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A CHARACTERIZATION AND SUM DECOMPOSITION FOR OPERATOR IDEALS

BY

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ABSTRACT. Let $L(H)$ be the ring of bounded operators on a separable Hilbert space. Assuming the continuum hypothesis, we prove that in $L(H)$ every two-sided ideal that contains an operator of infinite rank is the sum of two smaller two-sided ideals. The proof involves a new combinatorial description of ideals of $L(H)$. This description is also used to deduce some related results about decompositions of ideals. Finally, we discuss the possibility of proving our main theorem under weaker assumptions than the continuum hypothesis and the impossibility of proving it without the axiom of choice.

1. Introduction and notational conventions. Let H be a separable, infinite-dimensional, complex Hilbert space, and let $L(H)$ be the ring of bounded linear operators on H. Assuming the continuum hypothesis, we shall prove that every two-sided ideal I of $L(H)$ that properly includes the ideal of finite-rank operators can be decomposed as the sum $I = J_1 + J_2$ of two-sided ideals J_i properly included in I. The question whether such a decomposition exists for the ideal $K(H)$ of compact operators was raised by Brown, Pearcy and Salinas [**1**]. We shall also show that the decomposition of $K(H)$ is necessarily nonconstructive, in the sense that there need not be any *definable* ideals $\subsetneq K(H)$ whose sum is $K(H)$.

Our proof is in two parts. The first part develops a new characterization of ideals in terms of sequences of natural numbers and thereby reduces the problem to a combinatorial one. The second part uses the continuum hypothesis to solve this combinatorial problem. The two parts are presented in §§2 and 3, respectively. §4 contains some additional comments.

Before starting the proof, we adopt some notational conventions. By the sum of two sequences of real numbers, we mean the sequence obtained by adding corresponding terms; similarly, one sequence is $\leqslant$ another if every

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term of the former is $\leqslant$ the corresponding term of the latter. The sum of two sets consists of all possible sums of two members, one from each set. (We have already used this convention in writing $J_1 + J_2$ above.) A subset A of the set $\mathbf{N}$ of natural numbers is said to be almost included in another such set B if $A - B$ is finite; a sequence of such sets is said to be almost decreasing if each term is almost included in every earlier term. By an ideal we always mean a two-sided proper ideal of $L(H)$. We write $K(H)$ and $F(H)$ for the ideals of compact operators and of finite-rank operators, respectively. ($K(H)$ is the largest ideal, and $F(H)$ is the smallest nonzero ideal.) Finally i is a variable taking only the values 1 and 2. With these conventions, our main theorem takes the following form.

THEOREM 1. *Assume the continuum hypothesis. For every ideal* $I \underset{\neq}{\supset} F(H)$, *there exist two ideals* $J_i \underset{\neq}{\subset} I$ *such that* $J_1 + J_2 = I$.

2. The characterization of ideals. We begin by recalling the description of ideals given by von Neumann and Calkin [2]. To each ideal I, associate the Calkin ideal set Calk(I), consisting of all the sequences obtainable by taking any nonnegative selfadjoint operator in I and listing (in any order) its eigenvalues, repeating multiple eigenvalues according to their multiplicities. Since $I \subseteq K(H)$, Calk(I) is a subclass of the class c_0^+ of sequences of nonnegative real numbers converging to zero. Furthermore, Calk(I) is easily seen to be closed under the relations **permute, decrease**, and **add** defined as follows:

Permute: A sequence x yields a sequence y by **permute** if the same positive numbers occur in x and y with the same multiplicites.

Decrease: x yields y by **decrease** if $y \leqslant x$.

Add: x and y yield $x + y$ by **add**.

It is shown in [2] that every nonempty subset of c_0^+ that is closed under **permute, decrease**, and **add** is Calk(I) for a unique ideal I. An easy corollary, which we shall need later, is that the lattice $\mathcal{I}$ of ideals is distributive. (Our formulation differs from Calkin's in that **permute** allows insertion or deletion of zeros. Calkin shows that this is redundant, but it will be convenient for our purposes.) In this characterization of Calkin ideal sets, **add** can be replaced by the following two relations.

Double: x yields y by **double** if $y = 2x$, i.e., $y_n = 2x_n$ for all n.

Mesh: x and y yield z by **mesh** if the multiplicity of occurence in z of each real number is the sum of its multiplicities in x and y.

Indeed, **double** is obviously a special case of **add**, while **mesh** is obtainable from **permute** and **add** (use **permute** to insert infinitely many zeros). Conversely, the sum of two sequences is obtainable by **double** and **decrease** from their (term-by-term) maximum, which is a subsequence of a sequence given by **mesh**. One can extract the subsequence by using **decrease** to replace the unwanted terms by zeros and then using **permute** to remove these zeros. Thus, the Calkin ideal sets are precisely the nonempty subsets of c_0^+ closed under **permute**, **decrease**, **double**, and **mesh**.

Let b be the subset of c_0^+ consisting of sequences all of whose nonzero terms are of the form $1/2^n$ with $n \in \mathbf{N}$. For each $x \in c_0^+$ there exists $y \in b$ such that $y_n \leqslant x_n \leqslant 2y_n$ for all but finitely many n (the exceptional values of n are those for which $x_n > 2$). Then any nonzero Calkin ideal set contains x if and only if it contains y. Thus, a Calkin ideal set is uniquely determined by its intersection with b. Furthermore, a nonempty subset of b is $b \cap \text{Calk}(I)$ for some ideal I if and only if it is closed under **permute**, **decrease** (restricted to b), **mesh**, and a modified form of **double** in which only terms < 1 are doubled while ones are left unchanged (so that the result still lies in b).

A sequence $x \in b$ can be specified uniquely, modulo **permute**, by giving the multiplicity of each power of $\frac{1}{2}$ in x. It is more convenient for our purposes, however, to use cumulative multiplicities; we define $x^*(n)$ to be the total number of occurrences of $1, 1/2, \ldots, 1/2^n$ in x. Thus, for each $x \in b$, x^* belongs to the set s of nondecreasing functions from $\mathbf{N}$ to $\mathbf{N}$. An ideal I is completely determined by $b \cap \text{Calk}(I)$, which is in turn completely determined by the subset

$$I^* = \{x^* | x \in b \cap \text{Calk}(I)\}$$

of s. By translating the characterization of sets of the form $b \cap \text{Calk}(I)$, we find that sets of the form I^* are characterized, among all subsets of s, by closure under **decrease** (restricted to s), **add**, and **shift**, where we define

Shift: f yields g by **shift** if $g(n) = f(n + 1)$ for all n.

We denote the lattice of these sets by $\mathbb{S}$.

It is obvious that the correspondences between ideals, their Calkin ideal sets, their intersections with b, and finally their $*$-transforms all preserve inclusion. Thus, we have the first part of the following result.

THEOREM 2. *The correspondence $I \mapsto I^*$ is an isomorphism from the lattice $\mathcal{I}$ of ideals to the lattice $\mathbb{S}$ of subsets of s closed under* **decrease**, **add**, *and* **shift**. *In both of these lattices, meet is set-theoretic intersection and join is sum.*

The proof of the last sentence of the theorem is trivial except, possibly, the verification that, if two subsets of s are closed under **decrease**, then so is their sum. This amounts to checking that if $f \leqslant g_1 + g_2$ then $f = h_1 + h_2$ for some $h_i \leqslant g_i$ (where f, g_i, h_i all belong to s); the required functions h_i are easily defined by induction. Once $h_i(n)$ have been defined for a particular n, the requirements for $h_i(n + 1)$ are $h_i(n) \leqslant h_i(n + 1) \leqslant g_i(n + 1)$ and $h_1(n + 1) + h_2(n + 1) = f(n + 1)$; these requirements can be satisfied because $h_1(n) + h_2(n) = f(n) \leqslant f(n + 1) \leqslant g_1(n + 1) + g_2(n + 1)$.

Theorem 2 immediately implies the following corollary, which reduces the proof of Theorem 1 to a combinatorial problem about s.

COROLLARY. *An ideal I is the sum of two ideals $J_i \subsetneq I$ if and only if I^* is the sum of two sets $S_i \subsetneq I^*$ in $\mathbb{S}$.*

For future reference, we record here that the ideals $F(H)$ and $K(H)$ correspond to $F(H)^*$, consisting of all bounded functions in s, and $K(H)^* = s$.

The first part of the proof of Theorem 1, introducing the combinatorial characterization of ideals, is complete. Before proceeding with the second part (§3), we present some other results obtainable by combining Theorem 2 with some classical facts about s. The facts we have in mind are contained in the following lemma, which is due to Hausdorff [3].

LEMMA. *Let f_k $(k = 1, 2, \ldots)$ be countably many functions in s. Then there exists $f \in s$ such that, for each k, the inequality $f_k(n) \leqslant f(n)$ holds for all but finitely many n. Furthermore, if each f_k is unbounded, then there exists an unbounded $g \in s$ such that, for each k, $g(n) \leqslant f_k(n)$ holds for all but finitely many n.*

PROOF. Set $f(n) = \max\{f_k(n) | k \leqslant n\}$. Set $g(n) = \max\{l|$ for all $k \leqslant l$, $f_k(n) \geqslant l\}$.

The following corollary slightly strengthens a result of Salinas [8].

COROLLARY. *$K(H)$ is not the union of countably many ideals $I_k \subsetneq K(H)$.*

PROOF. For each k, let $f_k \in s - I_k^*$. By the lemma, let $f \in s$ be such that $f_k \leqslant f$ almost everywhere (i.e., for all but finitely many values). Then f is in no I_k^*, and it follows that a nonnegative selfadjoint compact operator with eigenvalues $1/2^{n+1}$ with multiplicity $f(n+1) - f(n)$ is in no I_k.

The next corollary shows that the decomposition in Theorem 1 cannot be very simple.

COROLLARY. *If $K(H) = I_1 + I_2$ with $I_i \subsetneq K(H)$, then neither of the ideals I_i is countably generated.*

PROOF. Suppose I_1 were countably generated. Then I_1^* would also be countably generated, say by $f_1, f_2, \ldots$. We may suppose, without loss of generality, that all of the functions obtainable from the f_k's by **add** and **shift** are already among the f_k's. Thus

$$s = K(H)^* = I_1^* + I_2^*$$
$$= \{ g \in s | \text{ for some } h \in I_2^* \text{ and some } k, g \leqslant f_k + h\}.$$

(It is easy to see the last equality by checking that the right side is in $\mathbb{S}$ and recalling that, in $\mathbb{S}$, $+$ is join.)

By the lemma, let $f \in s$ be such that $f_k \leqslant f$ almost everywhere. For every function $g \in s$, we have $f + g \in s$, so, for some $h \in I_2^*$ and some k, $f + g \leqslant f_k + h$. The choice of f then implies that $g \leqslant h$ almost everywhere, so $g \in I_2^*$. But g was an arbitrary element of s, so $s = I_2^*$ and therefore $K(H) = I_2$, a contradiction.

3. The combinatorial construction. We now return to the proof of Theorem 1. Our goal is to find two sets $S_i \in \mathbb{S}$ whose sum is a prescribed $S \in \mathbb{S}$, with $S \supsetneq F(H)^*$, i.e., with S containing some unbounded function. From here on, we assume the continuum hypothesis.

The first (and main) step in the construction of the S_i is to construct (quite independently of the prescribed S) certain almost decreasing sequences of subsets of $\mathbf{N}$. We begin with the observation that, if (B_ξ) is an almost decreasing sequence of infinite sets, indexed by a countable well-ordered set, then there is an infinite set B almost included in all the B_ξ's. Indeed, this is trivial if the given sequence is empty (use $\mathbf{N}$ as B) or has a last term (use that term as B). Otherwise, we can extract a cofinal subsequence that can be re-indexed by natural numbers, so we need only consider the case where the given sequence is (B_n) indexed by $\mathbf{N}$. In this case, we define B to consist of one element from B_0, a different element from $B_0 \cap B_1$, yet another element from $B_0 \cap B_1 \cap B_2$, etc. Such elements can be found because $B_0 \cap \cdots \cap B_n$ almost includes B_n and is therefore infinite. It is obvious that B is almost included in each B_n.

Having made this preliminary observation (which, like the lemma above, goes back to [3]), we now proceed to construct two almost decreasing sequences, (B_α^1) and (B_α^2), of infinite subsets of $\mathbf{N}$, each indexed by the set of all countable ordinal numbers, and having the following property. For any $f \in s$, there is an index α such that, for all $n \in \mathbf{N}$,

$$\max\left\{\operatorname{next}(B_\alpha^1, n), \operatorname{next}(B_\alpha^2, n)\right\} \geqslant f(n), \tag{$*$}$$

where we use the notation $\operatorname{next}(B, n)$ for the smallest number in B that is $\geqslant n$.

Since the set s has the cardinality of the continuum and the set of countable ordinal numbers has cardinality $\aleph_1$, the continuum hypothesis asserts that there is a bijection between these two sets. Fix such a bijection. We shall define the sets B_α^i by transfinite induction on α, in such a way that $(*)$ holds whenever f and α correspond in our fixed bijection. For any countable ordinal number α, the αth step of the induction is as follows. Suppose B_ξ^i has been defined for all $\xi < \alpha$ (and for both values of i), and suppose f is the function in s that corresponds to α. We must find two infinite sets, B_α^1 and B_α^2, such that, first, $(*)$ holds, and, second, B_α^i is almost included in B_ξ^i for all $\xi < \alpha$. By our preliminary observation, we can satisfy the second requirement; there are infinite sets A^i almost included in B^i_ξ for all $\xi < \alpha$. By taking the B^i_α to be infinite subsets of A^i, we automatically satisfy the second requirement, so we direct our efforts toward the first, making (*) hold. We define an increasing sequence

$$a_0^1 < a_0^2 < a_1^1 < a_1^2 < a_2^1 < a_2^2 < \dots$$

of natural numbers, with $a_k^i \in A^i$, such that $a_k^2 > f(n)$ for all $n \leqslant a_k^1$ and such that $a_{k+1}^1 > f(n)$ for all $n \leqslant a_k^2$. Since the A^i's are infinite, there is no difficulty in inductively choosing the a_k^i's. We set $B^i_\alpha = \{a_k^i | k \in \mathbf{N}\}$; clearly this is an infinite subset of A^i. To verify (*), consider any $n \in \mathbf{N}$. If $a_k^1 < n \leqslant a_k^2$ for some k, then $\operatorname{next}(B^1_\alpha, n) = a_{k+1}^1 > f(n)$ by our choice of a_{k+1}^1. Otherwise, $a_{k-1}^2 < n \leqslant a_k^1$ for some k (or simply $n \leqslant a_0^1$ if $k = 0$), and $\operatorname{next}(B^2_\alpha, n) = a_k^2 > f(n)$ by our choice of a_k^2. In either case, (*) holds. This completes the construction of the sequences (B^i_α).

Temporarily fix an arbitrary unbounded function $h \in s$. (Eventually, we will select a particular h, depending on the prescribed $S \in \mathbb{S}$ that we want to decompose, but for the moment the choice of h is immaterial. Some of what follows may seem more natural if the reader pretends that h is the identity function.) We say that a function $f \in s$ is linearly bounded on a set $B \subseteq \mathbf{N}$ if there are constants a and b such that $f(n) \leqslant ah(n) + b$ for all $n \in B$. Let S_i' ($i = 1$ or 2) be the set of all $f \in s$ that are linearly bounded on at least one of the sets B^i_α defined above. Note that such an f is also linearly bounded on all later terms $B^i_\beta (\beta > \alpha)$ of the sequence, as these are almost included in B^i_α. It follows that S_i' is closed under **add** as well as **decrease**. It might not, however, be closed under **shift**. We define S_i to be the closure of S_i' under **shift**; this is obviously closed under **decrease** and is easily seen to be closed under **add** as well. (The verification uses the fact that every $f \in s$ is nondecreasing, so its n-fold shift is $\leqslant$ its m-fold shift if $n \leqslant m$.) Thus, S_1 and S_2 belong to $\mathbb{S}$.

We show next that $S_1 + S_2 = s$. Consider an arbitrary $g \in s$. Since h is unbounded, we can find $f \in s$ such that $h(f(n)) \geqslant g(n)$ for all n. By construction of the sequences (B^i_α), we can find an α such that (*) holds for α and f. Define

$$g_i(n) = h\big(\operatorname{next}(B^i_\alpha, n)\big).$$

These two functions g_i belong to the corresponding S_i, for g_i is linearly bounded on B^i_α (being equal to h there). And, because of (*) and the monotonicity of h,

$$\begin{aligned} g_1(n) + g_2(n) &\geqslant \max\{g_1(n), g_2(n)\} \\ &= h\big(\max\{\operatorname{next}(B^1_\alpha, n), \operatorname{next}(B^2_\alpha, n)\}\big) \\ &\geqslant h(f(n)) \geqslant g(n). \end{aligned}$$

Therefore, $g \in S_1 + S_2$.

For any prescribed $S \in \mathbb{S}$, we have, by distributivity of $\mathbb{S}$,

$$(S \cap S_1) + (S \cap S_2) = S \cap (S_1 + S_2) = S \cap s = S.$$

To complete the proof of Theorem 1, we must still show that, if S contains an unbounded function g, then h can be so chosen that neither S_i includes S (so $S \cap S_i \subsetneq S$). In fact, we shall find h such that g belongs to neither S_i, because g and all the functions from which g is obtainable by **shift** will be linearly bounded only on finite sets. Thus, we seek an unbounded $h \in s$ such that, for any a, b, c, the inequality

$$g(n - c) \leqslant ah(n) + b$$

holds for only finitely many values of n. In other words, we need an unbounded $h \in s$ such that, for each a, b, c,

$$h(n) \leqslant [\, g(n - c)/a + b/a]$$

for all but finitely many n. But such an h exists by the second part of Hausdorff's Lemma above, so the proof of Theorem 1 is complete.

4. Additional remarks.

4.1. Theorem 1 can also be proved by directly constructing the sets $S_i \subsetneq S$ such that $S_1 + S_2 = S$. The construction is an inductive one, of length $\aleph_1$. At each step of the construction, new functions are put into S_1 and S_2 to guarantee that some $f \in S$ belongs to $S_1 + S_2$. The continuum hypothesis is needed to insure that each $f \in S$ can be considered at some stage. The construction is so arranged that a particular unbounded $g \in S$ is never put into either S_i, so $S_i \subsetneq S$. The appropriate functions to put into S_i at each stage are given by the following "splitting lemma".

LEMMA. *Let S_1 and S_2 be countably generated members of $\mathbb{S}$, let $g_i \notin S_i$, $g_i \in s$, and let $f \in s$ be arbitrary. Then f can be written as $f_1 + f_2$, where $f_i \in s$, and where the sets $S_i' \in \mathbb{S}$ generated by $S_i \cup \{f_i\}$ do not contain the corresponding g_i.*

This lemma is perhaps most easily proved by means of a Baire category argument which establishes that "most" decompositions $f_1 + f_2$ of f (with $f_i \in s$) have the required property. A similar lemma can be proved (with somewhat more work) for the characteristic sets introduced by Salinas [7], and Theorem 1 can also be obtained in this way.

4.2. Theorem 2 can be rather neatly reformulated as follows. Define a binary relation $\leqslant$ on s by

> $f \leqslant g$ iff there exist natural numbers a and b such that $f(n) \leqslant ag(n + b)$ for all n.

This relation is obviously reflexive and transitive. We write $[f]$ for the equivalence class of f with respect to the associated equivalence relation and l for the set of equivalence classes. With the partial ordering induced by $\preccurlyeq$, l is a distributive lattice in which

$$[f] \wedge [g] = [\min(f, g)] \quad \text{and}$$
$$[f] \vee [g] = [\max(f, g)] = [f + g].$$

It is easy to verify that a subset S of s belongs to $\mathbb{S}$ if and only if (1) it is closed under **add** and (2) $f \preccurlyeq g \in S$ implies $f \in S$. Therefore, the sets $S \in \mathbb{S}$ correspond, via the projection $s \to l$, to the ideals of l. Theorem 2 thus asserts that the lattice of ideals (of the ring $L(H)$) is isomorphic to the lattice of ideals of the lattice l.

4.3. An obvious question about Theorem 1 is whether the continuum hypothesis is really needed. We have been unable to eliminate it entirely from the argument, but we can get by with certain weaker hypotheses. Readers familiar with Martin's axiom **[6]** will have no difficulty carrying out essentially the same argument (or the argument outlined in 4.1) on the basis of that assumption. The weakest hypothesis from which we have deduced the conclusion of Theorem 1 is

> There exist two nonprincipal ultrafilters D_i on $\mathbf{N}$ such that $f(D_1) \neq f(D_2)$ for all finite-to-one functions $f\colon \mathbf{N} \to \mathbf{N}$. (**)

The proof involves showing that two such ultrafilters have the property (*) required of the sets $\{B_\alpha^i | \alpha \text{ countable}\}$ in §3.

The hypothesis (**) is true if, for example, there exist two nonisomorphic rare ultrafilters (take these as the D_i) and also if there exists a rare non-Ramsey ultrafilter (take it as D_1 and any proper image of it as D_2). It is perhaps worth noting in this connection that Kunen's model **[5]** (generated by $\aleph_2$ random reals over a model of the continuum hypothesis), in which there are no Ramsey ultrafilters, nevertheless satisfies (**) because it has plenty of rare ultrafilters (measure algebra extensions preserve rarity). It seems possible that one could prove (**) (without any special assumptions), perhaps by an adaptation of Kunen's techniques in **[4]**, but we have been unable to do this.

4.4. Although the continuum hypothesis might be eliminable from Theorem 1, the axiom of choice is not. Furthermore, even if the axiom of choice and the (generalized) continuum hypothesis are assumed, there might not exist any *definable* ideals $I_i \subsetneq K(H)$ such that $I_1 + I_2 = K(H)$, not even if arbitrary real numbers and arbitrary ordinal numbers are allowed as parameters in the definitions of the I_i. Thus, Theorem 1 itself, not merely its present proof, is highly nonconstructive.

To prove these assertions, we begin by topologizing the set s as a subspace of the product of countably many copies of the discrete space N. More explicitly, for every finite nondecreasing sequence p of natural numbers, we define

$$[p] = \{f \in s \mid p \text{ is an initial segment of } f\},$$

and we use the sets $[p]$ as a basis for the topology. (We reserve the letters p, q, and r to stand for such sequences.) The space s so defined admits a complete metric ($d(f, g) = 1/(n + 1)$ if n is the smallest integer for which $f(n) \neq g(n)$) and therefore satisfies the Baire category theorem.

We shall need the following results, which are proved in **[9]** (under the assumption that the existence of an inaccessible cardinal is consistent). There is a model of set theory without the axiom of choice, in which the axiom of dependent choice holds, and in which every subset of s has the Baire property (i.e. differs from an open set by a meager set). There is another model of set theory, in which the axiom of choice and the generalized continuum hypothesis hold, and in which all definable subsets of s have the Baire property, even if real numbers and ordinal numbers occur as parameters in the definition. The axiom of dependent choice, mentioned in the first of these results, is a weak form of choice that suffices for the proofs of "nonpathological" results of real analysis, like the Baire category theorem and our Theorem 2, but not for the construction of "pathological" examples, like nonmeasurable sets or sets lacking the Baire property.

In view of these facts, the following proposition suffices to establish our claims about the nonconstructivity of Theorem 1.

PROPOSITION. *If $s = S_1 + S_2$ with $S_i \in \mathfrak{S}$, and $S_i \subsetneq s$, then at least one of the S_i lacks the Baire property.*

The proof of this proposition involves two lemmas about comeager subsets of s, so we begin by discussing such sets. By definition of the topology, every open set in s has the form

$$[D] = \bigcup_{p \in D} [p]$$

for some collection D of nondecreasing finite sequences p. $[D]$ is dense in s if and only if every q has an extension $p \in D$. More generally, $[D] \cap [r]$ is dense in the basic open set $[r]$ if and only if every extension q of r has an extension $p \in D$; in this case, we say that D is dense beyond r. A set $C \subseteq s$ is comeager in $[r]$ if and only if it includes the intersection of countably many sets $[D_n]$ where each D_n is dense beyond r.

LEMMA A. *If C is comeager in a nonempty open subset of s, then every function $f \in s$ is eventually majorized by the sum of two elements of C.*

PROOF. Without loss of generality, the nonempty open set in question is $[r]$ and C is $\bigcap_n [D_n]$, where each D_n is dense beyond r. Let an $f \in s$ be given. We shall construct $g_i \in C$ such that $f(k) \leqslant g_1(k) + g_2(k)$ for all but finitely many k. The construction will proceed in stages; after stage n, we shall have defined finite nondecreasing sequences g_1^n and g_2^n, of equal length, which are to be initial segments of the desired g_1 and g_2. We begin by setting $g_1^0 = g_2^0 = r$. At an odd-numbered stage, say $2n + 1$, we properly extend g_1^{2n} to a sequence $g_1^{2n+1} \in D_n$. This is possible, as D_n is dense beyond r and g_1^{2n} extends r. We also extend g_2^{2n} to g_2^{2n+1}, of the same length as g_1^{2n+1}, using values so large that $g_1 + g_2$ will majorize f on domain $(g_1^{2n+1}) - \text{domain}(g_1^{2n})$. (For example, the new values of g_2 could be taken to agree with f.) Symmetrically, at even-numbered stages, $2n + 2$, extend g_2^{2n+1} to $g_2^{2n+2} \in D_n$ and extend g_1^{2n+1} so that f is majorized. The functions g_1 and g_2 thus obtained are in C, for stage $2n + i$ guarantees that $g_i \in [D_n]$. And $g_1(k) + g_2(k) \geqslant f(k)$ for all k greater than the length of r. □

LEMMA B. *Let C be comeager in s. There exists an $f \in S$ such that, in any decomposition $f = h_1 + h_2$ with $h_1, h_2 \in s$, one of the summands h_i majorizes an element of C.*

PROOF. Without loss of generality, $C = \bigcap_n [D_n]$ with D_n dense (beyond the empty sequence). If $h \in s$ and p is a finite nondecreasing sequence, we say that h is (p, n)-small if h majorizes p but does not majorize any extension of p in D_n. (To be precise, "h majorizes p" means that the finite initial segment of h having the same length as p majorizes p.) If h is not (p, n)-small for any p and n, then we can construct $g \in C$ with $g \leqslant h$ by inductivity defining initial segments $g_n \in D_n$ majorized by h. The existence of g_{n+1} follows trivially from the facts that h majorizes g_n and h is not $(g_n, n + 1)$-small. Thus, it suffices to find an $f \in s$ such that, for any quadruple (p_1, n_1, p_2, n_2),

f has no decomposition $f = h_1 + h_2$ in which each h_i is (p_i, n_i)-small. (†)

By the Baire category theorem, it suffices to show that, for each fixed (p_1, n_1, p_2, n_2), the set of f satisfying (†) has dense interior, that is, that every finite nondecreasing sequence q has an extension q' such that every f extending q' satisfies (†).

Let p_i, n_i, and q be given. We may assume that q and both of the p_i have the same length l; otherwise, just extend the shorter ones by repeating their last terms. By density, we can find extensions $r_i \in D_{n_i}$ of p_i. Let a_i be the largest (i.e. last) term in r_i. Define q' to be the extension of q, of length $l + 1$, having $a_1 + a_2$ as its last term $q'(l)$. If f extends q' and $f = h_1 + h_2$, then either $h_1(l) \geqslant a_1$ or $h_2(l) \geqslant a_2$. As the h_i are nondecreasing, it follows, by the choice of a_i, that, for at least one of the two values of i, h_i majorizes r_i if it majorizes p_i. So, for at least one i, h_i is not (p_i, n_i)-small. □

We now complete the proof of the proposition. Suppose $s = S_1 + S_2$ with

$S_i \in \mathbb{S}$, $S_i \subsetneq s$. We remark that neither S_i is $\{0\}$, so any f that is majorized almost everywhere by an element of S_i must itself be in S_i. This remark, the closure of S_i under **add**, the assumption that $S_i \subsetneq s$, and Lemma A immediately imply that neither S_i is comeager in any nonempty open set. On the other hand, if both S_i were meager, we could apply Lemma B to the complement of their union. Remembering that the S_i are closed under **decrease**, we would obtain an $f \in s$ such that, in any decomposition of f, one of the summands is in neither of the S_i. This would contradict $f \in s = S_1 + S_2$.

Thus, at least one of the S_i is neither comeager in a nonempty open set nor meager. It therefore lacks the Baire property. □

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