} = I$.
- (iii') \Rightarrow (i) Since $I^{-\infty} = (I^{-\infty})^{-\infty}$ and $I^{-\infty} \subset \mathcal{L}_1$, it follows that $I^{-\infty} \subset I \subset \mathcal{L}_1$. But then $I = I \cap \mathcal{L}_1 = I^{-\infty}$ and hence I is am-∞ closed.
- (i) \Rightarrow (iv) $I = I_{-\infty} \subset J_{-\infty} \subset J$.
- (v) \Rightarrow (i) Since $I_{-\infty} = (I_{-\infty})_{-\infty}$ and $I_{-\infty} \subset \mathcal{L}_1$, it follows that $I \subset I_{-\infty}$ and hence $I = I_{-\infty}$, i.e., I is am-∞ closed.

(O_∞) Recall that ${}_{a_\infty} L \subset L^{o\infty} \subset L$ and $L \cap se(\omega) \subset L^{o\infty} \subset L_{a_\infty} \subset se(\omega)$ for every ideal L (see *ibid*).

- (i) \Rightarrow (ii') $I = I^{o\infty} \subset J^{o\infty} \subset J$.
- (iii') \Rightarrow (i) As $I^{o\infty} = (I^{o\infty})^{o\infty}$, so $I \subset I^{o\infty}$ hence $I = I^{o\infty}$, i.e., I is am-∞ open.
- (i) \Rightarrow (iv) As every am-∞ open ideal, $I \subset se(\omega)$. Moreover,

$$J = J \cap se(\omega) \subset J^{o\infty} \subset I^{o\infty} = I.$$

- (v) \Rightarrow (i) Since $I^{o\infty} = (I^{o\infty})^{o\infty}$ and $I^{o\infty} \subset se(\omega)$, so $I^{o\infty} \subset I \subset se(\omega)$. hence $I = I \cap se(\omega) = I^{o\infty}$, i.e., I is am-∞ open. \square

A reformulation of (C) in the above lemma is that I is not am-closed if and only if there is an ideal $L \not\subset I$ for which $I_a = L_a$, in which case I is contained in the strictly larger ideal $J := I + L$ with the same arithmetic mean as I . In case I is countably generated (resp. principal) the next proposition shows that we can choose that larger ideal J to also be countably generated (resp. principal). The same holds for the am-∞ case.

Proposition 6.2.

- (i) *If I is a countably generated ideal that is not am-stable, then there is a countably generated ideal $J \supsetneq I$ such that $J_a = I_a$. If I is principal, then J can be chosen to be principal.*
- (ii) *If I is a countably generated ideal that is not am- ∞ stable, then there is a countably generated ideal $J \supsetneq I$ such that $J_{a_\infty} = I_{a_\infty}$. If I is principal, then J can be chosen to be principal.*

PROOF. By [16, Corollaries 2.9 and 3.5], if I is countably generated and not am-stable (resp. am- ∞ stable,) then it is not am-closed (resp. am- ∞ closed). By the upper density of \mathcal{L}_{\aleph_0} and \mathcal{PL} in \mathcal{L} , there is a $J \in \mathcal{L}_{\aleph_0}$ such that $I \subsetneq J \subsetneq I^-$ (resp. $I \subsetneq J \subsetneq I^{-\infty}$), and if $I \in \mathcal{PL}$ then J can be chosen to be principal. But then

$$I_a \subset J_a \subset (I^-)_a = I_a \quad (\text{resp. } I_{a_\infty} \subset J_{a_\infty} \subset (I^-)_{a_\infty} = I_{a_\infty}).$$

□

The answer is different for the reverse inclusion question: If I is not am-stable, can we always find a smaller ideal $J \subsetneq I$ such that $J_a = I_a$? The next example shows that the answer is negative even when I is principal and that the same holds for the am- ∞ case.

Example 6.3.

- (i) *Let $\xi_i := \frac{1}{k!}$ for $((k-1)!)^2 < i \leq (k!)^2$ and all $k \geq 2$ and let $\xi_1 = 1$. Then the principal ideal (ξ) is not am-stable and $J \subsetneq (\xi)$ implies $J_a \neq (\xi)_a$.*
- (ii) *Let $\xi_i := \frac{1}{(k!)^2}$ for $(k-1)! < i \leq k!$ and all $k \geq 2$ and let $\xi_1 = 1$. Then the principal ideal (ξ) is not am- ∞ stable and $J \subsetneq (\xi)$ implies $J_{a_\infty} \neq (\xi)_{a_\infty}$.*

PROOF.

- (i) It is easy to verify that $\xi \in c_o^*$ and that ξ does not satisfy the $\Delta_{1/2}$ -condition and hence is not am-stable. Now assume by contradiction that there is an ideal $J \subsetneq (\xi)$ with $J_a = (\xi)_a$. Then $(\xi)_a = (\eta)_a$ for some $(\eta) \subset J$, that is, J can also be chosen to be principal. Without loss of generality, since ξ and every $MD_m\xi$ for $M > 0$ and $m \in \mathbb{N}$, generate the same ideal, we can assume that $\eta \leq \xi$. Since $(\eta) \neq (\xi)$, $\xi \neq O(D_j\eta)$ for every $j \in \mathbb{N}$, i.e., $\xi_{r_j} \geq j\eta_{\lceil \frac{r_j}{j} \rceil}$ for some strictly increasing sequence of integers r_j . Furthermore, we can assume that the intervals $((k_j-1)!)^2 < r_j \leq (k_j!)^2$ containing distinct r_j are disjoint, that is, k_j is strictly

increasing. Then $\eta_{\frac{(k_j!)^2}{j}} \leq \eta_{\lceil \frac{r_j}{j} \rceil} \leq \frac{1}{j} \xi_{r_j} = \frac{1}{j(k_j!)}$, and since $\sum_1^k n! \leq 2k!$,

$$\begin{aligned}
(k_j!)^2 (\eta_a)_{(k_j!)^2} &= \sum_{i=1}^{(k_j!)^2} \eta_i \leq \sum_{i=1}^{\frac{(k_j!)^2}{j}} \xi_i + \sum_{i=\frac{(k_j!)^2}{j}+1}^{(k_j!)^2} \eta_{\frac{(k_j!)^2}{j}} \\
&= 1 + \sum_{n=2}^{k_j-1} \frac{(n!)^2 - ((n-1)!)^2}{n!} + \\
&\quad + \left(\frac{(k_j!)^2}{j} - ((k_j-1)!)^2 \right) \frac{1}{k_j!} + \left((k_j!)^2 - \frac{(k_j!)^2}{j} \right) \eta_{\frac{(k_j!)^2}{j}} \\
&\leq 2(k_j-1)! + \frac{k_j!}{j} + \frac{k_j!}{j} \leq 4 \frac{k_j!}{j} \\
&= \frac{4}{j} (k_j!)^2 \xi_{(k_j!)^2} \\
&\leq \frac{4}{j} (k_j!)^2 (\xi_a)_{(k_j!)^2}.
\end{aligned}$$

But then $\xi_a \neq O(\eta_a)$, and hence $(\eta_a) \neq (\xi_a)$, a contradiction.

(ii) Since $\sum_{k=1}^{\infty} \frac{1}{(k!)^2} (k! - (k-1)!) < \infty$, by [15, Example 4.5(iii), Corollary 4.10 and Definition 4.11], (ξ) is not am- ∞ stable. As in the proof of part (i), reasoning by contradiction we can assume that $(\eta) \subsetneq (\xi)$ but $(\eta_{a_\infty}) = (\xi_{a_\infty})$ for some $\eta \in (\ell^1)^*$ with $\eta \leq \xi$, and we can choose a sequence $(k_j - 1)! < r_j \leq k_j!$ with strictly increasing k_j for which $\xi_{r_j} \geq j \eta_{\lceil \frac{r_j}{j} \rceil}$. Then $j \eta_{\frac{k_j!}{j}} \leq j \eta_{\lceil \frac{r_j}{j} \rceil} \leq \xi_{r_j} = \xi_{k_j!} = \frac{1}{(k_j!)^2}$, and since $\sum_k \frac{1}{n!} \leq \frac{e}{k!}$,

$$\begin{aligned}
k_{j^2}! (\eta_{a_\infty})_{\frac{k_{j^2}!}{j^2}} &= j^2 \left(\sum_{i=\frac{k_{j^2}!}{j^2}+1}^{k_{j^2}!} \eta_i + \sum_{i=k_{j^2}!+1}^{\infty} \eta_i \right) \\
&\leq j^2 \left(k_{j^2}! \eta_{\frac{k_{j^2}!}{j^2}} + \sum_{n=k_{j^2}+1}^{\infty} \frac{n! - (n-1)!}{n!^2} \right) \\
&\leq j^2 \left(\frac{1}{j^2 k_{j^2}!} + \sum_{n=k_{j^2}+1}^{\infty} \frac{1}{n!} \right) \\
&\leq j^2 \left(\frac{1}{j^2 k_{j^2}!} + \frac{e}{(k_{j^2}+1)!} \right) \leq \frac{4}{k_{j^2}!}.
\end{aligned}$$

On the other hand, for $j \geq 2$,

$$k_{j^2}!(\xi_{a_\infty})_{\frac{k_{j^2}}{j}} \geq j \sum_{i=\frac{k_{j^2}}{j}+1}^{k_{j^2}} \xi_i = j(k_{j^2}! - \frac{k_{j^2}!}{j}) \frac{1}{(k_{j^2}!)^2} \geq \frac{j}{2k_{j^2}}.$$

Therefore $(\xi_{a_\infty})_{\frac{k_{j^2}}{j}} \geq \frac{j}{8}(\eta_{a_\infty})_{\frac{k_{j^2}}{j}}$ and hence $\xi_{a_\infty} \neq O(D_m \eta_{a_\infty})$ for any integer m . Thus $(\eta_{a_\infty}) \neq (\xi_{a_\infty})$. As in part (i), this concludes the proof. \square

Remark 6.4. *That $(\eta)_a \supset (\xi)$ implies $(\eta)_a \supset (\xi)_a$ for the sequence ξ of part (i) of the above example (or equivalently, that $(\xi)^{oo} = (\xi)_a$) was proved in [16, Example 2.20]. This combined with Example 6.3(i) shows that if $(\eta) \subset (\xi) \subset (\eta)_a$, then $(\eta) = (\xi)$. That is, (ξ) cannot lie strictly between any principal ideal and its arithmetic mean.*

At an early stage of this project, Ken Davidson and the second named author found, for the case of principal ideals, a direct constructive proof of Proposition 6.2(i) or rather, equivalently, that there is a principal ideal $J \supsetneq I$ with $J^- = I^-$. The same result was obtained earlier by Allen and Shen [2] by different methods. The proof that we presented here is a consequence of the identity $I_- = {}_a I$ for countably generated ideals (see [16, Theorem 2.9]).

To handle the am-interior (resp. am- ∞ interior) analogs of Proposition 6.2 for principal ideals, a direct constructive proof seems needed. This we present in the next proposition.

Proposition 6.5.

(i) *Let $(\rho) \neq \{0\}$ be a principal ideal that is not am-stable. Then there are two principal ideals $(\eta^{(1)})$ and $(\eta^{(2)})$ with $(\eta^{(1)})$ possibly zero, such that*

$$(\eta^{(1)}) \subsetneq (\rho) \subsetneq (\eta^{(2)}), \quad (\eta^{(1)})^{oo} = (\rho)^o, \quad \text{and} \quad (\rho)^{oo} = (\eta^{(2)})^o.$$

(ii) *Let $(\rho) \neq \{0\}$ be a principal ideal that is not am- ∞ stable. Then there are two principal ideals $(\eta^{(1)})$ and $(\eta^{(2)})$ such that*

$$(\eta^{(1)}) \subsetneq (\rho) \subsetneq (\eta^{(2)}), \quad (\eta^{(1)})^{oo\infty} = (\rho)^{o\infty}, \quad \text{and} \quad (\rho)^{oo\infty} = (\eta^{(2)})^{o\infty}.$$

PROOF.

(i) By [16, Lemma 2.14], $(\rho)^o$ and $(\rho)^{oo}$ are principal. Also $(\rho)^o = \{0\}$ precisely when $(\omega) \not\subset (\rho)$. If (ρ) is not am-open and hence $(\rho)^o \subsetneq (\rho) \subsetneq (\rho)^{oo}$, then choosing $(\eta^{(1)}) := (\rho)^o$ and $(\eta^{(2)}) := (\rho)^{oo}$ satisfies the requirement. Thus assume that (ρ) is am-open and hence $(\rho)^o = (\rho)^{oo} = (\rho)$. By [16, Lemma 2.13] ρ is equivalent to ξ_a for some $\xi \in c_o^*$. Thus ξ_a is irregular and by [10, Remark 3.11] (see also Corol-

lary 5.3(i)), ξ too is irregular, i.e., $\xi_a \neq O(\xi)$. Choose an increasing sequence of indices n_k such that $(\xi_a)_{n_k} \geq k\xi_{n_k}$ and without loss of generality assume that $n_{k+1} > kn_k$. Then

$$kn_k(\xi_a)_{kn_k} = \sum_{j=1}^{n_k} \xi_j + \sum_{j=n_k+1}^{kn_k} \xi_j \leq n_k(\xi_a)_{n_k} + kn_k\xi_{n_k} \leq 2n_k(\xi_a)_{n_k}$$

and hence $(\xi_a)_{n_k} \geq \frac{k}{2}(\xi_a)_{kn_k}$. Define

$$\eta_j^{(1)} = \begin{cases} (\xi_a)_{kn_k} & \text{for } n_k \leq j \leq kn_k \\ (\xi_a)_j & \text{for } kn_k \leq j < n_{k+1} \end{cases}$$

$$\eta_j^{(2)} = \begin{cases} (\xi_a)_{n_k} & \text{for } n_k \leq j \leq kn_k \\ (\xi_a)_j & \text{for } kn_k < j \leq n_{k+1}. \end{cases}$$

Then $\eta^{(i)} \in c_o^*$ for $i = 1, 2$ and $\eta^{(1)} \leq \xi_a \leq \eta^{(2)}$. Thus $(\eta^{(1)}) \subset (\xi_a) \subset (\eta^{(2)})$ and since (ξ_a) is am-open, $(\eta^{(1)})^{oo} \subset (\xi_a) \subset (\eta^{(2)})^o$. We need to prove strict containments in the first pair of inclusions and equalities in the second pair. Since

$$\left(\frac{\xi_a}{\eta^{(1)}} \right)_{n_k} = \left(\frac{\eta^{(2)}}{\xi_a} \right)_{kn_k} = \frac{(\xi_a)_{n_k}}{(\xi_a)_{kn_k}} \geq \frac{k}{2},$$

i.e., $\xi_a \neq O(\eta^{(1)})$ and $\eta^{(2)} \neq O(\xi_a)$, and since ξ_a satisfies the $\Delta_{1/2}$ -condition, it follows that $(\eta^{(1)}) \neq (\xi_a) \neq (\eta^{(2)})$.

By [16, Lemma 2.14], $(\eta^{(1)})^{oo}$ is principal, i.e., $(\eta^{(1)})^{oo} = (\mu_a)$ for some $\mu \in c_o^*$ [ibid, Lemma 2.13]. Since $(\eta^{(1)}) \subset (\mu_a)$ and μ_a satisfies the $\Delta_{1/2}$ -condition, by multiplying μ by a constant if necessary, one can assume without loss of generality that $\mu_a \geq \eta^{(1)}$. For all $n_k \leq j \leq kn_k$,

$$j(\mu_a)_j \geq (n_k - 1)(\mu_a)_{n_k-1} \geq (n_k - 1)\eta_{n_k-1}^{(1)} = (n_k - 1)(\xi_a)_{n_k-1}$$

$$\geq (n_k - 1)(\xi_a)_{n_k} \geq \frac{k}{2}(n_k - 1)(\xi_a)_{kn_k} \geq \frac{1}{4}kn_k(\xi_a)_{kn_k} \geq \frac{1}{4}j(\xi_a)_j.$$

For all $kn_k \leq j < n_{k+1}$ we have $j(\mu_a)_j \geq j\eta_j^{(1)} = j(\xi_a)_j$. Thus $\xi_a = O(\mu_a)$ and hence $(\xi_a) \subset (\mu_a) = (\eta^{(1)})^{oo}$, whence $(\xi_a) = (\eta^{(1)})^{oo}$. Similarly there is a $\nu_a \leq \eta^{(2)}$ such that $(\nu_a) = (\eta^{(2)})^o$. Then for all $n_k \leq j \leq kn_k$,

$$j(\nu_a)_j \leq (kn_k + 1)(\nu_a)_{kn_k+1} \leq (kn_k + 1)\eta_{kn_k+1}^{(2)} = (kn_k + 1)(\xi_a)_{kn_k+1}$$

$$\leq 2kn_k(\xi_a)_{kn_k} \leq 4n_k(\xi_a)_{n_k} \leq 4j(\xi_a)_j.$$

For all $kn_k < j \leq n_{k+1}$ we have $j(\nu_a)_j \leq j\eta_j^{(2)} = j(\xi_a)_j$. Thus $\nu_a = O(\xi_a)$ and hence $(\nu_a) \subset (\xi_a)$, whence $(\eta^{(2)})^o = (\xi_a)$.

(ii) We consider separately the cases when (ρ) is and when it is not am- ∞ open.

First assume that (ρ) is not am- ∞ open and hence $(\rho)^{o\infty} \subsetneq (\rho) \neq (\rho)^{oo\infty}$. Recall that by [16, Lemma 3.9(i)], $(\rho)^{o\infty}$ is not principal precisely when $(\omega) \subset (\rho)$, in which case $(\rho)^{o\infty} = se(\omega)$. By [16, Lemmas 3.9(ii) and 3.16(ii)], $(\rho)^{oo\infty}$ is not principal precisely when $(\rho) \not\subset se(\omega)$, in which case $(\rho)^{oo\infty} = se(\omega)$. Hence if $(\rho) \subset se(\omega)$, then both $(\rho)^{o\infty}$ and $(\rho)^{oo\infty}$ are principal and $(\rho)^{o\infty} \subsetneq (\rho) \subsetneq (\rho)^{oo\infty}$, so it suffices to choose $(\eta^{(1)}) = (\rho)^{o\infty}$ and $(\eta^{(2)}) = (\rho)^{oo\infty}$. And otherwise if $(\rho) \not\subset se(\omega)$, then choose a sequence $\eta^{(2)}$ sufficiently large so that $(\rho + \omega) \subsetneq (\eta^{(2)})$. Then $(\eta^{(2)})^{o\infty} = se(\omega) = (\rho)^{oo\infty}$. To obtain $\eta^{(1)}$ in the case when $(\omega) \subset (\rho)$, choose $0 \neq \eta^{(1)} \leq \omega$ but $\eta^{(1)} \neq o(\omega)$. Then $\omega \neq O(\eta^{(1)})$ hence $(\eta^{(1)}) \subsetneq (\omega)$, and $(\eta^{(1)}) \not\subset se(\omega)$ (which follows since ω satisfies the $\Delta_{1/2}$ -condition). Then $(\eta^{(1)}) \subsetneq (\rho)$ and $(\eta^{(1)})^{oo\infty} = se(\omega) = (\rho)^{o\infty}$. In the case when $(\omega) \not\subset (\rho)$, choose $(\eta^{(1)}) = (\rho)^{o\infty}$ since then the latter ideal is principal.

Thus, as in part (i), it remains to consider only the case that (ρ) is am- ∞ open and hence $(\rho) = (\xi_{a_\infty})$ for some $\xi \in \ell_1^*$ (see [16, Lemma 3.7]). By [15, Theorem 4.12(i) and (iv)], since (ρ) is not am- ∞ stable, ξ is am- ∞ irregular, and again by [ibid, Theorem 4.12(i) and (ii)], $\xi_{a_\infty} \neq O(\xi)$. Thus we can find an increasing sequence of indices n_k such that $(\xi_{a_\infty})_{n_k} \geq 2k\xi_{n_k}$, and without loss of generality we can assume that $n_{k+1} > kn_k$. Then

$$\begin{aligned} n_k(\xi_{a_\infty})_{n_k} &= \sum_{j=n_k+1}^{kn_k} \xi_j + \sum_{j=kn_k+1}^{\infty} \xi_j \leq kn_k\xi_{n_k} + kn_k(\xi_{a_\infty})_{kn_k} \\ &\leq \frac{1}{2}n_k(\xi_{a_\infty})_{n_k} + kn_k(\xi_{a_\infty})_{kn_k} \end{aligned}$$

and hence $(\xi_{a_\infty})_{n_k} \leq 2k(\xi_{a_\infty})_{kn_k}$. As for the am-case, define

$$\begin{aligned} \eta_j^{(1)} &= \begin{cases} (\xi_{a_\infty})_{kn_k} & \text{for } n_k \leq j \leq kn_k \\ (\xi_{a_\infty})_j & \text{for } kn_k \leq j < n_{k+1} \end{cases} \\ \eta_j^{(2)} &= \begin{cases} (\xi_{a_\infty})_{n_k} & \text{for } n_k \leq j \leq kn_k \\ (\xi_{a_\infty})_j & \text{for } kn_k < j \leq n_{k+1}. \end{cases} \end{aligned}$$

Then $\eta^{(i)} \in \mathfrak{c}_o^*$ for $i = 1, 2$ and $\eta^{(1)} \leq \xi_{a_\infty} \leq \eta^{(2)}$, whence it follows that $(\eta^{(1)}) \subset (\xi_{a_\infty}) \subset (\eta^{(2)})$. Moreover, since $\xi_{a_\infty} \in \Sigma(se(\omega))$ it follows also that $(\eta^{(1)}) \subset se(\omega)$, and since $(\frac{\omega}{\eta^{(2)}})_{n_k} = (\frac{\omega}{\xi_{a_\infty}})_{n_k} \rightarrow \infty$, $(\eta^{(2)}) \not\subset (\omega)$. Thus both $(\eta^{(1)})^{oo\infty}$ and $(\eta^{(2)})^{o\infty}$ are principal and $(\eta^{(1)})^{oo\infty} \subset (\xi_{a_\infty}) \subset (\eta^{(2)})^{o\infty}$ [16, Lemma 3.9(i)-(ii)]. For $m \in \mathbb{N}$ and all $k \geq m$ we have

$$\left(\frac{\xi_{a_\infty}}{D_m \eta^{(1)}} \right)_{kn_{k^2}} \geq \left(\frac{\xi_{a_\infty}}{D_k \eta^{(1)}} \right)_{kn_{k^2}} = \frac{(\xi_{a_\infty})_{kn_{k^2}}}{\eta_{n_{k^2}}^{(1)}} = \frac{(\xi_{a_\infty})_{kn_{k^2}}}{(\xi_{a_\infty})_{k^2 n_{k^2}}} \geq k$$

since, as is easy to verify, $(\zeta_{a_\infty})_m \geq k(\zeta_{a_\infty})_{km}$ for any positive integers k, m and $\zeta \in (\ell^1)^*$. Thus $(\xi_{a_\infty}) \neq (\eta^{(1)})$. Similarly, for all $k \geq m$,

$$\left(\frac{\eta^{(2)}}{D_m \xi_{a_\infty}} \right)_{k^2 n_{k^2}} \geq \left(\frac{\eta^{(2)}}{D_k \xi_{a_\infty}} \right)_{k^2 n_{k^2}} = \frac{(\eta^{(2)})_{n_{k^2}}}{(\xi_{a_\infty})_{kn_{k^2}}} \geq k,$$

whence $(\eta^{(2)}) \neq (\xi_{a_\infty})$. Since $(\eta^{(1)}) \subset (\eta^{(1)})^{oo\infty}$ and the latter is am- ∞ open and principal, by [15, Lemma 4.1(i)] we can choose one of its generators μ_{a_∞} so that $\mu_{a_\infty} \geq \eta^{(1)}$. Then for all $n_k \leq j \leq kn_k$, using $(\xi_{a_\infty})_{n_k} \leq 2k(\xi_{a_\infty})_{kn_k}$,

$$j(\mu_{a_\infty})_j \geq kn_k(\mu_{a_\infty})_{kn_k} \geq kn_k \eta_{kn_k}^{(1)} = kn_k(\xi_{a_\infty})_{kn_k} \geq \frac{1}{2}n_k(\xi_{a_\infty})_{n_k} \geq \frac{1}{2}j(\xi_{a_\infty})_j,$$

and for all $kn_k \leq j < n_{k+1}$ we have $j(\mu_{a_\infty})_j \geq j(\eta^{(1)})_j = j(\xi_{a_\infty})_j$. Thus $\xi_{a_\infty} = O(\mu_{a_\infty})$, hence $(\xi_{a_\infty}) \subset (\mu_{a_\infty}) = (\eta^{(1)})^{oo\infty}$, whence $(\xi_{a_\infty}) = (\eta^{(1)})^{oo\infty}$.

Similarly, by the proof of [16, Lemma 3.7], there is a generator ν_{a_∞} for $(\eta^{(2)})^{oo\infty}$ such that $\nu_{a_\infty} \leq \eta^{(2)}$. If $n_k \leq j \leq kn_k$, then

$$j(\nu_{a_\infty})_j \leq n_k(\nu_{a_\infty})_{n_k} \leq n_k \eta_{n_k}^{(2)} = n_k(\xi_{a_\infty})_{n_k} \leq 2kn_k(\xi_{a_\infty})_{kn_k} \leq 2j(\xi_{a_\infty})_j.$$

If $kn_k < j \leq n_{k+1}$, then $j(\nu_{a_\infty})_j \leq j\eta_j^{(2)} = j(\xi_{a_\infty})_j$. Thus $\nu_{a_\infty} = O(\xi_{a_\infty})$, hence $(\eta^{(2)})^{oo\infty} = (\nu_{a_\infty}) \subset (\xi_{a_\infty})$, whence $(\eta^{(2)})^{oo\infty} = (\xi_{a_\infty})$. \square

Remark 6.6.

- (i) *The existence of the sequence of indices n_k constructed above can also be derived for case (i) from [26, Lemma 1] and for case (ii) from [15, Theorem 4.12(v)].*
- (ii) *In Proposition 6.5, the non am-stability condition for the principal ideal (ρ) is necessary. Indeed if (ρ) were am-stable, hence am-open, then condition $(\eta^{(1)})^{oo} = (\rho)$ or $(\eta^{(2)})^o = (\rho)$ would imply (Theorem 5.1): $(\eta^{(1)}) = (\eta^{(1)})^{oo} = (\rho)$ or $(\eta^{(2)}) = (\eta^{(1)})^o = (\rho)$. Likewise for the am- ∞ case and argument.*

Theorem 6.7.

- (A) *Let $I \neq \{0\}$ be a principal ideal and let J be an arbitrary ideal. Then the following are equivalent*
 - (i) *I is am-stable*
 - (ii) *${}_a J = {}_a I$ (or, equivalently, $J^o = I^o$) implies $J = I$*
 - (iii) *$J_a = I_a$ (or, equivalently, $J^- = I^-$) implies $J = I$*
 - (iv) *$J^{oo} = I^{oo}$ implies $J = I$*
 - (v) *$J_- = I_-$ implies $J = I$*

(B) Let $I \neq \{0\}$ be a principal ideal and let J be an arbitrary ideal. Then the following are equivalent

- (i) I is am- ∞ stable
- (ii) ${}_a\infty J = {}_a\infty I$ (or, equivalently, $J^{o\infty} = I^{o\infty}$) implies $J = I$.
- (iii) $J_{a_\infty} = I_{a_\infty}$ (or, equivalently, $J^{-\infty} = I^{-\infty}$) implies $J = I$
- (iv) $J^{oo\infty} = I^{oo\infty}$ implies $J = I$
- (v) $J_{-\infty} = I_{-\infty}$ implies $J = I$

PROOF. As the am-case is similar and slightly simpler than the am- ∞ case, we prove only the latter. Recall from [15, Corollary 4.10] that I is am- ∞ stable if and only if $I = {}_a\infty I$, or, equivalently, if and only if $I \subset \mathcal{L}_1$ and $I = I_{a_\infty}$, in which case also

$$I = I_{-\infty} = I^{-\infty} = I^{o\infty} = I^{oo\infty} \subset st_{a_\infty}(\mathcal{L}_1) \subsetneq \mathcal{L}_1.$$

(i) \Rightarrow (ii)-(v) Since I is am- ∞ open (resp. am- ∞ closed), if the hypothesis of (ii) (resp. (v)) are satisfied, then in either case, $I \subset J$ by Lemma 6.1. If the hypothesis of (iv) (resp. (iii)) are satisfied, then $J^{oo\infty} \neq se(\omega)$ (resp. $J^{-\infty} \neq \mathcal{L}_1$), hence by [16, Lemma 3.16 (ii)], $J \subset se(\omega)$ (resp. by [16, Lemma 3.16 (i)], $J \subset \mathcal{L}_1$ and thus in either case, $J \subset I$ by Lemma 6.1.

Now assume by contradiction that $I \neq J$. Then in cases (ii) and (v), by the upper density of $\mathcal{S}_\infty \mathcal{P}\mathcal{L}$ in \mathcal{L} (see Corollary 3.7(ii) and Theorem 3.13) there is an ideal $L \in \mathcal{S}_\infty \mathcal{P}\mathcal{L}$ strictly between I and J . In cases (iii) (resp. (iv)), there is a principal ideal $(\eta) \subset J$ for which $(\eta)_{a_\infty} = I_{a_\infty} = I$ (resp. $(\eta)^{oo\infty} = I^{oo\infty} = I$). Since $I \neq (\eta)$ by the assumption $I \neq J$, by Theorem 3.13 there is again an ideal $L \in \mathcal{S}\mathcal{P}\mathcal{L}$ strictly between I and (η) . Since $L = {}_a\infty L = L_{-\infty} = L^{o\infty} = L_{a_\infty}$, this yields an immediate contradiction for all four cases. (ii),(iv) \Rightarrow (i) Since $(I^{o\infty})^{o\infty} = I^{o\infty}$ (resp. $(I^{oo\infty})^{oo\infty} = I^{oo\infty}$), it follows that I is am- ∞ open. The conclusion is now immediate from Proposition 6.5(ii).

(iii),(v) \Rightarrow (i) If I is not am- ∞ stable, then it is not am- ∞ closed [16, Theorem 3.5] and so it cannot coincide with $I^{-\infty}$ (resp. with $I_{-\infty}$), but $(I^{-\infty})^{-\infty} = I^{-\infty}$ (resp. $(I_{-\infty})_{-\infty} = I_{-\infty}$), contradicting the hypothesis. \square

Examples 5.4 and 5.5 show that the first order equality cancelations of Proposition 6.7 can fail for countably generated am-stable ideals. Indeed, ${}_a N = N_-$ is am-stable yet ${}_a N = {}_{a_2} N \not\neq N = {}_a N$ and $N_- = (N_-)_- \not\neq N = N_-$. Similarly, $L_a = L^{oo}$ is am-stable yet $L_a = L_{a_2}$ does not imply $L = L_a$ and $L^{oo} = (L^{oo})^{oo}$ does not imply $L = L^{oo}$.

First order cancelations involving inclusions are less simple even for principal ideals. As shown in Lemma 6.1, necessary and sufficient conditions on the ideal

I for the cancelations ${}_aI \subset {}_aJ \Rightarrow I \subset J$ (resp. $J_a \subset I_a \Rightarrow J \subset I$) are straightforward: I must be am-open (resp. am-closed). The “opposite” implications, ${}_aJ \subset {}_aI \Rightarrow J \subset I$ and $I_a \subset J_a \Rightarrow I \subset J$ do not hold in general even when I is principal and am-stable. For the latter, Corollaries 6.15 and 6.16 show that this cancelation fails for every $I = (\omega^p)$ ($0 < p < 1$) and Example 6.18 provides a principal ideal where this cancelation does hold. For the former, the cancelation fails for every principal ideal I , as shown by the next proposition which actually proves more.

Proposition 6.8. *Let I be an ideal. Then for every principal ideal (ξ) there is a principal ideal $(\zeta) \not\subset (\xi)$ with ${}_a(\zeta) \subset {}_aI$.*

PROOF. Assume first that ${}_aI \neq 0$ and hence $\mathcal{L}_1 \subset {}_aI$ (by [16, Lemma 6.2(iii)-(iv)]). Let n_k be an increasing sequence of integers such that $\xi_{\lceil \frac{n_{k+1}-1}{k} \rceil} \leq \frac{1}{kn_k}$. Define the sequence $\zeta_j := \frac{1}{n_k}$ for $j \in [n_k, n_{k+1})$. Then $\zeta \in \mathfrak{c}_o^*$ and since for every k , $\zeta_{n_{k+1}-1} \geq k(D_k \xi)_{n_{k+1}-1}$, it follows that $(\zeta) \not\subset (\xi)$. Now let $\rho \in \Sigma({}_a(\zeta))$, that is, $\rho_a \in \Sigma((\zeta))$, and since ρ_a satisfies the $\Delta_{1/2}$ -condition, assume without loss of generality that $\rho_a \leq \zeta$. Then $\sum_{i=1}^{n_k} \rho_i = n_k(\rho_a)_{n_k} \leq 1$ and hence $\rho \in \ell^1$. Thus ${}_a(\zeta) \subset \mathcal{L}_1 \subset {}_aI$.

For the case when ${}_aI = \{0\}$, let n_k be a strictly increasing sequence of integers such that $\xi_{\lceil \frac{n_{k+1}-1}{k} \rceil} \leq \frac{1}{k^2 n_k}$. Define the sequence $\zeta_j := \frac{1}{kn_k}$ for $j \in [n_k, n_{k+1})$. Then it is easy to verify that $(\zeta) \not\subset (\xi)$ and $(\omega) \not\subset (\zeta)$, and hence ${}_a(\zeta) = \{0\}$. \square

As a consequence of Proposition 6.8, recalling from [16, Theorem 2.9] that $(\zeta)_- = {}_a(\zeta)$ for every principal ideal (ζ) , we see that the cancelation $J_- \subset I_- \Rightarrow J \subset I$ also fails for every principal ideal I .

The proof of Proposition 6.8 can be adapted to show that while principal ideals have principal am-interiors, the converse does not hold.

Example 6.9. *There is a non-principal ideal with principal nonzero am-interior.*

PROOF. Define the sequences $\zeta_j^{(p)} = \frac{1}{(2^p k)!}$ for $j \in [(2^p k)!, (2^p(k+1))!)$ for every $p \in \mathbb{N}$. Then it is easy to verify that $(\omega) \subset (\zeta^{(p)}) \subsetneq (\zeta^{(p+1)})$ for all p , from which it follows that the ideal $I := \bigcup_p (\zeta^{(p)})$ is not principal. On the other hand, if $\{0\} \neq (\rho_a) \subset (\zeta^{(p)})$ is an am-open ideal, and without loss of generality, $\rho_a \leq \zeta^{(p)}$, then the inequality $(2^p k)! (\rho_a)_{(2^p k)!} \leq 1$ shows that $\rho \in \ell^1$. Thus, as in the proof of the above proposition, $(\rho_a) = (\omega)$, that is, $(\zeta^{(p)})^o = (\omega)$. By [16, Theorem 2.17], $I^o = \bigcup_p (\zeta^{(p)})^o = (\omega)$, which is a principal ideal. \square

Proposition 6.8 implies that for any given ideal I , there is never a “least upper bound” principal ideal for the solutions J of the inclusion ${}_a J \subset {}_a I$. Intuitively, a candidate might be the union of such ideals but this, in general, is not an ideal. In contrast however, the intersection of ideals being always an ideal, for every given ideal I , the “greatest lower bound,” \widehat{I} , for the solutions of the inclusion $J_a \supset I_a$, i.e., the largest ideal for which $J_a \supset I_a \Rightarrow J \supset \widehat{I}$, is given by:

Definition 6.10. $\widehat{I} := \bigcap \{J \mid J_a \supset I_a\}$

We will identify \widehat{I} for principal ideals and show that it is itself principal (when I is principal) and that it may be strictly smaller than I (even when I is am-stable), though not always (see Corollary 6.16(ii), Example 6.18 and Proposition 6.17).

The motivation for considering \widehat{I} comes from the still open question from [10, Section 7]: If the class $[I, B(H)]_1$ of single commutators of operators in I with operators in $B(H)$, contains a finite rank operator with nonzero trace, must $\Sigma(I)$ contain $\sqrt{\omega}$? It was shown in [10, Theorem 7.3] that $\Sigma(I^-) \ni \sqrt{\omega}$, i.e., there is an $\eta \in \Sigma(I)$ with $\eta_a \geq (\sqrt{\omega})_a \asymp \sqrt{\omega}$. However, we will show in Corollary 6.16(ii) (for $p = \frac{1}{2}$) that the latter condition does not imply $(\eta) \supset (\sqrt{\omega})$ but only that $(\eta) \supset (\omega)$ and that this “lower bound” (ω) is sharp for this implication. For general ξ it can be shown that there is never a minimal $\widehat{\xi}$ for which $\eta_a \geq \xi_a$ implies $\eta \geq \widehat{\xi}$, but as Theorem 6.14(i) shows there is a minimal one asymptotically.

Definition 6.11. For $\xi \in c_o^* \setminus \ell^1$, let $\nu(\xi)_n := \min\{k \in \mathbb{N} \mid \sum_{i=1}^k \xi_i \geq n\xi_1\}$ and define $\widehat{\xi} := \langle (\xi_a)_{\nu(\xi)_n} \rangle$.

Lemma 6.12. If $\xi \in c_o^* \setminus \ell^1$, then $\widehat{\xi} \in c_o^*$ and $(1 - \frac{1}{\nu(\xi)_n})\widehat{\xi}_n < \frac{n}{\nu(\xi)_n}\xi_1 \leq \widehat{\xi}_n$.

PROOF. Since $\xi \notin \ell^1$, the distribution sequence $\nu(\xi)$ is well-defined, nondecreasing, and tends to infinity since $\nu(\xi)_n \geq n$ because $n\xi_1 \leq \sum_{i=1}^{\nu(\xi)_n} \xi_i \leq \nu(\xi)_n \xi_1$. Thus $\widehat{\xi} \in c_o^*$. For each fixed m and all n for which $\nu(\xi)_n \geq m$,

$$n\xi_1 \leq \sum_{i=1}^{\nu(\xi)_n} \xi_i = \sum_{i=1}^{m-1} \xi_i + \sum_{i=m}^{\nu(\xi)_n} \xi_i \leq \sum_{i=1}^{m-1} \xi_i + \nu(\xi)_n \xi_m$$

so $0 \leq \overline{\lim}_n \frac{n}{\nu(\xi)_n} \xi_1 \leq \overline{\lim}_n \frac{\sum_{i=1}^{m-1} \xi_i}{\nu(\xi)_n} + \xi_m = \xi_m$.

Since m is arbitrary, $\frac{n}{\nu(\xi)_n} \rightarrow 0$. Moreover,

$$(\nu(\xi)_n - 1)\widehat{\xi}_n \leq (\nu(\xi)_n - 1)(\xi_a)_{\nu(\xi)_{n-1}} = \sum_{i=1}^{\nu(\xi)_{n-1}} \xi_i < n\xi_1 \leq \sum_{i=1}^{\nu(\xi)_n} \xi_i = \nu(\xi)_n \widehat{\xi}_n,$$

whence the claim. □

Notice $\widehat{\xi}$ is asymptotic to $\langle \frac{n}{\nu(\xi)_n} \xi_1 \rangle$. Also the next lemma is left to the reader.

Lemma 6.13. *For all $\xi, \eta \in c_o^* \setminus \ell^1$ and all $t > 0$ we have*

- (i) $\nu(t\xi) = \nu(\xi)$ and $t\widehat{\xi} = t\widehat{\xi}$
- (ii) $\min\{\nu(\xi), \nu(\eta)\} \leq \nu(\xi + \eta) \leq \max\{\nu(\xi), \nu(\eta)\}$ and $\widehat{\xi + \eta} \leq \widehat{\xi} + \widehat{\eta}$

Theorem 6.14. *Let $\xi \in c_o^* \setminus \ell^1$.*

- (i) *If $\eta_a \geq \xi_a$ and $\eta \in c_o^*$, then for every $\varepsilon > 0$, $\eta_n \geq (1 - \varepsilon)\widehat{\xi}_n$ for n sufficiently large.*
- (ii) *For every $\rho \in c_o^* \setminus \Sigma((\widehat{\xi}))$, there is an $\eta \in c_o^*$ for which $\eta_a \geq \xi_a$ and $\rho \notin \Sigma((\eta))$.*

PROOF.

- (i) By Lemma 6.13(i), assume without loss of generality that $\xi_1 = 1$. For every n large enough so that $(\eta_a)_n \leq \frac{\varepsilon}{2}$ and $\nu(\xi)_n > \max(n, \frac{2}{\varepsilon})$, we have

$$n \leq \sum_{i=1}^{\nu(\xi)_n} \xi_i \leq \sum_{i=1}^{\nu(\xi)_n} \eta_i \leq \sum_{i=1}^n \eta_i + (\nu(\xi)_n - n)\eta_n \leq \frac{\varepsilon}{2}n + (\nu(\xi)_n - n)\eta_n$$

and thus

$$\eta_n \geq (1 - \frac{\varepsilon}{2}) \frac{n}{\nu(\xi)_n - n} \geq (1 - \frac{\varepsilon}{2}) \frac{n}{\nu(\xi)_n} > (1 - \frac{\varepsilon}{2})^2 \widehat{\xi}_n > (1 - \varepsilon) \widehat{\xi}_n$$

where the third inequality follows from Lemma 6.12.

- (ii) Assume without loss of generality that $\xi_1 = 1$ and, again by Lemma 6.13(i), that $\rho_1 = 1$. We construct inductively an increasing sequence of integers n_k and three derived sequences $l_k = \lceil \frac{4k(k+1)}{\rho_{n_k}} \rceil$, $m_k := \lceil \frac{n_k}{k} \rceil$, and $p_k := \lceil \frac{n_k}{l_{k-1}} \rceil$ starting with $l_0 = n_1 = 1$. Assume the construction up to $k-1 \geq 1$. Since $\rho \neq O(D_{l_{k-1}}\widehat{\xi})$, we can choose an integer n_k large enough so that

- (a) $\rho_{n_k} \geq k(D_{l_{k-1}}\widehat{\xi})_{n_k} = k(\widehat{\xi})_{p_k}$
- (b) $n_k \geq l_{k-1}$
- (c) $\rho_{n_k} \leq \frac{k^2}{2l_{k-1}}$
- (d) $m_k \geq 2m_{k-1}$
- (e) $m_k > \nu(\xi)_{p_{k-1}}$

Now define the sequence $\eta_j := \frac{1}{k}\rho_{n_k}$ for $m_k \leq j < m_{k+1}$. Then $\eta \in c_o^*$ and $\rho \notin \Sigma((\eta))$ because $\rho_{n_k} = k(D_k\eta)_{n_k}$ for all k . The key to prove that $\eta_a \geq \xi_a$ is the following lower bound for $\nu(\xi)_{p_k}$:

$$\nu(\xi)_{p_k} \geq \frac{p_k}{\widehat{\xi}_{p_k}} \geq \frac{kn_k}{l_{k-1}\rho_{n_k}} \geq 2\frac{n_k}{k} > m_k,$$

where the first inequality follows from Lemma 6.12, the second from (a) and the

definition of p_k , the third from (c) and the fourth from the definition of m_k . Thus by (e), $m_k < \nu(\xi)_{p_k} < m_{k+1}$. If $m_k \leq j < \nu(\xi)_{p_k}$, then

$$\begin{aligned} \sum_{i=1}^j \eta_i - \sum_{i=1}^j \xi_i &\geq \sum_{i=m_k-1}^{m_k-1} \eta_i - \sum_{i=1}^{\nu(\xi)_{p_k}-1} \xi_i && \text{(obvious)} \\ &> (m_k - m_{k-1})\eta_{m_k-1} - p_k && \text{(by the definition of } \nu(\xi)_{p_k} \text{)} \\ &> \frac{(m_k - m_{k-1})}{k-1} \rho_{n_{k-1}} - \frac{n_k}{l_{k-1}} - 1 && \text{(by the definitions of } \rho \text{ and of } p_k \text{)} \\ &> \frac{n_k}{2(k-1)k} \rho_{n_{k-1}} - 2 \frac{n_k}{l_{k-1}} && \text{(by (d) and (b) and since } m_k \geq \frac{n_k}{k} \text{)} \\ &\geq \frac{n_k}{2(k-1)k} \frac{4(k-1)k}{l_{k-1}} - 2 \frac{n_k}{l_{k-1}} = 0 && \text{(by the definition of } l_k \text{)}. \end{aligned}$$

If now $\nu(\xi)_{p_k} \leq j < m_{k+1}$, then

$$\begin{aligned} \sum_{i=1}^j \eta_i - \sum_{i=1}^j \xi_i &= \sum_{i=1}^{\nu(\xi)_{p_k}-1} \eta_i - \sum_{i=1}^{\nu(\xi)_{p_k}-1} \xi_i + \sum_{i=\nu(\xi)_{p_k}}^j \eta_i - \sum_{i=\nu(\xi)_{p_k}}^j \xi_i \\ &\geq \sum_{i=\nu(\xi)_{p_k}}^j \eta_i - \sum_{i=\nu(\xi)_{p_k}}^j \xi_i && \text{(by the above inequality)} \\ &\geq (j - \nu(\xi)_{p_k} + 1) \left(\frac{1}{k} \rho_{n_k} - \xi_{\nu(\xi)_{p_k}} \right) && \text{(by the monotonicity of } \eta \text{ and } \xi \text{)} \\ &\geq (j - \nu(\xi)_{p_k} + 1) \left(\frac{1}{k} \rho_{n_k} - (\xi_a)_{\nu(\xi)_{p_k}} \right) \\ &= (j - \nu(\xi)_{p_k} + 1) \left(\frac{1}{k} \rho_{n_k} - \widehat{\xi}_{p_k} \right) \geq 0 && \text{(by the definition of } \widehat{\xi} \text{ and (a)).} \end{aligned}$$

Therefore $\eta_a \geq \xi_a$, which completes the proof. □

Using Theorem 6.14 to compute \widehat{I} (see Definition 6.10) for principal ideals I :

Corollary 6.15. *Let $0 \neq \xi \in c_o^*$, then $\widehat{(\xi)} = \begin{cases} F & \text{for } \xi \in \ell^1 \\ \widehat{(\xi)} & \text{for } \xi \notin \ell^1 \end{cases}$*

PROOF. If $\xi \in \ell^1$, then $(\xi_a) = (\omega) = F_a$, thus $\widehat{(\xi)} = F$. If $\xi \notin \ell^1$, then for every ideal J such that $J_a \supset (\xi)_a$, there is an $\eta \in \Sigma(J)$ such that $\xi_a \leq \eta_a$. By Theorem 6.14(i), $\widehat{(\xi)} \subset (\eta) \subset J$, hence $\widehat{(\xi)} \subset \widehat{(\eta)}$. By Theorem 6.14(ii), for every $\rho \in c_o^* \setminus \Sigma(\widehat{(\xi)})$, there is an $\eta \in c_o^*$ such that $(\eta)_a \supset (\xi)_a$ but $\rho \notin \Sigma((\eta))$. But then $\rho \notin \Sigma(\bigcap \{J \mid J_a \supset (\xi)_a\}) = \Sigma(\widehat{(\xi)})$, whence the claim. □

Corollary 6.16.

(i)

$$\widehat{(\omega)}^p = \begin{cases} F & \text{for } p > 1 \\ \langle \frac{n}{e^n} \rangle & \text{for } p = 1 \\ (\omega)^{p'} & \text{for } 0 < p < 1, \text{ where } \frac{1}{p} - \frac{1}{p'} = 1 \end{cases}$$

(ii) $\widehat{(\omega)}^p$ is am-stable if and only if $0 < p < \frac{1}{2}$.

PROOF.

(i) For $p > 1$, ω^p is summable and hence, by Corollary 6.15, $\widehat{(\omega)}^p = F$. A routine computation shows $\nu(\omega)_n \asymp e^n$ and therefore, again by Corollary 6.15 and Lemma 6.12, one has $\widehat{(\omega)} = \widehat{(\omega)} = \langle \frac{n}{e^n} \rangle$. Similarly, for $0 < p < 1$, $(\nu(\omega)^p)_n \asymp n^{\frac{1}{1-p}}$, hence one has $\widehat{(\omega)}^p = \langle \frac{n}{n^{\frac{1}{1-p}}} \rangle = (\omega)^{p'}$.

(ii) The ideals F and $\langle \frac{n}{e^n} \rangle$ are clearly not am-stable, and $(\omega)^{p'}$ is am-stable if and only if $0 < p' < 1$. □

So, in particular $\widehat{(\omega)}^{1/2} = (\omega)$. By definition, $\widehat{(\xi)} \subset (\xi)$, but as the case of $(\omega)^{1/2}$ illustrates, the inclusion can be proper even when ξ is regular, i.e., when (ξ) is am-stable. As the following proposition shows, the inclusion is certainly proper when $(\xi) \neq F$ is not am-stable.

Proposition 6.17. *If $F \neq (\xi) = \widehat{(\xi)}$ with $\{0\} \neq \xi \in c_o^*$, then $(\xi) = (\xi)_a$.*

PROOF. Assume otherwise. Then there is an increasing sequence of integers n_k , starting with $n_1 = 1$, such that $(\xi_a)_{n_k} \geq k^2 \xi_{n_k}$ and for $k > 1$, n_k can be chosen large enough so that for $m_k := \max\{j \in \mathbb{N} \mid \xi_j > k \xi_{n_k}\}$, $m_k \geq 2kn_{k-1}$, and also $(\xi_a)_{\lceil \frac{m_k}{k} \rceil} \leq \frac{1}{2k} \xi_{n_{k-1}}$. First we show that $\sum_{i=1}^{n_k} \xi_i \leq 2 \sum_{i=1}^{m_k} \xi_i$. Indeed,

$$\sum_{i=m_k+1}^{n_k} \xi_i \leq n_k \xi_{m_k+1} \leq kn_k \xi_{n_k} \leq \frac{n_k}{k} (\xi_a)_{n_k} = \frac{1}{k} \sum_{i=1}^{n_k} \xi_i,$$

thus

$$\sum_{i=1}^{n_k} \xi_i = \sum_{i=1}^{m_k} \xi_i + \sum_{i=m_k+1}^{n_k} \xi_i \leq \sum_{i=1}^{m_k} \xi_i + \frac{1}{k} \sum_{i=1}^{n_k} \xi_i,$$

and hence $\sum_{i=1}^{n_k} \xi_i \leq \frac{1}{1-\frac{1}{k}} \sum_{i=1}^{m_k} \xi_i \leq 2 \sum_{i=1}^{m_k} \xi_i$. Now, define the sequence

$$\eta_j = \begin{cases} \xi_{n_{k-1}} & \text{for } j \in [n_{k-1}, \lceil \frac{m_k}{k} \rceil) \\ \xi_{n_k} & \text{for } j \in [\lceil \frac{m_k}{k} \rceil, n_k] \end{cases}.$$

Clearly, $\eta \in \mathfrak{c}_o^*$ and if $j \in [\lceil \frac{m_k}{k} \rceil, n_k]$, then

$$\begin{aligned} j(\eta_a)_j &= \sum_{i=1}^j \eta_i \geq \sum_{i=n_{k-1}+1}^{\lceil \frac{m_k}{k} \rceil} \eta_i = (\lceil \frac{m_k}{k} \rceil - n_{k-1})\xi_{n_{k-1}} \geq \frac{1}{2} \lceil \frac{m_k}{k} \rceil 2k(\xi_a)_{\lceil \frac{m_k}{k} \rceil} \\ &\geq m_k(\xi_a)_{\lceil \frac{m_k}{k} \rceil} \geq m_k(\xi_a)_{m_k} \geq \frac{1}{2} \sum_1^{n_k} \xi_i \geq \frac{1}{2} \sum_1^j \xi_i = \frac{1}{2} j(\xi_a)_j. \end{aligned}$$

By using this inequality, if $j \in [n_{k-1}, \lceil \frac{m_k}{k} \rceil]$, then also

$$j(\eta_a)_j = \sum_{i=1}^{n_{k-1}} \eta_i + \sum_{i=n_{k-1}+1}^j \eta_i \geq \frac{1}{2} \sum_{i=1}^{n_{k-1}} \xi_i + \sum_{i=n_{k-1}+1}^j \xi_i \geq \frac{1}{2} \sum_{i=1}^j \xi_i = \frac{1}{2} j(\xi_a)_j.$$

This proves that $(\eta)_a \supset (\xi)_a$ and hence $(\eta) \supset (\widehat{\xi})$. On the other hand,

$$\xi_{m_k} > k\xi_{n_k} = k\eta_{\lceil \frac{m_k}{k} \rceil} = k(D_k\eta)_{m_k}$$

and hence $(\eta) \not\supset (\xi)$, that is, $(\xi) \neq (\widehat{\xi})$, against the hypothesis. □

It remains to show that there exist principal ideals $I = \widehat{I}$ with $I \neq F$.

Example 6.18. Let $\xi_j = \frac{1}{k}$ for $j \in ((k-1)!, k!]$. Then $(\xi) = (\widehat{\xi})$.

PROOF. It is enough to show that if $\eta \in \mathfrak{c}_o^*$ and $(\xi) \not\subset (\eta)$, then $(\xi)_a \not\subset (\eta)_a$. Assume without loss of generality that $\eta_1 = 1$. Since $\xi \notin \Sigma((\eta))$, choose an increasing sequence $r_j \in \mathbb{N}$ for which $\xi_{r_j} \geq j\eta_{\lceil \frac{r_j}{j} \rceil}$. Let $r_j \in ((k_j-1)!, k_j!]$ and assume without loss of generality that k_j is strictly increasing. Then

$$\frac{1}{k_j} = \xi_{k_j!} = \xi_{r_j} \geq j\eta_{\lceil \frac{r_j}{j} \rceil} \geq j\eta_{\frac{k_j!}{j}}$$

and hence

$$\begin{aligned} (k_j+1)!(\eta_a)_{(k_j+1)!} &\leq \sum_{i=1}^{\frac{k_j!}{j}} \eta_i + \sum_{i=\frac{k_j!}{j}+1}^{(k_j+1)!} \eta_{\frac{k_j!}{j}} \leq \frac{k_j!}{j} + (k_j+1)\eta_{\frac{k_j!}{j}} \\ &\leq \left(1 + \frac{k_j+1}{k_j}\right) \frac{k_j!}{j} \leq 3 \frac{k_j!}{j}. \end{aligned}$$

On the other hand,

$$(k_j+1)!(\xi_a)_{(k_j+1)!} \geq \sum_{i=k_j!+1}^{(k_j+1)!} \frac{1}{k_j+1} = \frac{k_j}{k_j+1} k_j! \geq \frac{1}{2} k_j!.$$

Combining the two inequalities, $(\xi_a)_{(k_j+1)!} \geq \frac{j}{6}(\eta_a)_{(k_j+1)!}$, whence $\xi_a \neq O(\eta_a)$ and hence $(\xi_a) \not\subset (\eta_a)$. \square

REFERENCES

- [1] Aljančić, S. and Arandelović, D. *O*-regularly varying functions, Publ. Inst. Math. (Beograd) (N.S.) **22(36)** (1977), 5–22.
- [2] Allen, G. D. and Shen, L. C., *On the structure of principal ideals of operators*, Trans. AMS **238** (1978), pp. 253–270.
- [3] Bingham, N. H. , Goldie, C. M., and Teugels, J. L., *Regular Variation*, Encyclopedia of Mathematics and its Applications (27), Cambridge University Press, 1989.
- [4] Blass, A., *Applications of superperfect forcing and its relatives*, Set Theory and its Applications, Lecture Notes in Mathematics 1401, Springer-Verlag (1989), 18–40.
- [5] Blass, A., *Groupwise density and related cardinals*, Arch. Math. Logic 30 (1990) 1–11.
- [6] Blass, A. and Weiss, G., *A Characterization and Sum Decomposition of Operator Ideals*, Trans. Amer. Math. Soc. **246** (1978), 407–417.
- [7] Brown, A., Percy, C., and Salinas, N., *Ideals of compact operators on Hilbert space*, Trans. Amer. Math. Soc. **18** (1971), 373–384.
- [8] Calkin, J. W., *Two-sided ideals and congruences in the ring of bounded operators in Hilbert space*, Ann. of Math. (2) **42** (1941), pp. 839–873.
- [9] Connes, A., *Non Commutative Geometry*, San Diego Academic Press, 1994.
- [10] Dykema, K., Figiel, T., Weiss, G., and Wodzicki, M., *The commutator structure of operator ideals*, Adv. Math., 185/1 pp. 1–79.
- [11] Dykema, K. J. and Kalton, N. J., *Spectral characterization of sums of commutators. II*, J. Reine Angew. Math. **504** (1998), pp. 127–137.
- [12] Dykema, K. J. and Kalton, N. J., *Sums of commutators in ideals and modules of type II factors* Ann. Inst. Fourier (Grenoble) **55** 3 (2005), pp. 931–971.
- [13] Dykema, K., Weiss, G. and Wodzicki, M., *Unitarily invariant trace extensions beyond the trace class*, Complex analysis and related topics (Cuernavaca, 1996), Oper. Theory Adv. Appl. **114** (2000), pp. 59–65.
- [14] Kaftal, V. and Weiss, G., *Traces, ideals, and arithmetic means*, Proc. Natl. Acad. Sci. USA **99** (11) (2002), pp. 7356–7360.
- [15] Kaftal, V. and Weiss, G., *Traces on operator ideals and arithmetic means*, J. Operator Theory, **63** (1) (2010), pp. 3–46.
- [16] Kaftal, V. and Weiss, G., *Soft ideals and arithmetic mean ideals*, Integral Equations and Operator Theory **58** (2007), pp. 363–405.
- [17] Kaftal, V. and Weiss, G., *2nd order arithmetic means and cancellation for operator ideals*, Operators and Matrices **1** (2) (2007), pp. 235–256.
- [18] Kaftal, V. and Weiss, G., *A survey on the interplay between arithmetic mean ideals, traces, lattices of operator ideals, and an infinite Schur-Horn majorization theorem*, Hot topics in operator theory, Theta Ser. Adv. Math., 9, Theta, Bucharest, (2008), pp. 101–135.
- [19] Kaftal, V. and Weiss, G., *An infinite dimensional Schur-Horn Theorem and majorization theory*. J. Functional Analysis **259** (2010), pp. 3115–3162.
- [20] Kaftal, V. and Weiss, G., *Majorization theory and arithmetic mean ideals*. Preprint.
- [21] Kalton, N. J., *Unusual traces on operator ideals*, Math. Nachr. **134** (1987), pp. 119–130.

- [22] Kalton, N. J., *Trace-class operators and commutators*, J. Funct. Anal. **86** (1989), pp. 41–74.
- [23] Kalton, N. J., *Spectral characterization of sums of commutators I*, J. Reine Angew. Math. **504** (1998), pp. 115–125.
- [24] Mildenberger, H., *Groupwise dense families*, Arch. Math. Logic **40** (2001), no. 2, pp. 93–112.
- [25] Salinas, N., *Symmetric norm ideals and relative conjugate ideals*, Trans. Amer. Math. Soc. **188** (1974), pp. 213–240.
- [26] Varga, J., *Traces on irregular ideals*, Proc. Amer. Math. Soc. **107** 3 (1989), pp. 715–723.
- [27] Weiss, G., *Commutators and Operators Ideals*, dissertation (1975), University of Michigan Microfilm.
- [28] Weiss, G., *Commutators of Hilbert-Schmidt operators II*, IEOT **3** (4) (1980), pp. 574–600.
- [29] Weiss, G., *Commutators of Hilbert-Schmidt operators I*, IEOT **9** (6) (1986), pp. 877–892.
- [30] Weiss, G., *B(H)-commutators: a historical survey*, Recent advances in operator theory, operator algebras, and their applications, pp. 307–320, Oper. Theory Adv. Appl., 153, Birkh" auser, Basel, 2005.
- [31] Wodzicki, M., *Algebraic K-theory and functional analysis*, First European Congress of Mathematics, Vol. II (Paris,1992) **120** (1994), pp. 485–496.
- [32] Wodzicki, M., *Vestigia investiganda*, Mosc. Math. J. **4** (2002), pp. 769–798, 806.

Received July 23, 2007

Revised version received January 7, 2009

UNIVERSITY OF CINCINNATI, DEPARTMENT OF MATHEMATICS, CINCINNATI, OH, 45221-0025,
USA

E-mail address: victor.kaftal@uc.edu

E-mail address: gary.weiss@uc.edu