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Some C_4 and C_6 Norm Inequalities Related to the Paving Problem

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§1. **Introduction and notations.** The central problem in this area is the extension problem (Kadison and Singer, 1959, **[6]**): Does every pure state of the atomic masa D of the diagonal operators on a separable Hilbert space H have a unique extension to a pure state of $B(H)$?

Equivalent to it is a formulation in terms of finite matrices called the paving problem (Anderson, **[1]**, **[2]**): Is there an integer k such that for every finite matrix x with zero diagonal, there is a decomposition of the identity into k mutually orthogonal diagonal projections p_m such that

$$\left\| \sum_{m=1}^{k} p_m x p_m \right\| \leq \frac{1}{2}\|x\|?$$

In this paper we consider the paving problem with respect to the Schatten C_p-norms: Is there a (minimal) integer $k = k(p)$ such that, for every finite matrix x with zero diagonal, there is a decomposition of the identity into k mutually orthogonal diagonal projections p_m such that

$$\left\| \sum_{m=1}^{k} p_m x p_m \right\|_p \leq \frac{1}{2}\|x\|_p?$$

We show that the answer is affirmative for $p = 4$ and $p = 6$.

The crucial question, however, is: How does the (minimal) number $k = k(p)$ depend on p? Indeed, since $\lim_{p\to\infty} \|x\|_p = \|x\|$ for all finite matrices x, determining that $\sup k(p) < \infty$ would provide a positive answer to the paving problem (and hence to the extension problem). This connection was initially our main motivation. We found, however, that the investigation, even for low values of p, leads to some interesting and hard matrix norm inequalities, which appear to be worth pursuing for their own merit.

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Our main result is that, for $p = 4$ and $p = 6$ and any finite selfadjoint matrix x with zero diagonal, there is a decomposition of the identity into k mutually orthogonal diagonal projections p_m (with $k \geq 4$ a perfect square) such that

$$\left\| \sum_{m=1}^{k} p_m x p_m \right\|_p \leq a_p k^{-[(1/4)+(1/2p)]} \|x\|_p,$$

where $a_4 = 2^{1/4}$ (Theorem 5) and $a_6 = 5.3^{1/6}$ (Theorem 8).

From these inequalities for selfadjoint matrices, one can easily derive analogous inequalities for nonselfadjoint matrices. Or, to obtain better upper bounds, one can reproduce the selfadjoint proof, but with an increase in computational complexity. Therefore, we shall consider here only selfadjoint matrices.

Our techniques are a blend of combinatorial and probabilistic methods. We would like to sketch them in this introduction and illustrate why we feel that both are needed.

Throughout this paper x will denote an $n \times n$ selfadjoint matrix with zero diagonal and entries x_{ij}, and $k > 1$ will be a fixed positive integer that does not depend on x (nor on p). In Theorems 5 and 8, we shall furthermore assume that k is a perfect square, hence $k \geq 4$.

Every diagonal projection corresponds to (and will be identified with) a subset of $\{1, 2, \ldots, n\}$. For a decomposition of the identity into k mutually orthogonal diagonal projections p_m (a k-decomposition, or a k-coloring of $\{1, 2, \ldots, n\}$ in the language of combinatorics) we call

$$x' = \sum_{m=1}^{k} p_m x p_m$$

a k-paving of x or simply a paving of x if k is understood. If p is even, then

$$\|x\|_p^p = \operatorname{tr} x^p = \sum x_{i_1 i_2} x_{i_2 i_3} \cdots x_{i_p i_1}. \tag{1}$$

By decomposing this sum into apropriate pieces (see §4 on trees for indices), we can explicitly compute the average $E\|x'\|_p^p$ over all the possible k-decompositions. An upper bound for this average will of course provide an upper bound for the minimal value of $\|x'\|_p$, which is what we are seeking.

While this simple averaging approach may work, i.e., yield a reasonable upper bound, when the matrices are 'diffuse', e.g., have entries with constant modulus, a moment's reflection shows that it cannot work alone in the general case. Indeed, if we go to the 'other extreme' and we take a matrix with at most one nonzero entry per row and column and with zero diagonal (a 'very sparse matrix' henceforth), then we know from **[6**, Theorem 3] that there is a paving $x' = 0$. This paving, however, may be unique (e.g., the nonselfadjoint unitary shift for n even and $k = 2$), while $E\|x'\| \approx 1$ and the average $E\|x'\|_p^p$ also may be relatively 'large'. In the case where x is very sparse and selfadjoint, (12) provides a precise formula for $E\|x'\|_4^4$.

The existence of such a paving $x' = 0$, which is easy to prove for very sparse matrices, can also be derived from the following more general 'combinatorial' result, which has already played an important role in work on the paving problem [**1**, Theorem 2], [**3**, Corollary 3.3], [**5**, Theorem 1.3′]:

(2) There is a k-paving x' such that $\|x'e_i\| \leq k^{-1/2}\|xe_i\|$ for all i (here $\{e_i\}$ denotes the basis of H for which the elements of D are diagonal). If x were not selfadjoint, then we would replace $k^{-1/2}$ by $2^{1/2}k^{-1/2}$.

Reducing the row norms of a matrix x by passing to the paving x' given by (2) will—very roughly speaking—eliminate the peak entries of the matrix x. But then we can average the further pavings x'' of this matrix x' (which are themselves pavings of x, but for a finer diagonal decomposition) and we will obtain in general a better upper bound. We make this heuristic argument precise in §3 where we show how our purely 'probabilistic' estimates are indeed improved by this method.

We present here in detail the computations for the case $p = 4$. Since the computational complexity grows exponentially, in keeping with the expository purpose of this paper, we only sketch our argument in the $p = 6$ case and provide examples of the key subcases that best illustrate the nature of the proof.

Not surprisingly, this method will not reproduce the best results for the 'extreme' cases, but this is the price to pay for having a unified approach. This fact is immediately obvious for very sparse matrices, because, as we have remarked above, there may exist a unique paving equal to zero. For 'diffuse' matrices, reducing combinatorially the row norms first also is not the most efficient strategy: The purely probabilistic method will yield somewhat better results. To illustrate this point, and because they are important on their own, we analyze in §6 the pavings of Hadamard matrices h (unitary $n \times n$ matrices with entries $\pm n^{-1/2}$ and in our case also selfadjoint and with diagonal removed, i.e., set equal to zero). We obtain in (24) that if $p = 4$ the estimate $k^{-3/8}$ in Theorem 5 can be decreased (asymptotically for $n \to \infty$) to $k^{-1/2}$, and for $p = 6$, we prove in Theorem 9 that the estimate $k^{-1/3}$ in Theorem 8 can be decreased to $k^{-1/2}$ and 5.3 can be decreased to 5 (asymptotically). These upper bounds can be compared with $k^{-1/2}$, which is the lower bound (asymptotically) for the C_p-norm of any paving h' for $p \geq 2$ [**3**, Theorem 4.2].

In our proofs we shall often use the following two inequalities: If x is any positive operator and e is any unit vector in the Hilbert space, then we have for $p \geq 1$ [7, Lemma 2.1]:

$$(x^p e, e) \geq (xe, e)^p. \tag{3}$$

For any finite matrix x, when $p \geq 2$,

$$\|x\|_p^p \geq \sum \|xe_i\|^p \geq \sum |x_{ij}|^p. \tag{4}$$

The first inequality in (4) follows from (3), and the proof of the second inequality is obvious.

We wish to thank W. Bryc for useful suggestions.

§2. Averaging and the cases $p = 2$ and $p = 4$.

DEFINITION 1. The operation E denotes the average over the collection of all k-decompositions.

Equivalently, E is the average over the collection (of cardinality k^n) of all partitions (or colorings) of $\{1, \dots, n\}$ into k subsets (some of which may be empty).

Notice that if x' is a paving of x, then

$$(5) \qquad x'_{ij} = \begin{cases} x_{ij} & \text{if } i, j \in p_m \quad \text{for some } m, \text{ i.e., have the same color} \\ 0 & \text{otherwise.} \end{cases}$$

LEMMA 2. *If $(i_1, i_2, i_3, \dots, i_p)$ is a p-tuple of the integers $\{1, 2, \dots, n\}$ with precisely q distinct entries, then*

$$E(x'_{i_1 i_2} x'_{i_2 i_3} \cdots x'_{i_p i_1}) = k^{-(q-1)} x_{i_1 i_2} x_{i_2 i_3} \cdots x_{i_p i_1}.$$

PROOF. By (5), the only colorings contributing to the average

$$E(x'_{i_1 i_2} x'_{i_2 i_3} \cdots x'_{i_p i_1})$$

are those for which each pair $\{i_1, i_2\}, \{i_2, i_3\}, \dots, \{i_{p-1}, i_p\}$ has the same color, i.e., for which $\{i_1, i_2, i_3, \dots, i_p\}$ all have the same color; and for each of these colorings we have $x'_{i_1 i_2} x'_{i_2 i_3} \cdots x'_{i_p i_1} = x_{i_1 i_2} x_{i_2 i_3} \cdots x_{i_p i_1}$. Thus

$$E(x'_{i_1 i_2} x'_{i_2 i_3} \cdots x'_{i_p i_1}) = c x_{i_1 i_2} x_{i_2 i_3} \cdots x_{i_p i_1},$$

where c is the probability that $\{i_1, i_2, i_3, \dots, i_p\}$ all have the same color. Clearly, c is also the probability that $\{1, \dots, q\}$ all have the same color, hence $c = k^{-(q-1)}$.

REMARK. Lemma 2 does not require x to have zero diagonal nor to be selfadjoint.

CASE $p = 2$ (warm-up). By Lemma 2, $E|x'_{ij}|^2 = k^{-1}|x_{ij}|^2$ if $i \neq j$, i.e., $q = 2$. The same holds also when $i = j$ by (5) and the assumption that x has a 0-diagonal. So, by summing over all i and j, we obtain $E\|x'\|_2^2 = k^{-1}\|x\|_2^2$. Thus, by the minimum principle,

$$\min \|x'\|_2^2 \leq E\|x'\|_2^2 = k^{-1}\|x\|_2^2,$$

where the minimum is taken over all k-decompositions. Therefore, there is a k-paving x' such that (see [**3**, Theorem 4.4])

$$(6) \qquad \|x'\|_2 \leq k^{-1/2}\|x\|_2.$$

Notice that in general $\|x\|_2^2 = \sum |x_{ij}|^2$, and since exactly the same argument holds, there would be no advantage here in assuming that x is selfadjoint.

CASE $p = 4$. By (1), $\operatorname{tr} x^4 = \sum x_{ij}x_{jk}x_{kl}x_{li}$. Since $x_{ii} = 0$ for all i, we may assume that $i \neq j$, $j \neq k$, $k \neq l$, $l \neq i$, i.e., that consecutive indices are distinct. Then we can split this sum into sums with 4, 3, or 2 distinct indices:

$$\|x\|_4^4 = \sum^{(4)}(x) + 2\sum^{(3)}(x) + \sum^{(2)}(x), \tag{7}$$

where

$$\sum^{(4)}(x) = \sum\{x_{ij}x_{jk}x_{kl}x_{li}\colon 1 \leq i, j, k, l \leq n \text{ and are distinct}\},$$
$$\sum^{(3)}(x) = \sum\{|x_{ij}|^2|x_{ik}|^2\colon 1 \leq i, j, k \leq n \text{ and are distinct}\},$$
$$\sum^{(2)}(x) = \sum\{|x_{ij}|^4\colon 1 \leq i, j \leq n \text{ and are distinct}\}.$$

In other words, all monomials $x_{ij}x_{jk}x_{kl}x_{li}$ with 4 distinct indices are summed in $\sum^{(4)}(x)$; those with 2 distinct indices must have $i = k$ and $j = l$, and so are summed in $\sum^{(2)}(x)$. Finally, those with 3 distinct indices have two cases, namely, $i = k$ but $j \neq l$, or $i \neq k$ but $j = l$. Since the corresponding sums are equal, this accounts for $\sum^{(3)}(x)$ occurring with a coefficient of 2 in equation (7). This decomposition can be described also in terms of a certain tree, which we shall discuss in §4. Notice, moreover, that

$$\sum^{(3)}(x) + \sum^{(2)}(x) = \sum \|xe_i\|^4 = \|D(x^2)\|_2^2 \leq \|x\|_4^4, \tag{8}$$

where $D(x)$ denotes the diagonal of x, and

$$\sum^{(4)}(x) + \sum^{(3)}(x) = \|x^2 - D(x^2)\|_2^2, \tag{9}$$

which together yield

$$\|x\|_4^4 = \|x^2 - D(x^2)\|_2^2 + \|D(x^2)\|_2^2. \tag{10}$$

The inequality in (8) is just (4). (10) also has a simple operator theoretic proof, namely, write $x^2 = (x^2 - D(x^2)) + D(x^2)$, square both sides, and apply the trace. Lemma 2 applied to equation (7) for x' yields

PROPOSITION 3. $E\|x'\|_4^4 = k^{-3}\sum^{(4)}(x) + 2k^{-2}\sum^{(3)}(x) + k^{-1}\sum^{(2)}(x)$.

Notice that this expansion of $E\|x'\|_4^4$ into powers of k^{-1} has no constant term, due to the fact that $D(x) = 0$ and so there is no $\sum^{(1)}(x)$ term in the expansion (7) of $\|x\|_4^4$. We rewrite this expansion in operator theoretic terms using equations (7) and (8):

$$\begin{aligned} E\|x'\|_4^4 &= k^{-3}\|x\|_4^4 + (2k^{-2} - 2k^{-3})\sum^{(3)}(x) + (k^{-1} - k^{-3})\sum^{(2)}(x) \\ &= k^{-3}\|x\|_4^4 + 2k^{-2}(1 - k^{-1})\sum \|xe_i\|^4 + k^{-1}(1 - k^{-1})^2\sum^{(2)}(x). \end{aligned}$$

Using (4) we thus have

$$E\|x'\|_4^4 \le k^{-3}\|x\|_4^4 + (k^{-1} - k^{-3})\sum \|xe_i\|^4. \tag{11}$$

So, by using (4) again, we have $E\|x'\|_4^4 \le k^{-1}\sum \|x\|_4^4$, and hence by the minimum principle applied to (11), we obtain

PROPOSITION 4. *There is a k-paving x' of x such that $\|x'\|_4 \le k^{-1/4}\|x\|_4$.*

§**3. Integration of combinatorial and probabilistic methods for $p = 4$.** Proposition 4 already enables us to answer in the affirmative the paving problem for C_4, as it immediately yields $k(4) \le 16$ (where $k(4)$ denotes the mimimal integer k so that $\|x'\|_4 \le \frac{1}{2}\|x\|_4$).

We can, however, improve on this estimate. Indeed, in the expansion of $E\|x'\|_4^4$ into powers of k^{-1} given in Proposition 3, the coefficients of the powers of k^{-1} are a priori all of the order of magnitude of $\|x'\|_4^4$ so that the $k^{-1}\sum^{(2)}(x)$ term could be the dominant one.

This indeed is the case if the matrix x is very sparse (e.g., a finite direct sum of $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$), because then

$$\sum^{(4)}(x) = \sum^{(3)}(x) = 0 \quad \text{and} \quad \sum^{(2)}(x) = \sum \|xe_i\|^4 = \|x\|_4^4.$$

Thus we see that the upper bound given in Proposition 4 is now attained:

$$E\|x'\|_4^4 = k^{-1}\|x\|_4^4. \tag{12}$$

But in the case of very sparse matrices, the gap between average and minimum for $\|x'\|_4^4$ is particularly large, because, as we have noticed in the introduction, combinatorial methods, either directly or by using (2), show that $\min \|x'\|_4 = 0$.

On the other hand, the averaging method works fairly well for matrices with $|x_{ij}| = \text{constant}$, since then the coefficient $\sum^{(2)}(x)$ of k^{-1} is much smaller than the other coefficients (see §6 and [**3**, Proposition 4.6]).

We can now integrate the combinatorial method (2) with the averaging process used above to improve the estimate in Proposition 4.

Assume k is a perfect square and choose the (possibly unique) paving x' of x into $k^{1/2}$ 'colors' given by (2) such that $\|x'e_i\| \le k^{-1/4}\|xe_i\|$ for all i. Keeping this x' fixed, average all pavings x'' of x' into a further $k^{1/2}$ colors. Then, by (11) (applied to x' and $k^{1/2}$ colors), the fact that $\|x'\|_4 \le \|x\|_4$, and by (4), we have

$$\begin{aligned} E\|x''\|_4^4 &\le k^{-3/2}\|x'\|_4^4 + (k^{-1/2} - k^{-3/2})\sum \|x'e_i\|^4 \\ &\le k^{-3/2}\|x\|_4^4 + (k^{-1/2} - k^{-3/2})k^{-1}\sum \|xe_i\|^4 \\ &\le 2k^{-3/2}\|x\|_4^4. \end{aligned}$$

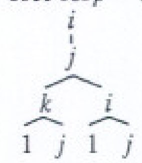
Now every $k^{1/2}$-paving x'' of x' is also a k-paving of x, so that from the above inequality and from the minimum principle we get

THEOREM 5. *There is a k-paving x' such that $\|x'\|_4 \leq 2^{1/4}k^{-3/8}\|x\|_4$.*

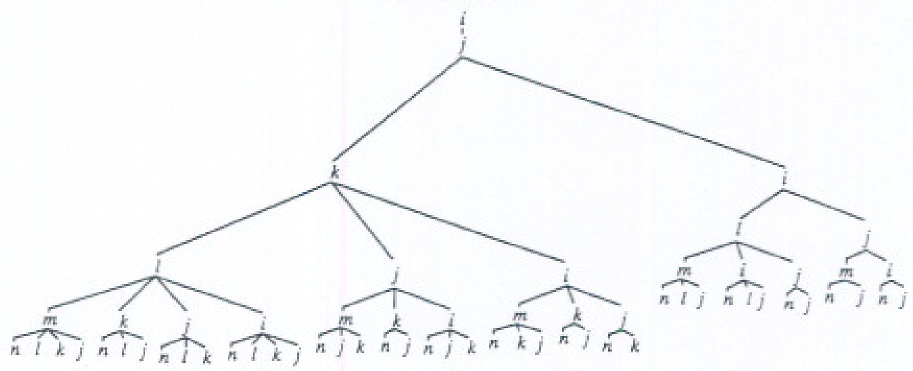
This is a better estimate than the one obtained in Proposition 4 for all k, (since $k \geq 4$) but particularly so for large values of k. Notice for instance that an upper bound for $k(4)$ is now 11.

§4. Trees for indices. In order to generalize (7) to the case $p = 6$ (or larger even integers) we need to split the sum $\|x\|_p^p = \operatorname{tr} x^p = \sum x_{i_1 i_2} x_{i_2 i_3} \cdots x_{i_p i_1}$ into sub-sums corresponding to different groupings of their associated p-tuples $(i_1, i_2, i_3, \ldots, i_p)$. To describe these groupings we introduce a tree for the p-tuples and we sum over the individual branches of the tree. In the tree, distinct letters denote distinct values. Instead of a rigorous definition of the tree, it is simpler to present the self-explanatory tree for the case of $p = 4$, where we have already seen the decomposition, and for $p = 6$.

Tree for $p = 4$



Tree for $p = 6$



Thus the $p = 4$ tree has four branches: The branch i, j, k, l (all values distinct) that yields the sum $\sum^{(4)}(x)$, the branch i, j, i, j that yields the sum $\sum^{(2)}(x)$, and the two branches i, j, k, j and i, j, i, l that yield the same sum $\sum^{(3)}(x)$, as one sum is obtained from the other through an elementary change of variables in the summation indices. Notice that this change of variables corresponds to the action of a permutation matrix on x. We shall call these two branches of the tree isomorphic; also we shall call the sums $\sum^{(4)}(x)$, $\sum^{(3)}(x)$, and $\sum^{(2)}(x)$ the invariant sums.

The $p = 6$ tree is already considerably more complex because it has 41 branches. We list below the 12 invariant sums, and in parentheses, their isomorphism classes, where we number the branches left to right.

$$\begin{aligned}
\textstyle\sum^{(6)}(x) &= \textstyle\sum x_{ij}x_{jk}x_{kl}x_{lm}x_{mn}x_{ni} && (1),\\
\textstyle\sum^{(5,1)}(x) &= \textstyle\sum x_{ij}x_{ji}x_{il}x_{lm}x_{mn}x_{ni} && (2,4,5,11,15,30),\\
\textstyle\sum^{(5,2)}(x) &= \textstyle\sum x_{ij}x_{jk}x_{ki}x_{im}x_{mn}x_{ni} && (3,8,23),\\
\textstyle\sum^{(4,1)}(x) &= \textstyle\sum x_{ij}x_{ji}x_{ij}x_{jm}x_{mn}x_{ni} && (6,12,14,18,32,38),\\
\textstyle\sum^{(4,2)}(x) &= \textstyle\sum x_{ij}x_{ji}x_{il}x_{lj}x_{jn}x_{ni} && (9,13,16,25,26,36),\\
\textstyle\sum^{(4,3)}(x) &= \textstyle\sum x_{ij}x_{ji}x_{il}x_{lm}x_{ml}x_{li} && (7,20,31),\\
\textstyle\sum^{(4,4)}(x) &= \textstyle\sum x_{ij}x_{jk}x_{ki}x_{ij}x_{jn}x_{ni} && (10,24,28),\\
\textstyle\sum^{(4,5)}(x) &= \textstyle\sum x_{ij}x_{ji}x_{il}x_{li}x_{in}x_{ni} && (17,33),\\
\textstyle\sum^{(3,1)}(x) &= \textstyle\sum x_{ij}x_{ji}x_{ij}x_{ji}x_{in}x_{ni} && (19,22,34,35,39,40),\\
\textstyle\sum^{(3,2)}(x) &= \textstyle\sum x_{ij}x_{ji}x_{il}x_{lj}x_{jl}x_{li} && (21,27,37),\\
\textstyle\sum^{(3,3)}(x) &= \textstyle\sum x_{ij}x_{jk}x_{ki}x_{ij}x_{jk}x_{ki} && (29),\\
\textstyle\sum^{(2)}(x) &= \textstyle\sum x_{ij}x_{ij}x_{ij}x_{ij}x_{ij}x_{ij} && (41).
\end{aligned}$$

§5. **The case $p = 6$.** From the list of invariant sums with their multiplicities we generalize equation (7):

$$\begin{aligned}
(13)\qquad \|x\|_6^6 = \sum^{(6)}(x) + 6\sum^{(5,1)}(x) + 3\sum^{(5,2)}(x) \\
+ 6\sum^{(4,1)}(x) + 6\sum^{(4,2)}(x) + 3\sum^{(4,3)}(x) + 3\sum^{(4,4)}(x) + 2\sum^{(4,5)}(x) \\
+ 6\sum^{(3,1)}(x) + 3\sum^{(3,2)}(x) + \sum^{(3,3)}(x) + \sum^{(2)}(x).
\end{aligned}$$

Since $\sum^{(m,n)}(x')$ is the sum of monomials, each with m distinct entries, we apply Lemma 2 to obtain

$$(14)\qquad E\sum^{(m,n)}(x') = k^{-(m-1)}\sum^{(m,n)}(x).$$

Combining (13) and (14) we thus obtain

PROPOSITION 6.

$$\begin{aligned}
E\|x'\|_6^6 = k^{-5}\sum^{(6)}(x) + k^{-4}\left[6\sum^{(5,1)}(x) + 3\sum^{(5,2)}(x)\right] \\
+ k^{-3}\left[6\sum^{(4,1)}(x) + 6\sum^{(4,2)}(x) + 3\sum^{(4,3)}(x) + 3\sum^{(4,4)}(x) + 2\sum^{(4,5)}(x)\right] \\
+ k^{-2}\left[6\sum^{(3,1)}(x) + 3\sum^{(3,2)}(x) + \sum^{(3,3)}(x)\right] + k^{-1}\sum^{(2)}(x).
\end{aligned}$$

Next we pass, not without work, to the following estimate for $E\|x'\|_6^6$:

LEMMA 7.

$$\begin{aligned} E\|x'\|_6^6 &\\ &\le k^{-5}\|x\|_6^6 + 9k^{-4}(1-k^{-1})(\sum_i \|xe_i\|^6)^{1/3}\|x\|_6^4 \\ &\quad + 6k^{-3}(1-k^{-1})^2(\sum_i \|xe_i\|^6)^{1/2}\|x\|_6^3 \\ &\quad + k^{-1}(1+4k^{-1}-9k^{-2}+4k^{-4})\sum_i \|xe_i\|^6. \end{aligned}$$

SKETCH OF PROOF. The proof of this lemma requires several steps. First, we decompose the sums $\sum^{(6)}(x)$, $\sum^{(5,1)}(x), \dots, \sum^{(2)}(x)$ in the expansion of $E\|x'\|_6^6$ of Proposition 6 into the following 'ordinary sums' (i.e., sums where the indices range freely, not subject to the condition of being distinct).

$$\begin{aligned} (15) \quad E\|x'\|_6^6 = k^{-5}\|x\|_6^6 & \\ &+ 3k^{-4}(1-k^{-1})\sum_i ((x^3)_{ii})^2 \\ &+ 6k^{-4}(1-k^{-1})\sum_i \|xe_i\|^2\|x^2e_i\|^2 \\ &+ 6k^{-3}(1-k^{-1})^2\sum_{ij} |x_{ij}|^2 x_{ij}(x^3)_{ji} \\ &+ 3k^{-3}(1-k^{-1})^2\sum_{ij} (x_{ji})^2((x^2)_{ij})^2 \\ &+ 6k^{-3}(1-k^{-1})^2\sum_{ij} |x_{ij}|^2|(x^2)_{ij}|^2 \\ &+ 3k^{-3}(1-k^{-1})^2\sum_{ij} \|xe_i\|^2|x_{ij}|^2\|xe_j\|^2 \\ &+ 2k^{-3}(1-k^{-1})(1-2k^{-1})\sum_i \|xe_i\|^6 \\ &+ k^{-2}(1-k^{-1})^3\sum_{ijl} (x_{ij})^2(x_{jl})^2(x_{li})^2 \\ &+ 3k^{-2}(1-k^{-1})^3\sum_{ijl} |x_{ij}|^2|x_{il}|^2|x_{jl}|^2 \\ &+ k^{-1}(1-6k^{-1}+13k^{-2}+6k^{-3}-14k^{-4})\sum_{ij} |x_{ij}|^6 \\ &+ 6k^{-2}(1-4k^{-1}+2k^{-2}+k^{-3})\sum_{ij} |x_{ij}|^4\|xe_j\|^2. \end{aligned}$$

Two examples of the identities linking invariant sums and ordinary sums used to obtain (15) are

$$(16)\ \sum^{(3,1)}(x) = \sum x_{ij}x_{ji}x_{ij}x_{ji}x_{in}x_{ni} = \sum_{i,j,n\neq j} |x_{ij}|^4|x_{in}|^2$$
$$= \sum_{ij}\left(\sum_n |x_{ij}|^4|x_{in}|^2 - |x_{ij}|^6\right) = \sum_{ij}|x_{ij}|^4\|xe_i\|^2 - \sum_{ij}|x_{ij}|^6,$$

and the hardest identity:

$$(17)\ \sum^{(5,1)}(x) = \sum x_{ij}x_{ji}x_{il}x_{lm}x_{mn}x_{ni} = \sum |x_{ij}|^2 x_{il}x_{lm}x_{mn}x_{ni}$$
$$= \sum_i \|xe_i\|^2\|x^2e_i\|^2 - 2\sum_{ij}|x_{ij}|^2 x_{ij}(x^3)_{ji}$$
$$- \sum_{il}\|xe_i\|^2|x_{il}|^2\|xe_l\|^2 + 2\sum_{ij}|x_{ij}|^4\|xe_j\|^2 - \sum_i \|xe_i\|^6$$
$$+ \sum_{ij}|x_{ij}|^6 - \sum_{ij}|x_{ij}|^2|(x^2)_{ij}|^2 + \sum_{ijl}|x_{ij}|^2|x_{il}|^2|x_{jl}|^2.$$

Using (15) we can verify that the 'ordinary sums' are all real, but they are not necessarily positive. We majorize their absolute values in terms of $\|x\|_6^6$ and $\sum_i \|xe_i\|^6$ by applying the Hölder, Minkowski, (3), (4), and other inequalities. The following string of inequalities exemplifies our work:

(18)

$$0 \leq \sum^{(2)}(x) = \sum_{ij}|x_{ij}|^6 \leq \sum_{ij}|x_{ij}|^4\|xe_j\|^2 \leq \sum_{il}\|xe_i\|^2|x_{il}|^2\|xe_l\|^2$$
$$\leq \sum_{il}|x_{il}|^2(\|xe_i\|^4 + \|xe_l\|^4)/2 = \sum_i \|xe_i\|^6 \leq \sum_i \|xe_i\|^2\|x^2e_i\|^2$$
$$\leq \sum_i \|xe_i\|^2\|x^2e_i\|^2 = \sum_i \|xe_i\|^2(x^4e_i, e_i)$$
$$\leq \left(\sum_i \|xe_i\|^6\right)^{1/3}\left(\sum_i (x^4e_i, e_i)^{3/2}\right)^{2/3} \leq \left(\sum_i \|xe_i\|^6\right)^{1/3}\left(\sum_i (x^6e_i, e_i)\right)^{2/3}$$
$$= \left(\sum_i \|xe_i\|^6\right)^{1/3}\|x\|_6^4.$$

We collect all the necessary inequalities in (19):

(19a)
$$\left|\sum_i ((x^3)_{ii})^2\right| \le \sum_i \|xe_i\|^2\|x^2e_i\|^2 \le \left(\sum_i \|xe_i\|^6\right)^{1/3}\|x\|_6^4$$
(19b)
$$\left|\sum_{ij} |x_{ij}|^2 x_{ij}(x^3)_{ji}\right| \le \left(\sum_i \|xe_i\|^6\right)^{1/2}\|x\|_6^3$$
(19c)
$$\left|\sum_{ij} (x_{ji})^2((x^2)_{ij})^2\right| \le \sum_{ij} |x_{ij}|^2|(x^2)_{ij}|^2 \le \sum_{ij} \|xe_i\|^2|x_{ij}|^2\|xe_j\|^2 \le \sum_i \|xe_i\|^6$$
(19d)
$$\left|\sum_{ijl} (x_{ij})^2(x_{jl})^2(x_{li})^2\right| \le \sum_{ijl} |x_{ij}|^2|x_{il}|^2|x_{jl}|^2 \le \sum_i \|xe_i\|^6$$
(19e)
$$\sum_{ij} |x_{ij}|^6 \le \sum_{ij} |x_{ij}|^4\|xe_j\|^2 \le \sum_i \|xe_i\|^6.$$

Notice that in the expansion (15) of $E\|x'\|_6^6$ all but the last polynomial in k^{-1} are non-negative for all $k \ge 2$. Furthermore, by (19e) the sum of the last two terms in (15) is majorized by

$$k^{-1}(1 - 11k^{-2} + 18k^{-3} - 8k^{-4})\sum_i \|xe_i\|^6,$$

where this polynomial in k^{-1} is now non-negative for all $k \ge 2$. Thus, by passing when necessary to the absolute values of the other 'ordinary sums', and by using all the inequalities in (19) we obtain the inequality in Lemma 7.

By using (4) in Lemma 7 we can furthermore obtain (for $k \ge 2$)

$$E\|x'\|_6^6 \le k^{-1}(1 + 4k^{-1} - 3k^{-2} - 3k^{-3} + 2k^{-4})\|x\|_6^6,$$

and thus the minimum principle gives us an upper bound for $\min \|x'\|_6$.

As in the $p = 4$ case, we can improve this purely probabilistic estimate by also using the combinatorial method (2). Therefore, assuming that k is a perfect square (and so $k \ge 4$), we choose the (possibly unique) paving x' of x into $k^{1/2}$ colors given by (2) such that $\|x'e_i\| \le k^{-1/4}\|xe_i\|$ for all i. Then we average all the pavings x'' of x' into a further $k^{1/2}$ colors to obtain, by

Lemma 7, that

$$(20)\quad E\|x''\|_6^6 \le k^{-5/2}\|x'\|_6^6 + 9k^{-2}(1-k^{-1/2})\left(\sum_i \|x'e_i\|^6\right)^{1/3}\|x'\|_6^4 + 6k^{-3/2}(1-k^{-1/2})^2\left(\sum_i \|x'e_i\|^6\right)^{1/2}\|x'\|_6^3 + k^{-1/2}(1+4k^{-1/2}-9k^{-1}+4k^{-2})\left(\sum_i \|x'e_i\|^6\right).$$

Now we use (4) and the facts that $\|x'\|_6 \le \|x\|_6$, $\|x'e_i\| \le k^{-1/4}\|xe_i\|$ to obtain

$$\sum_i \|x'e_i\|^6 \le k^{-3/2}\sum_i \|xe_i\|^6 \le k^{-3/2}\|x\|_6^6,$$

and then

$$(21)\quad E\|x''\|_6^6 \le k^{-2}(1+6k^{-1/4}+14k^{-1/2}-12k^{-3/4}-18k^{-1}+6k^{-5/4}+4k^{-2})\|x\|_6^6.$$

A simple computation shows that the quantity enclosed in the parentheses has maximum (attained for $k=9$) $(301+72\sqrt{3})/81 \le 5.3$ and tends asymptotically to 1 for $k \to \infty$. Since every $k^{1/2}$-paving x'' of x' is also a k-paving of x, from (21) and the minimum principle we obtain (for k a perfect square) Theorem 8.

THEOREM 8. *There is a k-paving x' such that* $\|x'\|_6 \le 5.3^{1/6}k^{-1/3}\|x\|_6$.

§**6. Hadamard matrices.** Let h denote a selfadjoint $n \times n$ Hadamard matrix (a unitary operator with entries $\pm n^{-1/2}$) with its diagonal removed, i.e., with $D(h) = 0$. Then for $p \ge 2$, $\|h\|_p \approx n^{1/p}$ (asymptotically, as $n \to \infty$). In [**3**, Proposition 4.2] we also found that for every k-paving h' of h and for $p \ge 2$ we have

$$(22)\quad \|h'\|_p \ge \frac{k^{-1/2}-n^{-1/2}}{1+n^{-1/2}}\|h\|_p.$$

To obtain an upper bound for $\min \|h'\|_p$ we compute (asymptotically as $n \to \infty$) the coefficients of the powers of k^{-1} in the expansion of $E\|h'\|_4^4$ in Proposition 3, namely

$$\|h\|_4^4 \approx n, \quad \sum^{(3)}(h) \approx \sum_i \|he_i\|^4 \approx n, \quad \sum^{(2)}(h) \approx 1,$$

and hence

$$\sum^{(4)}(h) = \|h\|_4^4 - 2\sum^{(3)}(h) - \sum^{(2)}(h) \approx -n.$$

Therefore,

$$(23)\quad E\|h'\|_4^4 \approx (-k^{-3}+2k^{-2})\|h\|_4^4 \le 2k^{-2}\|h\|_4^4.$$

It is essentially through this reasoning that we obtained in [3, Proposition 4.6, and Corollary 4.5] that there is a k-paving h' such that

$$\|h'\|_4 \leq [2^{1/4}k^{-1/2} - \varepsilon(n,k)]\|h\|_4 \tag{24}$$

where for each k, $\varepsilon(n,k) \to 0$ as $n \to \infty$.

Likewise for $p = 6$, most of the invariant sums in (13) are negligible for n large:

$$\sum^{(5,1)}(h) \approx -n, \quad \sum^{(4,3)}(h) \approx n, \quad \sum^{(4,5)}(h) \approx n,$$

and the rest of the invariant sums are $o(n)$. Hence, using equation (13), we also have $\sum^{(6)}(h) \approx 2n$. Therefore, by Proposition 6,

$$\begin{aligned} E\|h'\|_6^6 &\approx 2nk^{-5} - 6nk^{-4} + 5nk^{-3} \\ &\approx k^{-3}(5 - 6k^{-1} + 2k^{-2})n \leq 5k^{-3}n. \end{aligned} \tag{25}$$

Since $\|h\|_6^6 \approx n$, we have proven Theorem 9.

THEOREM 9. *There is a k-paving h' such that*

$$\|h'\|_6 \leq (5^{1/6}k^{-1/2} - \varepsilon(n,k))\|h\|_6$$

where for each k, $\varepsilon(n,k) \to 0$ as $n \to \infty$.

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