Unitarily Invariant Trace Extensions Beyond the Trace Class

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Abstract

The existence of unitarily invariant trace extensions of the standard trace on the ideal of finite rank operators in $B(\mathcal{H})$ past the trace class to a strictly larger ideal is proven using matrix forms and a certain trace obstruction.

1 Introduction

This paper proves the existence of unitarily invariant trace extensions of the standard trace from the ideal of finite rank operators to an ideal strictly larger than the ideal of trace class operators. The proof presented here is our early matricial solution of the question about sums of commutators described below and proved in Theorem 2, the main theorem. This question arose in connection with the larger question: Which ideals have traces nonvanishing on the ideal of finite rank operators? Theorem 2 also follows as a consequence of the general characterization found in [DFWW].

All ideals herein are 2-sided ideals of $B(\mathcal{H})$, the class of all bounded linear operators on a separable, infinite-dimensional, complex Hilbert space $\mathcal{H}$. Let $\mathcal{F}$ denote the ideal of all finite rank operators on $\mathcal{H}$ and let $\mathcal{K}$ denote the ideal of all compact operators on $\mathcal{H}$. It is well-known that for all ideals $\mathcal{I}$, $\{0\} \subset \mathcal{F} \subset \mathcal{I} \subset \mathcal{K} \subset B(\mathcal{H})$ and that the standard trace, Tr, is a unitarily invariant linear functional on the trace class, C_1, hence on all ideals contained in C_1 and, in particular, on $\mathcal{F}$.

Ideals are determined by their characteristic sets $s(\mathcal{I})$ where $\mathcal{I} \leftrightarrow s(\mathcal{I}) := \{s(T) := \langle s_n(T)\rangle : T \in \mathcal{I}\}$ is a 1-1, onto, lattice preserving map from the class of ideals $\mathfrak{I} \leftrightarrow \mathfrak{S}$, the class of characteristic sets. Here, $\forall T \in \mathcal{K}$, define

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$s(T) := \langle s_n(T) \rangle$, the sequence of s-numbers of T. Characteristic sets, S, are those subsets of $c_o^{++} := \{x = \langle x_n \rangle : x$ is a decreasing sequence with $x_n \to 0\}$ that are closed under addition, scalar multiplication, ampliation ($x \in S$ implies $(x_1, x_1, x_2, x_2, x_3, \ldots) \in S$) and domination ($x_n \leq y_n \; \forall n \geq 1$ and $y = \langle y_n \rangle \in S$ implies $x \in S$) (see [G]).

A unitarily invariant trace, τ, on an ideal $\mathcal{I}$ of $B(\mathcal{H})$ is a unitarily invariant linear functional on $\mathcal{I}$, i.e., a linear functional where $\tau(T) = \tau(U^*TU) \; \forall T \in \mathcal{I}$ and for all unitary operators $U \in B(\mathcal{H})$.

A linear functional, τ, is unitarily invariant on $\mathcal{I}$ if and only if τ vanishes on the commutator space (also called the commutator ideal), $[\mathcal{I}, B(\mathcal{H})]$, the linear span of all $(\mathcal{I}, B(\mathcal{H}))$-single commutators $AB - BA$ with $A \in \mathcal{I}$ and $B \in B(\mathcal{H})$. This follows easily from the identity $[U^*, UT] = T - UTU^*$ and the fact that every $B(\mathcal{H})$ operator B is representable as a linear combination of 4 unitaries.

Every unitarily invariant trace τ on $\mathcal{I}$ is a unitarily invariant trace extension of some scalar multiple cTr from $\mathcal{F}$ to $\mathcal{I}$ since every ideal $\mathcal{I}$ contains the finite rank ideal $\mathcal{F}$; the restriction to $\mathcal{F}$ of any unitarily invariant trace τ on $\mathcal{I}$ is a unitarily invariant trace on $\mathcal{F}$; and the only unitarily invariant traces on $\mathcal{F}$ are the scalar multiples, cTr, of the standard trace Tr on $\mathcal{F}$. In other words, any unitarily invariant trace on $\mathcal{I}$ which is nonvanishing on $\mathcal{F}$ is, up to a nonzero scalar multiple, a unitarily invariant trace extension of Tr.

A necessary and sufficient condition for the existence of trace extensions from $\mathcal{J} \subset \mathcal{I}$ to $\mathcal{I}$ of a nonzero trace τ is that:

$$[\mathcal{I}, B(\mathcal{H})] \cap \mathcal{J} \subset \mathcal{J}^o,$$

where $\mathcal{J}^o := \{T \in \mathcal{J} : \tau(T) = 0\}$. To see this, first define τ on $\mathcal{J} + [\mathcal{I}, B(\mathcal{H})]$ via $\tau(J + S) := \tau(J) \; \forall J \in \mathcal{J}$ and $\forall S \in [\mathcal{I}, B(\mathcal{H})]$. Then τ is well-defined if and only if $[\mathcal{I}, B(\mathcal{H})] \cap \mathcal{J} \subset \mathcal{J}^o$. Extending τ from the linear subspace $\mathcal{J} + [\mathcal{I}, B(\mathcal{H})]$ nonuniquely to all of $\mathcal{I}$ is an easy Hamel basis argument.

Hence there is a trace on $\mathcal{I}$ which is nonvanishing on the ideal $\mathcal{F}$ of finite rank operators if and only if $[\mathcal{I}, B(\mathcal{H})] \cap \mathcal{F} \subset \mathcal{F}^o$.

The standard trace Tr on the trace class C_1 is a unitarily invariant trace extension from $\mathcal{F}$ to C_1. A natural ideal properly containing C_1 is the ideal, $\mathcal{I}_{o(1/n)}$, of operators with s-numbers $s_n(T) = o\left(\frac{1}{n}\right)$. The class $S(\mathcal{I}_{o(1/n)})$ satisfies the characteristic set axioms [G] and it properly contains C_1 since every decreasing summable sequence satisfies the $o\left(\frac{1}{n}\right)$ condition while the sequence $\left\langle \frac{1}{n \log n} \right\rangle$ is not summable but also satisfies the condition. This leads to the question of the existence of a unitarily invariant trace extension of Tr from $\mathcal{F}$ (beyond C_1) to $\mathcal{I}_{o(1/n)}$, or equivalently:

Question: If F is a finite rank operator which is a finite sum of commutators $AB - BA$ where $B \in B(H)$ and $A \in K$ with $s_n(A) = o\left(\frac{1}{n}\right)$, must $Tr\ F = 0$?

Note. There are no positive unitarily invariant trace extensions τ of Tr from $\mathcal{F}$ (beyond C_1) to a strictly larger ideal $\mathcal{I}$. Assuming otherwise, for any $T \in \mathcal{I} \backslash C_1$, multiply U^* in its polar decomposition $T = U|T|$ so that we can without loss of generality assume $T \geq 0$. Then $\tau(T) \geq \tau(P_n T) = Tr(P_n T) \to \|T\|_{C_1} = \infty$, where P_n denotes the projection onto the span of the eigenvectors of the largest n-eigenvalues of T. Hence $\tau(T) = \infty$, which is a contradiction.

Let $\left\langle \frac{1}{n} \right\rangle$ denote the sequence $(1, 1/2, 1/3, \ldots, 1/n, \ldots)$. The main theorem is:

Theorem 2. The ideal $\mathcal{I}_{o(1/n)}$ possesses unitarily invariant trace extensions of the standard trace on the ideal of finite rank operators.

Remark. A necessary and sufficient condition for an ideal $\mathcal{I}$ to possess unitarily invariant trace extensions from $\mathcal{F}$ to $\mathcal{I}$ which is nonvanishing on $\mathcal{F}$ is that $\operatorname{diag}\left\langle \frac{1}{n} \right\rangle \notin \mathcal{I}$ (i.e., $\left\langle \frac{1}{n} \right\rangle \notin S(\mathcal{I})$), but his requires more development (see [DFWW] and a variation on [PT] or a slight variation on the simpler [W1, pp. 34-35]).

History. The role of the classical trace Tr is well-known. In a series of recent papers, Alain Connes linked some exotic traces (Dixmier traces on the dual Macaev ideal) to questions in Physics [Co 1, 2], theory of noncommutative residue [Co 1], [Wod 1], theory of quasifuchsian groups [Co 3], [Co Sul], and questions about Hausdorff measure on Julia sets [Co 3]. Which ideals possess a nontrivial trace and the ability to describe all interesting exotic traces on such ideals often yield significant applications.

Traces provide a dual viewpoint on commutator spaces. Indeed the trace results in [DFWW] evolved from the study of commutator spaces. The history of commutators dates back to one of the formulations of Heisenberg's uncertainty principle. After the characterization of the class of single $(\mathcal{B}(H), \mathcal{B}(H))$-commutators [BrPe 1] (see also [AndSta], [BrHaPe], [BrPe 2], and [Ha 1-3]), Pearcy and Topping [PeTo 2] began the study of commutator spaces of operator ideals with several questions. One question was whether or not $[C_2, C_2] = C_1^o :=$ the trace zero operators. The second listed author obtained results in this area [W 1-4]. In particular, $[C_2, C_2] \neq C_1^o$, which follows from the characterization in [W 2-3] of those diagonal operators $\text{diag}(-1, d_1, d_2, ...) \in [C_2, C_2]$ with $\langle d_n \rangle$ positive and decreasing and $\Sigma\, d_n = 1$. Indeed, $\text{diag}(-1, d_1, d_2, ...) \in [C_2, C_2]$ if and only if $\Sigma\, \mathrm{d}_n \log n < \infty$. Then in [Kal 1], Kalton proved $[C_2, C_2] = [C_1, B(H)]$ and characterized this commutator space as follows. For each operator $T \in \mathcal{K}$, let $\langle \lambda_n \rangle$ denote its nonzero eigenvalue sequence arranged in order of decreasing moduli counting algebraic multiplicities and inserting zeros if T has only finitely many nonzero eigenvalues. Denote $\mathrm{C}(T) = \text{diag}\left\langle \frac{\lambda_1 + \ldots + \lambda_n}{n} \right\rangle$, the diagonal operator with entries the Cesaro sequence from the eigenvalue sequence, $\langle \lambda_n \rangle$, of T. Then $T \in [C_2, C_2] = [C_1, B(H)]$ if and only if $\mathrm{C}(T) \in C_1$, i.e., $\|\mathrm{C}(T)\|_1 = \sum \left| \frac{\lambda_1 + \ldots + \lambda_n}{n} \right| < \infty$. More recently in [DFWW] we proved, with Figiel, for arbitrary ideals I, J, that $[I, J] = [IJ, B(H)]$ and that this general commutator space is characterized as those operators $T \in IJ$ whose real and imaginary parts, Re T and Im T, satisfy $\mathrm{C}(\text{Re } T), \mathrm{C}(\text{Im } T) \in IJ$. Other important contributions were [And 1-2], [AndVas], [Kal 2] and [DykKal].

2 General block upper-Hessenberg forms.

Theorem 1. For every finite sequence of operators $T_1, T_2, \ldots, T_k$ with cyclic vector e (i.e., with $\{p(T_1, T_2, \ldots, T_k)e : p \text{ is a } k\text{-variable polynomial}\}$ dense in $\mathcal{H}$), e extends to a basis on which all $T_1, T_2, \ldots, T_k$ simultaneously have block tri-diagonal matrix forms with the same block sizes:

$$\begin{pmatrix} d_1 & b_1 & * & * & . \\ a_1 & d_2 & b_2 & * & . \\ 0 & a_2 & d_3 & . & . \\ 0 & 0 & . & . & . \\ . & . & . & . & . \end{pmatrix}$$

where a_n, b_n, d_n are finite rectangular matrices with height a_n = width d_{n+1} = height d_{n+1} = width $b_n = O(k^n)$.

Remark. In [WW] two additional related matrix forms are obtained.

(i) In the above matrix, one can replace the $*$'s with 0's as follows. For every finite sequence of operators $T_1, T_2, \ldots, T_k$ with $*$-cyclic vector e (i.e., with $\{p(T_1, T_1*, T_2, T_2*, \ldots, T_k, T_k*)e$: p is a $2k$-variable polynomial $\}$ dense in $\mathcal{H}$), e extends to a basis in which all $T_1, T_2, \ldots, T_k$ simultaneously have blocked tri-diagonal matrix forms with the same block sizes: height $a_n =$ width $d_{n+1} =$ height $d_{n+1} =$ width $b_n = O((2k)^n)$.

(ii) If $AB - BA$ is finite rank, then there exists A', B', and a projection P with $AP = A' \oplus 0$, $BP = B' \oplus 0$ with $A'B' - B'A'$ finite rank and $Tr(A'B' - B'A') = Tr(AB - BA)$ where A' and B' simultaneously have block upper-Hessenberg matrix form with the same block sizes: height $a_n =$ width $d_{n+1} =$ height $d_{n+1} =$ width $b_n = O(n)$.

Proof of Theorem 1. The set of n^{th}-order words in $T_1, T_2, \ldots, T_k$ is given by the disjoint union $W_n(T_1, T_2, \ldots, T_k) = \bigcup_{j=1}^{k} W_{n-1}(T_1, T_2, \ldots, T_k)T_j$ which leads to the recursive cardinality equation $|W_n(T_1, T_2, \ldots, T_k)| = k|W_{n-1}(T_1, T_2, \ldots, T_k)|$ with $|W_1(T_1, T_2, \ldots, T_k)| = k$, hence $|W_n(T_1, T_2, \ldots, T_k)| = k^n$.

Now denote the subspace $M_n = \operatorname{span}_{1 \leq j \leq n} W_j(T_1, T_2, \ldots, T_k)e$ having $\dim M_n \leq k^n$, and for all $1 \leq j \leq k$, $T_j M_n \subset M_{n+1}$. Then the subspaces $M_{n+1} \ominus M_n$ yield block upper-Hessenberg forms as above, all the T_j's with the same block sizes and with a_n, b_n, d_n matrix sizes $O(k^n)$.

3 Matrix forms and commutators.

The proof of Theorem 2, the main theorem, is obtained from the interplay between matrix forms and commutators, in particular, a certain diagonal trace obstruction.

Theorem 2. The ideal $\mathcal{I}_{o(1/n)}$ possesses unitarily invariant trace extensions of the standard trace on the ideal of finite rank operators.

Proof: As previously mentioned, Theorem 2 is equivalent to the inclusion: $[\mathcal{I}_{o(1/n)}, B(\mathcal{H})] \cap \mathcal{F} \subset \mathcal{F}^0$. So suppose $F = \sum_{k=1}^{m}(A_kB_k = B_kA_k)$ is a finite rank operator with each $A_k \in \mathcal{I}_{o(1/n)}$ and $B_k \in B(\mathcal{H})$ and assume to the contrary that $Tr\ F \neq 0$.

First we reduce to the case where $\langle A_k \rangle$, $\langle B_k \rangle$ has a cyclic vector. Use the general block tri-diagonal upper-Hessenberg forms in Theorem 1 to put all the operators in the finite sequences $\langle A_k \rangle$ and $\langle B_k \rangle$ simultaneously into an infinite direct sum (or finite direct sum according to the number of cyclic subspaces required) of block tri-diagonal forms. The restriction to each of these subspaces of $\sum_{k=1}^{m}(A_kB_k - B_kA_k)$ is again finite rank and retains this block matrix form. Since $\sum_{k=1}^{m}(A_kB_k-B_kA_k) \in C_1$, the sequence of the traces of these restrictions sums to $Tr\sum_{k=1}^{m}(A_kB_k = B_kA_k) \neq 0$. Therefore at least one restriction has non-zero trace. In other words, restricting to a common reducing subspace for all A_k, B_k, the finite sequences $\langle A_k\rangle \subset \mathcal{I}_{o(1/n)}$, $\langle B_k\rangle \subset B(\mathcal{H})$ then have a cyclic vector as defined in Theorem 1, the restriction of $\sum_{k=1}^{m}(A_kB_k - B_kA_k)$ also has block matrix form, $\sum_{k=1}^{m}(A_kB_k - B_kA_k)$ is finite rank, and the trace of the restriction $Tr\sum_{k=1}^{m}(A_kB_k-B_kA_k) \neq 0$. So, without loss of generality, we may assume the set $\langle A_k\rangle$, $\langle B_k\rangle$ has a cyclic vector and hence, for each $1 \leq k \leq m$,

$$A_k = \begin{pmatrix} d_1(k) & b_1(k) & * & * & . \\ a_1(k) & d_2(k) & b_2(k) & * & . \\ 0 & a_2(k) & d_3(k) & . & . \\ 0 & 0 & . & . & . \\ . & . & . & . & . \end{pmatrix}, \quad B_k = \begin{pmatrix} d_1'(k) & b_1'(k) & * & * & . \\ a_1'(k) & d_2'(k) & b_2'(k) & * & . \\ 0 & a_2'(k) & d_3'(k) & . & . \\ 0 & 0 & . & . & . \\ . & . & . & . & . \end{pmatrix}$$

simultaneously with block tri-diagonal forms all with the same block sizes and of order $O((2k)^n)$.

Computing the diagonal of $F = \sum_{k=1}^{m}(A_kB_k - B_kA_k)$, the first diagonal block entry $D_1 = \sum_{k=1}^{m}(d_1(k)d_1'(k) - d_1'(k)d_1(k) + b_1(k)a_1'(k) - b_1'(k)a_1(k))$, followed by the n^{th} block diagonal entry, for $n \geq 2$,

$$D_n = \sum_{k=1}^{m}(d_n(k)d_n'(k)-d_n'(k)d_n(k)+b_n(k)a_n'(k)+a_{n-1}(k)b_{n-1}'(k)-a_{n-1}'(k)b_{n-1}(k)-b_n'(k)a_n(k)).$$

Then

$$\begin{aligned} Tr\left(\sum_{j=1}^{n} D_j\right) &= \sum_{j=1}^{n} Tr\ D_j \\ &= \sum_{k=1}^{m}\sum_{j=1}^{n} Tr(d_j(k)d_j'(k) - d_j'(k)d_j(k)) + \sum_{k=1}^{m}\sum_{j=1}^{n} Tr(b_j(k)a_j'(k) - a_j'(k)b_j(k)) \end{aligned}$$

$$+\sum_{k=1}^{m}\sum_{j=1}^{n} Tr(a_j(k)b_j'(k) - b_j'(k)a_j(k)) + Tr(b_n(k)a_n'(k) - b_n'(k)a_n(k))$$

$$= Tr\sum_{k=1}^{m}(b_n(k)a_n'(k) - b_n'(k)a_n(k)),$$

since all matrices $a_n(k), b_n(k), d_n(k), a_n'(k), b_n'(k), d_n'(k)$ are finite rank and all commutators of finite rank operators have trace 0. But also $Tr\left(\sum_{j=1}^{n} D_j\right) \to Tr\ F \neq 0$. That is, for n sufficiently large,

$$\begin{aligned} 0 &< |Tr\ \mathcal{F}|/2 \\ &\leq \left|Tr\sum_{k=1}^{m}(b_n(k)a_n'(k) - b_n'(k)a_n(k))\right| \\ &\leq \sum_{k=1}^{m}(\|b_n(k)\|_1\|a_n'(k)\| + \|b_n'(k)\|\|a_n(k)\|_1) \\ &\leq \|B\|\sum_{k=1}^{m}(\|b_n(k)\|_1 + \|a_n(k)\|_1). \end{aligned}$$

So for n sufficiently large, $\sum_{k=1}^{m}(\|b_n(k)\|_1 + \|a_n(k)\|_1)$ is bounded below with $\sum_{k=1}^{m}(\|b_n(k)\|_1 + \|a_n(k)\|_1) \geq \varepsilon := |Tr\ \mathcal{F}|/2\|B\|$.

We next construct an operator $T \in \mathcal{I}_{o(1/n)}$ having matrix block diagonal $\left\langle \sum_{k=1}^{m}(|a_n(k)| + |b_n(k)|)\right\rangle$. It suffices to show how to obtain, for each fixed k, an operator $A' \in \mathcal{I}_{o(1/n)}$ with n^{th} block diagonal entry $|a_n(k)|$. Then constructing $B' \in \mathcal{I}_{o(1/n)}$ with matrix block diagonals $|b_n(k)|$ follows similarly and adding $A' + B'$ and summing over k yields T. Using the polar decompositions $a_n(k) = u_n|a_n(k)|$, $A' = A_k * \begin{pmatrix} 0 & 0 & 0 & 0 \\ u_1 & 0 & 0 & 0 \\ 0 & u_2 & 0 & . \\ 0 & 0 & . & . \end{pmatrix}$ has block diagonal entry $|a_n(k)|$. Although the blocks of A_k are not all square matrices and hence the polar decompositions are not all square matrices, still this product is well-defined in that all matrix sizes match up properly when performing the multiplication and $A' \in \mathcal{I}_{o(1/n)}$ is achieved.

Let P_n denote the projection onto the domain of $\begin{pmatrix} . & . & . & . & . \\ . & 0 & 0 & 0 & . \\ . & 0 & I & 0 & . \\ . & 0 & 0 & 0 & . \\ . & . & . & . & . \end{pmatrix}$ with block sizes the same as all the A_k's. Then $\dim P_n = O((2k)^n)$ and, for all sufficiently large n, say $n \geq N$, $0 < \varepsilon \leq \sum_{k=1}^{m}(\|b_n(k)\|_1 + \|a_n(k)\|_1) = Tr(P_nTP_n)$. Hence $\dim P_n \leq M(2k)^n$ for some M and, using Fan's Theorem [GK, II.4, Lemma 4.1], for all $n \geq N$,

$$\varepsilon(n-N) \leq \sum_{j=N}^{n} Tr(P_jTP_j) \leq \sum_{j=1}^{n} Tr(P_jTP_j) \leq \sum_{j=1}^{\dim P_1+\dim P_2+\cdots+\dim P_n} s_j(T) \leq \sum_{j=1}^{c(2k)^n} s_j(T).$$

From $\varepsilon(n-N) \leq \sum_{j=1}^{c(2k)^n} s_j(T)$ for all $n \geq N$, it is routine to show that, for some $\varepsilon' > 0$, $\varepsilon' \log n \leq \sum_{j=1}^{n} s_j(T)$, for all sufficiently large n. But $T \in \mathcal{I}_{o(1/n)}$ implies $s_j(T) = o\left(\frac{1}{j}\right)$, hence $\sum_{j=1}^{n} s_j(T) = o(\log n)$, which is a contradiction.

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